Long-Time Asymptotics for a Classical Particle Interacting with a Scalar Wave Field

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Abstract We consider the Hamiltonian system consisting of scalar wave field and a single particle coupled in a translation invariant manner. The point particle is subject to a confining external potential. The stationary solutions of the system are a Coulomb type wave field centered at those particle positions for which the external force vanishes. We prove that solutions of finite energy converge, in suitable local energy seminorms, to the set of stationary solutions in the long time limit $t \to \pm\infty$. The rate of relaxation to a stable stationary solution is determined by spatial decay of initial data.

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1 Introduction and Main Results

We consider the Hamiltonian system consisting of a real scalar field $\phi(x)$, $x \in \mathbb{R}^3$, and a point particle with position $q \in \mathbb{R}^3$. The field is governed by the standard linear wave equation. The point particle is subject to an external potential, V, which is confining in the sense that $V(q) \to \infty$ as $|q| \to \infty$. The interaction between the particle and the scalar field is local, translation invariant, and linear in the field. We would like to understand the long-time behavior of the coupled system. On a physical level one argues that the force due to the potential V accelerates the particle. Thereby energy is transferred to the wave field a part of which is eventually transported to infinity. Thus the particle feels a sort of friction and we expect that as $t \to \infty$ it will come to rest at some critical point q^* of V, where $\nabla V(q^*) = 0$. Our main achievement here is to give this argument a precise mathematical setting.

The by far most important physical realization of our system is the electromagnetic field governed by Maxwell's equations and coupled to charges by the Lorentz force. The physical mechanism just described is then known as radiation damping, an ubiquituous phenomenon. There is a huge literature on this subject [3, 5, 19, 24, 31, 32]. Somewhat surprisingly, there is however little mathematical work, notable exceptions being [1, 2]. In our paper we simplify somewhat by ignoring the vector character of the electromagnetic field and hope to come back to the full coupled Maxwell-Lorentz equations at some later time.

Let $\pi(x)$ be the canonically conjugate field to $\phi(x)$ and let p be the momentum of the particle. The Hamiltonian (energy functional) reads then

$$H(\phi, q, \pi, p) \equiv (1+p^2)^{1/2} + V(q) + \frac{1}{2}\int d^3x\, (|\pi(x)|^2 + |\nabla\phi(x)|^2) + \int d^3x\, \phi(x)\rho(x-q). \quad (1.1)$$

The mass of the particle and the propagation speed for ϕ have been set equal to 1. A relativistic kinetic energy has been chosen only to ensure that $|\dot{q}| < 1$. In spirit the interaction term is simply $\phi(q)$. This would result however in an energy which is not bounded from below. Therefore we smoothen out the coupling by the function $\rho(x)$, which is assumed to be radial and to have compact support. In analogy to the Maxwell-Lorentz equations we call $\rho(x)$ the "charge distribution". Taking formally variational derivatives in (1.1), the coupled dynamics becomes

$$\begin{aligned} &\dot{\phi}(x,t) = \pi(x,t), && \dot{\pi}(x,t) = \Delta\phi(x,t) - \rho(x-q(t)), \\ &\dot{q}(t) = p(t)/(1+p^2(t))^{1/2}, && \dot{p}(t) = -\nabla V(q(t)) + \textstyle\int d^3x \phi(x,t)\nabla\rho(x-q(t)). \end{aligned} \quad (1.2)$$

The stationary solutions for (1.2) are easily determined. We define for $q \in \mathbb{R}^3$

$$\phi_q(x) = -\int \frac{d^3y}{4\pi|y-x|}\rho(y-q).$$

Let $S = \{q^* \in \mathbb{R}^3 : \nabla V(q^*) = 0\}$ be the set of critical points for V. Then the set of stationary solutions, $\mathcal{S}$, is given by

$$\mathcal{S} = \{(\phi, q, \pi, p) = (\phi_{q^*}, q^*, 0, 0) =: Y_{q^*} | \ \ q^* \in S\}. \quad (1.3)$$

One natural goal is to investigate the domain of attraction for $\mathcal{S}$ and in particular to prove that a solution of (1.2) converges to some stationary state $Y_{q^*} = (\phi_{q^*}, q^*, 0, 0) \in \mathcal{S}$ in the long time limit $t \to \infty$. Since the total energy is conserved, the only meaningful notion of convergence is a local comparison, i.e. a comparison between the true time-dependent solution and the asymptotic stationary solution in suitable local norms.

More ambituously one would like to estimate the rate of convergence to Y_{q^*}. As a preliminary step one linearizes (1.2) at some stationary state Y_{q^*}, $q^* \in S$. One observes that the stability of Y_{q^*} is in correspondence with the "stability" of the potential $V(q)$ at the point q^*. In fact, if $d^2V(q^*) > 0$ as a quadratic form, then on the linearized level the relaxation to Y_{q^*} is exponentially fast. For small deviation from Y_{q^*} the linearized part should dominate the nonlinear part of (1.2) and one expects a full neighborhood of Y_{q^*} to be contracted to Y_{q^*} at an exponential rate in time. This should still hold if the initial data have an exponential decay in space. For a power decay of initial data in space one cannot hope for more than a power rate of contraction. On the other hand if $d^2V(q^*)$ has some negative eigenvalues then Y_{q^*} is linearly unstable. An interesting case is when $d^2V(q^*) \geq 0$ including a zero eigenvalue, which however will not be discussed here.

To state our main results we need some assumptions on V and ρ. The potential is in fact fairly arbitrary. We only need

$$V \in C^2(\mathbb{R}^3), \qquad \lim_{|q|\to\infty} V(q) = \infty. \tag{P}$$

For the charge distribution ρ we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3), \qquad \rho(x) = 0 \ \text{ for } \ |x| \geq R_\rho, \qquad \rho(x) = \rho_r(|x|). \tag{C}$$

However, as to be explained, we also need that all "modes" of the wave field couple to the particle. This is formalized by the Wiener condition

$$\hat\rho(k) = \int d^3x\, e^{ikx}\rho(x) \neq 0 \qquad \text{for} \quad k \in \mathbb{R}^3. \tag{W}$$

In particular the total charge $\hat\rho(0)$ does not vanish. If (W) is violated, then on the linearized level one can construct periodic solutions provided the coupling strength is adjusted to the zeros of $\hat\rho$. If such a periodic solution would persist for the full nonlinear equations (1.2), then global convergence to $\mathcal{S}$ would fail. In the Appendix we will construct examples of charge distributions satisfying both (C) and (W).

Next we must have a little closer look at (1.2). This means we have to introduce a suitable phase space and have to establish existence and uniqueness of solutions. A point in phase space is referred to as state. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with norm $|\!|\cdot|\!|$ and scalar product $(\cdot,\cdot)$, and let $D^{1,2}$ be the completion of real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\phi(x)\| = |\!|\nabla\phi(x)|\!|$. Equivalently, using Sobolev's embedding theorem, $D^{1,2} = \{\phi(x) \in L^6(\mathbb{R}^3) : \ |\nabla\phi(x)| \in L^2\}$ (see [20]). Let $|\!|\phi|\!|_R$ denote the norm in $L^2(B_R)$ for $R > 0$, where $B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Then the seminorms $\|\phi\|_R = |\!|\nabla\phi|\!|_R + |\!|\phi|\!|_R$ are continuous on $D^{1,2}$.

We denote by $\mathcal{E}$ the Hilbert space $D^{1,2} \oplus \mathbb{R}^3 \oplus L^2 \oplus \mathbb{R}^3$ with finite norm

$$\|\,Y\|_{\mathcal{E}} = \|\phi\| + |q| + |\!|\pi|\!| + |p| \quad \text{for} \quad Y = (\phi, q, \pi, p)\,.$$

For smooth $\phi(x)$ vanishing at infinity we have

$$\begin{aligned}-\frac{1}{8\pi}\int\!\!\int d^3x\; d^3y\; \frac{\rho(x)\rho(y)}{|x-y|} = \frac{1}{2}(\rho,\Delta^{-1}\rho) \qquad &\leq \frac{1}{2}|\!|\nabla\phi|\!|^2 + (\phi(x),\rho(x-q)) \\ &\leq \ |\!|\nabla\phi|\!|^2 - \frac{1}{2}(\rho,\Delta^{-1}\rho).\end{aligned} \tag{1.4}$$

Therefore $\mathcal{E}$ is the space of finite energy states and in particular $\|Y_q\|_{\mathcal{E}} < \infty$. Let us note that $D^{1,2}$ is not contained in L^2 and for instance $|\phi_q| = \infty$. The lower bound in (1.4) implies that the energy (1.1) is bounded from below. In the point charge limit this lower bound tends to $-\infty$. We define the local energy seminorms by

$$\|Y\|_R = \|\phi\|_R + |q| + |\pi|_R + |p| \quad \text{for} \quad Y = (\phi, q, \pi, p) \tag{1.5}$$

for every $R > 0$, and denote by $\mathcal{E}_F$ the phase space $\mathcal{E}$ equipped with the Fréchet topology induced by these local energy seminorms. Let dist_R denote the distance in the seminorm (1.5). Note that the spaces $\mathcal{E}_F$ and $\mathcal{E}$ are metrisable.

Proposition 1.1 *For every $Y^0 = (\phi^0, q^0, \pi^0, p^0) \in \mathcal{E}$ the Hamiltonian system (1.2) has a unique solution $Y(t) = (\phi(t), q(t), \pi(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ with $Y(0) = Y^0$.*

We refer to Section 2 where also the precise notion of solution is explained.

¿From physical intuition one is tempted to conjecture that every solution $Y(t)$ of finite energy will converge to some stationary state Y_{q^*} as $t \to \infty$. We do not achieve such a global result. First of all, the decay of initial fields at infinity should be as required by finite energy but with some additional smoothness. Secondly the set S need not be discrete. In this case $Y(t)$ may never settle to a definite Y_{q^*} but wander around to approach $\mathcal{S}$ only as a set.

Theorem 1.2 *Let (P), (C), (W) hold. Let the initial state $Y^0 = (\phi^0, q^0, \pi^0, p^0) \in \mathcal{E}$ have the following decay at infinity: For some $R_0 > 0$ the functions $\phi^0(x)$,$\pi^0(x)$ are C^2,C^1-differentiable respectively outside the ball B_{R_0} and for $|x| \to \infty$*

$$DY^0(x) = |\phi^0(x)| + |x|(|\nabla\phi^0(x)| + |\pi^0(x)|) + |x|^2(|\nabla\nabla\phi^0(x)| + |\nabla\pi^0(x)|) = \mathcal{O}(|x|^{-\sigma}) \tag{1.6}$$

with some $\sigma > 1/2$. Then for the solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to (1.2) with $Y(0) = Y^0$
(i) $Y(t)$ converges as $t \to \infty$ in Fréchet topology of the space $\mathcal{E}_F$ to the set $\mathcal{S}$, i.e. for every $R > 0$

$$\lim_{t\to\infty} \text{dist}_R(Y(t), \mathcal{S}) = 0\,. \tag{1.7}$$

(ii) If the set S is discrete, then there exists a point $q^* \in S$ such that

$$Y(t) \longrightarrow Y_{q^*} \quad in \quad \mathcal{E}_F \quad as \quad t \longrightarrow \infty. \tag{1.8}$$

Remarks. (i) Since the Hamiltonian system (1.2) is invariant under time-reversal, our results also hold for $t \to -\infty$.
(ii) The assumption (C) can be weakened to finite differentiability and some decay of $\rho(x)$ at infinity.

