Optical Solitons as Ground States of NLS in the Regime of Strong Dispersion Management

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Abstract

The dispersion-managed nonlinear Schrödinger equation (DM-NLS) is considered. After applying an exact transformation, the so called lens transformation, we derive a new Schrödinger-type equation with additional quadratic potential. In the case of strong dispersion management it is shown how the scales transform into the resulting equation. By constructing ground states of the averaged variational principle we can prove the existence of a standing wave solution of the averaged equation in the case of positive residual dispersion. In contrast to some former discussions of the problem we show the existence of a region where the potential is of trapping type. Due to the potential. the solution has a faster decay than the traditional soliton. Moreover we explain why the shape of the DM-soliton changes with increasing pulse energy from a sech-profile to a behavior with Gaussian core and oscillating tails and further to a flatter profile. Furthermore, we illustrate why the DM-soliton can propagate only for small values of the initial pulse width in the case of vanishing or even negative residual dispersion. This results seem to be a new analytical description of what is well-known from numerical simulations.

Keywords: dispersion management, optical solitons, lens transformation, nonlinear Schrödinger equation, harmonic oscillator

1 Introduction and Main Results

In this paper we present a mathematical investigation of the so-called dispersion managed soliton using an approach developed by Turitsyn et al. [16, 17, 18, 19, 20] which relies on the self-similar properties of the main peak of the DM-soliton, combined with a method previously used by Zharnitsky et al. [23] which permits an averaging procedure.

The idea of dispersion management is using a dispersion compensating fiber (DCF) to overcome the dispersion of the standard mono-mode fiber (SMF) which causes dispersive broadening of the pulse over long distances. At the end of each compensation section the pulse should be close to its shape at the beginning. During the compensation period the pulse undergoes breathing-like oscillations. Numerical simulations and experiments show that this pulse is stable over many compensation periods. In analogy to the traditional NLS it is called DM-soliton or breathing soliton. Large variation of the dispersion makes nonlinear effects very small and strictly modifies the soliton propagation, this situation is called strong dispersion management (DM). The case of weak DM is more or less clear by using the Lie-transform technique [1].

The model equation describing pulse propagation in optical fibers with dispersion management is given by the cubic nonlinear Schrödinger equation with periodically varying coefficients (DM-NLS):

$$iA_z(t,z) + D(z)A_{tt}(t,z) + c(z)|A(t,z)|^2 A(t,z) = 0. \tag{1}$$

Here, A is the complex envelope of the electric field, t is retarded time, z is propagation distance, D is the dispersion coefficient and $c(z) > 0$ is representing loss and influence of the amplifiers. Throughout the paper we assume c to be constant, which is reasonable in cases where the compensation period is much smaller than amplification distance (loss-less model). Moreover, the dispersion profile D is assumed to be periodic with normalized period 1 and we only consider the case of strong dispersion management, where the residual dispersion $\langle D \rangle$ is small compared to local dispersion, i.e. $\langle D \rangle << D$. Here $\langle \cdot \rangle$ denotes averaging over one period.

In contrast to the method of Zharnitsky et al. [23], our approach is based on a transformation similar to the lens transformation or pseudo-conformal transformation which is well-known in the theory of self-focusing in critical NLS. After applying lens transformation and a generalized variations of constants, we obtain an equation of Schrödinger-type which can be averaged. Although this leads to a slightly more complicated class of equations, we are able to show the existence of ground states for many situations of practical interest. The most important result is that the quadratic potential in the resulting equation is of trapping type in contrast to a former discussion of the problem (e.g. [1, 19, 20]). In the case of positive residual dispersion, we illustrate the different regimes where the shape of the pulse varies from a sech profile to a Gaussian shape and further to a flatter waveform [16]. Moreover, for vanishing or negative residual dispersion we present conditions under which the DM soliton can propagate. Our analysis includes the situation examined by Zharnitsky et al. [23].

