A NOTE ON A PAPER OF CELLINA CONCERNING DIFFERENTIAL EQUATIONS IN BANACH SPACES

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The investigation of differential equations in Banach spaces with a right-hand side being more general than Lipschitz continuous, compact, dissipative, or a sum of the before-mentioned types, started more than 20 years ago with Cellina's paper [1]. There were considered so-called α-dissipative right-hand sides, and the corresponding theorem claims the existence of a solution in the case that X^* is uniformly convex and some natural conditions are satisfied.

In the present paper we point out a mistake in this proof which seems not having been noted up to now. Then we shall sketch the proof of an existence theorem under the additional assumption that the right-hand side is uniformly continuous; this turns out to be necessary on proof-technical reasons. Finally we shall comment on the strong assumptions. [1]

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In what follows, X will denote an infinite-dimensional real Banach space with norm $|\cdot|$ and uniformly convex dual space X^*; this implies that the semi-inner product $(x,y)=(\mathcal{F}(y))(x)$ is uniformly continuous on bounded subsets of $X\times X$. Here $\mathcal{F}:X\to X^*$ is the duality map of X. Moreover, $B_b(x_0)$ will be the open ball with radius $b>0$, center $x_0\in X$ and closure $\overline{B}_b(x_0)$. Furthermore, α denotes Kuratowski's measure of noncompactness in X. All properties of $(\cdot,\cdot)$ and α which will be needed in the sequel may be found in Paragraphs 13.1 and 7.3 of [2].

Next, let us recall from [1] that for fixed $a,b>0$ and $x_0\in X$, a map $f:R=[0,a]\times\overline{B}_b(x_0)\to X$ is said to be α-dissipative if, given $\varepsilon>0$, there is a finite covering $R=\bigcup_{k=0}^{K}O^k$ of R such that

$$(f(t',y')-f(t'',y''),y'-y'')\leq\varepsilon \tag{1}$$

whenever (t',y') and (t'',y'') are in the same O^k.

Now the theorem on p.294 of [1] claimes the existence of a solution to the initial value problem $x'=f(t,x),x(0)=x_0$, on $[0,\min\{a,b/(4M)\}]$ in the case that $f:R\to X$ is continuous, α-dissipative, and $M=\sup_{(t,x)\in R}|f(t,x)|<\infty$.

However, the proof of this theorem is not correct as it stands: On p.295 the diameter of

$$\Xi^{rs}=\{x+\delta z : x\in V^r, z\in f(B_{2M\delta}(t_j,x)\cap R)\cap f(O^s)\}$$

had to be estimated, where O^s is such that (1) can be applied with a certain $\varepsilon>0$ whenever $(t',y'),(t'',y'')\in O^s$. For, there are chosen $x_i=x_i^0+\delta z_i$ according to the definition of Ξ^{rs} for $i=1,2$. Then

$$\begin{aligned}|x_1-x_2|^2-|x_1^0-x_2^0|^2 &= 2\int_0^\delta (z_1-z_2,y'-y'')\,dt\\ &\quad +2\int_0^\delta (\mathcal{F}(x_1^0+tz_1-[x_2^0+tz_2])-\mathcal{F}(y'-y''))(z_1-z_2)\,dt\,,\end{aligned}$$

for every $y',y''\in X$. [Note a misprint in [1]: $\int_{t_j}^{t_{j+1}}\ldots dt$ has to be replaced by $\int_0^\delta\ldots dt$; only $t_{j+1}-t_j=\delta$ is valid.] To estimate the first term on the right-hand side by means of (1), one should have $z_1=f(t',y')$ and $z_2=f(t'',y'')$ for some $(t',y'),(t'',y'')\in O^s$. Using the uniform continuity of $\mathcal{F}$ on bounded sets, the second integral would be small, if $|x_1^0+tz_1-[x_2^0+tz_2]-[y'-y'']|\leq 6\delta M$, cf. [1]; for, $|(t',y')-(t_j,x_1^0)|,|(t'',y'')-(t_j,x_2^0)|<2M\delta$, i.e. $(t',y')\in B_{2M\delta}(t_j,x_1^0)$ and $(t'',y'')\in B_{2M\delta}(t_j,x_2^0)$, is sufficient since f is bounded by M. Summing up, we should have $z_i\in f(B_{2M\delta}(t_j,x)\cap O^s)$ for $i=1,2$. But by definition of Ξ^{rs} only