To prove Theorem 1.2 we will estimate the energy dissipation by decomposing ϕ into a near and far field. Energy is dissipated in the far field. Since energy is bounded from below, such dissipation cannot go on forever and a certain energy dissipation functional has to be bounded. This dissipation functional can be written as a convolution. By a Tauberian theorem of Wiener, using (W), we conclude that $\lim_{t\to\infty} \ddot{q}(t) = 0$, and also $\lim_{t\to\infty} \dot{q}(t) = 0$ since $|q(t)|$ is bounded by some $q_0 < \infty$ due to (P). This implies that $\mathcal{A} = \{(\phi_q, q, 0, 0) : |q| \le q_0)\}$ is a compact attracting set. Relaxation and compactness reduce $\mathcal{A}$ to $\mathcal{S}$ as a minimal attractor.

To establish the rate of convergence in Theorem 1.2 the point $q^* \in S$ must be stable in the following sense.

Definition 1.3 *A point $q^* \in S$ is said to be stable if $d^2V(q^*) > 0$ as a quadratic form.*

Even for a stable $q^* \in S$, the slow decay of the initial fields in space will transform into a slow decay in time.

Theorem 1.4 *Let all assumptions of Theorem 1.2 hold, $V \in C^3(\mathbb{R}^3)$ and let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution of the system (1.2) converging to Y_{q^*} as in (1.8) with the stable point $q^* \in S$. Then*
i) for every $R, \varepsilon > 0$

$$\|Y(t) - Y_{q^*}\|_R = \mathcal{O}(t^{-\sigma+\varepsilon}) \quad as \quad t \to \infty. \tag{1.9}$$

ii) Let additionally

$$DY^0(x) \ = \mathcal{O}(e^{-\alpha|x|}) \quad as \quad |x| \to \infty \tag{1.10}$$

with some $\alpha > 0$. Then there exists a $\gamma^* = \gamma(q^*) > 0$ such that for every $R > 0$

$$\|Y(t) - Y_{q^*}\|_R = \mathcal{O}(e^{-\beta t}) \quad as \quad t \to \infty \tag{1.11}$$

with $\beta = \alpha$ if $\alpha < \gamma^*$ and with arbitrary $\beta < \gamma^*$ if $\alpha \geq \gamma^*$.

We will prove Theorem 1.4 in Sections 6 to 9 by controlling the nonlinear part of (1.2) by the linearized equation. For the linearized equation exponential convergence can be established by Paley-Wiener technics for complex Fourier transforms [23]. As a byproduct in Theorem 6.1 we will also establish exponential convergence for initial states not covered by Theorem 1.4.

Before entering into the proofs it may be useful to put our results in the context of related works. We establish here that solutions of a Hamiltonian system converge to an attractor, possibly consisting of an infinite number of points, in the long time limit. Such a behavior is familiar from dissipative systems. The mechanism is however completely different. For a dissipative system there is a local loss of "energy", whereas here energy is propagated to infinity. If the wave field in (1.2) would be enclosed in some finite volume, then Theorem 1.2 would not be valid. Propagation of energy to infinity is also the essence of scattering theory for Hamiltonian linear wave equations [17, 18, 21], [28]–[30] and for Hamiltonian nonlinear wave equations either with a unique "zero" stationary solution [4, 7, 8, 9, 22, 26, 27] or with small initial data [10, 11]. Note that the attractor consists then only of the zero solution in contrast to the case considered here.

Somewhat closer to our investigation are [12]–[14] where a one dimensional version of (1.2) is studied: the particle is coupled to an infinite string and moves only transversally subject to some confining external potential. The interaction with the string generates then a linear friction term for the dynamics of the particle and the attraction to stationary states can be studied by ordinary differential equation methods. [15] considers several such oscillators coupled to a string. In this case the effects of retarded interaction have to be controlled. For wave equations with local nonlinear terms a result similar to Theorem 1.4 is proved in [16].

2 Existence of dynamics, a priori estimates

We consider the Cauchy problem for the Hamiltonian system (1.2), which we write as

$$\dot{Y}(t) = F_0(Y(t)) + F_1(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y^0. \tag{2.1}$$

All derivatives are understood in the sense of distributions. Here $Y(t) = (\phi(t), q(t), \pi(t), p(t))$, $Y^0 = (\phi^0, q^0, \pi^0, p^0) \in \mathcal{E}$, and $F_0 : Y \mapsto (\pi, 0, \Delta\phi, 0)$. One is interested also in situations where the particle is allowed to travel to infinity, e.g. when the external potential $V(q)$ vanishes identically. The existence of dynamics and the relaxation of the acceleration $\ddot{q}(t)$ are in fact true under such more general conditions. We state then as a weaker form of (P),

$$V \in C^2(\mathbb{R}^3), \qquad V_0 := \inf_{q\in\mathbb{R}^3} V(q) > -\infty. \tag{$P_{\min}$}$$

Lemma 2.1 *Let (C) and $(P_{\min})$ hold. Then*
(i) For every $Y^0 \in \mathcal{E}$ the Cauchy problem (2.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
(ii) For every $t \in \mathbb{R}$ the map $W_t : Y^0 \mapsto Y(t)$ is continuous both on $\mathcal{E}$ and on $\mathcal{E}_F$.
(iii) The energy is conserved, i.e.

$$H(Y(t)) = H(Y^0) \quad for \quad t \in \mathbb{R}. \tag{2.2}$$

(iv) The energy is bounded from below, and

$$\inf_{Y\in\mathcal{E}} H(Y) = 1 + V_0 - \frac{1}{2(2\pi)^3} \int d^3k \, \frac{|\hat{\rho}(k)|^2}{|k|^2}. \tag{2.3}$$

(v) The speed is bounded,

$$|\dot{q}(t)| \le q_1 < 1 \quad for \quad t \in \mathbb{R}. \tag{2.4}$$

(vi) If (P) holds, then the time derivatives $q^{(k)}(t)$, $k = 0, 2, 3$, also are uniformly bounded, i.e. there are constants $q_k > 0$, depending only on the initial data, such that

$$|q^{(k)}(t)| \le q_k \quad for \quad t \in \mathbb{R}. \tag{2.5}$$

Proof Let us fix an arbitrary $b > 0$ and prove (i)–(iii) for $\|Y^0\|_{\mathcal{E}} \le b$ and $|t| \le \varepsilon = \varepsilon(b)$ for some sufficiently small $\varepsilon(b) > 0$.
ad (i) Fourier transform provides the existence and uniqueness of solution $Y_0(t) \in C(\mathbb{R}, \mathcal{E})$ to the linear problem (2.1) with $F_1 = 0$. Let $W_t^0 : Y^0 \mapsto Y_0(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then uniqueness of solution to the (inhomogeneous) linear problem implies that (2.1) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to

$$Y(t) = W_t^0 Y^0 + \int_0^t ds \, W_{t-s}^0 F_1(Y(s)), \tag{2.6}$$

because $F_1(Y(\cdot)) \in C(\mathbb{R}, \mathcal{E})$ in this case. The latter follows from a local Lipschitz continuity of the map F_1 in $\mathcal{E}$: for each $b > 0$ there exist a $\kappa = \kappa(b) > 0$ such that for all $Y, Z \in \mathcal{E}$ with $\|Y\|_{\mathcal{E}}, \|Z\|_{\mathcal{E}} \le b$

$$\|F_1(Y) - F_1(Z)\|_{\mathcal{E}} \le \kappa \|Y - Z\|_{\mathcal{E}}. \tag{2.7}$$

For example, we have

$$\left| \int d^3x \, (\phi_1(x) - \phi_2(x)) \nabla\rho(x - q) \right| \le |\nabla(\phi_1 - \phi_2)||\rho|.$$

Moreover, by the contraction mapping principle, (2.6) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{E})$ with $\varepsilon > 0$ depending only on b.

ad (ii) The map $W_t : Y^0 \mapsto Y(t)$ is continuous in the norm $\|\cdot\|_{\mathcal{E}}$ for $|t| \le \varepsilon$ and $\|Y^0\| \le b$. To prove continuity of W_t in $\mathcal{E}_F$, let us consider Picard's successive approximation scheme

$$Y^N(t) = W_t^0 Y^0 + \int_0^t ds\, W_{t-s}^0 F_1(Y^{N-1}(s)), \quad N = 1, 2, \ldots$$

The third equation in this system implies $|\dot{q}^N(t)| < 1$ and therefore $|q(t)| < |q^0| + |t|$. Now we fix $t \in \mathbb{R}$ and choose $R > |q^0| + |t| + R_\rho$ with R_ρ from (C). From the explicit solution of the free wave equation $W_t^0 Y^0$ we conclude that every Picard's approximation $Y^N(t)$ and hence the solution $Y(t) = (\phi(x,t), q(t), \pi(x,t), p(t))$ for $|x| < R$ depends only on the initial data $(\phi^0(x), q^0, \pi^0(x), p^0)$ with $|x| < R + |t|$. Thus the continuity of W_t in $\mathcal{E}_F$ follows from the continuity in $\mathcal{E}$.
ad (iii) For $k = 0, 1, \ldots$ denote by $C_0^k(\mathbb{R}^3)$ the space of functions $\phi(x) \in C^k(\mathbb{R}^3)$ with compact support. For initial data $(\phi_0, \pi_0) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$ the solution $\phi = \phi(x,t)$ satisfies $\phi \in C^2(\mathbb{R}^3 \times \mathbb{R})$. Indeed, this is well known for the solution $W_t^0 Y^0$ of the linear wave equation. The integral representation (2.6) then implies the same property for ϕ. In addition, let Y^0 have compact support as in (1.10). Since $|q(t)| < |q^0| + |t|$, (2.6) implies

$$\phi(x,t) = 0 \quad \text{for} \quad |x| \ge |t| + \max\{R_0, |q^0| + |t|\}.$$

Thus, for such initial data energy conservation can be shown by integration by parts. Hence (iii) follows from the continuity of W_t and the fact that $C_0^3(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus C_0^2(\mathbb{R}^3) \oplus \mathbb{R}^3$ is dense in $\mathcal{E}$.
ad (iv) The lower bound from (1.4) implies

$$\begin{aligned} \inf_Y H(Y) &\ge 1 + V_0 + \inf_Y \Big(-\frac{1}{2}(\phi, \Delta\phi) + (\phi, \rho(\cdot - q)) \Big) \\ &= 1 + V_0 + \frac{1}{2}(\rho, \Delta^{-1}\rho). \end{aligned} \tag{2.8}$$

Since the infimum is attained for $Y = (\phi_q, q, 0, 0)$ in the limit as $V(q) \to V_0$, we have shown (2.3).