The aim of this paper is to present this new approach in the theory of DM-solitons. Instead of giving all derivations exactly, we have focused on the description of the method and the main results. Moreover, we only state some qualitative properties of the derived equations. The detailed bifurcation analysis is beyond the scope of this discussion and will be done in a continuation of this paper [9].

2 Lens Transformation and Nonlinear TM-Equations

Throughout this paper we restrict ourselves to the following standard dispersion profile:

$$D(z) = D_{\text{loc}}(z) + \langle D \rangle = \begin{cases} d + \langle D \rangle & : 0 \le z \le \frac{1}{4}, \frac{3}{4} \le z \le 1 \\ -d + \langle D \rangle & : \frac{1}{4} < z < \frac{3}{4}, \end{cases} \tag{2}$$

with $\langle D_{\text{loc}} \rangle = 0$. In the case of strong dispersion management, $\langle D \rangle$ is small compared to d. Note that d is assumed to be positive without loss of generality (otherwise everything remains true by a shift of 1/2).

Remark 2.1. *One may treat nonlinearity and residual dispersion as perturbations in the regime of strong DM. In the previous studies [7, 23] they were considered to be both of the same order ϵ. We will explain in this paper why this seems not to be the correct model in all parameter constellations.*

2.1 The linear problem

Since both, nonlinearity and residual dispersion are perturbations, we first examine the unperturbed problem of (1) which is the linear Schrödinger equation

$$iA_z(t,z) + D_{\mathrm{loc}}(z)A_{tt}(t,z) = 0. \tag{3}$$

With Gaussian initial data $A(t,0) = N\exp(-\frac{1}{2}(\frac{t}{T_0})^2)$ equation (3) can be solved explicitly. For physical reasons we have $T_0, N > 0$. The solution is then given by

$$A(t,z) = N\frac{Q(t/T_{\mathrm{lin}}(z),z)}{\sqrt{T_{\mathrm{lin}}(z)}}\ \exp\left(it^2\frac{M_{\mathrm{lin}}(z)}{T_{\mathrm{lin}}(z)}\right), \tag{4}$$

where

$$\begin{aligned} R(z) &:= \int_0^z D_{\mathrm{loc}}(z')dz' \\ T_{\mathrm{lin}}(z) &:= \sqrt{T_0^2 + 4R^2(z)/T_0^2} \\ M_{\mathrm{lin}}(z) &:= \frac{1}{T_0^2}\frac{R(z)}{T_{\mathrm{lin}}(z)} \\ Q(x,z) &:= \exp\left(-\frac{x^2}{2}\right)\exp\left(-\frac{i}{2}\arctan(R(z))\right), \quad x = t/T_{\mathrm{lin}}(z). \end{aligned}$$

Here, R is the accumulated dispersion and M_{lin} and T_{lin} describe the optical pulse chirp (=time-dependent phase) and width. Differentiating T_{lin} resp. M_{lin} and using $R'(z) = D(z)$ we obtain

$$T'_{\mathrm{lin}}(z) = 4D_{\mathrm{loc}}(z)M_{\mathrm{lin}}(z),\ T_{\mathrm{lin}}(0) = T_0 > 0 \tag{5}$$

$$M'_{\mathrm{lin}}(z) = \frac{D_{\mathrm{loc}}(z)}{T^3_{\mathrm{lin}}(z)}, \qquad M_{\mathrm{lin}}(0) = 0. \tag{6}$$

Here, $(\cdot)'$ denotes d/dz. Note that T_{lin} and M_{lin} are 1-periodic if and only if the residual dispersion is zero. In this case $A(t,z)$ is also 1-periodic in z. The evolution of Gaussian initial data in the perturbed equation (1) should be close to the solution of the linear problem in the case of strong DM since nonlinear effects and residual dispersion are small.