$$z_i\in f(B_{2M\delta}(t_j,x)\cap R)\cap f(O^s)\supset f(B_{2M\delta}(t_j,x)\cap O^s)$$

is true, which is not enough. Hence it remains unclear whether or not the theorem in [1] holds without additional assumptions, and the same has to be said about [4, Proposition 2.1].

To make clearer where the principal problem arises, let us now sketch the proof of the existence of a solution on $J=[0,T]$ for some fixed $T<\min\{a,b/M\}$, if $f:R\to X$ is also assumed to be uniformly continuous; cf. [3, Theorem 3.6] for a related result.

1. We start with some preliminaries. Let $(\varepsilon_n)_{n\in\mathbb{N}}\subset(0,\min\{1,b/T-M\}]$ be decreasing to zero. For every $n\in\mathbb{N}$ we find a continuously differentiable approximate solution $u_n:J\to\overline{B}_b(x_0)$ such that

$$u_n'(t)=f(t,u_n(t))+r_n(t)\,,\;u_n(0)=x_0 \quad\text{and}\quad |r_n(t)|\le\varepsilon_n \tag{2}$$

for $t\in J$; cf. [4, Lemma 3.4]. Therefore u_n, $\Delta_{nm}=|u_n(\cdot)-u_m(\cdot)|$ and Δ_{nm}^2 are Lipschitz with a common constant $C>0$, hence differentiable a.e. in J. Since f and $(\cdot,\cdot):\overline{B}_{2M}(0)\times\overline{B}_{2b}(0)\to\mathbb{R}$ are uniformly continuous, $\omega:[0,\infty)\to\mathbb{R}$, defined by

$$\begin{aligned}w(r)=\sup\{&|(f(t,x)-f(t,y),x-y)-(f(\bar t,\bar x)-f(\bar t,\bar y),\bar x-\bar y)|\,:\,t,\bar t\in J,\\ &x,\bar x,y,\bar y\in\overline{B}_b(x_0)\,,\,|t-\bar t|\le r\,,\,|x-\bar x|,|y-\bar y|\le Cr\}\,,\end{aligned}$$

is bounded and continuous with $\omega(0)=0$. Furthermore, if we let

$$\phi_{nm}(t)=(f(t,u_n(t))-f(t,u_m(t)),u_n(t)-u_m(t))\,,$$

then [2, Proposition 13.1(e)] implies that

$$\Delta_{nm}^2(t)\le\Delta_{nm}^2(t-h)+2\int_{t-h}^{t}\phi_{nm}(s)\,ds+4bh(\varepsilon_n+\varepsilon_m) \tag{3}$$

is valid for all $n,m\in\mathbb{N}$, $0<t\in J$, and $h>0$ such that $t-h\in J$.

2. By the Arzelà-Ascoli theorem it is sufficient to find a subsequence $T_\infty\subset\mathbb{N}$ with $\alpha(\{u_n(t):n\in T_\infty\})=0$ on J. To do this and to exploit (1), for every $j\in\mathbb{N}$ we fix a partition $\mathcal{Z}_j=(\zeta_i^j)_{0\le i\le I(j)}$ of J such that

$$0=\zeta_0^j<\zeta_1^j<\ldots<\zeta_{I(j)}^j=T \quad\text{and}\quad \sup_{0\le i\le I(j)-1}(\zeta_{i+1}^j-\zeta_i^j)\le\varepsilon_j\,. \tag{4}$$

