Qualitative Bifurcation Analysis of a Non-Smooth Friction-Oscillator Model

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`Dedicated to Prof. E. Stein on the occasion of his 65`$^{\text{th}}$` birthday`

Abstract

We numerically study the bifurcation structure (in dependence of the amplitude and frequency of the forcing) of a non-smooth friction-oscillator sytem with dry friction.

1 Introduction

In many mechanical models the effects of impacts and/or dry friction on the dynamics of the system play an important rôle. There are plenty of examples of non-smooth systems like oscillators impacting with a wall or oscillators with dry friction. More realistic applications in mechanics include rotor models, turbine blade dampers or the noise generation in railway wheels, cf. e.g. [5, 32, 38, 33, 34]. In comparison with smooth systems, in non-smooth systems it is possible for trajectories to collide with some discontinuity surface in phase space. This often results in new bifurcation phenonena, which are generally adressed as "non-smooth bifurcations", cf. e.g. [9, 27, 7].

In this paper (which is descriptive in nature) we concentrated on the numerical analysis of non-smooth bifurcations for a particular dissipative system, namely a friction-oscillator described in more detail below. Compared to related results on this example as obtained e.g. in [31, 32, 37, 19, 20], we identified a lot of new and interesting bifurcation structures by reducing the analysis of the whole excited 3-dimensional system to the study of a suitable 1-dimensional Poincarè-map. We determined the various ranges of frequency and amplitude of the external forcing which correspond to differentiability and/or monotonicity of this Poincaré-map. Moreover, we found that besides familiar phenomena (like period doublings, periodic windows or intermittency), so-called "grazing bifurcations" lead to some remarkable consecutive period doublings and undoublings, cf. Fig. 8c below. Moreover, the corresponding Lyapunov exponent scales $\lambda \sim 1/|\ln\varepsilon|$ near the bifurcation point, if $\varepsilon > 0$ denotes the bifurcation parameter. This scaling agrees with what was observed before in an impact oscillator system at intermittend transition to chaos, cf. [9, 23].

2 A single-degree-of-freedom friction-oscillator with excitation

We start by decribing the physical and mathematical setup of the system. Afterwards we are going to analyze the bifurcations corresponding to the change of some parameters.

The systems consists of a mass on a rough belt which turns around with constant velocity $v_0 = 1$. The mass is attached to a spring and periodically excited by a forcing, as is shown in Fig. 1.

Fig. 1: The friction-oscillator

Fig. 2: Qualitative sketch of the friction characteristic μ from (2)

To derive the governing equation for this system, three kinds of forces have to be taken into account:

1. the force created by the excitation, which is choosen to be $\gamma \cos(\eta t)$ with $\gamma \geq 0$ and $\eta > 0$,
2. the restoring force of the spring which is (by Hooke's law) proportional to $-x(t)$, and
3. the friction force due to dry fiction of the mass with the belt. By Coulomb's law, this friction force equals the negative of the constant normal force multiplied by the friction coefficient and by the sign of the relative velocity $(\dot{x}(t) - 1)$ of the mass.

The balance of forces thus results in a discontinuous equation, which after a suitable normalization (also ensuring that $(x, \dot{x}) = (0, 0)$ is a stationary solution of the unforced system with $\gamma = 0$) may be written as

$$\ddot{x}(t) + x(t) + a[\mu(1) + \mu(\dot{x}(t) - 1)\,\mathrm{sgn}(\dot{x}(t) - 1)] = \gamma \cos(\eta t) \tag{1}$$

for $t \in \mathbb{R}$ with a parameter $a > 0$; cf. [31, 32, 37]. Here μ is the friction characteristic, which is taken as

$$\mu(v) = \frac{\mu_0 - \mu_1}{1 + \lambda_0 |v|} + \mu_1 + \lambda_1 |v|^2 \tag{2}$$

for $v \in \mathbb{R}$, so that in particular $\mu(-v) = \mu(v)$. We shall also assume that $\mu_0 > \mu_1 > 0$, i.e. the static friction coefficient exceeds the kinetic friction coefficient, cf. Fig. 2 above. In (1) resp. (2),

$$a = 10, \quad \mu_0 = 0.4, \quad \mu_1 = 0.1, \quad \lambda_0 = 1.42 \quad \text{and} \quad \lambda_1 = 0.01$$

always will be fixed (cf. [32]), whereas the amplitude γ and the frequency $2\pi/\eta$ of the forcing are allowed to vary; they will later be used as bifurcation parameters.

