Higher regularity of the “tangential” fields in the relativistic Vlasov-Maxwell system

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Abstract

It is shown that the “tangential” electric and magnetic fields, in the Glassey-Strauss representation formulas, are in fact bounded in $L^\infty_{\mathrm{loc},\,t}L^{2+\delta}_x$ for some $\delta > 0$.

1 Introduction and main result

The relativistic Vlasov-Maxwell system describes the time evolution of a plasma with particles moving at high velocities (close to the speed of light which is taken to be $c = 1$). The Vlasov equation

$$\partial_t f + v \cdot \nabla f + (E + v \wedge B) \cdot \nabla_p f = 0 \tag{1.1}$$

governs the evolution of the scalar density function $f = f(t, x, p) \geq 0$, depending on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$, and momentum $p \in \mathbb{R}^3$; here ∇ always means ∇_x. The velocity $v \in \mathbb{R}^3$ associated to p is

$$v = \frac{p}{\sqrt{1+p^2}}\,, \quad \text{thus} \quad p = \frac{v}{\sqrt{1-v^2}}\,,$$

where $p^2 = |p|^2$ and $v^2 = |v|^2$ for brevity. The Lorentz force

$$L = L(t, x, v) = E(t, x) + v \wedge B(t, x) \in \mathbb{R}^3$$

is obtained from the electric field $E = E(t, x) \in \mathbb{R}^3$ and the magnetic field $B = B(t, x) \in \mathbb{R}^3$, which in turn satisfy the Maxwell equations

$$\partial_t E = \nabla \wedge B - j, \quad \nabla \cdot E = \rho, \tag{1.2}$$

and

$$\partial_t B = -\nabla \wedge E, \quad \nabla \cdot B = 0. \tag{1.3}$$

The coupling of (1.1) to (1.2), (1.3) is realized through the charge density $\rho = \rho(t, x) \in \mathbb{R}$ and the current density $j = j(t, x) \in \mathbb{R}^3$ via

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, p)\, dp \quad \text{and} \quad j(t, x) = \int_{\mathbb{R}^3} v\, f(t, x, p)\, dp.$$

Furthermore, initial data

$$f(t=0) = f^{(0)}, \quad E(t=0) = E^{(0)}, \quad \text{and} \quad B(t=0) = B^{(0)}$$

are prescribed such that the constraint equations

$$\nabla \cdot E^{(0)} = \rho^{(0)} = \int_{\mathbb{R}^3} f^{(0)}\, dp \quad \text{and} \quad \nabla \cdot B^{(0)} = 0$$

are satisfied.

There has been quite some activity concerning the relativistic Vlasov-Maxwell over the years, but nonetheless the question whether (for instance smooth) initial data will yield a global in time solution still remains open. See [2] and [14] for a general introduction and overview, [9] for a summary of results up to approximately 2015 and [12] for some newer and further refined criteria concerning unrestricted global existence, generalizing both [10, 11] and [9]; the full global existence problem has only been settled in two dimensions [4, 5] and in "two-and-one-half-dimensions" $x \in \mathbb{R}^2$, $p \in \mathbb{R}^3$ [3].

To explain the observation which is the subject of the present paper, we go back to the pioneering work [6], where Glassey and Strauss noted that a bound on the momentum support of f does yield global existence; this result was later reproved by [8] and [1], using different methods. First recall that the energy

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1+p^2}\, f(t,x,p)\, dx\, dp + \frac{1}{2} \int_{\mathbb{R}^3} (|E(t,x)|^2 + |B(t,x)|^2)\, dx \tag{1.4}$$

is conserved along solutions of (1.1), (1.2), and (1.3); note that $\nabla_p \sqrt{1+p^2} = v$. Therefore one gets a bound

$$E, B \in L^\infty_t L^2_x \tag{1.5}$$

in terms of the initial data for free. Next, defining $E^{(1)}(x) = \partial_t E(0,x)$ and $B^{(1)}(x) = \partial_t B(0,x)$, E and B are the solutions to the wave equations

$$\Box E = -(\partial_t j + \nabla \rho) = -\int_{\mathbb{R}^3} (v\, \partial_t + \nabla) f\, dp, \quad E(0) = E^{(0)}, \quad \partial_t E(0) = E^{(1)}, \tag{1.6}$$

$$\Box B = \nabla \wedge j = \nabla \wedge \int_{\mathbb{R}^3} v\, f\, dp, \qquad B(0) = B^{(0)}, \quad \partial_t B(0) = B^{(1)}. \tag{1.7}$$

In [6], Glassey and Strauss noted that (1.6) and (1.7) can be used to derive representation formulas for the fields as follows. Write

$$S = \partial_t + v \cdot \nabla, \quad T_j = -\omega_j \partial_t + \partial_{x_j}.$$

Then ∂_t and ∇ can be expressed in terms of S and T, since

$$\partial_t = (1 + v\cdot\omega)^{-1}(S - v \cdot T), \tag{1.8}$$

$$\partial_{x_j} = T_j + (1 + v \cdot \omega)^{-1} \omega_j (S - v \cdot T). \tag{1.9}$$

