On the existence of infinitely many modes of a nonlocal nonlinear Schrödinger equation related to dispersion-managed solitons

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Abstract

We present a comprehensive study of an NLS with additional quadratic potential and general, possibly highly nonlocal, cubic nonlinearity. In particular this equation arises in a variety of applications and is known as the Gross-Pitaevskii equation in context of Bose-Einstein condensates with parabolic traps or as a model equation describing average pulse propagation in dispersion-managed fibers. Both, global and local, bifurcation behavior is determined showing the existence of infinitely many symmetric modes of the equation. In particular our theory provides a strict theoretical proof of the existence of a symmetric bi-soliton which recently was found by numerical simulations.

Keywords: nonlinear Schrödinger equation, harmonic potential, dispersion management, global bifurcation theorem

1 Introduction and main results

In this paper we consider the nonlinear Schrödinger equation (NLS) with additional quadratic potential

$$iu_t + u_{xx} - x^2 u = F(u),\ x \in \mathbb{R},\ t \geq 0, \tag{1}$$

where $F(u)$ is a cubic, possibly nonlocal, nonlinearity satisfying some assumptions given later in this paper. Nonlocality of the nonlinearity is an important factor in many applications, often approximated by a simpler local nonlinearity.

Equation (3) models a variety of phenomena and is known as the Gross-Pitaevskii (GP) equation in context of Bose-Einstein condensates (BEC) with parabolic traps. Assuming a higly anisotropic trap Kishvar et al. [9] derived the one-dimensional GP-equation (1) with $F(u) = \pm|u|^2 u$ as model equation for the macroscopic dynamics of cooled atoms confined in a three-dimensional parabolic potential created by a magnetic trap. Using an approximation technique they explain the existence of infinitely many nonlinear modes of the equation. In the present paper we rigorously prove the existence of such modes identifying them as bifurcating solutions from the eigenvalues of the linear harmonic oscillator. Moreover, the shape of the mode is determined by the corresponding Gauss-Hermite eigenfunctions.

*This file is a preprint and differs from the submitted version.

Our main interest, however, lies in the context of nonlinear fiber optics. Modern optical transmission systems succesfully use the so-called dispersion management (DM) technique. The idea of DM is to use a dispersion-compensating fiber to overcome the dispersion of the standard monomode fiber which causes dispersive broadening of a pulse. If the residual dispersion is small the signal should evolve nearly periodical, this situation is called strong DM. Numerical and experimental results show that the corresponding pulse, the so-called DM-soliton, is stable over hundreds of periods. Using the so-called lens transformation and an averaging technique suggested by Zharnitsky et al. [22] we have shown in [12] that the master equation can be transformed into equation (1) with nonlocal nonlinearity of the following form (after normalization)

$$F(u) := -\int_0^1 S^{-1}(z)\left(\frac{1}{T(z)}|S(z)u|^2 S(z)u\right)dz, \tag{2}$$

with $S(z) := U(R^{\text{eff}}(z))$ where $U(z)$ denotes the group generated by the harmonic oscillator, $T(z)$ is a characteristic pulse width and $R^{\text{eff}}(z)$ is the effective residual dispersion (for details see section 3 below).

Regarding equation (1) we notice that the linear part is nothing else than the harmonic oscillator which has the well-known basis of Gauss-Hermite eigenfunctions. The presence of the quadratic potential will help us to overcome the problems due to the unboundedness of the underlying spatial domain.

It is natural to seek for standing wave solutions, the corresponding ansatz $u(t,x) = \exp(-\lambda t)v(x)$ results in the standard eigenvalue problem

$$-u_{xx} + x^2 u + F(u) = \lambda u, \tag{3}$$

where we have required $F(\exp(i\theta)u) = \exp(i\theta)F(u)$ to derive the equation. The natural space to consider equation (3) is the weighted Hilbert space [8, 11]

$$X := \{u \in H^1(\mathbb{R}) | \int_{\mathbb{R}} x^2|u|^2 dx < \infty\} \tag{4}$$

with inner product $\langle u, v\rangle_X := (u_x, v_x) + (xu, xv)$ where $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\mathbb{R})$ and corresponding energy norm

$$\|u\|_X^2 = \int_{\mathbb{R}} |u_x|^2 + x^2|u|^2 dx = \|u_x\|_2^2 + \|xu\|_2^2. \tag{5}$$

The main conditions on the nonlinearity are

($\mathcal{F}_1$) $F : X \to L^2(\mathbb{R}) : F(\exp(i\theta)u) = \exp(i\theta)F(u)$ and $\mathrm{u} : \mathbb{R} \to \mathbb{R} \Rightarrow \mathrm{F} \circ \mathrm{u} : \mathbb{R} \to \mathbb{R}$.

($\mathcal{F}_2$) There exist $0 < \alpha < 7/2, \beta \geq 4 - \alpha$ such that
$$\|F(u) - F(v)\|_2^2 \leq C\left(\|u\|_X^\alpha \|u\|_2^\beta + \|v\|_X^\alpha \|v\|_2^\beta\right)\|u - v\|_X^{1/2}\|u - v\|_2^{3/2} \ \forall u, v \in X.$$

($\mathcal{F}_3$) F is sufficiently smooth, i.e. $F \in C^1(X, L^2)$.

In virtue of assumption ($\mathcal{F}_1$) we have

$$\langle F(u), u\rangle \in \mathbb{R} \ \forall u \in X_{\mathbb{C}} = \{u \in H^1(\mathbb{R}, \mathbb{C}) | \int_{\mathbb{R}} x^2|u|^2 dx < \infty\}$$

and hence we consider throughout the paper only real-valued functions and consequently X instead of $X_{\mathbb{C}}$. Assumption $(\mathcal{F}_2)$ is a general growth condition for a cubic nonlinearity which appears quite natural resulting in $F(u) = \mathcal{O}(\|u\|_2^3)$ for $u \to 0$. $(\mathcal{F}_3)$ is a minimum smoothness condition which will later be replaced by some stronger condition in order to determine the local bifurcation behavior. Necessary for the validity of the variational approach is the potential property of F. Hence we require the following assumption:

$(\mathcal{F}_4)$ There exists $G \in \mathcal{C}^1(X, \mathbb{R})$ with $G(0) = 0$ such that $G'(u)v = (F(u), v) \forall u, v \in X$.

Later we will restrict ourselves to the practical relevant case, where ground states exist. In order to determine the direction of bifurcation one has to fix the sign of the nonlinearity, that is to consider only "focussing" nonlinearities with an additional technical assumption, i.e.

$(\mathcal{F}_5)$ $G(u) < 0 \; \forall u \in X \setminus \{0\}$ and $(\mathrm{F(su), u}) \geq \mathrm{s}^{\delta}(\mathrm{F(u), u})$ for $0 < \mathrm{s} < 1$ and $\delta > 1$.

Moreover, we are interested in the symmetry of the solutions. Thus we consider at some stage only symmetric potentials

$(\mathcal{F}_6)$ $G(u(x)) = G(u(-x))$ and $G(u(x)) = G(-u(-x))$.

The above assumptions allow it to determine the direction of bifurcation and orbital stability of the solution. It should be noted that in our applications $G(u)$ is nothing else than $(F(u), u)/4$ which is typically of one sign, but much more general nonlinearities can be treated as well.