Now the problem is that (1) wouldn't make sense for $\dot{x}(t)=1$ if one would define $\mathrm{sgn}(0)=0$. To see this, we suppose that a solution stays in the discontinuity surface $\mathcal{D}=\{(x,v,t)\in\mathbb{R}^3 : v=1\}$ for a non zero time interval, i.e. $\dot{x}(t)=1$ for $t\in(t_0,t_1)$. In physical terms this means that the mass sticks on the belt for some time. Then (1) and (2) together with $x(t)=(t-t_0)+x(t_0)$ for $t\in(t_0,t_1)$ imply

$$(t-t_0)+x(t_0)+a[\mu(1)+\mu_0\,\mathrm{sgn}(0)]=\gamma\cos(\eta t) \quad \text{for} \quad t\in(t_0,t_1),$$

a contradiction. So a common device (cf. [14, 15]) to obtain a precise mathematical setup for this type of problems is to replace the sgn-function by its multivalued version

$$\mathrm{Sgn}(v)=\begin{cases}\{v/|v|\} & : \quad v\neq 0\\ [-1,1] & : \quad v=0\end{cases}, \tag{3}$$

so that $\mathrm{Sgn}(v)\subset\mathbb{R}$ now is a set for every $v\in\mathbb{R}$. Moreover, we also relax (1) to the corresponding differential inclusion

$$\ddot{x}(t)+x(t)+a\mu(1)\in\gamma\cos(\eta t)-a\mu(\dot{x}(t)-1)\,\mathrm{Sgn}(\dot{x}(t)-1) \quad \text{a.e. in} \quad \mathbb{R}. \tag{4}$$

For $\dot{x}(t)\neq 1$, inclusion (4) is the same as equation (1), but for $\dot{x}(t)=1$ this inclusion (4) only requires

$$\ddot{x}(t)+x(t)+a\,\mu(1)\in[-a\,\mu_0+\gamma\cos(\eta t),\,a\,\mu_0+\gamma\cos(\eta t)]. \tag{5}$$

We remark that for differential inclusions there is a well-established theory about the existence of solutions. (In our case, a solution is understood to be C^1 with absolutely continuous $\dot{x}$ and locally integrable $\ddot{x}$.) By means of [14, Corollary 5.2] or [15, Theorem 1, p.77] we hence can assume that for every $(x_*,v_*,t_*)\in\mathbb{R}^3$ there exists a solution x of (4), unique to the right of t_*, with $(x(t_*),\dot{x}(t_*))=(x_*,v_*)$.

Note that we cannot expect solutions to be unique to the left, since a solution of the inclusion may stay in the discontinuity surface $\mathcal{D}$ for some time. If e.g. the mass sticks on the belt for $t\in(t_0,t_1)$, then (5) gives $\varphi^-(t)\le x(t)\le\varphi^+(t)$ in (t_0,t_1) with

$$\varphi^{\pm}(t)=-a\mu(1)\pm a\mu_0+\gamma\cos(\eta t), \quad \text{hence} \quad \varphi^+(t)-\varphi^-(t)=2a\mu_0>0, \tag{6}$$

and the solution moves on a straight line in $\mathcal{S}=\{(x,1,t)\in\mathcal{D} : \varphi^-(t)\le x\le\varphi^+(t)\}$ until it can't stay in the sliding-motion region $\mathcal{S}$ any longer. Fig. 3 below shows in (x,v,t)-coordinates an example of a trajectory which starts in $\mathcal{D}^+=\{(x,v,t)\in\mathbb{R}^3 : v>1\}$ at time $t=t_0$, then enters $\mathcal{S}$ at time $t=t_1$, afterwards stays in $\mathcal{S}$ up to time $t=t_2$, and then exits to $\mathcal{D}^-=\{(x,v,t)\in\mathbb{R}^3 : v<1\}$.

After having described the setup for the problem we now study bifurcations w.r.t. $\gamma\ge 0$ and $\eta\ge 0$. We remark that, contrary to the related results in e.g. [31, 32], our aim is to localize the various different regions in $\{(\eta,\gamma) : \eta,\gamma\ge 0\}$ which correspond to qualitatively different behaviour of the system. In this way we will obtain a more complete picture of the dynamics instead of particular examples.