Note that with $\omega = \frac{y-x}{|y-x|}$:

$$\nabla_y \left[f(t - |y-x|, y, p)\right] = (-\omega\, \partial_t + \nabla) f(\ldots) = (Tf)(\ldots),$$

and, for instance,

$$\begin{aligned} E &= -\Box^{-1} \int_{\mathbb{R}^3} dp\,(\nabla + v\,\partial_t) f \\ &\cong -\int_{\mathbb{R}^3} dp \int_{|y-x|\le t} \frac{dy}{|y-x|}\,(\nabla + v\,\partial_t) f. \end{aligned}$$

First one uses (1.9) and (1.8) for the right-hand side and then one integrates $(Tf)(\ldots) = \nabla_y[\ldots]$ by parts in y. After a lengthy calculation one finds

$$E = E_D + E_{DT} + E_T + E_S \tag{1.10}$$

(and a similar expression for B), where E_D and E_{DT} are data terms,

$$E_T(t,x) = -\int_{|y|\le t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} dp\, K_{E,T}(\omega, v) f(t-|y|, x+y, p), \tag{1.11}$$

$$E_S(t,x) = -\int_{|y|\le t} \frac{dy}{|y|} \int_{\mathbb{R}^3} dp\, K_{E,S}(\omega, v)\,(Lf)(t-|y|, x+y, p), \tag{1.12}$$

and the integral kernels $K_{E,T}(\omega,v) \in \mathbb{R}^3$ and $K_{E,S}(\omega,v) \in \mathbb{R}^{3\times 3}$ behave as follows:

$$|K_{E,T}(\omega,v)| \le C(1+p^2)^{-1}(1+v\cdot\omega)^{-3/2}, \tag{1.13}$$

$$|K_{E,S}(\omega,v)z| \le C(1+p^2)^{-1/2}(1+v\cdot\omega)^{-1}|z| \quad (z\in\mathbb{R}^3). \tag{1.14}$$

See Section 3 below for the precise form of the kernels and a recap of the proof of (1.13) and (1.14). Relation (1.10) is the Glassey-Strauss representation formula for the electric field E and, together with its counterpart for B, it has become an indipensible tool for proving existence results for the relativistic Vlasov-Maxwell system. Variants of it have been used for related systems as well.

Assuming initial data of compact support, certainly the data terms E_D and E_{DT} in (1.10) will behave well. Thus, in the light of (1.5), it is natural to ask what could be said about the terms E_T and E_S individually. We will call E_T the tangential part and we are going to prove the following result.

Theorem 1.1 *Consider initial data of compact support. Then $E_T, B_T \in L^\infty_{\mathrm{loc},t} L^{2+\delta}_x$ for some $\delta > 0$.*

Remark 1.2 (a) Since the argument for B_T is the same as for E_T, we will only consider the latter in what follows.
(b) The number $\delta > 0$ will be a uniform constant, for instance $\delta = \frac{2}{17}$ is a possible choice. As this result is mainly understood to be a "proof of concept", certainly the regularity that is gained here will not be optimal.
(c) By $E_T \in L^\infty_{\mathrm{loc},t} L^{2+\delta}_x$ we mean the following: There is a continuous function $C = C(t) : [0,\infty[\to [1,\infty[$ which only depends on t and the initial energy $\mathcal{E}(0)$, the initial mass $\mathcal{M}(0) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^{(0)}(x,p)\,dx\,dp$ and $\|f^{(0)}\|_\infty$ such that $\|E_T(t,\cdot)\|_{L^{2+\delta}_x(\mathbb{R}^3)} \le C(t)$ for $t \in [0, T_{\max}[$, where $T_{\max} > 0$ denotes the maximal time of existence of the solution. A constant denoted by C will always be one which only depends on $\mathcal{E}(0)$, $\mathcal{M}(0)$ and $\|f^{(0)}\|_\infty$.
(d) Due to Theorem 1.1 and (1.5) one has $E_S \in L^\infty_{\mathrm{loc},t} L^2_x$, but we are not able to derive this bound directly from (1.12).

2 Proof of Theorem 1.1

According to (1.11) and (1.13) we have

$$\begin{aligned}|E_T(t,x)| &\le C\int_{|y|\le t}\frac{dy}{|y|^2}\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,\frac{1}{(1+v\cdot\omega)^{3/2}}\,f(t-|y|,x+y,p)\\ &=: Cu(t,x).\end{aligned}$$

The Fourier transform of u is

$$\begin{aligned}\hat{u}(t,\xi) &= \int_{\mathbb{R}^3} e^{-i\,\xi\cdot x}\,u(t,x)\,dx\\ &= \int_{|y|\le t}\frac{dy}{|y|^2}\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,\frac{1}{(1+v\cdot\omega)^{3/2}}\int_{\mathbb{R}^3}dx\,e^{-i\,\xi\cdot x}\,f(t-|y|,x+y,p)\\ &= \int_{\mathbb{R}^3}\frac{dp}{1+p^2}\int_{|y|\le t}\frac{dy}{|y|^2}\,e^{i\,\xi\cdot y}\,\frac{1}{(1+v\cdot\omega)^{3/2}}\,\hat{f}(t-|y|,\xi,p)\\ &= \int_{\mathbb{R}^3}\frac{dp}{1+p^2}\int_0^t ds\,\hat{f}(t-s,\xi,p)\int_{|\omega|=1}dS(\omega)\,\frac{e^{is\,\xi\cdot\omega}}{(1+v\cdot\omega)^{3/2}}.\end{aligned}$$