Note that the nonlinearity $F(u) = \sigma|u|^2 u$ with $\sigma < 0$ satisfies assumptions $(\mathcal{F}_1)$-$(\mathcal{F}_6)$. Equation (3) with standard cubic nonlinearity was investigated by several authors in the past: Existence and stability of the solutions of equation (3) was discussed by Fukuizumi [4], Oh [16] and Zhang [23]. Kishvar et al. [9] observe the existence of infinitely many nonlinear modes of (3), but they did not give a theoretical explanation. The NLS with quadratic potential is also discussed in the book of Cazenave [1]. Both, global and local, bifurcation results are obtained by Kunze et al. [11]. Moreover, the corresponding solutions decay very fast, i.e. Gaussian-like and there exists a positive solution [5, 8].

Allowing the nonlinearity to be nonlocal we will explain in this paper that some results and methods can be adapted, whereas other properties of the solutions are lost, e.g. the Gaussian decay. The main result of this paper can be summarized as follows: In each eigenvalue of the harmonic oscillator bifurcates an unbounded branch of localized solutions in the sense of the global bifurcation theorem of Rabinowitz which give rise to the existence of infinitely many nonlinear modes. Under slightly more restrictive conditions on the nonlinearity the bifurcating solutions can be characterized as minimizers resp. saddle points of the corresponding energy functional. Moreover, there exist infinitely many even resp. odd solutions. Furthermore, stability and decay properties of the solutions are discussed. This assertions can be visualized in a bifurcation diagram as shown in Figure 1, the direction of bifurcation depends on the sign of the nonlinearity.

In the context of fiber optics the DM-soliton is characterized as the ground state of the corresponding energy functional, the other branches correspond to modes of arbitrary order. Our method guarantees the uniqueness (up to a phase factor) of the DM-soliton close to the bifurcation point for fixed energy and shows that it is even. This is a new theoretical result, well-supported by numerical simulations. The DM-soliton as a ground state of a macroscopic quantum oscillator as (3) has recently been studied by Schäfer et al. [20] , but they consider only reduced models and consequently our results are a verification of their approximation method.

Of great practical interest is the theoretical verification that DM-systems support the bi-soliton in addition to the well-known single-soliton. The bi-soliton was recently numerically observed by Maruta et al. [14], see also the paper of Pare et Belanger [18]. It is a promising candidate for the improvement of todays systems and will help to increase transmission rates by using new encoding schemes.

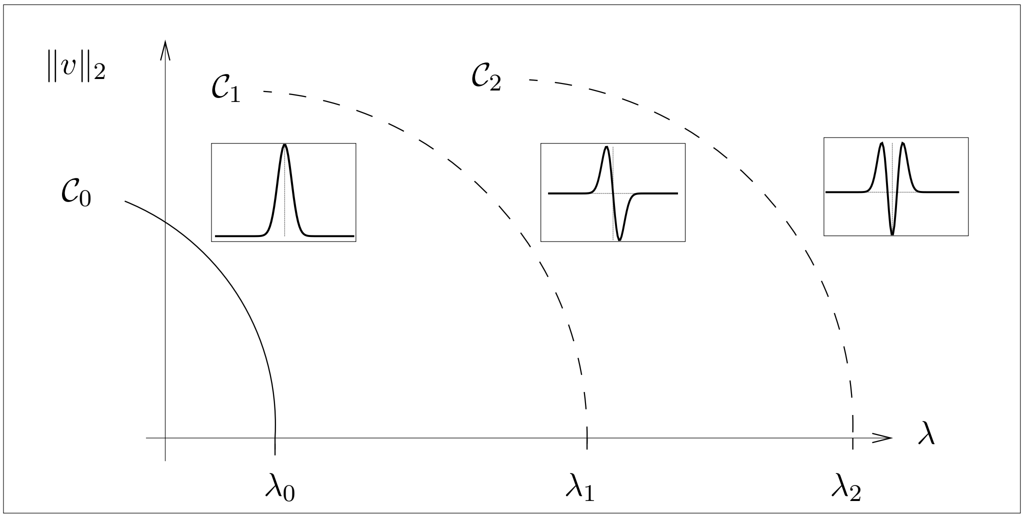

Figure 1: Bifurcation diagram of $-v_{xx} + x^2 v + F(v) = \lambda v$.

2 Analysis of the Gross-Pitaevskii equation

In this section we present the analysis of the Gross-Pitaevskii equation (3) and state our main results. Thereby the developed theory is sufficiently general to cover both applications, Bose-Einstein condensates and dispersion-managed optical fibers. However, our main goal is to generalize the bifurcation result of Kunze et al. [11] to nonlocal nonlinearities and to characterize the bifurcating solutions more precisely by variational arguments. In order not to bother the reader with too many technical details some of the proofs are postponed in an appendix.

2.1 Bifurcation analysis

In this subsection we investigate the bifurcation behavior of equation (3). We strongly rely on the paper by Kunze et al. [11]

2.1.1 Preliminaries

The key property of the space X is the following [23]

Lemma 2.1. *The embedding $X \hookrightarrow L^q(\mathbb{R})$ is compact for $2 \leq q < \infty$.*

Next we consider the linear problem corresponding to (3), i.e.

$$-u_{xx} + x^2 u - \lambda u = 0. \tag{6}$$

The following properties of the linear harmonic oscillator are well-known [11]:

Lemma 2.2. *Let $\lambda_n = 2n+1, n \in \mathbb{N}_0$ and*

$$u_n(x) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp(-x^2/2) H_n(x),$$

where H_n is the n-th Hermite polynomial

$$H_n(x) := (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$$

(i) *λ_n are exactly the eigenvalues of (6). They are simple and the corresponding eigenfunctions are given by $u_n \in X$.*

(ii) *The eigenfunctions u_n of (6) form a complete orthonormal system of $L^2(\mathbb{R})$.*

2.1.2 Global bifurcation behavior

In this section we apply the global bifurcation theorem of Rabinowitz [19] to equation (3). The proof is similar to the one by Kunze et al. [11]. Due to the potential property of F we can define a weak solution. Let

$$S_0 := \{(\lambda, u) \in \mathbb{R} \times X : u \neq 0 \text{ is a weak solution of } -\mathrm{u_{xx}} + \mathrm{x^2 u} + \mathrm{F(u)} = \lambda \mathrm{u}\}$$

denote the set of all nontrivial solutions of (3) and $S = \overline{S_0}^{\mathbb{R}\times X}$ its closure. It is clear that $u \in X \subset H^1(\mathbb{R})$ implies $u \in C_0^0(\mathbb{R})$, the set of continuous functions on $\mathbb{R}$ vanishing for $x \pm \infty$. Using a bootstrapping argument it follows then that $u \in C^\infty(\mathbb{R})$, cf. [22] and consequently a weak solution is a classical solution.

Our global bifurcation result then reads as follows

Theorem 2.3. *Let F satisfy assumptions $(\mathcal{F}_1)$-$(\mathcal{F}_4)$. Then for all $n \in \mathbb{N}_0$, $(0, \lambda_n)$ is a bifurcation point. Let $\mathcal{C}_n$ denote the component of $\mathcal{S}$ with $(0, \lambda_n) \in \mathcal{C}_n$. Then the following alternative holds: Either*

(i) *$\mathcal{C}_n$ is unbounded in $\mathbb{R} \times X$ or*

(ii) *$\mathcal{C}_n$ is compact and there exists $m \neq n$ such that $(0, \lambda_m) \in \mathcal{C}_n$.*

Proof. Using the Green's function $g(x, \xi)$ the problem is transformed to an integral problem, see [11] for details. To apply the global bifurcation theorem of Rabinowitz the resulting nonlinearity should be compact and of higher order. Roughly speaking, this is guaranteed by the growth condition together with the compact embedding. In particular, assumptions $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ yield the assertions of Lemma 5 in [11], the proof then is along the lines of [11]. □

Usually the second alternative is ruled out by nodal arguments, see [11] for details. These arguments are no longer valid for nonlocal nonlinearities. However, in the next section we will show by variational arguments that the bifurcating branches are unbounded.