We use now energy conservation to ensure the existence of a global solution and its continuity. As in (2.8) we have

$$H(Y) \ge \frac{1}{2}|\pi|^2 + \frac{1}{4}|\nabla\phi|^2 + \sqrt{1+p^2} + V(q) + (\rho, \Delta^{-1}\rho),$$

and by energy conservation, for $|t| \le \varepsilon$,

$$\begin{aligned} \frac{1}{2}|\pi(t)|^2 + \frac{1}{4}|\nabla\phi(t)|^2 + \sqrt{1+p^2(t)} + V(q(t)) + (\rho, \Delta^{-1}\rho) \\ \le H(Y(t)) = H(Y^0). \end{aligned} \tag{2.9}$$

Therefore $(P_{\min})$ implies the a priori estimate

$$\|\phi(t)\| + |\pi(t)| + |p(t)| \le B \quad \text{for} \quad t \in \mathbb{R} \tag{2.10}$$

with B depending only on the norm $\|Y^0\|_{\mathcal{E}}$ of the initial data. Properties (i)–(iii) for arbitrary $t \in \mathbb{R}$ now follow from the same properties for small $|t|$ and from the a priori bound (2.10).
ad (v) Note first that (2.10) implies $|p(t)| \le p_0 < \infty$. Hence

$$|\dot{q}(t)|/(1 - \dot{q}^2(t))^{1/2} = |p(t)| \le p_0 < \infty,$$

which yields $|\dot{q}(t)| \le q_1 < 1$.
ad (vi) (P) and (2.9) imply (2.5) with $k = 0$. Since $|q(t)| \le q_0$ and $|\dot{q}(t)| \le q_1 < 1$, the last equation in (1.2) implies $|\ddot{q}(t)| \le q_2 < \infty$. Differentiating the last equation in (1.2) and using $|q^{(k)}(t)| \le q_k$ with $k = 0, 1, 2$, we finally obtain $|q^{(3)}(t)| \le q_3 < \infty$ for $t \in \mathbb{R}$. □

3 Energy dissipation functional

In this section we establish a lower bound on the total energy radiated to infinity in terms of the energy dissipation integral (3.1). Since the energy is bounded a priori, this integral has to be finite, which is then our main input for proving Theorem 1.2.

Let S^2 denote the unit sphere $|\omega| = 1$ in $\mathbb{R}^3$ with surface element area $d^2\omega$.

Proposition 3.1 *Let* $(P_{\min})$, (C) *hold and let* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *be a solution to (1.2) with initial state* $Y^0 = Y(0) \in \mathcal{E}$ *satisfying (1.6). Then*

$$\int_0^\infty dt \int_{S^2} d^2\omega \Big| \int d^3y\, \rho(y - q(t + \omega \cdot y)) \frac{\omega \cdot \ddot{q}(t + \omega \cdot y)}{(1 - \omega \cdot \dot{q}(t + \omega \cdot y))^2} \Big|^2 < \infty. \tag{3.1}$$

Proof Step 1. For $R > |q(t)| + R_\rho$ the energy $E_R(t)$ in the ball B_R at time $t > 0$ is defined by

$$\begin{aligned} E_R(t) &= \frac{1}{2} \int_{B_R} d^3x (|\pi(x,t)|^2 + |\nabla\phi(x,t)|^2) \\ &\quad + (1 + p^2(t))^{1/2} + V(q(t)) + \int d^3x \phi(x,t) \rho(x - q(t))\,. \end{aligned} \tag{3.2}$$

Let us denote $\omega(x) = x/|x|$, d^2x the surface area element of ∂B_R and $\overline{R}_0 = \max(R_0, |q^0| + R_\rho)$. Let $\Delta_R = [\overline{R}_0 + R, (R - \overline{R}_0)/q_1]$. Since $0 < q_1 < 1$, Δ_R is a nonempty interval of the length $|\Delta_R| \sim R(1 - q_1)/q_1$ for large R.

(1.6) implies that the solution $\phi(x,t)$ is C^1-differentiable in the region $t > R_0 + |x|$. Moreover, the estimate (2.4) insures that $|q(t)| < |q^0| + q_1 t$ for $t > 0$. Hence we get, similarly to (2.2),

$$\frac{d}{dt} E_R(t) = \int_{\partial B_R} d^2x \;\; \omega(x) \cdot \nabla\phi(x,t)\; \pi(x,t) \quad \text{for} \quad t \in \Delta_R. \tag{3.3}$$

The differentiability follows from the integral representation of the solution. Namely,

$$\phi(x,t) = \phi_r(x,t) + \phi_0(x,t) \quad \text{for} \;\; x \in \mathbb{R}^3, \;\; t > 0, \tag{3.4}$$

where $\phi_r(x,t)$ is the retarded potential and $\phi_0(x,t)$ is the Kirchhoff integral,

$$\phi_r(x,t) = -\frac{1}{4\pi} \int \frac{d^3y\; \theta(t - |x - y|)}{|x - y|} \rho(y - q(t - |x - y|)), \tag{3.5}$$

$$\phi_0(x,t) = \frac{1}{4\pi t} \int_{S_t(x)} d^2y\; \pi^0(y) + \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{S_t(x)} d^2y\; \phi^0(y) \right). \tag{3.6}$$

$S_t(x)$ denotes here the sphere $\{y : \;|y - x| = t\}$.

Step 2. We have, $\nabla\phi(x,t) = \nabla\phi_r(x,t) + \nabla\phi_0(x,t)$ and $\pi(x,t) = \pi_r(x,t) + \pi_0(x,t)$, where $\pi_r(x,t) = \dot{\phi}_r(x,t)$ and $\pi_0(x,t) = \dot{\phi}_0(x,t)$. Hence (3.3) reads

$$\frac{d}{dt} E_R(t) = \int_{\partial B_R} d^2x \;\; \omega(x) \cdot (\nabla\phi_r\; \pi_r + \nabla\phi_r\; \pi_0 + \nabla\phi_0\; \pi_r + \nabla\phi_0\; \pi_0) \;\; \text{for} \;\; t \in \Delta_R.$$

We separate the first term which turns out to be negatively definite from the remainder which is controlled simply by Cauchy-Schwarz. We have

$$\frac{d}{dt}E_R(t) \le \int_{\partial B_R} d^2x\Big(\omega(x)\cdot\nabla\phi_r\ \pi_r + \frac{1}{4}(|\nabla\phi_r|^2+|\pi_r|^2) + 2(|\nabla\phi_0|^2+|\pi_0|^2)\Big) \text{ for } t\in\Delta_R.$$

Integrating in t we obtain for $\overline{R}_0 < T < T_R = (R-\overline{R}_0)/q_1 - R \sim R(1-q_1)/q_1$

$$E_R(T+R) - E_R(\overline{R}_0+R) \le \int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ \ \Big(\omega(x)\cdot\nabla\phi_r\ \pi_r + \frac{1}{4}(|\nabla\phi_r|^2+|\pi_r|^2) + 2(|\nabla\phi_0|^2+|\pi_0|^2)\Big). \quad (3.7)$$

Step 3. The radiated energy must be bounded from below. Indeed, by conservation of energy, $E_R(\overline{R}_0+R) \le H(Y(t)) = H(Y_0)$. On the other hand, the local energy $E_R(T+R)$ is bounded from below since

$$E_R(t) \ge H(Y(t)) - \frac{1}{2}(\|\pi(t)\|^2 + \|\nabla\phi(t)\|^2) \ge -C_1(H(Y_0)-1-V_0) + C_2(\rho,\Delta^{-1}\rho),$$

cf. (2.8) and (2.9). Hence

$$E_R(T+R) - E_R(\overline{R}_0+R) \ge -I$$

with a constant $I<\infty$ not depending on R,T. Therefore, (3.7) implies

$$\begin{aligned} &-\int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ \ \Big((\omega(x)\cdot\nabla\phi_r\ \pi_r + \frac{1}{4}(|\nabla\phi_r|^2+|\pi_r|^2)\Big) \\ &\le I + 2\int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ (|\nabla\phi_0|^2+|\pi_0|^2) \quad \text{for } \ \overline{R}_0 < T < T_R. \end{aligned} \quad (3.8)$$

Step 4. In order to finish the proof we have to show that in the limit $R\to\infty$ and subsequently $T\to\infty$, the left hand side dominates the dissipation integral (3.1), while the right hand side remains bounded. This follows from next two lemmas.

Lemma 3.2 *For every fixed $T>\overline{R}_0$*

$$\nabla\phi_r(x,t) = -\pi_r(x,t)\ \omega(x) + \mathcal{O}(|x|^{-2}) \text{ in the region } \overline{R}_0 < t-|x| < T. \quad (3.9)$$

Lemma 3.3 *There exist $I_0<\infty$ such that*

$$\int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ (|\nabla\phi_0(x,t)|^2 + |\pi_0(x,t)|^2) \le I_0 < \infty \qquad \textit{for} \quad R,T>0. \quad (3.10)$$

Using Lemma 3.2 and 3.3 in (3.8), we obtain for every fixed $T>0$ and sufficiently large $R>R_T\sim Tq_1/(1-q_1)$

$$\int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ \ |\pi_r(x,t)|^2 \le C(I+I_0) + T\mathcal{O}(R^{-2}).$$

Hence, (3.5) implies

$$\int_{\overline{R}_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ \ |\int_{Q_T} d^3y \frac{1}{|x-y|}\frac{\partial}{\partial t}\rho(y-q(t-|x-y|))|^2 \le C(I+I_0) + T\mathcal{O}(R^{-2}), \quad (3.11)$$

where $Q_T = \{y \in \mathbb{R}^3 : \ |y| \le \max_{[0,T]} |q(t)| + R_\rho\}$. Furthermore, uniformly in $x \in \partial B_R$ and $y \in Q_T$

$$|x-y| \sim R \quad \text{and} \quad t + R - |x-y| = t + \omega \cdot y + \mathcal{O}(R^{-1}), \qquad \omega = x/|x|.$$

Thus taking the limit $R \to \infty$ in (3.11) we obtain

$$\int_{\overline{R}_0}^T dt \int_{S^2} d^2\omega \Big| \int d^3y \frac{\partial}{\partial t} \rho(y - q(t + \omega \cdot y)) \Big|^2 \le C(I + I_0).$$

Since this bound also holds in the limit $T \to \infty$, we are left with rewriting the integrand as in (3.1). For this purpose we note that

$$\int d^3y \frac{\partial}{\partial t} \rho(y - q(t + \omega \cdot y)) = - \int d^3y \, \dot{q}(t + \omega \cdot y) \cdot \nabla \rho(y - q(t + \omega \cdot y))$$

and

$$\dot{q}(t + \omega \cdot y) \cdot \nabla_y (\rho(y - q(t + \omega \cdot y))) = (1 - \omega \cdot \dot{q}(t + \omega \cdot y)) \dot{q}(t + \omega \cdot y) \cdot \nabla \rho(y - q(t + \omega \cdot y)).$$

Thus, by partial integration, and because of $|\dot{q}(t)| < 1$, we finally obtain

$$\int_Q d^3y \frac{\partial}{\partial t} \rho(y - q(t + \omega \cdot y)) = \int_Q d^3y \, \rho(y - q(t + \omega \cdot y)) \nabla_y \cdot (\dot{q}(t + \omega \cdot y)(1 - \omega \cdot \dot{q}(t + \omega \cdot y))^{-1}),$$

which agrees with the integrand in (3.1). □

We still have to prove Lemma 3.2, 3.3.