To evaluate the inner integral choose a unit vector $e\in\mathbb{R}^3$ such that $\{\bar{v},\bar{u},e\}$ is an orthonormal basis of $\mathbb{R}^3$, where $\bar{v}=v/|v|=p/|p|$ and $\bar{u}=\frac{\bar{\xi}-(\bar{\xi}\cdot\bar{v})\bar{v}}{\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}}$ are orthogonal unit vectors. Consider the matrix $A=\begin{pmatrix}e\\ \bar{u}\\ \bar{v}\end{pmatrix}\in\mathbb{R}^{3\times3}$, where the vectors are taken as rows. Then $A\bar{v}=e_3$ and $A\bar{u}=e_2$. It follows that $Av=|v|A\bar{v}=|v|e_3$ and $A\xi=|\xi|A\bar{\xi}=|\xi|A(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\bar{u}+(\bar{\xi}\cdot\bar{v})\bar{v})=|\xi|(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,e_2+(\bar{\xi}\cdot\bar{v})e_3)$. Thus if $\omega\in\mathbb{R}^3$ and $A\omega=\eta$, then by the invariance of the inner product,

$$\xi\cdot\omega=(A\xi)\cdot(A\omega)=|\xi|\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,e_2+(\bar{\xi}\cdot\bar{v})e_3\Big)\cdot\eta=|\xi|\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\eta_2+(\bar{\xi}\cdot\bar{v})\eta_3\Big),$$

and similarly $v\cdot\omega=|v|\eta_3$. Hence we can employ the change of variables $\eta=A\omega$, $dS(\eta)=dS(\omega)$, and afterwards pass to spherical coordinates to obtain

$$\begin{aligned}\int_{|\omega|=1}dS(\omega)\,\frac{e^{is\,\xi\cdot\omega}}{(1+v\cdot\omega)^{3/2}} &= \int_{|\eta|=1}dS(\eta)\,\frac{e^{is\,|\xi|(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\eta_2+(\bar{\xi}\cdot\bar{v})\eta_3)}}{(1+|v|\eta_3)^{3/2}}\\ &= \int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\,\sin\varphi\,\frac{e^{is\,|\xi|(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sin\theta\sin\varphi+(\bar{\xi}\cdot\bar{v})\cos\varphi)}}{(1+|v|\cos\varphi)^{3/2}}\\ &= \int_{-1}^{1}d\sigma\,\frac{e^{is\sigma\,\xi\cdot\bar{v}}}{(1+|v|\sigma)^{3/2}}\int_0^{2\pi}d\theta\,e^{is|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sin\theta\sqrt{1-\sigma^2}}\\ &= 2\pi\int_{-1}^{1}d\sigma\,\frac{e^{is\sigma\,\xi\cdot\bar{v}}}{(1+|v|\sigma)^{3/2}}J_0\Big(s|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sqrt{1-\sigma^2}\Big),\end{aligned}$$

where

$$J_0(r)=\frac{1}{2\pi}\int_0^{2\pi}e^{ir\sin\theta}\,d\theta$$

is the Bessel function of order zero. Its asymptotic expansion is

$$J_0(r) = \sqrt{\frac{2}{\pi r}}\cos\Big(r-\frac{\pi}{4}\Big) + O(r^{-3/2}), \quad r\to\infty, \tag{2.1}$$

see [7, p. 432], and also $|J_0(r)| \le 1$ is verified. Thus altogether we obtain

$$\hat{u}(t,\xi) = 2\pi\int_0^t ds\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,\hat{f}(t-s,\xi,p)\int_{-1}^1 d\sigma\,\frac{e^{is\sigma\,\xi\cdot\bar{v}}}{(1+|v|\sigma)^{3/2}}J_0\Big(s|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sqrt{1-\sigma^2}\Big).$$

Our estimates below will only use (2.1) and $|J_0(r)| \le 1$. It is tempting to assume that some improvement could be possible due to the oscillatory behavior of J_0, but we have not been able to do so.