2.1.3 Local bifurcation behavior and orbital stability

In order to determine the local behavior in the vicinity of $(0, \lambda_n)$ we introduce a nondegeneracy condition on the nonlinearity

$(\mathcal{F}_3^n)$ $F \in C^3(X, L^2)$ with $\langle \delta^3 F(0)[u_n]^3, u_n \rangle \neq 0$

Note that in virtue of $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ the first derivatives of F vanish, i.e. $\delta F(0)[u_n] = 0$ and $\delta^2 F(0)[u_n]^2 = 0$. The local bifurcation behavior then can be determined as follows:

Lemma 2.4. *Let F satisfy $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3^n)$. Then there exists $\epsilon > 0$ such that $(\lambda, u) \in \mathcal{C}_n \cap U_\epsilon(\lambda_n, 0)$ implies*

$$\lambda = \lambda_n + \lambda(s), \;\; u = su_n + sv_n(s),$$

where $0 < |s| < \epsilon$ and $\lambda(0) = 0$, $\lambda'(0) = 0$ and

$$\mathrm{sgn}\left(\lambda''(0)\right) = \mathrm{sgn}\left(\langle \delta^3 \mathrm{F}(0)[\mathrm{u_n}]^3, \mathrm{u_n}\rangle\right). \tag{7}$$

Moreover, $v_n(0) = 0$ with $(v_n(s), u_n)_X = 0$.

Proof. The assertions follow by standard Ljapunov-Schmidt theory since the eigenvalues are simple. □

Thus, the direction of bifurcation is determined by the sign of $\lambda''(0)$. In the case of negative sign the bifurcating solutions from the first eigenvalue are orbitally stable by the method of Rose and Weinstein, otherwise they are unstable. Instead of discussing this in detail we refer to the next section where the solutions are characterized as ground states of the energy functional in the situation where the nonlinearity is focussing.

2.2 Variational calculus

In this section we characterize the bifurcating solutions by variational arguments and identify them as minimizers or saddle points of the corresponding energy functional.

Considering the energy functional corresponding to equation (3)

$$J(u) = \frac{1}{2}\|u\|_X^2 + G(u), \tag{8}$$

we are in position to state our main theorem which characterizes the bifurcating solutions. In particular, it shows the existence of infinitely many nonlinear modes of arbitrary energy.

Theorem 2.5. *Suppose $(\mathcal{F}_1)$-$(\mathcal{F}_5)$ are satisfied and $\omega > 0$ is given. Then there exists an unbounded sequence $\{u_n^\omega\}_{n\in\mathbb{N}} \subset X$ of critical points of J with $\|u\|_2^2 = \omega$ and corresponding Lagrange multipliers $\lambda_n^\omega \leq \lambda_n$ with the following properties*

1. *u_0^ω is the ground state of the energy functional, it is even and orbitally stable. Moreover, $(u_0^\omega, \lambda_0^\omega) \in \mathcal{C}_0$.*
2. *u_1^ω is odd and minimizes J among all odd functions. Furthermore, we have $(u_1^\omega, \lambda_1^\omega) \in \mathcal{C}_1$.*
3. *The u_n^ω with $n > 1$ are saddle points with $(u_n^\omega, \lambda_n^\omega) \in \mathcal{C}_n$. Morever, u_{2k} is even and u_{2k+1} is odd.*

In case of a simple nonlinearity $F(u) = -|u|^2 u$ it is shown that the ground states are positive [5] using Kato's inequality and maximum principle. Moreover, the solutions of the simpler equation decay like a Gaussian and are unique. However, all these arguments need informations about nodal properties of the solutions which are not at hand for the nonlocal nonlinearity we have in mind. In fact, the DM-solution is not positive and decays exponentially fast with a Gaussian core.

In particular the second alternative of the global bifurcation theorem can be ruled out by the above theorem:

Corollary 2.6. *1. $\mathcal{C}_n$ is unbounded in both, u and λ.*

2. $\mathcal{C}_n \cap \mathcal{C}_m = \emptyset$ for $n \neq m$.

For some applications it may be of interest to fix the wave-number. Due to the direction of bifurcation we have

Corollary 2.7. *For all $\lambda \in \mathbb{R}$ equation (3) has infinitely many solutions u_n^λ.*

Remark 2.8. *Requiring $\|u\|_p \leq |G(u)|$ for some $p \geq 2$ one can verify that the sequence $\{u_n^\lambda\}$ is unbounded in X [13].*

3 Applications

In this section we explain the meaning of the derived results in the context of dispersion-managed optical fibers and discuss their practical relevance.

From a mathematical point of view the model equation describing pulse propagation in optical fibers with dispersion management is given by the cubic nonlinear Schrödinger equation (DM-NLS) with periodically varying coefficients

$$iA_z(z,t) + D(z)A_{tt}(z,t) + c|A(z,t)|^2 A(z,t) = 0. \tag{9}$$

Here, A is the complex envelope of the electric field, t is retarded time, z is propagation distance, D is the dispersion coefficient and $c > 0$ is representing loss and influence of the amplifiers and is assumed to be constant (loss-less model). The dispersion profile D is periodic with normalized period 1. In the case of strong dispersion management the residual dispersion $\langle D \rangle$ is small compared to local dispersion, i.e. $\langle D \rangle << D$, where $\langle \cdot \rangle$ denotes averaging over one period. Equation (9) is mostly studied for dispersion profiles having the form of a symmetric two-step map:

$$D(z) = D_{\text{loc}} + \langle D \rangle = \begin{cases} d + \langle D \rangle : 0 \leq z \leq L, 1 - L \leq z \leq 1 \\ -d + \langle D \rangle : L < z < 1 - L \end{cases} \tag{10}$$

with $\langle D_{\text{loc}} \rangle = 0$. Throughout the paper we restrict ourselves to the case of positive residual dispersion, i.e. we require $\langle D \rangle > 0$. In a series of papers Turitsyn and Gabitov, cf. [21], suggest to apply the following transformation to equation (9) which is known as lens transformation or pseudo-conformal transformation

$$A(z,t) = N\frac{Q(z, t/T(z))}{\sqrt{T(z)}} \exp\left(it^2 \frac{M(z)}{T(z)}\right). \tag{11}$$

Here (T, M) is a periodic solution of the so-called nonlinear TM-equations which arise in the context of lens transformation (see [12, 21] for details):

$$T'(z) = 4D(z)M(z), \qquad T(0) = T_0 > 0, \tag{12}$$

$$M'(z) = \frac{D(z)}{T(z)^3} - \frac{N^2}{T(z)^2}, \; M(0) = 0. \tag{13}$$

Thereby, T_0 has to be determined in such a way that for a given N^2 the corresponding solution is periodic or vice versa. T and M have the physical meaning of pulse width and chirp, N^2 is the pulse energy. In [12] it was shown that after applying lens transformation to the (strong) dispersion managed NLS (9) with a two-step map as in (10) and averaging of the resulting equation one arrives at a Schrödinger-type equation with additional quadratic potential, i.e.