Proof of Lemma 3.2 The representation (3.5) implies for $\overline{R}_0 < |t| - |x| < T$

$$\pi_r(x,t) = - \int_{Q_T} d^3y \frac{1}{4\pi|x-y|} \frac{\partial}{\partial t} \rho(y - q(t - |x-y|)), \tag{3.12}$$

$$\begin{aligned} \nabla \phi_r(x,t) = & \int_{Q_T} d^3y \frac{1}{4\pi|x-y|} \frac{\partial}{\partial t} \rho(y - q(t - |x-y|)) \, \omega(x) \\ & + \int_{Q_T} d^3y \frac{1}{4\pi|x-y|^2} \rho(y - q(t - |x-y|)) \, \nu(x,y) \\ & + \int_{Q_T} d^3y \frac{1}{4\pi|x-y|} \frac{\partial}{\partial t} \rho(y - q(t - |x-y|)) (\nu(x,y) - \omega(x)) \end{aligned} \tag{3.13}$$

with $\nu(x,y) = (x-y)/|x-y|$. For $|x| = R$, the second and third term are bounded by CR^{-2}. (3.12) and (3.13) imply then (3.9). □

Proof of Lemma 3.3 We deduce (3.10) from Kirchhoff formula (3.6) and assumption (1.6). (3.6) implies the representation, for $t > \overline{R}_0 + |x|$,

$$\nabla \phi_0(x,t) = \sum_{|\alpha| \le 1} t^{|\alpha|-2} \int_{S_t(x)} d^2y \, a_\alpha(x-y) \partial^\alpha \pi^0(y) + \sum_{|\alpha| \le 2} t^{|\alpha|-3} \int_{S_t(x)} d^2y \, b_\alpha(x-y) \partial^\alpha \phi^0(y). \tag{3.14}$$

Here all derivatives are understood in a classical sense, and the coefficients $a_\alpha(\cdot)$ and $b_\alpha(\cdot)$ are bounded. A similar representation holds for $\pi_0(x,t)$. The coefficients are homogeneous functions of order zero and smooth outside the origin. Hence, taking into account our assumption (1.6), we obtain from (3.14) for $t > \overline{R}_0 + |x|$

$$|\nabla \phi_0(x,t)| \le C \sum_{0 \le k \le 1} t^{k-2} \int_{S_t(x)} d^2y \, |y|^{-\sigma-1-k} + C \sum_{0 \le k \le 2} t^{k-3} \int_{S_t(x)} d^2y \, |y|^{-\sigma-k}. \tag{3.15}$$

We can always adjust σ such that $\sigma + k \neq 2$. Then by explicit computation a typical term reads,

$$I^k(x,t) := \int_{S_t(x)} d^2y\ |y|^{-\sigma-k} = \frac{2\pi t}{|x|(2-\sigma-k)}\Big((t+|x|)^{2-\sigma-k}\ -\ (t-|x|)^{2-\sigma-k}\Big).$$

Hence the contribution of the corresponding term from (3.14) in the left hand side of (3.10) can be majorized by

$$\begin{aligned} B^{\alpha}_{R,T} :=\quad & C_1 \int_{R_0+R}^{T+R} dt \int_{\partial B_R} d^2x\ \Big|t^{|\alpha|-3} I^{|\alpha|}(x,t)\Big|^2 \\ =\quad & C_2 \int_{R_0}^{T} dt\ (R+t)^{2(|\alpha|-2)} \Big|(2R+t)^{2-\sigma-|\alpha|}\ -\ t^{2-\sigma-|\alpha|}\Big|^2. \end{aligned} \tag{3.16}$$

We may adjust σ slightly larger than $1/2$. Then for $|\alpha| \leq 1$, we have $\sigma + |\alpha| \leq 2$ and (3.16) implies

$$B^{\alpha}_{R,T} \leq C \int_{R_0}^{T} dt\ (R+t)^{-2\sigma} \leq B^{\alpha} < \infty \text{ for } R,T \geq 0.$$

For $|\alpha| = 2$ the bound (3.16) implies

$$B^{\alpha}_{R,T} \leq C \int_{R_0}^{T} dt\ t^{-2\sigma} \leq B^{\alpha} < \infty \text{ for } R,T \geq 0.$$

All contributions of other terms from (3.14) can be majorized in a similar fashion, correspondingly for $\pi_0(x,t)$ on the left hand side of (3.10). □

Remark The representations (3.6), (3.14) and the corresponding representation for $\pi_0(x,t)$ together with (1.6) imply for every $R > 0$

$$\max_{|x|\leq R} \Big(|\phi_0(x,t)| + t|\pi_0(x,t)| + t|\nabla\phi_0(x,t)|\Big) = \mathcal{O}(t^{-\sigma}) \quad \text{as} \quad t \to \infty, \tag{3.17}$$

where the derivatives are understood in a classical sense.

4 Relaxation of the particle velocity

In this section we will deduce from Proposition 3.1 that $\dot q(t)$, $\ddot q(t) \to 0$ as $t \to \infty$ provided we add the assumptions (W) and (P).

Assumption (P) implies the bounds (2.4) and (2.5) with $k = 2, 3$ due to Lemma 2.1 (vi). Hence, the function

$$R_\omega(t) = \int d^3y\, \rho(y - q(t+\omega\cdot y)) \frac{\omega \cdot \ddot q(t+\omega\cdot y)}{(1-\omega\cdot\dot q(t+\omega\cdot y))^2} \tag{4.1}$$

is globally Lipschitz continuous in ω and t. Thus by Proposition 3.1

$$\lim_{t\to\infty} R_\omega(t) = 0 \tag{4.2}$$

uniformly in $\omega \in S^2$. Let $r(t) = \omega \cdot q(t) \in \mathbb{R}$, $s = \omega \cdot y$, and $\rho_a(q_3) = \int dq_1 dq_2 \rho(q_1, q_2, q_3)$. In (4.1) we decompose the y-integration along and transversal to ω. Then

$$\begin{aligned} R_\omega(t) &= \int ds\, \rho_a(s - r(t+s)) \frac{\ddot{r}(t+s)}{(1-\dot{r}(t+s))^2} \\ &= \int d\tau\, \rho_a(t - (\tau - r(\tau))) \frac{\ddot{r}(\tau)}{(1-\dot{r}(\tau))^2} \\ &= \int d\vartheta\, \rho_a(t-\vartheta) g_\omega(\vartheta) = \rho_a * g_\omega(t). \end{aligned} \tag{4.3}$$

Here we substituted $\vartheta = \vartheta(\tau) = \tau - r(\tau)$, which is a nondegenerate diffeomorphism since $|\dot{r}| \le q_1 < 1$ due to (2.4), and we set

$$g_\omega(\vartheta) = (1 - \dot{r}(\tau(\vartheta)))^{-3} \ddot{r}(\tau(\vartheta)).$$

Let us extend $q(t)$ smoothly to zero for $t < 0$. Then $\rho_a * g_\omega\,(t)$ is defined for all t and agrees with $R_\omega(t)$ for sufficiently large t. Hence (4.2) reads as a convolution limit

$$\lim_{t\to\infty} \rho_a * g_\omega(t) = 0. \tag{4.4}$$

Now note that (2.4) and (2.5) with $k = 2, 3$ imply that $g'_\omega(\vartheta)$ is bounded. Hence (4.4) and (W) imply by Pitt's extension to Wiener's Tauberian Theorem, cf. [25, Thm. 9.7(b)],

$$\lim_{\vartheta\to\infty} g_\omega(\vartheta) = 0. \tag{4.5}$$

Because $\omega \in S^2$ is arbitrary and $\vartheta(t) \to \infty$ as $t \to \infty$, we have proved

Lemma 4.1 *Let all assumptions of Proposition 3.1 hold, and the potential V satisfy (P). If (W) holds, then*

$$\lim_{t\to\infty} \ddot{q}(t) = 0. \tag{4.6}$$

Remarks. (i) For a point charge $\rho(x) = \delta(x)$, (4.4) implies (4.5) directly.
(ii) Parseval's identity, (4.3) and (3.1) give

$$\int_{S^2} d^2\omega \int d\xi\, |\hat{\rho}_a(\xi)\hat{g}_\omega(\xi)|^2 < \infty.$$

If $|\hat{\rho}_a(\xi)| \ge C > 0$, then $\int d^2\omega \int d\vartheta\, |g_\omega(\vartheta)|^2 < \infty$, and (4.5) would follow from the Lipschitz continuity of g_ω. Thus, the main difficulty results from the rapid decay of the Fourier transform ("symbol") $\hat{\rho}_a$, due to the smoothness of the kernel ρ_a.
(iii) Condition (W) is necessary. Indeed, if (W) is violated, then $\hat{\rho}_a(\xi) = 0$ for some $\xi \in \mathbb{R}$, and with the choice $g(\vartheta) = \exp(i\xi\vartheta)$ we have $\rho_a * g(\vartheta) = 0$ whereas g does not decay to zero.