2.2 Lens Transformation

In a series of papers, Turitsyn and Gabitov (see for example [16, 17, 18, 19]) suggest the following exact transformation which is similar to the so called lens transformation also known as pseudo-conformal transformation:

$$A(t,z) = N\frac{Q(t/T(z),z)}{\sqrt{T(z)}}\exp\left(it^2\frac{M(z)}{T(z)}\right). \tag{7}$$

Here, the rapid oscillations of pulse width and chirp are included in T and M. Inserting the above transformation into the master equation (1) they obtain a reduced variational problem and after suitable scaling they derive the following ordinary differential equation describing the evolution of T and M as periodic solutions of

$$T'(z) = 4D(z)M(z), \qquad T(0) = T_0 > 0 \tag{8}$$

$$M'(z) = \frac{D(z)}{T^3(z)} - \frac{N^2}{T^2(z)},\ M(0) = 0. \tag{9}$$

Thereby, T_0 has to be determined in such a way that for a given N^2 the corresponding solution is periodic or vice versa. N^2 has the physical meaning of pulse energy. To simplfy notation c should be normalized to 1 (that means c is included in N^2). Note that the above equations are some nonlinear version of equations (5) and (6) and are therefore often called "nonlinear TM-equations". Numerical simulations show that transformation (7) describes the DM soliton very well. A typical shape of periodic T and M can be found in Figure 1.

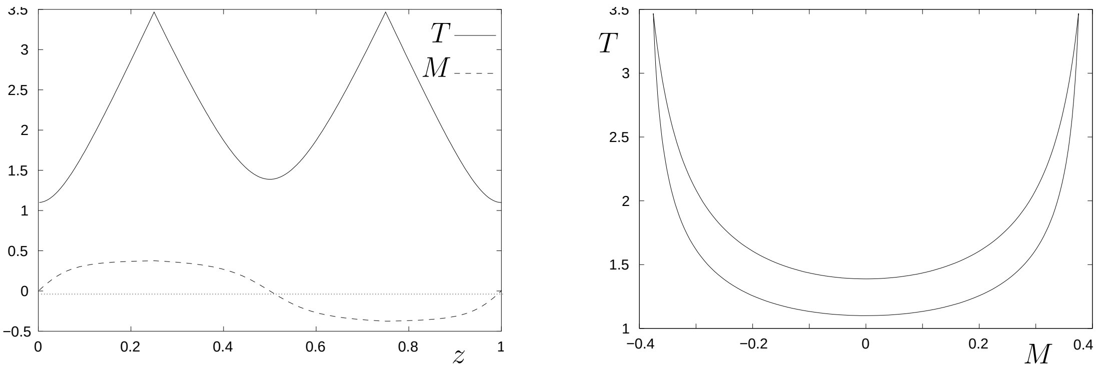


Figure 1: Periodic solutions T (solid line) and M (dashed line) and periodic orbit in the MT-plane

System (8), (9) raises several questions:

- Do periodic solutions always exist ?
- How is N^2 related to $T_0, \langle D\rangle$ and d ? Can we find approximative formulas ?
- If the residual dispersion is vanishing, are there other solutions besides $N^2 = 0$? Does there occur some bifurcation ?

Applying the lens transformation (7) we obtain after some algebraic manipulations the following equation for $Q = Q(x, z)$ in the new variable $x = t/T(z)$:

$$iQ_z + \frac{D(z)}{T^2(z)}(Q_{xx} - x^2 Q) + \frac{N^2}{T(z)}(x^2 Q + |Q|^2 Q) = 0. \tag{10}$$

Although (10) seems to be more complicated than the master equation (1), it is more suitable for our purposes. The major differences are:

- The linear part of the equation has now an additional term corresponding to a quadratic potential.
- Instead of D, which is typically piecewise constant, the "effective dispersion" D/T^2 is not explicitly known in general.

Of course the "sign" of the quadratic potential is of fundamental importance for the whole approach. Naive averaging of equation (10) gives a repelling potential (e.g. [1, 19, 20]). Thus, it is essential to do a careful averaging procedure. Numerical investigations of (10) in the case of positive residual dispersion using the basis of the Gauss-Hermite functions show that the real DM-soliton can be approximated with high accuracy using only the first modes [14, 16].