To begin with, the unforced system with $\gamma=0$ is described in Fig. 4 below by a phase portrait from [40]. Here we have an asymptotically stable limit cycle, and the stationary solution $x=0$ is unstable. This is obvious from the figure (it also may be proved by using a non-smooth version of the Poincaré-Bendixon theorem, cf. [15]), and was observed before in [32, Fig. 3].

Fig. 3: Example of a trajectory of (4)

Fig. 4: Phase portrait for $\gamma = 0$

If the forcing is turned on ($\gamma > 0$), we have studied the additional effect of the excitation on the dynamics by investigation of a 1-dimensional Poincaré-map. To define this map, let

$$\mathcal{R} = \{(\varphi^+(t), 1, t) : t \in \mathbb{R}\} \subset \mathcal{S} \tag{7}$$

denote the "right-hand side" of $\mathcal{S}$. It may be seen that solutions starting in $\mathcal{R}$ at time $t = t_0$ will return to $\mathcal{R}$ at some time $t_1 > t_0$. Since solutions are unique to the right, the Poincaré-map

$$P : \mathcal{R} \to \mathcal{R}, \quad (\varphi^+(t_0), 1, t_0) \mapsto (\varphi^+(t_1), 1, t_1) \mapsto \text{next exit point from } \mathcal{R} \text{ after } \mathrm{t} = \mathrm{t}_1 \tag{8}$$

is therefore well-defined, cf. Fig. 5.

Fig. 5: Construction of P and h

Fig. 6: The different regions in $\{(\eta, \gamma) : \eta, \gamma \geq 0\}$ which correspond to qualitatively different behaviour of h

Note that due to the geometry of $\mathcal{S}$, at some boundary points it may be impossible to exit from $\mathcal{S}$, so that the next possible exit point $(x_2, 1, t_2)$ may in fact have $t_2 > t_1$. We also remark that the construction of P is analogous to the construction of the "impact map" in impacting systems, cf. [8, 39, 10]. Because the forcing $\gamma\cos(\eta t)$ has period $2\pi/\eta$, the map P from (8) naturally carries over to a one-dimensional $h : [0, 2\pi/\eta] \to [0, 2\pi/\eta]$, as is explained in Fig. 5 above. Of course, h depends on γ and η, and will be thought of as a map from $[0, 2\pi]$ to itsself by indentifying h with the rescaled $\tilde{h} : [0, 2\pi] \to [0, 2\pi]$, $\tau \mapsto \eta h(\tau/\eta)$. Our aim was to classify the properties of h in dependence of parameters $p = (\eta, \gamma)$. The result is described by Fig. 6 above from [40].

To explain the typical behaviour in the different regions P_{dm}, P_{c}, P_{n}, $P_{\mathrm{d}} \cup P_{\mathrm{n}}$ and P_{p} in Fig. 6, let us start with P_{dm}. For parameters $p = (\eta, \gamma) \in P_{\mathrm{dm}}$ the map h is smooth and monotonously increasing. (This holds of course only if h is considered as a map into $\mathbb{R}$, since due to periodicity h may have jumps of height 2π.) Therefore one can use the p-dependent rotation numbers

$$\mathrm{rot}_h(\tau) := \frac{\eta}{2\pi} \lim_{n\to\infty} \frac{h^n(\tau) - \tau}{n} \tag{9}$$

(cf. [2, Chapter 3] for details) to obtain information about the attractor. Because h is monotonic, rot_h is independent of τ, and since h is continous, $\mathrm{rot}_h \in \mathbb{Q}$ if and only if h has a periodic orbit, i.e. $(h^n(\tau_0))_{n\in\mathbb{N}}$ is periodic for some τ_0. Moreover, in that case all periodic orbits are structurally stable and of the same (minimal) period. On the other hand, if $\mathrm{rot}_h \in \mathbb{R} \setminus \mathbb{Q}$, then all orbits are dense since h is monotonic and C^2. It is also important that the rotation numbers depend continuously on the parameter $p \in P_{\mathrm{dm}}$, since this excludes the possibility of period doublings. To illustrate these results, we consider the example $\gamma = 0.46$. Then $(\eta, \gamma) \in P_{\mathrm{dm}}$ for all $\eta \in [0, 2]$, cf. Fig. 6.