Corollary 4.2 *Let all assumptions of Theorem 1.2 hold. Then*

$$\lim_{t\to\infty} \dot{q}(t) = 0. \tag{4.7}$$

Proof Since $|q(t)| \le q_0$ due to (2.5) with $k = 0$, (4.6) implies (4.7). □

Remark Lemma 4.1 holds even under an assumption on the potential, weaker than (P),

$$V \in C^2(\mathbb{R}^3), \quad \inf_{q\in\mathbb{R}^3} V(q) > -\infty \qquad \text{and} \qquad \sup_{q\in\mathbb{R}^3} |\partial^\alpha V(q)| < \infty \ \text{ for } |\alpha| = 1 \text{ and } 2. \tag{P_w}$$

The proof of Lemma 4.1 with (P_w) instead of (P) remains unchanged. Firstly, (P_w) includes $(P_{\min})$, hence it provides (3.1) and (2.4). Secondly, (P_w) implies the bounds (2.5) with $k = 2, 3$ similarly to (P) in Lemma 2.1 *(vi)*.

5 A compact attracting set, proof of Theorem 1.2

Definition 5.1 *Let* $\mathcal{A} = \{Y_q : q \in \mathbb{R}^3, \quad |q| \le q_0\}$, *where* Y_q *is defined in (1.3).*

Since $\mathcal{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$, $\mathcal{A}$ is compact in $\mathcal{E}_F$.

Lemma 5.2 *Let all assumptions of Theorem 1.2 hold. Then* $Y(t) \to \mathcal{A}$ *in* $\mathcal{E}_F$ *as* $t \to \infty$.

Proof. For every $R > 0$

$$\begin{aligned}\mathrm{dist}_R(Y(t), \mathcal{A}) &= \inf_{Y_q \in \mathcal{A}} \|Y(t) - Y_q\|_R \\ &= |p(t)| + |\pi(t)|_R + \inf_{|q| \le q_0} (|\nabla(\phi(t) - \phi_q)|_R + |\phi(t) - \phi_q|_R + |q(t) - q|). \end{aligned} \tag{5.1}$$

Let us estimate each term separately.
i) (4.7) implies $|p(t)| \to 0$ as $t \to \infty$.
ii) Let us denote $R_\infty = q_0 + R_\rho$. Then (3.12) implies for $t > R + R_\infty$ and $|x| < R$

$$|\pi_r(x,t)| \le \ C \max\{|\dot{q}(\tau)| : t - R - R_\infty \le \tau \le t\} \int_{|y| < R_\infty} d^3y \, \frac{1}{|x-y|} |\nabla \rho(y - q(t - |x-y|))| \, .$$

The integral is bounded uniformly in $t > R + R_\infty$ and $x \in B_R$, and therefore (4.7) implies $|\pi_r(t)|_R \to 0$ as $t \to \infty$. Then also $|\pi(t)|_R \to 0$ due to (3.4) and (3.17).
iii) To estimate the infimum over q in (5.1), we may substitute $q(t)$ for q. Then the last term vanishes, and (3.5) implies for $t > R + R_\infty$ and $|x| < R$

$$\phi_r(x,t) - \phi_{q(t)}(x) = -\int_{|y| < R_\infty} d^3y \frac{1}{4\pi |x-y|} \Big(\rho(y - q(t - |x-y|)) - \rho(y - q(t))\Big) \, .$$

The difference $\rho(y - q(t - |x-y|)) - \rho(y - q(t))$ may be written as an integral depending only on $\dot{q}(\tau)$ for $\tau \in [t - R - R_\infty, t]$, which tends to zero as $t \to \infty$ uniformly in $x \in B_R$ due to (4.7). Hence $|\phi_r(t) - \phi_{q(t)}|_R \to 0$ as $t \to \infty$. Then similarly, $|\phi(t) - \phi_{q(t)}|_R \to 0$ due to (3.4) and (3.17). This proves the claim, since $|\nabla(\phi(t) - \phi_{q(t)})|_R$ may be estimated in a similar way.
□

Proof of Theorem 1.2 *ad (i)* For a solution $Y(\cdot)$ let Ω be the set of all points $\overline{Y} \in \mathcal{E}$ such that for some sequence $t_k \to \infty$ we have $Y(t_k) \to \overline{Y}$ in $\mathcal{E}_F$. Then by continuity of W_t in $\mathcal{E}_F$, also $W_t Y(t_k) \to W_t \overline{Y}$ in $\mathcal{E}_F$, hence $W_t \overline{Y} \in \Omega$ for arbitrary $t \in \mathbb{R}$, i.e. Ω is an invariant set. We first prove

Lemma 5.3 Ω *is a subset of* $\mathcal{S}$.

Proof Since $\mathcal{A}$ is an attracting set, clearly $\Omega \subset \mathcal{A}$. This means for $\overline{Y} \in \Omega$ there exists a C^2-curve $t \mapsto Q(t) \in \mathbb{R}^3$ such that $W_t \overline{Y} = Y_{Q(t)}$, according to Definition 5.1. For $Y_{Q(t)}$ to be a solution of (1.2) we must have $\dot{Q}(t) = 0$, hence $Q(t) \equiv q^*$ with $\nabla V(q^*) = 0$. Therefore $\overline{Y} = Y_{q^*}$ with $q^* \in S$. □

We prove now (1.7) by contradiction. So let us assume that $\mathrm{dist}_R(Y(t_k), \mathcal{S}) \ge \varepsilon > 0$ for some $R, \varepsilon > 0$ and a sequence $t_k \to \infty$. Then Lemma 5.2 and the compactness of $\mathcal{A}$ imply that, for a suitable subsequence, $Y(t_{k'}) \to \overline{Y}$ in $\mathcal{E}_F$, where $\overline{Y} \in \mathcal{A}$. Then $\overline{Y} \in \Omega$ by definition. Since $\mathrm{dist}_R(\overline{Y}, \mathcal{S}) \ge \varepsilon > 0$ we obtain a contradiction to $\Omega \subset \mathcal{S}$.
ad (ii) Lemma 5.2 together with (1.7) imply that $Y(t) \to \mathcal{A} \cap \mathcal{S}$ in $\mathcal{E}_F$ as $t \to \infty$. If the set S of critical points of V is discrete, then the set $\mathcal{S}$ is discrete in $\mathcal{E}_F$. Moreover, the set $\mathcal{S}$ is closed in $\mathcal{E}_F$. Hence, the intersection $\mathcal{A} \cap \mathcal{S}$ is a finite set, as the intersection of a compact set and of a closed discrete set. In (1.7) the states $Y(t)$ approach the finite subset $\mathcal{A} \cap \mathcal{S}$ in the metrisable space $\mathcal{E}_F$. Therefore the continuity of $Y(t)$ implies (1.8). □

6 Linearization around a stationary state

If the particle is close to a stable minimum of V, we expect the nonlinear evolution to be dominated by the linearized dynamics. As to be explained in Section 7 the linearized dynamics has exponentially fast convergence provided the initial fields have compact support. For the nonlinear dynamics this corresponds to an initial state of the form Y_{q^*} plus perturbation of compact support. In such a situation we expect then exponential convergence to Y_{q^*}. The precise estimate is given in Theorem 6.1 below. In Theorem 1.4 the support of the perturbation in general is not a compact set. Thus it still requires some effort to prove Theorem 1.4, which is deferred to Section 9. In this section we establish the required estimate.

Theorem 6.1 *Let (C), (W) hold, and $V \in C^3(\mathbb{R}^3)$. Let $q^* \in S$ be a stable point and let the initial state $Y^0 \in \mathcal{E}$ be such that*

$$\phi^0(x) = \phi_{q^*}(x), \quad \pi^0(x) = 0 \quad for \quad |x| \geq M \tag{6.1}$$

with some $M > 0$. Let $Y(t) \in C(\mathbb{R}, \mathcal{E})$ be corresponding solution to (1.2) with $Y(0) = Y^0$. Then there exist a $\delta = \delta(M) > 0$ and a $\gamma^* = \gamma(q^*) > 0$, such that if $\|Y^0 - Y_{q^*}\|_{\mathcal{E}} \leq \delta$, the bounds hold for every $R > 0$ and every $\gamma < \gamma^*$

$$\|Y(t) - Y_{q^*}\|_R \leq C_R e^{-\gamma t}, \; t \geq 0 \tag{6.2}$$

with some suitable constant $C_R = C_R(M, \gamma) > 0$.

For notational simplicity we also assume isotropy in the sense that

$$\partial_i \partial_j V(q^*) = \omega_0^2 \delta_{ij}, \; i, j = 1, 2, 3, \; \omega_0 > 0\,. \tag{6.3}$$

Without loss of generality we take $q^* = 0$. Let $Y_{q^*} = Y_0 = (\phi_0, 0, 0, 0)$ be the stationary state of (1.2) corresponding to $q^* = 0$, and $Y^0 = (\phi^0, q^0, \pi^0, p^0) \in \mathcal{E}$ be an arbitrary state satisfying (6.1). We denote by $Y(t) = (\phi(x,t), q(t), \pi(x,t), p(t)) \in \mathcal{E}$ the solution to (1.2) with $Y(0) = Y^0$.

To linearize (1.2) at Y_0, we set $\psi(x,t) = \phi(x,t) - \phi_0(x)$. Then (1.2) becomes

$$\begin{array}{rlrl} \dot\psi(x,t) = \pi(x,t), & & \dot\pi(x,t) = & \Delta\psi(x,t) + \rho(x) - \rho(x - q(t)), \\ \dot q(t) = p(t)/(1 + p^2(t))^{1/2}, & & \dot p(t) = & -\nabla V(q(t)) + \int d^3x \; \psi(x,t) \, \nabla\rho(x - q(t)) \\ & & & + \int d^3x \; \phi_0(x)[\nabla\rho(x - q(t)) - \nabla\rho(x)]\,. \end{array} \tag{6.4}$$

We define the linearization by

$$\begin{array}{rlrl} \dot\Psi(x,t) = \Pi(x,t), & & \dot\Pi(x,t) = \Delta\Psi(x,t) + \nabla\rho(x) \cdot Q(t), \\ \dot Q(t) = P(t), & & \dot P(t) = -(\omega_0^2 + \omega_1^2)Q(t) + \int d^3x \, \Psi(x,t) \nabla\rho(x)\,. \end{array} \tag{6.5}$$

Here

$$\omega_1^2 \delta_{ij} = \frac{1}{3} \, \|\rho\|^2 \delta_{ij} = -\int d^3x \, \partial_i\phi_0(x) \partial_j\rho(x)\,, \tag{6.6}$$

where the factor 1/3 is due to a spherical symmetry of $\rho(x)$ (see (C)). We rewrite (6.5) as

$$\dot Z(t) = AZ(t), \quad t \in \mathbb{R}. \tag{6.7}$$

Here $Z(t)=(\Psi(\cdot,t),Q(t),\Pi(\cdot,t),P(t))$ and A is the linear operator defined by

$$A:\ Z=(\Psi,Q,\Pi,P)\mapsto(\Pi,P,\Delta\Psi+\nabla\rho\cdot Q,-\omega^2Q+\int d^3x\,\Psi(x)\nabla\rho(x)),$$

where $\omega^2=\omega_0^2+\omega_1^2$. (6.7) is a formal Hamiltonian system with the quadratic Hamiltonian

$$H_0(Z)=\frac{1}{2}\Big(P^2+\omega^2Q^2+\int d^3x\,(|\Pi(x)|^2+|\nabla\Psi(x)|^2-2\Psi(x)\nabla\rho(x)\cdot Q)\Big),\tag{6.8}$$

which is the formal Taylor expansion of $H(Y_0+Z)$ up to second order at $Z=0$.