Let $T_1=\mathbb{N}$. Since $\{(s,u_n(s)):s\in\mathcal{Z}_1,n\in T_1\}\subset R$ and $\mathcal{Z}_1$ is finite, we may use (1) to find a subsequence $T_2\subset T_1$ with $T_2\subset\{l\in\mathbb{N}:l\ge2\}$ as well as $\phi_{nm}(s)\le\varepsilon_1$ for all $n,m\in T_2$ and $s\in\mathcal{Z}_1$. By iterated application of (1) we therefore obtain a sequence $(T_j)_{j\in\mathbb{N}}\subset 2^{\mathbb{N}}$ such that each T_j is an infinite, increasingly ordered subset of $\mathbb{N}$ (to be interpreted as subsequence) with

$$T_{j+1}\subset T_j\,,\,T_j\subset\{l\in\mathbb{N}:l\ge j\} \quad\text{and}\quad \phi_{nm}(s)\le\varepsilon_j \tag{5}$$

for all $j\in\mathbb{N}$, $n,m\in T_{j+1}$ and $s\in\mathcal{Z}_j$. Define $\varphi_j(t)=\alpha(\{u_n(t):n\in T_j\})$ on J. Then $\varphi_j(0)=0$ and φ_j is Lipschitz with constant C; cf. [4, Lemma 3.3]. If we let $T_\infty:=\{n_1,n_2,\ldots\}$ denote the diagonal sequence of the T_j, where $n_j\in\mathbb{N}$ is the $j's$ element of T_j, then $\{n_j,n_{j+1},\ldots\}\subset T_j$. It follows that

$$\alpha(\{u_n(t):n\in T_\infty\})=\alpha(\{u_n(t):n\in\{n_j,n_{j+1},\ldots\}\})\le\varphi_j(t)\,,$$

and hence for all $t\in J$

$$\alpha(\{u_n(t):n\in T_\infty\})\leq\varphi_\infty(t)=\inf_{j\in\mathbb{N}}\varphi_j(t)=\downarrow\lim_{j\to\infty}\varphi_j(t).$$

Therefore it is sufficient to show $\varphi_\infty(t)=0$ on J.

3. Fix $j\in\mathbb{N}$, $n,m\in T_{j+1}$, $0<t\in J$, and $h>0$ such that $t-h\in J$. By inserting possible intermediate points from $\mathcal{Z}_j$ in $[t-h,h]$, the definition of ω and (5) imply

$$\int_{t-h}^{t}\phi_{nm}(s)\,ds\leq(\varepsilon_j+\omega(\varepsilon_j))\,h, \tag{6}$$

hence

$$\Delta^2_{nm}(t)\leq\Delta^2_{nm}(t-h)+2\,(\varepsilon_j+\omega(\varepsilon_j))\,h+8\,b\,h\,\varepsilon_{j+1}$$

by (3) and (5). Therefore standard covering arguments show

$$\varphi^2_{j+1}(t)\leq\varphi^2_{j+1}(t-h)+2\,(\varepsilon_j+\omega(\varepsilon_j))\,h+8\,b\,h\,\varepsilon_{j+1}.$$

As $j\to\infty$ this yields $\varphi^2_\infty(t)\leq\varphi^2_\infty(t-h)$, and from $\varphi_\infty(0)=0$ we obtain $\varphi_\infty(t)=0$ on J. Hence the proof is complete.