Next we introduce a standard Littlewood-Paley decomposition of u. For, fix $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_0(\xi) = 1$ for $|\xi| \le 1$ and $\varphi_0(\xi) = 0$ for $|\xi| \ge 2$. For $j \in \mathbb{N}$ put $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$. Then $\varphi_j(\xi) = 0$ for $|\xi| \le 2^{j-1}$ and for $|\xi| \ge 2^{j+1}$. Furthermore, $\sum_{j=0}^\infty \varphi_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. Henceforth we shall consider $u_j = u_j(t,x)$ given by $\hat{u}_j(t,\xi) = \varphi_j(\xi)\hat{u}(t,\xi)$ for $j \in \mathbb{N}_0$. In this way we obtain

$$u = \sum_{j=0}^\infty u_j$$

for

$$\hat{u}_j(t,\xi) = 2\pi\int_0^t ds\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,\hat{f}_j(t-s,\xi,p)\int_{-1}^1 d\sigma\,\frac{e^{is\sigma\,\xi\cdot\bar{v}}}{(1+|v|\sigma)^{3/2}}J_0\Big(s|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sqrt{1-\sigma^2}\Big),$$

where $\hat{f}_j(t,\xi,p) = \varphi_j(\xi)\hat{f}(t,\xi,p)$; the Fourier transform of f only refers to the variable x. Then

$$\|f_j(t,\cdot,p)\|_{L^q_x(\mathbb{R}^3)} \le C\|f(t,\cdot,p)\|_{L^q_x(\mathbb{R}^3)}, \quad j\in\mathbb{N}_0, \quad q\in[1,\infty], \tag{2.2}$$

uniformly in t and p; the constant $C > 0$ does only depend on q. Since $\mathrm{supp}\hat{f}_j(t,\cdot,p) \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$, Bernstein's inequality (or a direct estimate) moreover leads to

$$\|f_j(t,\cdot,p)\|_{L^2_x(\mathbb{R}^3)} \le C2^{3j/2}\|f_j(t,\cdot,p)\|_{L^1_x(\mathbb{R}^3)}, \tag{2.3}$$

uniformly in t and p. Denote by $(\psi_j)_{j\in\mathbb{N}_0}$ a partition of unity on $]0,1]$ such that $\mathrm{supp}\,\psi_0 \subset [\frac{1}{3},1]$ and $\mathrm{supp}\,\psi_j \subset [2^{-(j+2)}, 2^{-j+1}]$ for $j\in\mathbb{N}$. Accordingly we decompose

$$\hat{u}_j(t,\xi) = \sum_{k=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty \hat{u}_{jkmn}(t,\xi), \tag{2.4}$$

where

$$\begin{aligned}\hat{u}_{jkmn}(t,\xi) &= 2\pi\int_0^t ds\,\psi_k\Big(\frac{s}{t}\Big)\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,\hat{f}_j(t-s,\xi,p)\,\psi_n\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\Big)\\ &\quad\times\int_{-1}^1 d\sigma\,\frac{e^{is\sigma\,\xi\cdot\bar{v}}}{(1+|v|\sigma)^{3/2}}J_0\Big(s|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sqrt{1-\sigma^2}\Big)\,\psi_m(\sqrt{1-\sigma^2}).\end{aligned}$$

The next lemma is the main technical tool for the proof of Theorem 1.1.

Lemma 2.1 *For $j \in \mathbb{N}$ and $k, m, n \in \mathbb{N}_0$,*

$$\|\hat{u}_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} \le Ct \min\Big\{1, \frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}\, 2^{-k} \min\Big\{2^{-2m}\, 2^{3j/2}, (\sqrt{n}+\sqrt{j})\, 2^{-n}\Big\}. \tag{2.5}$$

Proof: Observe that by (2.1) always

$$\begin{aligned}
&\psi_k\Big(\frac{s}{t}\Big)\psi_n\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\Big)\psi_m(\sqrt{1-\sigma^2})\,\Big|J_0\Big(s|\xi|\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\,\sqrt{1-\sigma^2}\Big)\Big| \\
&\le\ C\psi_k\Big(\frac{s}{t}\Big)\psi_n\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\Big)\psi_m(\sqrt{1-\sigma^2})\ \min\Big\{1, \frac{1}{s^{1/2}|\xi|^{1/2}(1-(\bar{\xi}\cdot\bar{v})^2)^{1/4}\,(1-\sigma^2)^{1/4}}\Big\} \\
&\le\ C\psi_k\Big(\frac{s}{t}\Big)\psi_n\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\Big)\psi_m(\sqrt{1-\sigma^2})\ \min\Big\{1, \frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\hat{u}_{jkmn}(t,\xi)| \\
&\quad\le\ C\min\Big\{1, \frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}\int_0^t ds\,\psi_k\Big(\frac{s}{t}\Big)\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,|\hat{f}_j(t-s,\xi,p)|\,\psi_n\Big(\sqrt{1-(\bar{\xi}\cdot\bar{v})^2}\Big) \\
&\qquad\qquad\times\int_{-1}^1 d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\,\psi_m(\sqrt{1-\sigma^2}).
\end{aligned} \tag{2.6}$$

From (2.2) and (2.3) we deduce that

$$\|\hat{f}_j(t,\cdot,p)\|_{L^2_\xi(\mathbb{R}^3)} \le C\|f_j(t,\cdot,p)\|_{L^2_x(\mathbb{R}^3)} \le C2^{3j/2}\|f_j(t,\cdot,p)\|_{L^1_x(\mathbb{R}^3)} \le C2^{3j/2}\|f(t,\cdot,p)\|_{L^1_x(\mathbb{R}^3)}, \tag{2.7}$$

and also

$$\|\hat{f}_j(t,\cdot,p)\|^2_{L^2_\xi(\mathbb{R}^3)} \le C\|f_j(t,\cdot,p)\|^2_{L^2_x(\mathbb{R}^3)} \le C\|f(t,\cdot,p)\|^2_{L^2_x(\mathbb{R}^3)}, \tag{2.8}$$

both uniformly in t and p.