Introducing $X(t)=Y(t)-Y_0=(\psi(t),q(t),\pi(t),p(t))\in C(\mathbb{R},\mathcal{E})$, we rewrite the nonlinear system (6.4) in the form

$$\dot X(t)=AX(t)+B(X(t)).\tag{6.9}$$

Due to (6.6) the nonlinear part is given by

$$\begin{aligned}B(X)\ &=\ \Big(0,\,p/(1+p^2)^{1/2}-p,\,\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q,\\&\quad-\nabla V(q)+\omega_0^2q+\int d^3x\,\psi(x)[\nabla\rho(x-q)-\nabla\rho(x)]\\&\quad+\int d^3x\,\nabla\phi_0(x)[\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q]\Big)=:(\psi^1(x),q^1,\pi^1(x),p^1)\end{aligned}\tag{6.10}$$

for $X=(\psi,q,\pi,p)\in\mathcal{E}$. This definition immediately implies

Lemma 6.2 *Let (C) hold, $V\in C^3(\mathbb{R}^3)$, and $b>0$ be some fixed number. Then for $|q|\le b$*
(i) with notations (6.10),

$$\psi^1(x)=\pi^1(x)=0\quad for\quad |x|\ge R_\rho+b\ ;\tag{6.11}$$

(ii) for every $R>0$

$$\|B(X)\|_R\le C_b\,\|X\|^2_{R_\rho+b}.\tag{6.12}$$

Let us consider the Cauchy problem for the linear equation (6.7) with initial condition

$$Z|_{t=0}=Z^0.\tag{6.13}$$

Lemma 6.3 *Let (C) hold. Then*
(i) For every $Z^0\in\mathcal{E}$ the Cauchy problem (6.7), (6.13) has a unique solution $Z(\cdot)\in C(\mathbb{R},\mathcal{E})$.
(ii) For every $t\in\mathbb{R}$ the map $U(t):Z^0\mapsto Z(t)$ is continuous both on $\mathcal{E}$ and on $\mathcal{E}_F$.
(iii) For $Z^0\in\mathcal{E}$ the energy H_0 is finite and conserved, i.e.

$$H_0(Z(t))=H_0(Z^0)\quad for\quad t\in\mathbb{R}.\tag{6.14}$$

iv) For $Z^0\in\mathcal{E}$

$$\|Z(t)\|_{\mathcal{E}}\le B\quad for\quad t\in\mathbb{R}\tag{6.15}$$

with B depending only on the norm $\|Z^0\|_{\mathcal{E}}$ of the initial state.

The proof of this lemma is almost identical with the proof of the Lemma 2.1. In fact, for the linearized system the Hamiltonian is nonnegative, since (6.8) with the definition (6.6) implies

$$2H_0(Z)=P^2+\omega_0^2Q^2+\int d^3x\,(|\Pi(x)|^2+||\nabla|\Psi(x)-|\nabla|^{-1}(\nabla\rho(x)\cdot Q)|^2)\ge 0.$$

Thus (6.15) follows from (6.14) because of $\omega_0>0$. □

7 Decay estimates for the linearized system

We prove the local decay of solutions $Z(t)$ to the linearized system (6.7).

Proposition 7.1 *Let (C) and (W) hold, and $\omega_0 > 0$. Let $Z(t) \in C(\mathbb{R}, \mathcal{E})$ be a solution to the Cauchy problem (6.7), (6.13) and the initial state $Z^0 = (\Psi^0, Q^0, \Pi^0, P^0) \in \mathcal{E}$ have compact support,*

$$\Psi^0(x) = \Pi^0(x) = 0 \quad for \quad |x| \geq M \tag{7.1}$$

with some $M > 0$. Then there exist a $\gamma^* > 0$ such that for every $R > 0$ and every $\gamma < \gamma^*$

$$\|Z(t)\|_R \leq C_R e^{-\gamma t} \|Z^0\|_{\mathcal{E}} \qquad for \quad t \geq 0 \tag{7.2}$$

with suitable $C_R = C_R(M, \gamma) > 0$.

To prove this proposition we solve the Cauchy problem (6.7), (6.13) explicitly through Laplace transform

$$\tilde{Z}(\lambda) = \int_0^\infty dt\; e^{-\lambda t} Z(t), \quad \operatorname{Re}\lambda > 0.$$

Note that, by the bound (6.15), $\tilde{Z}(\lambda)$ is an analytic function in complex right half plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ with values in the Hilbert space $\mathcal{E}$. We have

$$\begin{array}{rcl} \tilde{\Psi}(\lambda) & = & (-\Delta + \lambda^2)^{-1} (\lambda \Psi^0 + \Pi^0) + (-\Delta + \lambda^2)^{-1} (\nabla \rho(x) \cdot \tilde{Q}(\lambda)), \\ \tilde{Q}(\lambda) & = & (a(\lambda))^{-1} \Big(\lambda Q^0 + P^0 + \int d^3 y \left[(-\Delta + \lambda^2)^{-1} (\lambda \Psi^0 + \Pi^0) \right](y) \cdot \nabla \rho(y) \Big) \end{array} \tag{7.3}$$

provided $\operatorname{Re}\lambda > 0$. Here $a(\lambda)$ is a matrix $a_{ij}(\lambda) = \alpha(\lambda)\delta_{ij}$ with $i, j = 1, 2, 3$,

$$\begin{aligned} a_{ij}(\lambda) &= (\lambda^2 + \omega_0^2 + \omega_1^2)\delta_{ij} + ((-\Delta + \lambda^2)^{-1} \partial_i \rho, \partial_j \rho) \\ &= \Big(\lambda^2 \Big[1 + \frac{1}{3} ((-\Delta + \lambda^2)^{-1} \rho, \rho) \Big] + \omega_0^2 \Big) \delta_{ij} = \alpha(\lambda)\delta_{ij} \qquad \text{for} \ \ \operatorname{Re}\lambda > 0 \end{aligned} \tag{7.4}$$

due to (6.6). In order to estimate the decay of $Z(t)$ we first have to investigate the zeros of $\alpha(\lambda)$. Denote $\mathbb{C}_\beta = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \beta\}$ for $\beta \in \mathbb{R}$.

Lemma 7.2 *Let α be defined by (7.4) for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ and let (C) hold. Then*

(i) α has an analytic continuation to $\mathbb{C}$.

(ii) For every $\beta < 0$ there exists $d_\beta > 0$ such that $|\alpha(\lambda)| \geq |\lambda|^2/2$ for $\lambda \in \mathbb{C}_\beta$ with $|\lambda| \geq d_\beta$.

(iii) If the Wiener condition (W) holds, then there exists $\gamma > 0$ such that $\alpha(\lambda) \neq 0$ for $\lambda \in \mathbb{C}_{-\gamma}$.

Proof *ad (i)* (7.4) and $(-\Delta + \lambda^2)\Big(e^{-\lambda|x|}/(4\pi|x|)\Big) = \delta(x)$ imply

$$\alpha(\lambda) = \lambda^2 \left(1 + \frac{1}{3} \int \int d^3x\, d^3x' \, \frac{\rho(x)\rho(x')}{4\pi|x - x'|} \, e^{-\lambda|x - x'|} \right) + \omega_0^2. \tag{7.5}$$

The right-hand side of this expression is defined and analytic in all of $\mathbb{C}$ and is thus an analytic continuation of α.
ad (ii) The assertion follows from (7.5), because

$$I(\lambda) = \int \int d^3x\, d^3x' \, \frac{\rho(x)\rho(x')}{4\pi|x - x'|} \, e^{-\lambda|x - x'|} \to 0 \quad \text{as} \ \ |\lambda| \to \infty \ \ \text{with} \ \ \lambda \in \mathbb{C}_\beta. \tag{7.6}$$

ad (iii) Since $\alpha(\lambda) \neq 0$ for $\mathrm{Re}\,\lambda > 0$ due to (7.3) and (6.15), we only have to exclude that $\alpha(\lambda)$ has zeros on the imaginary axis. For this the Wiener condition (W) will be needed. For $\lambda = iy, \quad y \in \mathbb{R}$, we obtain from (7.4)

$$\alpha(iy) = -y^2\Big(1 + \frac{1}{3(2\pi)^3}\int d^3k \frac{|\hat{\rho}(k)|^2}{k^2 - (y - i0)^2}\Big) + \omega_0^2. \tag{7.7}$$

Clearly, $\alpha(0) = \omega_0^2 > 0$. Let $y \neq 0$ and define

$$D(y) = \int d^3k \frac{|\hat{\rho}(k)|^2}{k^2 - (y - i0)^2} = \int_0^\infty d\nu \frac{g(\nu)}{\nu^2 - (y - i0)^2} = \int_0^\infty d\nu \frac{g(\nu)}{(\nu - y + i0)(\nu + y - i0)}$$

Because of (W)

$$g(\nu) = \int_{|k|=\nu} d^2k\, |\hat{\rho}(k)|^2 > 0 \quad \text{ for } \ \nu > 0.$$

Furthermore, because of (C), $g(\nu) \in C^\infty([0,\infty))$ and $\max_{0<\nu<\infty} |g(\nu)|(1 + \nu^p) < \infty$ for every $p > 0$. Therefore by the Plemelj formula [6]

$$D(y) = -i\pi \frac{g(|y|)}{2y} + \mathrm{pv}\int_0^\infty d\nu \frac{g(\nu)}{(\nu - y)(\nu + y)}$$

and $\mathrm{Im}\, D(y) \neq 0$. Then (7.7) implies that $\mathrm{Im}\,\alpha(iy) \neq 0$. □

Remarks. (i) For $\rho = 0$ the zeros of α are at $\pm i\omega_0$. If $\rho = \varepsilon\rho_0$ with some fixed ρ_0 satisfying (C), we can follow perturbatively how these zeros move to the left of the imaginary axis as the coupling strength ε is turned on.
(ii) If (W) does not hold, then g and thus the imaginary part of D vanish at isolated points. To make also the real part of D vanish at some of these points we need to vary at most "one parameter" in ρ. In this sense, for $\alpha(\lambda)$ to have some zeros on the imaginary axis is a codimension one property in the space of ρ's.