Fig. 7a: Bifurcation diagram for $\gamma = 0.46$ and $\eta \in [0, 2]$

The bifurcation diagram Fig. 7a shows, depending on $\eta \in [0, 2]$, the attractor of $h = h_\eta : [0, 2\pi] \to [0, 2\pi]$. By comparing the bifurcation diagram with h itsself, one thus can obtain rather complete information about the dynamics. As an example, for $\eta = 0.9$ an asymptotically stable fixed point $\tau_0 \approx 2.72$ of h is found (cf. Fig. 7c below), and this fixed point of course corresponds to an asymptotically stable periodic solution of the original friction-oscillator system. In Fig. 7a the 1-periodic orbits with integer rotation number 1 and 2 can be seen, cf. Fig. 7b below. Since they are structurally stable, they remain unchanged in large intervals of η.

Fig. 7b: Rotation numbers for $\gamma = 0.46$ and $\eta \in [0,2]$

Fig. 7c: h for $\gamma = 0.46$ and $\eta = 0.9$

The picture changes for parameters $p = (\eta, \gamma) \in P_{\mathrm{c}}$, since then h is still continuous, but loses its monotonicity. Therefore the rotation numbers from (9) may not be well-defined, and thus only bifurcation diagrams or Lyapunov exponents can be investigated. As an example, we consider $\gamma = 0.7$ and $\eta \in [1, 1.5]$. Then we have $p = (\eta, \gamma) \in P_{\mathrm{c}}$, cf. Fig. 6. The corresponding bifurcation diagram is shown in Fig. 8a.

Fig. 8a: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1, 1.5]$

Fig. 8b: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1.086, 1.094]$

Due to the stronger forcing the structurally stable regions of the periodic solutions are quite large, and contrary to the case $p \in P_{\mathrm{dm}}$) also period doublings leading to chaos are possible for $p \in P_{\mathrm{c}}$. This is illustrated by Fig. 8b above, an enlargement of the region $\eta \in [1.086, 1.094]$ from Fig. 8a.

Next we want to have a closer look on the parameter region $\eta \in [1.15, 1.25]$ from Fig. 8a, which is enlarged in Fig. 8c below. It can be seen that, starting from $\eta_* \approx 1.175$, consecutive period-doublings and period-undoublings from a 2-periodic orbit to a 4-periodic orbit and back are established. To our knowledge, this kind of bifurcation has not been observed before, and we remark that it results from "grazing", i.e. from a tangential arrival of trajectories at the discontinuity surface. To explain this in greater detail, the 2^{nd} resp. 4^{th} iterate of $h_\eta : [0, 2\pi] \to [0, 2\pi]$ for $\eta = \eta_*$ and $\eta = 1.19$ together with the corresponding phase portraits are given in Fig. 8d below, where we have $\gamma = 0.7$. The fixed point at $\tau_* \approx 1.681$

of $h^2_{\eta_*}$ corresponds to the 2-periodic solution shown in the phase portrait above. Since the trajectories arrive tangentially at the discontinuity surface, obviously the system cannot be structurally stable for $\gamma = 0.7$ and $\eta = \eta_*$.

Fig. 8c: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1.15, 1.25]$

Fig. 8d (I): h^2_η for $\eta = \eta_*$ h^2_η for $\eta = 1.19$ h^4_η for $\eta = \eta_*$ h^4_η for $\eta = 1.19$

Fig. 8d (II): Phase portrait for $\gamma = 0.7$ and $\eta = \eta_*$ Phase portrait for $\gamma = 0.7$ and $\eta = 1.19$

It turns out that for $\eta \neq \eta_*$ to the right of and near to η_*, the corresponding fixed point τ near τ_* of h^2_η has lost its stability (invisible in the graph of h^2_η), but τ has become stable for h^4_η, so that a stable 4-periodic orbit appears. This is possible since the slope of $h^4_{\eta_*}$ at

$\tau = \tau_*$ equals -1, cf. Fig. 8d (I). Approaching the next critical $\eta \approx 1.198$ from the left, this process is reversed, and so on.

We note that grazing bifurcations also were studied before in impacting oscillator systems, cf. [27, 6, 13] and the references therein, but a bifurcation structure like in Fig. 8c as a consequence of grazing seems to be new.