To begin with the estimate of (2.6), the support of $\psi_m(\sqrt{1-\sigma^2})$ is contained in

$$\sigma_- = \sqrt{1-2^{-2m+2}} \le |\sigma| \le \sqrt{1-2^{-2(m+2)}} = \sigma_+.$$

Then $\sigma_+ - \sigma_- \le C2^{-2m}$ and it follows that

$$\begin{aligned}
&\int_{-1}^1 d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\,\psi_m(\sqrt{1-\sigma^2}) \\
&\le\ \frac{2}{|v|}\Big(\frac{1}{\sqrt{1+\sigma_-|v|}} - \frac{1}{\sqrt{1+\sigma_+|v|}} + \frac{1}{\sqrt{1-\sigma_+|v|}} - \frac{1}{\sqrt{1-\sigma_-|v|}}\Big) \\
&\le\ C(\sigma_+-\sigma_-)\Big(1+(1+p^2)^{3/2}\Big) \le C2^{-2m}(1+p^2)^{3/2}.
\end{aligned}$$

Thus, below taking $R = 2^m \ge 1$ and using that $\int_{-1}^1 \frac{d\sigma}{(1+|v|\sigma)^{3/2}} = \frac{2}{|v|}(\frac{1}{\sqrt{1-|v|}} - \frac{1}{\sqrt{1+|v|}}) \le \frac{2\sqrt{2}}{|v|}\sqrt{1+p^2}$, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,|\hat{f}_j(t-s,\xi,p)|\int_{-1}^1 d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\,\psi_m(\sqrt{1-\sigma^2}) \\
&=\ \int_{|p|\le R}\frac{dp}{1+p^2}\,(\dots) + \int_{|p|\ge R}\frac{dp}{1+p^2}\,(\dots)
\end{aligned}$$

$$
\begin{aligned}
&\le C2^{-2m}\int_{|p|\le R}\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp + C\int_{|p|\ge R}\frac{dp}{\sqrt{1+p^2}}\,\frac{1}{|v|}\,|\hat f_j(t-s,\xi,p)| \\
&\le C2^{-2m}\int_{|p|\le R}\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp + CR^{-2}\int_{|p|\ge R}\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp \\
&\le C2^{-2m}\int_{\mathbb{R}^3}\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp.
\end{aligned}
$$

As a consequence, by (2.7) and energy conservation (1.4),

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,|\hat f_j(t-s,\cdot,p)|\int_{-1}^{1}d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\,\psi_m(\sqrt{1-\sigma^2})\Big\|_{L^2_\xi(\mathbb{R}^3)} \\
&\le C2^{-2m}\int_{\mathbb{R}^3}\sqrt{1+p^2}\,\|\hat f_j(t-s,\cdot,p)\|_{L^2_\xi(\mathbb{R}^3)}\,dp \\
&\le C2^{-2m}\,2^{3j/2}\int_{\mathbb{R}^3}dp\,\sqrt{1+p^2}\int_{\mathbb{R}^3}dx f(t-s,x,p) \\
&\le C\,\mathcal{E}(0)\,2^{-2m}\,2^{3j/2} \\
&= C\,2^{-2m}\,2^{3j/2}.
\end{aligned}
$$

Using this and $\psi_n(\dots)\le 1$ in (2.6), we obtain

$$
\begin{aligned}
\|\hat u_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} &\le C\min\Big\{1,\frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}\,2^{-2m}\,2^{3j/2}\int_0^t ds\,\psi_k\Big(\frac{s}{t}\Big) \\
&\le Ct\min\Big\{1,\frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}\,2^{-k}\,2^{-2m}\,2^{3j/2}. \qquad (2.9)
\end{aligned}
$$

This finishes the proof of the first part of (2.5). Secondly, the support of $\psi_n(\sqrt{1-\tau^2})$ is contained in

$$
\tau_- = \sqrt{1-2^{-2n+2}} \le |\tau| \le \sqrt{1-2^{-2(n+2)}} = \tau_+
$$

and $\tau_+ - \tau_- \le C2^{-2n}$. Thus, for all $\tilde R\ge 1$,

$$
\begin{aligned}
\int_{1\le|p|\le\tilde R}\frac{dp}{(1+p^2)^{3/2}}\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big) &= \int_{1\le|p|\le\tilde R}\frac{dp}{(1+p^2)^{3/2}}\,\psi_n\Big(\sqrt{1-\bar v_3^2}\Big) \\
&\le C\int_1^{\tilde R}dr\,\frac{r^2}{(1+r^2)^{3/2}}\int_0^{\pi}d\varphi\,\sin\varphi\,\psi_n(\sqrt{1-\cos^2\varphi}) \\
&\le C\ln(1+\tilde R)\int_{-1}^{1}\psi_n(\sqrt{1-\tau^2})\,d\tau \\
&\le C\ln(1+\tilde R)\,(\tau_+-\tau_-) \\
&\le C\ln(1+\tilde R)\,2^{-2n},
\end{aligned}
$$

and similarly

$$
\int_{|p|\le 1}\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\,dp \le C\,2^{-2n}.
$$