$$iu_z + au_{xx} - bx^2u + \int_0^1 S^{-1}(z)\left(\frac{N^2}{T(z)}|S(z)u|^2 S(z)u\right)dz = 0,\ z \geq 0,\ x \in \mathbb{R}, \quad (14)$$

where

$$\begin{aligned} a &= \langle \tfrac{D}{T^2} \rangle - N^2 \langle \tfrac{1-\cos(4R^{\text{eff}})}{2T} \rangle, \\ b &= \langle \tfrac{D}{T^2} \rangle - N^2 \langle \tfrac{1+\cos(4R^{\text{eff}})}{2T} \rangle. \end{aligned}$$

In equation (14), $S(z)$ is defined as $S(z) = U(R^{\text{eff}}(z))$, where R^{eff} has the physical meaning of accumulative effective dispersion, i.e. $R^{\text{eff}}(z) = \int_0^z D_{\text{loc}}/T_{\text{lin}}^2$. Furthermore, T_{lin} is the periodic solution of the linear TM-equations

$$\begin{aligned} T'_{\text{lin}}(z) &= 4D_{\text{loc}}(z)M_{\text{lin}}(z),\ T_{\text{lin}}(0) = T_0 > 0 \\ M'_{\text{lin}}(z) &= \frac{D_{\text{loc}}(z)}{T_{\text{lin}}(z)^3}, \qquad M_{\text{lin}}(0) = 0 \end{aligned}$$

which is explicitly known. $U(z)$ denotes the group generated by the harmonic oscillator, i.e. $U(z) = \exp(iAz)$ with $Au = \triangle u - x^2u$. It is essential for the whole approach that $S(z)$ is 1-periodic since $\langle D_{\text{loc}}/T_{\text{lin}}^2 \rangle = 0$. Equation (14) describes averaged pulse propagation in a strong dispersion-managed system after lens transformation. In the region of moderate energy values $N^2 < \overline{N^2}(\langle D \rangle)$ we have shown by numerical simulations in [12] that

$$a, b > 0$$

in contrast to former discussions of the problem (see cf. [21] for details) and hence the potential is attracting. In order to transform (14) to the standard bifurcation problem (3) we consider steady-state solutions of the following form

$$u(x,z) = \phi(\gamma x)\exp(-i\sqrt{ab}\lambda z),\ \text{with } \gamma = \left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{1/4}.$$

In the new variable $\xi = \gamma x$ we have $-\phi_{\xi\xi} + \xi^2\phi + F(\phi) = \lambda\phi$ with

$$F(\phi) := -\frac{N^2}{\sqrt{ab}} \int_0^1 S^{-1}(z)\left(\frac{1}{T(z)}|S(z)\phi|^2 S(z)\phi\right)dz. \quad (15)$$

Writing again u instead of ϕ and x instead of ξ we end up at (3) with a highly nonlocal nonlinearity which can be expressed in terms of Mehler's kernel and satisfies all the assumptions $(\mathcal{F}_1)$ throughout $(\mathcal{F}_6)$, see appendix. Hence we can apply the theory developed in the previous section.

Since N^2 in the nonlinear TM-equations has the physical meaning of pulse energy, we are interested in unit-norm solutions of the DM-NLS after lens tranformation. Accordingly we apply our main theorem with $\omega = 1$ to obtain

Theorem 3.1. *There exists a sequence $\{u_n, \lambda_n\}$ of solutions of the averaged equation (14) having the form*

$$u_n(x,z) = \phi_n(\xi)\exp(-i\sqrt{ab}\lambda z),\ \xi = \gamma x,$$

where ϕ_{2k} is even and ϕ_{2k+1} is odd. Moreover, there exists $C > 0$ such that

$$|u_n(x)| + |u_n'(x)| < C \exp\left(- \left(\frac{b}{a}\right)^{1/4} \frac{|x|}{3} \right).$$

Proof. It remains to verify the exponential decay, see appendix. Thereby, the well-known Gaussian decay which occurs for $F(u) = -|u|^2 u$ is lost due to the nonlocal properties of F which prevents nodal arguments or a maximum principle. □

Thus we have shown the existence of infinitely many even resp. odd solutions of the equation (3). Note that symmetry of the DM-soliton was not rigorously proven up to now. Kunze [10] considered only the case of two spatial dimensions, but in context of nonlinear optics space and time variables are interchanged and hence the one-dimensional case is of practical interest.

Before discussing the relevance of our results to dispersion-managed solitons, we add some comments on the mathematical differences to the original problem and the relevant publications. Due to the lens transformation the continuous spectrum of the DM-NLS becomes discrete. Therefore we have been able to apply standard bifurcation theory to our problem. With this method uniqueness of the DM-soliton in the vicinity of the bifurcation point is at hand in contrast to the original equation, where only variational arguments are used to ensure the bifurcation from the essential spectrum. However, for the original problem the solutions are of the following form

$$|A(z,t)|^2 = \frac{N^2}{T(z)} \left| S(z) \left\{ \exp(i\lambda z) u(t/T(z)) \right\} \exp\left(it^2 \frac{M(z}{T(z)} \right) \right|^2,$$

where u decays exponentially fast. Note that the ground state obtained by Zharnitsky et al. [22] was only shown to be in $H^1(\mathbb{R})$. The relevance of our theoretical results is summarized in the following concluding remark

Remark 3.2.

- *The ground state ϕ_0 corresponds to the DM-soliton; we have shown that it is an even function at least for small input energies. This is a new theoretical result already known from numerical simulations.*
- *Moreover, it is shown that the DM-soliton decays exponentially fast and has a Gaussian core. Numerical simulations show the existence of an "optimal" energy N^2, where γ and accordingly the decay rate is maximized [13].*
- *Uniqueness of the DM-soliton is still an open question, reduced models indicate that the DM-solitons form a one-parameter-family. We have shown in the present paper the uniqueness of the DM-soliton close to the bifurcation point (that is for small energies), but there could exist a secondary (symmetry-breaking) bifurcation.*
- *The odd solution ϕ_1 corresponds to the bi-soliton which was first observed by Maruta et al. [15] by numerical simulations, see also the work of Pare et Belanger [18]. It minimizes the energy functional with respect to all odd functions and is hence stable against odd perturbations. It is a promising candidate for the reduction of intra-channel interactions which play an important role in todays multi-channel systems. Numerical simulations [14] show that the bi-soliton propagates stable over long distances and the bit rate is increased significantly by a new encoding scheme.*

- *Furthermore, we have verified the existence of modes of arbitrary order, a fact which was unknown up to now. Similar to the basis of Gauss-Hermite functions for the linear oscillator there exists a family of nonlinear modes with shape close to the corresponding eigenfunction. Maruta et al. [14] also observed a tri-soliton which corresponds to solutions on the third branch in the bifurcation diagram. They conjectured the existence of a periodic pulse of arbitrary order which is guaranteed by our result.*