The Lyapunov exponents corresponding to Fig. 8c are given in Fig. 8e. They also indicate the change of stability at $\eta_* \approx 1.175$.

Fig. 8e: Lyapunov exponents for $\gamma = 0.7$ and $\eta \in [1.15, 1.25]$

These Lyapunov exponents numerically were calculated as

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |h'(h^i(\tau_0))| \tag{10}$$

with τ_0 being a point of the attractor, cf. [18]. We remark that in numerical experiments the convergence of (10) is rather fast. To obtain the scaling of Lyapunov exponents near the bifurcation point $\eta_* \approx 1.175$, we plotted

$$\lambda(\eta_* + \varepsilon)/|\ln \varepsilon| \quad \text{for} \quad \varepsilon = 10^{-2}, \ldots, 10^{-9}.$$

Here $\lambda(\eta)$ denotes the exponent corresponding to the parameter η. The result is summarized in Fig. 8f.

Fig. 8f: Scaling of the Lyapunov exponent near $\eta_* \approx 1.175$ with $\gamma = 0.7$

Fig. 8g: Scaling of the Lyapunov exponent near $\eta_0 \approx 1.243$ with $\gamma = 0.7$

So it seems that $\lambda \sim 1/|\ln\varepsilon|$ near η_*, a scaling which was obtained at transition to chaos in impact oscillator systems, cf. [9, 23]. Like in these systems, also "intermittency" can arise in our example of the friction-oscillator. Here this term describes the situation when long phases of regular behaviour (the so-called "laminar phases") are suddenly interrupted by short irregular phases without previous period-doublings, cf. e.g. [29]. In the present example, intermittency results from the loss of stability of periodic orbits, e.g. at $\eta_0 \approx 1.243$, cf. Fig. 8c. For the case of impacting oscillators without friction, this phenomenon, which is due to the collision of a periodic orbit with the discontinuity surface, is well-understood, cf. [4, 17, 8, 24], and it is also remarkable that in our present example of the friction-oscillator the corresponding scaling of Lyapunov exponents at these kind of bifurcation points is the same as it was observed before for the impacting oscillators in [9]. To be more precise, numerically it seems that also $\lambda \sim 1/|\ln\varepsilon|$ near $\eta_0 \approx 1.243$, cf. Fig. 8g above. Note that now we have $\lambda(\eta_0+\varepsilon) < 0$ already for $\varepsilon = 10^{-6}$, so we only considered $\varepsilon = 10^{-7}, 10^{-8}, 10^{-9}$.

Next we turn to parameters $p = (\eta, \gamma) \in P_{\mathrm{n}}$, cf. Fig. 6. For those parameters, the map h is neither continuous nor monotone (in particular, h is not a diffeomorphism of the circle), and oscillations of higher order (i.e. loops) are possible below the critical velocity $v_0 = 1$ of the belt, cf. the phase portrait from [40] in Fig. 9a.

Fig. 9a: Oscillation of order two for $\gamma = 0.7$ and $\eta = 1$

Fig. 9b: Oscillation of order zero for $\gamma = 1.3$ and $\eta = 1$

Additionally, for $p \in P_{\mathrm{d}}$ there can arise oscillations of order zero, cf. Fig. 9b above (from [40]), where $p = (1.3, 1) \in P_{\mathrm{n}} \cup P_{\mathrm{d}}$. The corresponding h is shown below.

Fig. 9b: h for $\gamma = 1.3$ and $\eta = 1$

Fig. 9c: A dead zone

A remarkable new feature for parameters $p \in P_{\mathrm{d}}$ is that h is not onto, i.e. there are "dead zones". This means that due to the geometry of the sliding-motion region $\mathcal{S}$ some points on the right end $\mathcal{R}$ of $\mathcal{S}$ cannot be exit points, and hence are not in $h([0,2\pi])$ by definition of h. Again this phenomenon is due to "grazing", here in a 2-dimensional form, since trajectories are tangentially to $\mathcal{R}$, cf. Fig. 9c above. To explain the physically meaning of a dead zone, we consider the differential inclusion (4), which by means of (6) may be written as

$$\ddot{x}(t) \in [\varphi^{-}(t) - x(t), \varphi^{+}(t) - x(t)] .$$

Hence, if a trajectory reaches at some t_1 the right boundary $\mathcal{R}$ where one has $\varphi^{+}(t_1) = x(t_1)$, then necessarily $\ddot{x}(t_1) \leq 0$. Since for an exit from the surface the velocity has to become different from $v = 1$, the value of $\ddot{x}$ (or at least of some derivative of higher order) has to be non-zero. Consequently, in boundary points like p_1 in Fig. 9c, the mass is not accelerated by the forcing and therefore has to stick for some more time.