Hence if we take $\tilde R = 2^{n/2}\,2^{3j/4}\ge 1$, then

$$
\int_{\mathbb{R}^3}\frac{dp}{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\int_{-1}^{1}d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}
$$

$$
\begin{aligned}
&= \int_{|p|\le 1} \frac{dp}{1+p^2}\,(\dots) + \int_{1\le|p|\le\tilde R} \frac{dp}{1+p^2}\,(\dots) + \int_{|p|\ge\tilde R} \frac{dp}{1+p^2}\,(\dots)\\
&\le C\int_{|p|\le 1} dp\,\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\\
&\quad + C\int_{1\le|p|\le\tilde R} \frac{dp}{\sqrt{1+p^2}}\,|\hat f_j(t-s,\xi,p)|\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\\
&\quad + C\int_{|p|\ge\tilde R} \frac{dp}{\sqrt{1+p^2}}\,|\hat f_j(t-s,\xi,p)|\\
&\le C\Big(\int_{|p|\le 1} \psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\,dp\Big)^{1/2}\Big(\int_{|p|\le 1} |\hat f_j(t-s,\xi,p)|^2\,dp\Big)^{1/2}\\
&\quad + C\Big(\int_{1\le|p|\le\tilde R} \frac{dp}{(1+p^2)^{3/2}}\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\Big)^{1/2}\Big(\int_{1\le|p|\le\tilde R} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|^2\,dp\Big)^{1/2}\\
&\quad + C\tilde R^{-2}\int_{|p|\ge\tilde R} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp\\
&\le C\,2^{-n}\Big(\int_{\mathbb{R}^3} |\hat f_j(t-s,\xi,p)|^2\,dp\Big)^{1/2}\\
&\quad + C(\ln(1+\tilde R))^{1/2}\,2^{-n}\Big(\int_{\mathbb{R}^3} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|^2\,dp\Big)^{1/2}\\
&\quad + C\tilde R^{-2}\int_{\mathbb{R}^3} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp\\
&\le C(\ln(1+\tilde R))^{1/2}\,2^{-n}\Big(\int_{\mathbb{R}^3} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|^2\,dp\Big)^{1/2}\\
&\quad + C\tilde R^{-2}\int_{\mathbb{R}^3} \sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|\,dp.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^3} \frac{dp}{1+p^2}\,|\hat f_j(t-s,\cdot,p)|\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\int_{-1}^1 d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\Big\|_{L^2_\xi(\mathbb{R}^3)}\\
&\qquad\le C(\ln(1+\tilde R))^{1/2}\,2^{-n}\Big(\int_{\mathbb{R}^3} d\xi\int_{\mathbb{R}^3} dp\,\sqrt{1+p^2}\,|\hat f_j(t-s,\xi,p)|^2\Big)^{1/2}\\
&\qquad\quad + C\tilde R^{-2}\int_{\mathbb{R}^3} \sqrt{1+p^2}\,\|\hat f_j(t-s,\cdot,p)\|_{L^2_\xi(\mathbb{R}^3)}\,dp.
\end{aligned}
$$

For the first term, we use (2.8), $\|f(t)\|_\infty \le \|f^{(0)}\|_\infty$ and (1.4), whereas we invoke (2.7) and (1.4) for the second. This leads to

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^3} \frac{dp}{1+p^2}\,|\hat f_j(t-s,\cdot,p)|\,\psi_n\Big(\sqrt{1-(\bar\xi\cdot\bar v)^2}\Big)\int_{-1}^1 d\sigma\,\frac{1}{(1+|v|\sigma)^{3/2}}\Big\|_{L^2_\xi(\mathbb{R}^3)}\\
&\qquad\le C(\ln(1+\tilde R))^{1/2}\,2^{-n} + C\tilde R^{-2}\,2^{3j/2}\\
&\qquad\le C\,(\ln(1+2^{n/2}\,2^{3j/4}))^{1/2}\,2^{-n}\\
&\qquad\le C\,(\sqrt{n}+\sqrt{j})\,2^{-n}.
\end{aligned}
$$

Due to (2.6), and dropping $\psi_m(\dots)\le 1$, it follows that

$$
\|\hat u_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} \le C(0)\min\Big\{1,\frac{2^{(k+m+n-j)/2}}{t^{1/2}}\Big\}(\sqrt{n}+\sqrt{j})\,2^{-n}\int_0^t ds\,\psi_k\Big(\frac{s}{t}\Big)
$$

$$\leq \quad C(0)t\min\left\{1, \frac{2^{(k+m+n-j)/2}}{t^{1/2}}\right\}(\sqrt{n}+\sqrt{j})\, 2^{-k}\, 2^{-n}. \tag{2.10}$$