- *A very effective way to derive approxmiations of the DM-soliton is to use a Hermite-Gaussian ansatz in the lens-transformed equation, cf. [20]. From the results derived in this paper it is now clear why this method works. We have shown that the DM-soltion in the averaged equation is close two the first eigenmode. In [20], u is expanded in terms of the Gauss-Hermite eigenfunctions, the expansion is truncated after a few modes. Bearing the bifurcation result in mind it is now obvious that the error is small although infinitely many modes are omitted. With this method it should also be possible to obtain approximations for the solutions bifurcating from the other eigenvalues by considering the corresponding eigenfunction and its neighbors as perturbations.*

A Proofs of the main theorems

In this section we will give the proofs of the main theorems. Since the ground states of the equation correspond to the DM-soliton in our application they are of fundamental importance and we will discuss them first. Define Γu as $\Gamma u(x) = u(-x)$. Due to the symmetry in equation (3) we can conclude that, if u is a solution, then also Γu and hence the functional J is invariant under Γ, see [10] for a similar result. With $X_\Gamma := \{u \in X | \Gamma u = u\}$ the following lemma holds

Lemma A.1 (Characterization of the ground state). *For all $\omega > 0$ the minimization problem*

$$J_\Gamma^\omega \quad = \min\{J(u) | u \in X_\Gamma, \| u \|_2^2 = \omega\} \tag{16}$$

has a nontrivial solution $u^\omega \in X_\Gamma$ which corresponds to a weak solution of (3). Moreover, u^ω is orbitally stable as the ground state of the equation, that is

$$J(u^\omega) = \min\{J(u) | u \in X, \| u \|_2^2 = \omega\}. \tag{17}$$

Furthermore, $\{(u^\omega, \lambda^\omega) | \omega \in (0, \infty)\} \subset \mathcal{C}_0$.

Proof. The proof relies on the principle of symmetric criticality [17] which allows us to reduce the problem to even functions, i.e. a minimizer of problem (16) is a critical point for the whole problem. At first we need to verify the following

Lemma A.2. *For $u, v \in X$ the following estimate holds with $\alpha' := \alpha/2 + 1/4 < 2$ and $\beta' = \beta/2 + 3/4$*

$$|G(u) - G(v)| \leq C(|u\|_X^{\alpha'} + \|v\|_X^{\alpha'})(|u\|_2^{\beta'} + \|v\|_2^{\beta'})\|u - v\|_2. \tag{18}$$

Proof. We calculate as follows

$$\begin{aligned}|G(u)-G(v)| &= |\int_0^1 \frac{d}{ds}G(su+(1-s)v)ds| = |\int_0^1 G'(su+(1-s)v)(u-v)ds| \\ &= \int_0^1 |(F(su+(1-s)v),u)|ds \le C\int_0^1 \|F(su+(1-s)v)\|_2\|u-v\|_2 ds \\ &\le C\int_0^1 \|su+(1-s)v\|_X^{\alpha/2+1/4}\|su+(1-s)v\|_2^{\beta/2+3/4}\|u-v\|_2 ds \\ &\le C(\|\|u\|_X^{\alpha'}+\|v\|_X^{\alpha'})(\|u\|_2^{\beta'}+\|v\|_2^{\beta'})\|u-v\|_2\end{aligned}$$

which is the assertion of the lemma. □

The above lemma together with assumption $(\mathcal{F}_2)$ shows

$$J(u) \;=\; \frac{1}{2}\|u\|_X^2 + G(u) \ge \frac{1}{2}\|u\|_X^2 - C\|u\|_X^{\alpha'}\omega^{(\beta'+1)/2} \tag{19}$$

which is bounded from below since $\alpha' < 2$. Making use of the fact that the embedding $X_\Gamma \subset\subset L^p(\mathbb{R})$ is compact for $p \ge 2$ we are able to show that a minimizer on X_Γ exists: Let $\{u_n\}$ be a minimizing sequence, that is $\|u_n\|_2^2 = \omega$ and $J(u_n) \to J_\Gamma^\omega$ which implies that $J(u_n)$ is bounded, say $J(u_n) \le M$, and using equation (19) we conclude that u_n is bounded in X. Passing to a subsequence we may assume that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L^2(\mathbb{R})$ which yields $\|u\|_2^2 = \omega$. From (18) we obtain

$$|G(u_n) - G(u)| \le C\|u - v\|_2 \tag{20}$$

since u_n, u are bounded in X. Finally it follows $J(u) \le \lim_{n\to\infty} J(u_n) = J_\Gamma^\omega$ and accordingly $u \in X_\Gamma$ is the desired minimizer. The principle of symmetric criticality then reveals that u is a critical point of J and, consequently, a weak solution of equation (3) with Lagrange-multiplier λ^ω.

$$-u_{xx}^\omega + x^2u^\omega + F(u^\omega) = \lambda^\omega u. \tag{21}$$

Next we show $\lim_{\omega\to 0}\lambda^\omega = \lambda_0$, where λ_0 is the eigenvalue of the harmonic osillator corresponding to u_0.

Therefore we take the inner product of the Euler-Lagrange-equation (21) with u^ω to obtain

$$\| u^\omega \|_X^2 + (F(u^\omega),\omega) = \lambda^\omega\omega, \text{ hence } \lambda^\omega = \frac{\| \mathrm{u}^\omega \|_{\mathrm{X}}^2}{\omega} + \frac{(\mathrm{F}(\mathrm{u}^\omega),\mathrm{u}^\omega)}{\omega}. \tag{22}$$

Note that λ_0 can be characterized by the Rayleigh quotient as follows

$$\lambda_0 = \|u_0\|_X^2 = \inf\{\frac{\|\mathrm{u}\|_{\mathrm{X}}^2}{\|\mathrm{u}\|_2^2}|\mathrm{u}\in\mathrm{X},\mathrm{u}\ne 0\} = \inf\{\frac{\|\mathrm{u}\|_{\mathrm{X}}^2}{\|\mathrm{u}\|_2^2}|\mathrm{u}\in\mathrm{X}_\Gamma,\mathrm{u}\ne 0\},$$

where the last equality holds since u_0 is even. Accordingly since $\beta' > 3/4$

$$\begin{aligned}\lambda^\omega &\ge \lambda_0 + \frac{(F(u^\omega),u^\omega)}{\omega} \ge \lambda_0 - \frac{\|F(u^\omega)\|_2}{\sqrt{\omega}} \\ &\ge \lambda_0 - \|u^\omega\|_X^{\alpha'}\|u^\omega\|_2^{\beta'-1/2} \ge \lambda_0 - \omega^{\beta'/2-1/4} \to \lambda_0 \text{ for } \omega \to 0.\end{aligned}$$

On the other hand G can be written as

$$0 > G(u) = \int_0^1 (F(su), u)ds \geq \int_0^1 s^\gamma ds(F(u), u) = \frac{1}{\gamma+1}(F(u), u)$$

which shows in particular $(F(u), u) < 0$. By definition of J^ω we can write

$$\begin{aligned} \lambda^\omega &= 2\frac{J(u^\omega) - G(u^\omega)}{\omega} + \frac{(F(u^\omega), \omega)}{\omega} \leq 2\frac{J(\sqrt{\omega}u_0)}{\omega} + \frac{(F(u^\omega), \omega) - 2G(u^\omega)}{\omega} \\ &= \lambda_0 + \frac{2G(\sqrt{\omega}u_0) - 2G(u^\omega) + (F(u^\omega), u^\omega)}{\omega} \\ &\leq \lambda_0 + \frac{2G(\sqrt{\omega}u_0)}{\omega} + (1 - \frac{2}{\gamma+1})(F(u^\omega), u^\omega) \leq \lambda_0. \end{aligned} \tag{23}$$

Thus we have shown $\lambda^\omega \to \lambda_0$ for $\omega \to 0$ which means that the u^ω are bifurcating from the first eigenvalue. Moreover, due to $\lambda^\omega \leq \lambda_0$ the direction of bifurcation is determined. In a similar way to the argument of Zhang [23] it follows that the bifurcating solutions are orbitally stable.