Let us finally point out where the principal problem arises when trying to prove a result without the assumptions of the uniform convexity of X^* and the uniform continuity of f. Both conditions were only needed to introduce the modulus of continuity ω. This ω in turn was used to estimate

$$\int_{t-h}^{t}\phi_{nm}(s)\,ds=\int_{t-h}^{t}(f(s,u_n(s))-f(s,u_m(s)),u_n(s)-u_m(s))\,ds \tag{7}$$

in (6) by relating $\phi_{nm}(s)$ to the value $\phi_{nm}(\zeta_i^j)$ at a near partition point $\zeta_i^j\in\mathcal{Z}_j$. Here lies the difficulty: from (1) we only were able to get information about $\phi_{nm}(s)$ uniformly in $n,m\in T_{j+1}$ at a finite number of partition points $s\in\mathcal{Z}_j$. On the other hand, to estimate the integral in (7) without ω, we would need information about $s\in B_j$, where $B_j\subset J$ should be a "big" set (in some sense). Now consider e.g. the first iteration step. From (1) we obtained

$$\{(s,u_n(s)):s\in J,n\in\mathbb{N}\}\subset R=\bigcup_{k=0}^{K}O^k$$

for some $K\in\mathbb{N}_0$ and $O^k\subset R$ with $(f(t',y')-f(t'',y''),y'-y'')\leq\varepsilon_1$ for all (t',y') and (t'',y'') in the same O^k. Therefore we would like to find a subsequence $T_2\subset\mathbb{N}$ and a "big" set $B_1\subset J$ such that $n,m\in T_2$ and $s\in B_1$ imply $(s,u_n(s)),(s,u_m(s))\in O^k$ for a common $k\in\{0,\ldots,K\}$. If we assume $J=[0,1]$ and $K=1$ for simplicity, the problem of finding B_1 may be reformulated as a set-theoretical problem by taking $A_k=\{(s,n)\in[0,1]\times\mathbb{N}:(s,u_n(s))\in O^k\}$ as follows:

Let $[0,1]\times\mathbb{N}=A_0\cup A_1$. Does there exist a subsequence T of $\mathbb{N}$ and a "big" set $B\subset[0,1]$ such that for all $s\in B$ there is an $k\in\{0,1\}$ with $\{s\}\times T\subset A_k$?

Unfortunately, in general this is not the case as is shown by the following simple example (contributed by D.Donder, München). Consider the $s \in [0,1]$ in duadic expansion $s = \sum_{n=1}^{\infty} s_n 2^{-n}$ with digits $s_n \in \{0,1\}$ and $s_n = 1$ only for finitely many $n \in \mathbb{N}$ if $s < 1$. Let $A_k := \{(s,n) \in [0,1] \times \mathbb{N} : s_n = k\}$ and $B_k(T) := \{s \in [0,1] : \{s\} \times T \subset A_k\}$ for $k = 0,1$ and every subsequence $T \subset \mathbb{N}$. Then $[0,1] \times \mathbb{N} = A_0 \cup A_1$. However, the "good" set $B(T) = B_0(T) \cup B_1(T) \subset [0,1]$ is of measure zero, hence in particular not big enough to estimate an integral. This may be seen, e.g. for $B_0(T)$, by considering the sets $B_n := \{s \in [0,1] : s_n = 0\}$ for all $n \in \mathbb{N}$. Then $B_0(T) = \bigcap_{n \in T} B_n$ and $\lambda(\bigcap_{n \in I} \overline{B_n}) = 2^{-|I|}$ for every finite $I \subset \mathbb{N}$, as may be shown by induction; here λ denotes the Lebesgue measure on $[0,1]$. Hence

$$\lambda(\overline{B_0(T)}) = \lambda\left(\overline{\bigcap_{n \in T} B_n}\right) \leq \lambda\big(\bigcap_{n \in T} \overline{B_n}\big) = \lim_{I \uparrow T} \lambda\big(\bigcap_{n \in I} \overline{B_n}\big) = 0\,.$$

In particular, $B(T)$ is nowhere dense.

Of course this does not imply that both uniform conditions in the theorem are really necessary. But it may indicate at least the difficulties that arise when dealing with right-hand sides of differential equations which have properties that are formulated in terms of possible partitions of sets.

Since sums of condensing and dissipative operators are a special form of the type considered above, it seems to us that finding a proof or a counter-example is as difficult as solving a related long-time open problem for such sums, cf. e.g. the final question in [5].

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