Definition 7.3 *For a stable $q^* \in S$ denote by $\alpha(\lambda)$ the function defined in (7.5) and (6.3), by $\Lambda(q^*)$ the set of zeros $\{\lambda \in \mathbb{C} : \alpha(\lambda) = 0\}$, and by*

$$\gamma^* = \gamma(q^*) = -\max_{\lambda \in \Lambda(q^*)} \mathrm{Re}\,\lambda. \tag{7.8}$$

To prove the exponential decay of $Y(t)$, we need the following lemma about the inverse Laplace transform of $1/\alpha(\lambda)$ given for arbitrary $\gamma < \gamma^*$ by

$$h(t) = \frac{1}{2\pi}\int_{\mathrm{Re}\,\lambda = -\gamma} d\lambda\, \frac{e^{\lambda t}}{\alpha(\lambda)} \quad \text{ for } \ t > 0.$$

Lemma 7.4 *For $j = 0, 1, 2, \ldots$ and every $\gamma < \gamma^*$*

$$|h^{(j)}(t)| = \mathcal{O}(e^{-\gamma t}) \quad \textit{as} \quad t \to \infty\ . \tag{7.9}$$

Proof By Lemma 7.2 (ii) and (iii), the bound on h follows. To prove the same bound for the derivatives $h^{(j)}(t)$, we consider corresponding Laplace transforms $\lambda^j/\alpha(\lambda)$. For large $N = 1, 2, 3, \ldots$ we are going to establish that

$$|(\frac{d}{d\lambda})^N \frac{\lambda^j}{\alpha(\lambda)}| \leq \frac{C_N(\gamma)}{(1+|\lambda|)^{\overline{N}}} \quad \text{for} \quad \mathrm{Re}\,\lambda = -\gamma, \tag{7.10}$$

where $\overline{N} \to \infty$ as $N \to \infty$. This implies then

$$\max_{0 \leq t < \infty} |t^N e^{\gamma t} h^{(j)}(t)| < \infty$$

for large N and (7.9) follows.

To prove (7.10), we note that $\alpha(\lambda) = \lambda^2(1 + I(\lambda)/3 + \omega_0^2/\lambda^2)$. Hence $1/(\lambda^2\alpha(\lambda))$ can be expanded due to (7.6) as a power series in $I(\lambda)$ and $1/\lambda$ for $\mathrm{Re}\,\lambda = -\gamma$ with large $|\lambda|$. Therefore it suffices to establish the bounds of type (7.10) for $I(\lambda)$,

$$|(\frac{d}{d\lambda})^N I(\lambda)| \leq \frac{C_N}{(1+|\lambda|)^{\overline{N}}} \quad \text{for} \quad \mathrm{Re}\,\lambda = -\gamma,$$

where $\overline{N} \to \infty$ as $N \to \infty$. These estimates follow from the representation (7.6) of the function $I(\lambda)$. □

Proof of Proposition 7.1 Using Lemma 7.3 we estimate the decay of $Z(t)$ in the local energy seminorms. From (7.3) we obtain

$$\begin{aligned}
\Psi(x,t) &= \Psi_1(x,t) + \int_0^t ds\, h(t-s)\Psi_2(x,s) \\
&\quad + \int_0^t ds\, h(t-s) \int_0^s ds'\, (\nabla\rho, \Psi_1(s-s')) \cdot \Psi_3(x,s'), \\
Q(t) &= \dot{h}(t)Q^0 + h(t)P^0 + \int_0^t ds\, h(t-s)(\Psi_1(s), \nabla\rho).
\end{aligned} \tag{7.11}$$

Here Ψ_1 is the solution of the homogeneous wave equation with initial data (Ψ^0, Π^0), Ψ_2 the solution with initial data $(Q^0 \cdot \nabla\rho, P^0 \cdot \nabla\rho)$, and Ψ_3 the solution with initial data $(0, \nabla\rho)$. Due to the strong Huygens principle $\|\Psi_1(\cdot,t)\|_R$, $\|\Psi_2(\cdot,t)\|_R$, $\|\Psi_3(\cdot,t)\|_R$ and $(\nabla\rho, \Psi_1(t))$ vanish for sufficiently large t. Therefore, the claimed estimate (7.2) follows from energy estimate (6.15) for the solutions $\Psi_{1,2,3}$ and from bounds (7.9) with $j = 0, 1, 2$. □

8 Proof of Theorem 6.1

In essence we follow [16]. We assume (6.3) and a stable $q^* = 0$.

Let us note that (6.1) implies that $X(0) = Y(0) - Y_0$ has a compact support, i.e. the property (7.1) holds for $Z^0 = X(0)$. So we have to prove for the solution $X(t) = Y(t) - Y_0 \in C(\mathbb{R}, \mathcal{E})$ of (6.9) with initial values of compact support and with small $\|X(0)\|_{\mathcal{E}}$,

$$\|X(t)\|_R \leq C_R e^{-\gamma t} \quad \text{for} \quad t \geq 0. \tag{8.1}$$

Let us note that $B(X(t)) \in C(\mathbb{R}, \mathcal{E})$ for such $X(t)$ because of (6.12). Therefore the integrated version of (6.9) holds

$$X(t) = U(t)X(0) + \int_0^t ds\, U(t-s)B(X(s)). \tag{8.2}$$

Let us fix an arbitrary number $b > 0$. We restrict ourselves to $0 \le t < t_b$, where $t_b = \sup\{t \ge 0 : \ |q(t)| \le b\} > 0$ for small $\|X(0)\|_{\mathcal{E}}$. Then Lemma 6.2 (ii) implies for all $R > 0$

$$\|B(X(t))\|_R \le C_b \, \|X(t)\|^2_{R_\rho + b} \quad \text{for} \ \ t < t_b.$$

Hence, Proposition 7.1, (6.11), and (8.2) imply the integral inequality

$$\|X(t)\|_R \le C_R(M, b) \left(e^{-\gamma t} \|X(0)\|_{\mathcal{E}} + \int_0^t ds\, e^{-\gamma(t-s)} \|X(s)\|^2_{R_\rho + b} \right) \quad \text{for} \ \ t < t_b. \tag{8.3}$$

Therefore, it suffices to prove that for sufficiently small $\delta = \|X(0)\|_{\mathcal{E}}$

(a) $t_b = \infty$, i.e. $|q(t)| \le b$ for all $t \ge 0$, and

(b) (8.1) with $R = R(b) = R_\rho + b$.

For this purpose we denote $n(t) = e^{\gamma t} \|X(t)\|_{R(b)}$. Then (8.3) with $R = R(b)$ implies that

$$n(t) \le C(M, b) \left(\delta + \int_0^t ds\, e^{-\gamma s}\, n^2(s) \right) \quad \text{for} \ \ t < t_b. \tag{8.4}$$

Further we denote $m(t) = \max_{0 \le s \le t} n(s)$ for $t \ge 0$. Then (8.4) implies the quadratic inequality

$$m(t) \le C(M, b, \gamma) \left(\delta + m^2(t) \right) \quad \text{for} \ \ t < t_b. \tag{8.5}$$

Let us choose $\delta > 0$ so small that the quadratic equation

$$m = C(M, b, \gamma) \left(\delta + m^2 \right)$$

has two positive roots $m_1 < m_2$. We may assume $C(M, b, \gamma) \ge 1$. Then $m(0) = \delta \le m_1$, hence the quadratic inequality (8.5) implies $m(t) \le m_1$ for all $t < t_b$ by continuity. Moreover, $m_1 \to 0$ as $\delta \to 0$, hence $|q(t)| \le m(t) \le m_1 < b$ holds for all $t < t_b$, provided δ be sufficiently small. Then (a) follows, and hence $m(t) \le m_1$ for all $t \ge 0$, which implies (b). □

9 Proof of Theorem 1.4

Let us consider a solution $Y(t)$ converging to some limit stationary state Y_{q^*} with stable q^*. We prove i) because ii) follows similarly. We assume that the initial state Y^0 satisfies (1.6), and set $q^* = 0$, as above.

Let M_* be a fixed number, $M_* > 3R_\rho + 1$.

Lemma 9.1 *For arbitrary $\delta > 0$ there exist $t_* > 0$ and a solution*

$$Y_*(t) = (\phi_*(x,t), q_*(t), \pi_*(x,t), p_*(t)) \in C([t_*, \infty), \ \mathcal{E})$$

to the system (1.2) such that
(i) $Y_*(t)$ coincides with $Y(t)$ in some future cone,

$$\begin{array}{llll}\phi_*(x,t) & =\phi(x,t) & for & |x|<t-t_*, \\ q_*(t) & =q(t) & for & t>t_*.\end{array} \tag{9.1}$$

(ii) $Y_*(t_*)$ admits a decomposition $Y_*(t_*) = Y_0 + W^0 + Z^0$, where $Z^0 = (\Psi^0\ ,Q^0\ ,\Pi^0\ ,P^0)$ satisfies

$$\Psi^0(x) = \Pi^0(x) = 0 \quad for \quad |x| \geq M_*\ , \tag{9.2}$$

$$\|Z^0\|_{\mathcal{E}} \leq \delta. \tag{9.3}$$

W^0 satisfies for every $R>0$ and every $\gamma<\gamma(q^*)$

$$\|U(\tau)W^0\|_R \leq C_R\Big((|t_*+\tau|+1)^{-\sigma} + e^{-\gamma\tau}(|t_*|+1)^{-\sigma}\Big) \quad for \quad \tau>0, \tag{9.4}$$

where $C_R = C_R(\gamma)$ does not depend on δ.