The uniqueness of the bifurcating solutions from the global bifurcation theorem(up to phase translation) and the fact that the same arguments apply for the (possibly non-symmetric) ground state $\tilde{u}^\omega$ lead to $u^\omega = \tilde{u}^\omega$ at least for small ω and hence the ground states are even. In addition the solutions u^ω exist for arbitrary $\omega > 0$ and consequently the branch is unbounded in $L^2(\mathbb{R})$ and hence also in X which rules out the second alternative of the global bifurcation theorem. □

Next we will show that the solutions bifurcating in the second eigenvalue λ_1 (which correspond to the bi-solition) are odd. It should be noted that the bifurcating solutions are somewhat orbitally stable among all odd functions which yields their practical relevance. In order to find odd solutions we introduce the action $\Gamma_2 u(x) = -u(-x)$ and the corresponding space X_{Γ_2} of fixed points of Γ_2 and apply the principle of symmetric criticality again to obtain

Lemma A.3. *For all $\omega > 0$ the minimization problem*

$$J^\omega_{\Gamma_2} = \min\{J(u) | u \in X_{\Gamma_2}, \| u \|_2^2 = \omega\} \tag{24}$$

has a nontrivial solution $u^\omega \in X_{\Gamma_2}$ which corresponds to a weak solution of (3) Furthermore, $\{(u^\omega, \lambda^\omega) | \omega \in (0, \infty)\} \subset \mathcal{C}_1$.

Proof. The proof of the existence of a minimizer can be adapted by replacing X_Γ with X_{Γ_2} from Lemma A.1. It remains to verify the behavior for $\omega \to 0$. Note that λ_1 can be characterized as

$$\lambda_1 = \|u_1\|_X^2 = \inf\{\frac{\|\mathrm{u}\|_\mathrm{X}^2}{\|\mathrm{u}\|_2^2} | \mathrm{u} \in \mathrm{X}_{\Gamma_2}, \mathrm{u} \neq 0\}.$$

and using (22) it follows

$$\lambda^\omega \geq \lambda_1 - \|u^\omega\|_X^{\alpha'} \|u^\omega\|_2^{\beta' - 1/2} \geq \lambda_1 - \omega^{\beta'/2 - 1/4} \to \lambda_1 \text{ for } \omega \to 0.$$

In the same way as in (23) we then can verify the direction of bifurcation, i.e. $\lambda^\omega \leq \lambda_1$. Hence $\lambda^\omega \to \lambda_1$. □

Next we show the existence of infinitely many modes of the equation which correpond to saddle points of the energy functional J. Obviously this includes the previously discussed situations, but the proof is more technical and we have therefeore discussed the previous cases separately.

Lemma A.4. *For arbitrary $\omega > 0$ there exists an unbounded sequence of even (odd) solutions $u_n \in X$ of equation (3) with $\| u \|_2^2 = \omega$.*

The proof relies on the following result [7] which is a generalization of the theorem of Ljusternik-Schnirelmann for infinite-dimensional Hilbert spaces:

Theorem A.5. *Let X be an infinite-dimensional Hilbert space, $J, K \in C^1(X, \mathbb{R})$ with $K'(v) \neq 0$ $\forall v \in X - \{0\}$ and $S := \{v \in X : K(v) = 0\}$. If $J|_S$ is even, bounded from below and satisfies the Palais-Smale condition on S then there exist infinitely many critical values, that is $c_k \in \mathbb{R}$ with $\lim_{k\to\infty} c_k = \infty$ and for all $k \in \mathbb{N}$ there exists a pair $(v_k, \lambda_k) \in S \times \mathbb{R}$ with $J(v_k) = c_k$ and $J'(v_k) - \lambda_k K'(u_k) = 0$. Moreover, $c_k \to \infty$.*

Proof of Lemma A.4. We apply the theorem with $K(u) = \|u\|_2^2 - \omega$ and $J(u)$ defined as in (8) for $X = X_\Gamma$ resp. $X = X_{\Gamma_2}$ separately. It suffices to show that J satisfies the Palais-Smale-condition on S. Let (u_n, λ_n) be a Palais-Smale sequence, that is

$$J(u_n) \to c \in \mathbb{R}, \qquad J'(u_n) - \lambda_n K'(u_n) \to 0 \text{ in X}',$$

where X' denotes the dual space of X. We have to show the existence of a strongly convergent subsequence. As in the proof of Lemma A.1 we can extract a subsequence still denoted as (u_n, λ_n) with $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in X. It remains to show $u_n \to u$ strongly in X. Therefore we calculate

$$\begin{aligned}
\|u_n - u\|_X^2 &= \big(J'(u_n) - J'(u)\big)(u_n - u) - (F(u_n) - F(u), u_n - u) \\
&\leq C\|J'(u_n)\|_{X'} + |J'(u)(u_n - u)| + \|F(u_n) - F(u)\|_2 \|u_n - u\|_2 \\
&\leq C\|J'(u_n)\|_{X'} + |J'(u)(u_n - u)| + C\|u_n - u\|_X^{1/4} \|u_n - u\|_2^{7/4}.
\end{aligned}$$

Thus, it follows $u_n \to u$ in X and from (22) it follows

$$\begin{aligned}
|\lambda_n - \lambda| &\leq \big|\|u_n\|_X^2 - \|u\|_X^2\big| + \|(F(u_n), u_n) - (F(u), u)| \\
&\leq \big|\|u_n\|_X^2 - \|u\|_X^2\big| + |(F(u_n) - F(u), u_n)| + |(F(u), u_n - u)|
\end{aligned}$$

which together with $(\mathcal{F}_2)$ and $u_n \to u \in X$ yields the desired convergence. □

We still have to show $(\lambda_n^\omega, u_n^\omega) \in \mathcal{C}_n$. Instead of discussing this in detail we refer to the characterization of the critical values as

$$c_n^\omega = \inf_{A \in \mathcal{B}_n} \max_{u \in A} \frac{J(u)}{\omega}, \tag{25}$$

where $\mathcal{B}_n$ is a certain family of sets A, cf. [7]. Using (22) and calculating as in (23) gives $\lambda_n^\omega \leq 2c_n^\omega$. The assertion then again follows from $G(u) < 0$ combined with the min-max characterization of the eigenvalues, i.e [3]

$$\inf_{A \in \mathcal{B}_n} \max_{v \in A} \frac{\| u \|_X^2}{\| u \|_2^2} = \min_{X_{n-1} \subset X} \max_{u \in X_{n-1}} \frac{\| u \|_X^2}{\| u \|_2^2} = \max_{X_{n-1} \subset X} \min_{u \in X_{n-1}^\perp} \frac{\| u \|_X^2}{\| u \|_2^2} = \lambda_n.$$

Again, this implies $\lambda_0 \leq \lim_{\omega \to 0} \lambda_n^\omega \leq \lambda_n$. Hence λ_n^ω must converge to an eigenvalue which must be λ_n by induction.
It remains to show Corollary 2.7:

Proof. Since all the branches are unbounded in both, u and λ, the existence of infinitely many solutions u_n with fixed wave-number λ is obvious. Note that due to $\lambda_n^\omega \leq \lambda_n$ we have $\lambda_n^\omega \to -\infty$ for $\omega \to \infty$. □

B Verification of assumptions

In this section we show that F as in (2) satisfies assumptions $(\mathcal{F}_1)$ to $(\mathcal{F}_6)$. The same holds obivously for the nonlinearity $F(u) = \sigma|u|^2u$ with $\sigma < 0$.