Therefore if we summarize (2.9) and (2.10), we have shown (2.5). □

Lemma 2.2 *For $j \in \mathbb{N}$,*

$$\|u_j(t,\cdot)\|_{L^2_x(\mathbb{R}^3)} \leq C(t+\sqrt{t})\, 2^{-\frac{j}{11}}.$$

Proof: By (2.4),

$$\|u_j(t,\cdot)\|_{L^2_x(\mathbb{R}^3)} \leq C\|\hat{u}_j(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} \leq C\sum_{k=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty \|\hat{u}_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)}.$$

In the following, Lemma 2.1 will be used to bound the right-hand side for fixed $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Let $\alpha = \frac{16}{15} > 1$ and $\varepsilon = \frac{1}{20} \in]0,1[$. Then by Lemma 2.1,

$$\begin{aligned}
\sum_{m=0}^\infty\sum_{n=0}^\infty \|\hat{u}_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} &\leq \sum_{m,\,n}\mathbf{1}_{\{m>\alpha\frac{3j}{4}\}}\,\|\hat{u}_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} + \sum_{m,\,n}\mathbf{1}_{\{m\leq\alpha\frac{3j}{4}\}}\,\|\hat{u}_{jkmn}(t,\cdot)\|_{L^2_\xi(\mathbb{R}^3)} \\
&\leq Ct\,2^{-k}\sum_{m=[\alpha\frac{3j}{4}]-1}^\infty\ \sum_{n=0}^\infty (2^{-2m}\,2^{3j/2})^{1-\varepsilon}\,((\sqrt{n}+\sqrt{j})\,2^{-n})^\varepsilon \\
&\quad + C\sqrt{t}\,2^{-k/2}\sum_{m=0}^{[\alpha\frac{3j}{4}]+1}\ \sum_{n=0}^\infty 2^{(m+n-j)/2}\,(\sqrt{n}+\sqrt{j})\,2^{-n} \\
&\leq Ct\,2^{-k}\,2^{3(1-\varepsilon)j/2}\,j^{\varepsilon/2}\sum_{m=[\alpha\frac{3j}{4}]-1}^\infty 2^{-2(1-\varepsilon)m} \\
&\quad + C\sqrt{t}\,2^{-k/2}\,2^{-j/2}\,\sqrt{j}\sum_{m=0}^{[\alpha\frac{3j}{4}]+1} 2^{m/2} \\
&\leq Ct\,2^{-k}\,2^{3(1-\varepsilon)j/2}\,j^{\varepsilon/2}\,2^{-2(1-\varepsilon)\alpha\frac{3j}{4}} + C\sqrt{t}\,2^{-k/2}\,2^{-j/2}\,\sqrt{j}\,2^{\alpha\frac{3j}{8}} \\
&= Ct\,2^{-k}\,j^{\varepsilon/2}\,2^{-(1-\varepsilon)(\alpha-1)\frac{3j}{2}} + C\sqrt{t}\,2^{-k/2}\,\sqrt{j}\,2^{-\frac{j}{2}(1-\frac{3}{4}\alpha)} \\
&= Ct\,2^{-k}\,j^{\frac{1}{40}}\,2^{-\frac{19}{200}j} + C\sqrt{t}\,2^{-k/2}\,\sqrt{j}\,2^{-\frac{j}{10}} \\
&\leq C(t+\sqrt{t})\,2^{-k/2}\,2^{-\frac{j}{11}}.
\end{aligned}$$

Summation on $k \in \mathbb{N}_0$ concludes the proof of the lemma. □

Now we are in position to finish the proof of Theorem 1.1. To summarize, we have seen that

$$|E_T(t,x)| \leq Cu(t,x), \quad u = \sum_{j=0}^\infty u_j, \quad \|u_j(t,\cdot)\|_{L^2_x} \leq C(t+\sqrt{t})\,2^{-\frac{j}{11}}$$

for $j \in \mathbb{N}$. Clearly one also has $\|u_0(t,\cdot)\|_{L^2_x} \leq Ct$. Let $H^s_x(\mathbb{R}^3)$ denote the standard (inhomogeneous) L^2_x-based Sobolev space of order s. Then by the inhomogeneous Sobolev embedding theorem and by Plancherel's theorem, for $2 < q < \infty$, $s > 0$ and $\frac{1}{2} \leq \frac{1}{q} + \frac{s}{3}$:

$$\begin{aligned}
\|E_T(t,\cdot)\|_{L^q_x} &\leq C\|u(t,\cdot)\|_{L^q_x} \\
&\leq C\|u(t,\cdot)\|_{H^s_x} \\
&\leq C\Big[\|u_0(t,\cdot)\|_{L^2_x} + \Big(\sum_{j=1}^{\infty} 2^{2sj}\|u_j(t,\cdot)\|^2_{L^2_x}\Big)^{1/2}\Big] \\
&\leq C(t)\Big[1 + \Big(\sum_{j=1}^{\infty} 2^{2j(s-\frac{1}{11})}\Big)^{1/2}\Big] \\
&\leq C(t),
\end{aligned}$$

provided that $s < \frac{1}{11}$. Hence $q = 2 + \delta$ is possible, and for instance $s = \frac{1}{12}$ and $\delta = \frac{2}{17}$ is a suitable choice. □