This lemma leads to

Proof of Theorem 1.4 (1.9) follows from (9.1) provided we establish that for every $R,\varepsilon>0$

$$\|Y_*(t)-Y_0\|_R = \mathcal{O}(t^{-\sigma+\varepsilon}) \quad \text{as} \quad t\to\infty. \tag{9.5}$$

We generalise the integral inequality method used in the proof of Theorem 6.1 of the previous Section. We set $X(\tau) = Y_*(t_*+\tau) - Y_0$. Then $X(0) = W^0 + Z^0$ and (8.2) reads

$$X(\tau) = U(\tau)W^0 + U(\tau)Z^0 + \int_0^\tau ds\, U(\tau-s)B(X(s)), \tag{9.6}$$

since $U(\tau)$ is a linear operator. (9.4) implies an integral inequality similar to (8.3),

$$\begin{aligned}\|X(\tau)\|_R \ \leq\ & C_R\Big((|t_*+\tau|+1)^{-\sigma} + e^{-\gamma\tau}(|t_*|+1)^{-\sigma} \\ & + e^{-\gamma\tau}\|Z^0\|_{\mathcal{E}} + \int_0^\tau ds\, e^{-\gamma(\tau-s)}\|X(s)\|^2_{R_\rho+b}\Big) \quad \text{for} \quad \tau<\tau_b.\end{aligned} \tag{9.7}$$

Here $C_R = C_R(M_*,b)$, $b>0$ is an arbitrary fixed number and $\tau_b = \sup\{\tau\geq 0:\ |q(t_*+\tau)\leq b\} > 0$ for sufficiently large t_* due to (1.8). Denoting $\mu(\tau) = (|\tau|+1)^{-\sigma+\varepsilon}$ and $n(\tau) = \|X(\tau)\|_{R(b)}/\mu(\tau)$ with $R(b) = R_\rho + b$, we rewrite (9.7) as

$$n(\tau) \leq C^1_R\Big(\frac{(|t_*+\tau|+1)^{-\sigma}}{(|\tau|+1)^{-\sigma+\varepsilon}} + \frac{e^{-\gamma\tau}(|t_*|+1)^{-\sigma}}{(|\tau|+1)^{-\sigma+\varepsilon}} + \delta + \int_0^\tau ds\, \frac{\mu(\tau-s)\mu^2(s)}{\mu(\tau)} n^2(s)\Big), \quad \tau<\tau_b. \tag{9.8}$$

We use now the basic inequality in [16]

$$\sup_{\tau>0}\int_0^\tau ds \frac{\mu(\tau-s)\mu^2(s)}{\mu(\tau)} \leq B < \infty \quad \text{for} \quad \sigma-\varepsilon > \frac{1}{2}. \tag{9.9}$$

A further remark is that for every $\varepsilon,\gamma>0$

$$\sup_{\tau>0}\left(\frac{(|t_*+\tau|+1)^{-\sigma}}{(|\tau|+1)^{-\sigma+\varepsilon}} + \frac{e^{-\gamma\tau}(|t_*|+1)^{-\sigma}}{(|\tau|+1)^{-\sigma+\varepsilon}}\right) \to 0 \quad \text{as} \quad t_*\to\infty. \tag{9.10}$$

Finally, let us denote by $m(\tau) = \max_{0\le s\le\tau} n(s)$ and let us choose $0 < \varepsilon < \sigma - 1/2$, $0 < \gamma < \gamma(q^*)$, and t_* sufficiently large. Then (9.8)-(9.10) lead to a quadratic inequality of the form (8.5) and the proof can be continued as in Section 8. □

Proof of Lemma 9.1 The convergence (1.8) with $q^* = 0$ implies that for every $\varepsilon > 0$ there exist t_ε such that

$$|q(t)| + |\dot q(t)| < \varepsilon \quad \text{for} \quad t > t_\varepsilon. \tag{9.11}$$

Let us denote

$$t_{0,\varepsilon} = t_\varepsilon + R_\rho, \quad t_{1,\varepsilon} = t_{0,\varepsilon} + 1, \quad t_{2,\varepsilon} = t_{1,\varepsilon} + \varepsilon + R_\rho, \quad t_{3,\varepsilon} = t_{2,\varepsilon} + \varepsilon + R_\rho. \tag{9.12}$$

Then there exist a function $q_\varepsilon(\cdot) \in C^1(\mathbb{R})$ such that

$$q_\varepsilon(t) = \begin{cases} q(t), & t > t_{1,\varepsilon}, \\ 0, & t < t_{0,\varepsilon}, \end{cases} \quad \text{and} \quad |q_\varepsilon(t)| + |\dot q_\varepsilon(t)| < \varepsilon \quad \text{for all} \quad t \in \mathbb{R} \tag{9.13}$$

by suitable interpolation. Further, we use Kirchhoff formula (3.4)-(3.6) to define the modification $\phi_\varepsilon(x,t)$ of the solution (3.4),

$$\phi_\varepsilon(x,t) = \phi_{r,\varepsilon}(x,t) + \phi_0(x,t) \quad \text{for} \quad x \in \mathbb{R}^3 \quad \text{and} \quad t > 0, \tag{9.14}$$

where

$$\phi_{r,\varepsilon}(x,t) = -\int \frac{d^3y\ \theta(t - |x-y|)}{4\pi|x-y|}\rho(y - q_\varepsilon(t - |x-y|))\ . \tag{9.15}$$

Then $\phi_\varepsilon(x,t)$ is a solution to the wave equation

$$\ddot\phi_\varepsilon(x,t) = \Delta\phi_\varepsilon(x,t) - \rho(x - q_\varepsilon(t)) \quad \text{for} \quad t > 0\ . \tag{9.16}$$

By (9.15), (3.5), and (9.13) we have

$$\phi_{r,\varepsilon}(x,t) = \phi_r(x,t) \qquad \text{for} \quad |x| < t - t_{2,\varepsilon}\ , \tag{9.17}$$

$$\phi_{r,\varepsilon}(x,t) = \phi_0(x) \qquad \text{for} \quad |x| > t - t_\varepsilon. \tag{9.18}$$

Moreover, $\phi_{r,\varepsilon}(\cdot,\cdot) \in C^1(\mathbb{R}^4)$ and (9.13) implies

$$\sup_{x\in\mathbb{R}^3, t\in\mathbb{R}} (|\dot\phi_{r,\varepsilon}(x,t)| + |\nabla\phi_{r,\varepsilon}(x,t) - \nabla\phi_0(x)| + |\phi_{r,\varepsilon}(x,t) - \phi_0(x)|) = \mathcal{O}(\varepsilon). \tag{9.19}$$

Let us define

$$\begin{gathered} Y_*(t) = (\phi_\varepsilon(\cdot,t), q(t), \dot\phi_\varepsilon(\cdot,t), p(t)) \quad \text{for} \quad t > t_{3,\varepsilon} = t_*, \\ W^0 = (\phi_0(\cdot,t_*), 0, \dot\phi_0(\cdot,t_*), 0), \quad Z^0 = (\phi_{r,\varepsilon}(\cdot,t_*) - \phi_0(x), q(t_*), \dot\phi_{r,\varepsilon}(\cdot,t_*), p(t_*)). \end{gathered} \tag{9.20}$$

It is easy to check that t_* and $Y_*(t)$, W^0, Z^0 satisfy all requirements of Lemma 9.1, provided $\varepsilon > 0$ be sufficiently small.

Firstly, $Y_*(t)$ is a solution of the system (1.2) for $t > t_*$. Indeed, (9.17) and (9.12) imply

$$\phi_\varepsilon(x,t) = \phi(x,t) \quad \text{for} \quad |x| < \varepsilon + R_\rho \quad \text{and} \quad t > t_{3,\varepsilon}.$$

Hence, $Y_*(t)$ together with $Y(t)$ is a solution to the system (1.2) in the region $|x| < \varepsilon + R_\rho$. On the other hand, (9.11) implies

$$\rho(x - q(t)) = 0 \ \text{ for } \ |x| > \varepsilon + R_\rho \ \text{ and } \ t > t_\varepsilon.$$

Hence, $Y_*(t)$ satisfies the equations (1.2) in the region $|x| > \varepsilon + R_\rho$ by (9.16). In addition,
ad (i) (9.1) follows from (9.17) and (9.20).
ad (ii) (9.2) for $M_* = 3R_\rho + 2\varepsilon + 1$ follows from (9.18). (9.3) follows from (9.2) and (9.19). We deduce (9.4) from the bounds (3.17) and the representation (7.11) for the linearized dynamics $U(\tau)$.

Denote by $U(\tau)W^0 = (\Psi(x,\tau), Q(\tau), \Pi(x,\tau), P(\tau))$ and let us prove the bounds of the type (9.4) for $(\Psi(\tau), \dot\Psi(\tau))$, for instance. Let us rewrite the representation (7.11) for $\Psi(x,\tau)$ in the form

$$\Psi(x,\tau) = \Psi_1(x,\tau) + \int_0^\tau ds\, h(\tau - s)\Psi_2(x,s) + \Psi_{1,3}(x,\tau)$$

and consider every term separately. At first, $(\Psi_1(x,\tau), \dot\Psi_1(x,\tau)) = w^0_\tau(\phi_0(\cdot,t_*), \dot\phi_0(\cdot,t_*))$ where w^0_τ is a dynamical group of the free wave equation. On the other hand $(\phi_0(\cdot,t_*), \dot\phi_0(\cdot,t_*)) = w^0_{t_*}(\phi^0, \pi^0)$ due to the Kirchhoff formula (3.6). Therefore $(\Psi_1(x,\tau), \dot\Psi_1(x,\tau)) = w^0_{t_*+\tau}(\phi^0, \pi^0)$ and the bound $C_R(|t_* + \tau| + 1)^{-\sigma}$ for $\|\Psi_1(x,\tau)\|_R + \|\dot\Psi_1(x,\tau)\|_R$ follows from (3.17).

Secondly, $(\Psi_2(x,\tau), \dot\Psi_2(x,\tau)) = w^0_\tau(0,0) = 0$. Finally, $(\Psi_3(x,\tau), \dot\Psi_3(x,\tau)) = w^0_\tau(0, \nabla\rho(x))$ and then $\|\Psi_3(x,\tau)\|_R = 0$ for $|\tau| > R + R_\rho$. Therefore the representation (7.11) of the term $\Psi_{1,3}(x,\tau)$ with $\Psi_3(x,s')$, together with (7.9) and bounds for Ψ_1, imply the bound

$$\|\Psi_{1,3}(x,\tau)\|_R \le C_R \int_0^\tau ds\ e^{-\tilde\gamma(\tau-s)} \int_0^{R+R_\rho} ds'\ (|t_* + s - s'| + 1)^{-\sigma}$$

with $\gamma < \tilde\gamma < \gamma(q^*)$. □

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10 Appendix: Densities of Wiener type

It is not completely evident that (C) and (W) can be satisfied simultaneously. To construct a generic example fix a real $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(s) \not\equiv 0$. Then $\hat\varphi$ may be extended as an analytic function to the whole complex plane and there exists an $\alpha \in \mathbb{R}$ such that $\hat\varphi(\xi + i\alpha) \neq 0$ for all $\xi \in \mathbb{R}$. By replacing $\varphi(s)$ with $\varphi(s)\exp(\alpha s)$ we may assume that $\alpha = 0$. Clearly $\phi(x_1)\phi(x_2)\phi(x_3)$ satisfies (W) and (C) except for rotation invariance. Since by rotational averaging we could pick up a zero, we first let $\rho_1 = \varphi * \psi$ with $\psi(s) = \varphi(-s)$. Then again $\rho_1 \in C_0^\infty(\mathbb{R})$ and $\hat\rho_1(\xi) = |\hat\varphi(\xi)|^2 > 0$ for all $\xi \in \mathbb{R}$. Let ρ be the rotational average of $\rho_1(x_1)\rho_1(x_2)\rho_1(x_3)$. Then ρ satisfies both (C) and (W).

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