Finally the Lyapunov exponents, calculated by means of (10), for $\gamma = 1.3$ and $\eta \in [0.8, 1.8]$, hence $(\eta, \gamma) \in P_{\mathrm{d}} \cup P_{\mathrm{n}}$, are given in Fig. 9d.

Fig. 9d: Lyapunov exponents for $\gamma = 1.3$ and $\eta \in [0.8, 1.8]$

Fig. 9e: A pierce for $\gamma = 3$ and $\eta = 1$

The last parameter-region from Fig. 6 we have to explain is P_{p}. Here the forcing is strong enough to allow pierces, cf. Fig. 9e above. A more detailled analysis for $p = (\eta, \gamma) \in P_{\mathrm{p}}$ may be carried out analogously to what we described before.

3 Concluding Remarks

(i) The friction-oscillator system considered is 3-dimensional, and hence there are three Lyapunov exponents. Because the motion always returns to the lower-dimensional discontinuity surface, a test-volume degenerates. Therefore one of the Lyapunov exponents has to equal $-\infty$. Since a second one is expected to be zero, the sign of the third Lyapunov exponent should completely determine the behaviour of the system. That only one Lyapunov exponent is of importance corresponds to the fact that it is possible to reduce the study of the dynamics to the investigation of a 1-dimensional Poincaré-map.

It should also be noted that there is a difference between smooth and non-smooth dynamical system concerning the existence of Lyapunov exponents from a theoretical point of view. In smooth systems $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and corresponding (semi-) flow

φ (i.e. $x(t) = \varphi(t, x_0)$ solves the differential equation and has $x(0) = x_0$), usually the multiplicative ergodic theorem of Oseledets (cf. [28, 1, 21]) is applied to ensure the existence of the Lyapunov exponents. In the smooth case this is possible, since $T(t,x) = \partial_x \varphi(t,x)$ defines a canonical cocycle for the semiflow. Because in non-smooth systems the semiflow can't be expected to be differentiable w.r. to the initial value, a suitable substitute has to be constructed, cf. [22] and also [26] for a more formal approach.

(ii) We considered only the continuous friction characteristic μ from (2), but there are other choices. For the simpler discontinuous

$$\mu(v) = \begin{cases} \mu_1 & : \quad v \neq 0 \\ \mu_0 & : \quad v = 0 \end{cases},$$

i.e. a characteristic which distinguishes only between the static friction coefficient μ_0 and the kinetic friction coefficient $0 < \mu_1 < \mu_0$, a similiar systematical classification of the different parameter-regions in $\{(\eta, \gamma) : \eta, \gamma \geq 0\}$ corresponding to different behaviours of h were obtained in [40]. One may also consult [31, 32, 37], where the dynamics were analyzed for some specific values of γ and η.

(iii) The above model may be extended by placing more additional masses on the belt, cf. [31, 32, 16]. If these masses are coupled by means of springs and if there is no external forcing, this is called the Burridge-Knopoff model in geology and thought of to describe earthquakes; the masses are representing the different tectonic plates, cf. [11, 12]. It will be interesting to investigate in more detail the resulting dynamics depending on the different parameters of the system by means of Lyapunov exponents. Of course these dynamics are expected to be more complicated then in the case of a single mass.

(iv) We remark that a lot of features of the bifurcation structure of the friction-oscillator are still not well-understood, cf. e.g. Fig. 8c, where at $\eta \approx 1.209$ suddenly the cascade of consecutive period doublings and undoublings is interrupted, and a 3-periodic orbit appears.

(v) Of course a lot of other non-smooth systems have been studied before, cf. e.g. [30] and the references therein, and also [3, 35, 36]. One of the most interesting ones consists of a rotor touching a rigid boundary, since in this example it is possible to have infinitely many impacts in a finite time, cf. the extensive study of this model in [38], and also [25].

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