$(\mathcal{F}_1)$: By definition we have $S(z) = U(R^{\text{eff}}(z))$, where $U(t)$ denotes the group of the harmonic oscillator. Using

$$\overline{U(-t)u} = U(t)\overline{u}$$

together with $-R^{\text{eff}}(z) = R^{\text{eff}}(z+1/2)$ implies

$$S(z)\overline{u} = U(R^{\text{eff}}(z))\overline{u} = \overline{U(-R^{\text{eff}}(z))u} = \overline{U(R^{\text{eff}}(z+1/2))u}.$$

Hence, by $S^{-1}(z) = U(-R^{\text{eff}}(z))$:

$$S^{-1}(z)\left(\frac{1}{T(z)}|S(z)\overline{u}|^2S(z)\overline{u}\right) = \overline{S^{-1}(z+\frac{1}{2})\left(\frac{1}{T(z)}|S(z+\frac{1}{2})u|^2S(z+\frac{1}{2})u\right)}.$$

Due to symmetry $T(z) = T(z+1/2)$ we can conclude

$$\begin{aligned} F(\overline{u}) &= -\int_0^1 \overline{S^{-1}(z+1/2)\left(\frac{1}{T(z+1/2)}|S(z+1/2)u|^2S(z+1/2)u\right)}dz \\ &= \overline{-\int_{1/2}^{3/2} S^{-1}(z)\left(\frac{1}{T(z)}|S(z)u|^2S(z)u\right)dz} = \overline{F(u)}, \end{aligned}$$

where the last equality holds due to the 1-periodicy of S and T which gives the assertion for real-valued u.

$(\mathcal{F}_2)$: Due to $\| u \|_{L^1(0,1)} \leq \| u \|_{L^2(0,1)}$ we can estimate in the following way

$$\begin{aligned} \| F(u) - F(v) \|_2^2 &\leq C \int_{\mathbb{R}}\left(\int_0^1 |S^{-1}(z)\left(|S(z)u|^2S(z)u - |S(z)v|^2S(z)v\right)|\,dz\right)^2 dx \\ &\leq \int_{\mathbb{R}}\left(\int_0^1 \left|S^{-1}(z)\left(|S(z)u|^2S(z)u - |S(z)v|^2S(z)v\right)\right|^2 dz\right) dx \\ &= \int_0^1 \||S(z)u|^2S(z)u - |S(z)v|^2S(z)v\|_2^2\, dz. \end{aligned}$$

Making use of the inequality $\left||a|^2a - |b|^2b\right| \leq \frac{3}{2}(|a|^2 + |b|^2)|a-b|$ which holds for $a, b \in \mathbb{C}$ and the Cauchy-Schwarz inequality we conclude

$$\begin{aligned} & \| \, |S(z)u|^2S(z)u - |S(z)v|^2S(z)v \,\|_2^2 \\ \leq\ & C \,\| \left(|S(z)u|^2 + |S(z)v|^2\right)|S(z)u - S(z)v| \,\|_2^2 \\ \leq\ & C \,\| (|S(z)u|^2 + |S(z)v|^2)^2 \|_2 \| \, |S(z)(u-v)|^2 \,\|_2 \\ \leq\ & C \,\left(\| \, |S(z)u|^4 \|_2 + \| \, |S(z)v|^4 \,\|_2\right) \| \, |S(z)(u-v)|^2 \,\|_2 \\ =\ & C \,\left(\| S(z)u \|_8^4 + \| S(z)v \|_8^4\right) \| S(z)(u-v) \|_4^2\,. \end{aligned}$$

From Oh [16] it is known that $S(z)$ is a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, where $q = p/(p-1)$. Since X is compactly embedded in $L^p(\mathbb{R})$ for all $p \geq 2$ we have $S(z)u \in L^q(\mathbb{R})$ for all $q \geq 2$. Estimating the terms separately we interpolate

$$\| S(z)u \|_8 \leq \| S(z)u \|_4^{\lambda} \| S(z)u \|_q^{1-\lambda},$$

where

$$\lambda = \frac{1/8 - 1/q}{1/4 - 1/q}.$$

Using the estimate of Sobolev type $\|v\|_4^4 \leq C\|v_x\|_2 \|v\|_2^3$ we obtain

$$\begin{aligned} \| S(z)u \|_8 &\leq C(\| S(z)u_x \|_2 \| S(z)u \|_2^3)^{\lambda/4} \| S(z)u \|_X^{1-\lambda} \\ &\leq C\|u\|_X^{\lambda/4} \|u\|_X^{1-\lambda} \|u\|_2^{3\lambda/4} = C\|u\|_X^{1-3\lambda/4} \|u\|_2^{3\lambda/4}. \end{aligned}$$

In the same way it follows

$$\| S(z)(u-v) \|_4^2 \leq C\|u-v\|_X^{1/2} \|u-v\|_2^{3/2}.$$

Hence by collecting all the terms

$$\| F(u) - F(v) \|_2^2 \leq C \left(\|u\|_X^{4-3\lambda} \|u\|_2^{3\lambda} + \|v\|_X^{4-3\lambda} \|v\|_2^{3\lambda} \right) \|u-v\|_X^{1/2} \|u-v\|_2^{3/2}.$$

We still have the freedom to choose q. In order to satisfy $\alpha = 4 - 3\lambda < 7/2$ we need $\lambda > 1/6$ which holds for $q > 10$. Note that a choice $q = 16$ would give $\alpha = 3, \beta = 1$.

$(\mathcal{F}_3)$: It is easy to observe that $F \in C^3(X, L^2)$ with $\delta F(0)[u_n] = 0$, $\delta^2 F(0)[u_n]^2 = 0$ and

$$\delta^3 F(0)[u_n]^3 = -6 \int_0^1 S^{-1}(z) \left(\frac{1}{T(z)} |S(z)u_n|^2 S(z)u_n \right) dz. \tag{26}$$

For a normalized eigenfunction ($\| u_n \|_2 = 1$) we have

$$\begin{aligned} (\delta^3 F(0)[u_n]^3, u_n)_{L^2} &= -6 \int_0^1 \left(S^{-1}(z) \left(\tfrac{1}{T(z)} |S(z)u_n|^2 S(z)u_n \right), u_n \right)_{L^2} dz \\ &= -6 \int_0^1 \tfrac{1}{T(z)} \| S(z)u_n \|_4^4 \; dz < 0. \end{aligned}$$

$(\mathcal{F}_4)$: Defining $G(u) = (F(u), u)/4$ gives

$$G(u) = -\frac{1}{4} \int_0^1 \frac{1}{T(z)} \|S(z)u\|_4^4 dz$$

with $G'(u)v = (F(u), v)$.

$(\mathcal{F}_5)$: This assumption is fulfilled with $\gamma = 3$.