3 Appendix: Explicit form of the kernels

To make this paper self-contained, we will include the following formulas; see [6, Section II] and [13, (A13), (A14), (A3)]. The fields E and B can be written as

$$\begin{aligned}
E &= E_D + E_{DT} + E_T + E_S, \\
B &= B_D + B_{DT} + B_T + B_S,
\end{aligned}$$

where

$$\begin{aligned}
E_D(t,x) &= \partial_t\Big(\frac{t}{4\pi}\int_{|\omega|=1} E^{(0)}(x+t\omega)\,d\omega\Big) \\
&\quad + \frac{t}{4\pi}\int_{|\omega|=1} \partial_t E(0,x+t\omega)\,d\omega, \\
E_{DT}(t,x) &= -\frac{1}{t}\int_{|y|=t}\int_{\mathbb{R}^3} K_{E,\,DT}(\omega,v) f^{(0)}(x+y,p)\,dp\,d\sigma(y), \\
B_D(t,x) &= \partial_t\Big(\frac{t}{4\pi}\int_{|\omega|=1} B^{(0)}(x+t\omega)\,d\omega\Big) \\
&\quad + \frac{t}{4\pi}\int_{|\omega|=1} \partial_t B(0,x+t\omega)\,d\omega, \\
B_{DT}(t,x) &= \frac{1}{t}\int_{|y|=t}\int_{\mathbb{R}^3} K_{B,\,DT}(\omega,v) f^{(0)}(x+y,p)\,dp\,d\sigma(y),
\end{aligned}$$

are the data terms. In addition,

$$\begin{aligned}
E_T(t,x) &= -\int_{|y|\leq t} \frac{dy}{|y|^2}\int_{\mathbb{R}^3} dp\,K_{E,T}(\omega,v) f(t-|y|,x+y,p), \\
E_S(t,x) &= -\int_{|y|\leq t} \frac{dy}{|y|}\int_{\mathbb{R}^3} dp\,K_{E,S}(\omega,v)\,(Lf)(t-|y|,x+y,p),
\end{aligned}$$

and

$$
\begin{aligned}
B_T(t,x) &= \int_{|y|\le t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} dp\, K_{B,T}(\omega,v) f(t-|y|, x+y, p),\\
B_S(t,x) &= \int_{|y|\le t} \frac{dy}{|y|} \int_{\mathbb{R}^3} dp\, K_{B,S}(\omega,v)\,(Lf)(t-|y|, x+y, p),
\end{aligned}
$$

defining $\omega = |y|^{-1}y$ and $L = E + v\wedge B$. The kernels are

$$
\begin{aligned}
K_{E,DT}(\omega,v) &= (1+v\cdot\omega)^{-1}(\omega-(v\cdot\omega)v),\\
K_{E,T}(\omega,v) &= (1+p^2)^{-1}(1+v\cdot\omega)^{-2}(v+\omega),\\
K_{E,S}(\omega,v) &= (1+p^2)^{-1/2}(1+v\cdot\omega)^{-2}\\
&\quad \Big[(1+v\cdot\omega) + ((v\cdot\omega)\omega - v)\otimes v - (v+\omega)\otimes\omega\Big] \in \mathbb{R}^{3\times 3},
\end{aligned}
$$

and

$$
\begin{aligned}
K_{B,DT}(\omega,v) &= -(1+v\cdot\omega)^{-1}(v\wedge\omega),\\
K_{B,T}(\omega,v) &= -(1+p^2)^{-1}(1+v\cdot\omega)^{-2}(v\wedge\omega),\\
K_{B,S}(\omega,v) &= (1+p^2)^{-1/2}(1+v\cdot\omega)^{-2}\\
&\quad \Big[(1+v\cdot\omega)\,\omega\wedge(\dots) - (v\wedge\omega)\otimes(v+\omega)\Big] \in \mathbb{R}^{3\times 3}.
\end{aligned}
$$

Proof of (1.13) and (1.14) : The bound (1.13) is immediate from

$$
|v+\omega| = (v^2 + 2(v\cdot\omega)+1)^{1/2} \le \sqrt{2}\,(1+v\cdot\omega)^{1/2}.
$$

Regarding (1.14), we use that

$$
\begin{aligned}
\Big[((v\cdot\omega)\omega - v)\otimes v - (v+\omega)\otimes\omega\Big]z &= (v\cdot z)((v\cdot\omega)\omega - v) - (\omega\cdot z)(v+\omega)\\
&= -(\omega-(v\cdot\omega)v)\cdot z\,(v+\omega) - (1+v\cdot\omega)(v\cdot z)\,v
\end{aligned}
$$

and

$$
\begin{aligned}
|\omega-(v\cdot\omega)v| &= (1-2(v\cdot\omega)^2 + (v\cdot\omega)^2v^2)^{1/2}\\
&\le (1-(v\cdot\omega)^2)^{1/2} \le \sqrt{2}\,(1+v\cdot\omega)^{1/2}.
\end{aligned}
$$

This yields the claim. □

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