$(\mathcal{F}_6)$: With $\Gamma u(x) = u(-x)$ we have to show $G(u) = G(\Gamma u)$. This is true since $S(z)$ and Γ commute which is shown as follows: Let $v(z) := \Gamma S(z)u$, then $v(0) = \Gamma u$ holds and with $Au = u_{xx} - x^2 u$ one can observe that

$$iv_z = i\Gamma \left(iR^{\mathrm{eff}}(z) A S(z) u \right) = -R^{\mathrm{eff}}(z) A \Gamma S(z) u$$

which gives the assertion for Γ and the same arguments apply for Γ_2.

C Exponential decay of the solutions

In this section we prove the exponential decay of the solutions of equation (3) with nonlocal nonlinearity (2). Note that for the simpler nonlinearity $F(u) = \sigma|u|^2 u$ the decay is Gaussian-like [8]. The proof is similar to Theorem 8.1.1 in [1], where exponential decay is verified without potential.

Proof. For $\epsilon > 0$ define the function

$$f^\epsilon(x) := \exp(\frac{x}{1+\epsilon x})$$

which has the following properties [1]

- f^ϵ is bounded $\forall \epsilon > 0$
- $f^\epsilon_x(x) \leq f^\epsilon(x)$
- $\lim_{\epsilon \to 0} f^\epsilon(x) = \exp(x)$

Multiplication of (3) with $f^\epsilon \overline{u}$ and integration gives in the real part

$$\int_{\mathbb{R}} (x^2 - \lambda) f^\epsilon |u|^2 dx \quad = \quad I - \Re\left(\int_{\mathbb{R}} u_x (f^\epsilon \overline{u})_x dx\right). \tag{27}$$

$$I := \Re\left(\int_0^1 \int_{\mathbb{R}} S^{-1}(z)\left(\frac{1}{T(z)}|S(z)u|^2 S(z)u\right) f^\epsilon \overline{u} dz dx\right) \tag{28}$$

At first we bound the left-hand side from below. Defining $R_1 := \sqrt{|\lambda| + 1}$ we find

$$\int_{\mathbb{R}} (x^2 - \lambda) f^\epsilon |u|^2 dx \quad \geq \int_{|x| \leq R_1} (x^2 - \lambda) f^\epsilon |u|^2 dx + (R_1^2 - \lambda) \int_{|x| > R_1} f^\epsilon |u|^2 dx \tag{29}$$

which due to $R_1^2 - \lambda \geq 1$ implies the inequality

$$\int_{|x| > R_1} f^\epsilon |u|^2 dx \quad \leq \quad \int_{\mathbb{R}} (x^2 - \lambda) f^\epsilon |u|^2 dx - \int_{|x| \leq R_1} f^\epsilon |u|^2 dx. \tag{30}$$

Next we estimate the right-hand side: The second term is bounded in ϵ, whereas for the first term regarding (27) and using $f^\epsilon_x \leq f^\epsilon$:

$$\begin{aligned}
\Re\left(\int_{\mathbb{R}} u_x (f^\epsilon u)_x dx\right) \quad &= \quad \Re\left(\int_{\mathbb{R}} f^\epsilon |u_x|^2 + f^\epsilon_x u_x \overline{u} dx\right) \geq \int_{\mathbb{R}} f^\epsilon |u_x|^2 - f^\epsilon |u||u_x| dx \\
&\geq \quad \int_{\mathbb{R}} f^\epsilon |u_x|^2 dx - \left(\int_{\mathbb{R}} f^\epsilon |u_x|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} f^\epsilon |u|^2 dx\right)^{1/2} \\
&\geq \quad \int_{\mathbb{R}} f^\epsilon |u_x|^2 dx - \frac{1}{2}\left(\int_{\mathbb{R}} f^\epsilon |u|^2 dx + \int_{\mathbb{R}} f^\epsilon |u_x|^2 dx\right).
\end{aligned}$$

Hence, by collecting all the terms and splitting the last integral we can conclude that

$$\frac{1}{2}\left(\int_{\mathbb{R}} f^\epsilon |u_x|^2 dx + \int_{|x| > R_1} f^\epsilon |u|^2 dx\right) \leq I - \int_{|x| \leq R_1} (x^2 - \lambda) f^\epsilon |u|^2 dx + \frac{1}{2} \int_{|x| \leq R_1} f^\epsilon |u|^2 dx$$

and it remains to estimate I. Splitting the integral into $|x| \le R$ and $|x| > R$ with $R > R_1$ to be determined later the following is true due to $f^\epsilon u \in X$ for all $\epsilon > 0$:

$$\begin{aligned}
&\left| \int_0^1 \int_{|x|>R} S^{-1}(z) \left(\frac{1}{T(z)} |S(z)u|^2 S(z)u \right) \; f^\epsilon \overline{u} dx dz \right| \\
\le\;& \max_{z\in[0,1]} \left(\| \frac{1}{T(z)} S(z)u \|^2_{L^\infty(x>R)} \right) \left| \int_0^1 \int_{x>R} S(z)u \overline{S(z)\,(f^\epsilon u)} dx dz \right| \\
=\;& \max_{z\in[0,1]} \left(\frac{1}{T(z)} \; \| S(z)u \|^2_{L^\infty(x>R)} \right) \int_{|x|>R} f^\epsilon |u|^2 dx.
\end{aligned}$$

Since $S(z)u \in L^2(\mathbb{R})$ in particular $S(z)u(x) \to 0$ for $x \to \pm\infty$. Accordingly there exists $R_2 > R_1$ with

$$\max_{z\in[0,1]} \left(\frac{1}{T(z)} \; \| S(z)u \|^2_{L^\infty(x>R_2)} \right) < \frac{1}{4}.$$

Using

$$\frac{1}{4} \int_{|x|>R_2} f^\epsilon |u|^2 dx \le \frac{1}{2} \int_{|x|>R_1} f^\epsilon |u|^2 dx - \frac{1}{4} \int_{|x|>R_2} f^\epsilon |u|^2 dx$$

the following estimate holds

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}} f^\epsilon |u_x|^2 dx + \frac{1}{4} \int_{|x|>R_2} f^\epsilon |u|^2 dx \le\; & -\textstyle\int_{|x|\le R_1} (x^2 - \lambda) f^\epsilon |u|^2 dx + \frac{1}{2} \int_{|x|\le R_1} f^\epsilon |u|^2 dx \\
& + \left| \textstyle\int_0^1 \int_{|x|\le R_2} S^{-1}(z) \left(\frac{1}{T(z)} |S(z)u|^2 S(z)u \right) dz f^\epsilon \overline{u} dx \right|.
\end{aligned}$$

On the right-hand side the limit $\epsilon \to 0$ is finite due to the boundedness of the domain of integration. Hence

$$\int_{x>R_2} \exp(|x|)|u(x)|^2 dx < \infty \text{ and } \int_{\mathbb{R}} \exp(|x|)|u_x|^2 dx < \infty$$

which implies

$$\int_{\mathbb{R}} \exp(|x|)(|u(x)|^2 + |u_x(x)|^2) dx < \infty. \tag{31}$$

Using the Lipschitz continuity of u we then are able to derive exactly as in [1] the existence of a $C > 0$ with

$$\exp(|x|)(|u(x)|^2 + |u_x(x)|^2) < C \;\forall x \in \mathbb{R}$$

which gives the desired decay estimate. □

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