In order to apply an averaging procedure to (10) we first have to introduce the correct scales in the case of strong DM. To understand how the scales change under lens transformation it is essential to examine the nonlinear TM-equations.

3 Periodic solutions of the nonlinear TM-equations

In this section we will discuss the existence of periodic solutions of the nonlinear TM-equations and derive approximative formulas for the value of N^2.

If the residual dispersion $\langle D\rangle$ is positive, the following theorem due to Kunze [4] holds:

Theorem 3.1 (Existence of periodic solutions). *Assume $C_1, C_2 > 0$ are given and $\langle D\rangle > 0$. Then the nonlinear TM-equations*

$$T'(z) = 4D(z)M(z), \qquad M'(z) = C_1\frac{D(z)}{T^3(z)} - \frac{C_2}{T^2(z)}$$

have a 1-periodic solution.

To prove this the author shows that there exists a T_0 such that the solution with $T(0) = T_0$ and $M(0) = 0$ is periodic. If (T, M) is a solution of the nonlinear TM-equations with $M(0) = 0$ then T is symmetric with respect to $z = \frac{1}{2}$. Of course the existence of a periodic orbit is essential for the whole approach.

The characteristic feature of the strong DM regime is $|\langle D\rangle| < d$. Assuming that $\langle D\rangle$ is of order ϵ, say $\langle D\rangle = \epsilon\alpha$, while D_{loc} is of order 1, Turitsyn et al. [16] establish the following relation by a perturbation method:

$$N^2 = \epsilon\frac{\alpha/T_0}{2/\sqrt{1+y^2} - y^{-1}\ln(y+\sqrt{1+y^2})} + O(\epsilon^2), \text{where } y = \frac{d}{2T_0^2}, \tag{11}$$

that is N^2 is of the same order as $\langle D\rangle$. The parameter y is well-known in the context of DM and is called "map strength". It should be mentioned that formula (11) gives a linear relation between $\langle D\rangle$ and N^2 and is a good approximation for small y corresponding to large T_0 in the case of positive residual dispersion and for large y corresponding to small T_0 in the case of negative residual dispersion. At a certain value of $y(=: \bar{y} \sim 3.319)$ the

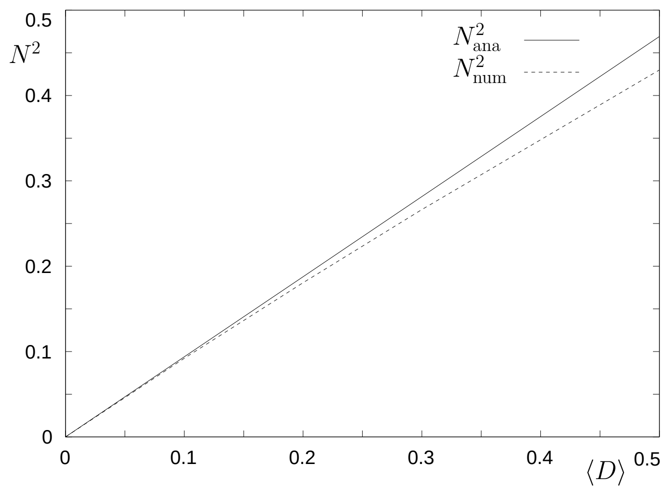


Figure 2: Numerical (solid line) vs. analytical (dotted line) values for N^2 in the case of large T_0 ($d = 8 \Rightarrow \overline{T_0} \sim 1.09767$ and $T_0 = 2.0$)

denominator becomes zero and (11) is no longer valid. Defining the corresponding critical pulse width as

$$\overline{T_0} = \sqrt{\frac{d}{2\overline{y}}} \tag{12}$$

it is natural to consider the case $T_0 \sim \overline{T_0}$ separately since relation (11) becomes singular. In a neighborhood of $\overline{T_0}$ we expand both, $\langle D\rangle$ and $T_0 - \overline{T_0}$, in powers of the artificial parameter ϵ and after cumbersome algebraic manipulations we obtain the following result:

Lemma 3.2. *If $T_0 \sim \overline{T_0}$ the following approximative formula holds:*

$$\begin{aligned} N^2 &= -\frac{b(T_0-\overline{T_0})}{2a} \pm \frac{\sqrt{b^2(T_0-\overline{T_0})^2 - 4ac\langle D\rangle}}{2a} \\ &\sim -4.665(T_0-\overline{T_0}) \pm \sqrt{21.766(T_0-\overline{T_0})^2 + 19.404\langle D\rangle}. \end{aligned} \tag{13}$$

Thereby, the constants a, b and c depend on $\overline{y}$ in the following way:

$$\begin{aligned} a: &= 4(\overline{y}^2-1)\frac{\overline{y}^3 + (\sqrt{1+\overline{y}^2}-1)(\overline{y}^2-\overline{y}+1)}{(1+\overline{y}^2)^{5/2}(\overline{y}+\sqrt{1+\overline{y}^2})} \sim 0.684 \\ b: &= 8(\overline{y}^2-1)\frac{\overline{y}}{(1+\overline{y}^2)^{3/2}} \sim 6.386 \\ c: &= -4\overline{y}\frac{\overline{y}^2+\overline{y}\sqrt{1+\overline{y}^2}+1}{(\overline{y}+\sqrt{1+\overline{y}^2})\sqrt{1+\overline{y}^2}} \sim -13.28. \end{aligned}$$

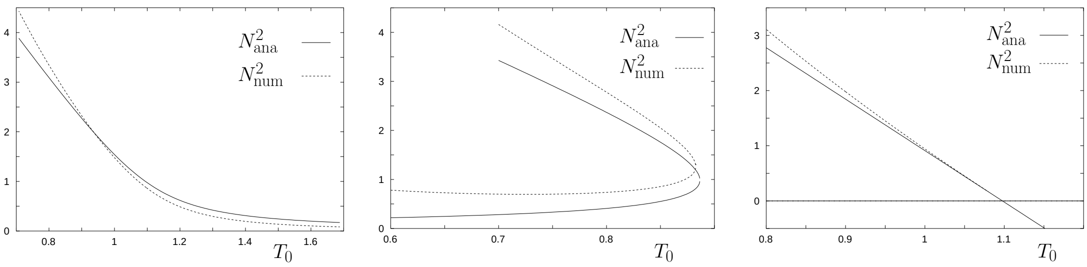
The above result implies the existence of three qualitatively different situations depending on the sign of $\langle D\rangle$.

- $\langle D\rangle > 0$: one positive ($N^2 > 0$) solution for all T_0.
- $\langle D\rangle < 0$: two positive solutions if $T_0 < T_0^*$, where

$$T_0^* := \overline{T_0} - 2\sqrt{ac\langle D\rangle}/b \sim \overline{T_0} - 0.944\sqrt{-\langle D\rangle}.$$

- $\langle D\rangle = 0$: one positive solution if $T_0 < \overline{T_0}$.

Figure 3 illustrates this result and compares analytical and numerical values in the different parameter constellations.



(a) (b) (c)

Figure 3: The relation between N^2 and T_0 in the case (a) $\langle D\rangle = 0.05$, (b) $\langle D\rangle = -0.05$ and (c), $\langle D\rangle = 0$. The local dispersion was chosen as $d = 8$. Analytical values of N^2 are compared to values of N^2 leading to fixed points of the nonlinear TM-equations found by numerical simulations.

Due to the different formulas for N^2 we have to distinguish between two situations in the case of positive residual dispersion:

- $T_0 >> \overline{T_0}$, where the linear relation (11) is valid.
- $T_0 \sim \overline{T_0}$, where formula (13) is valid.

In the next section we will show the existence of a ground state of the averaged Hamiltonian in the case of positive residual dispersion and $T_0 > \overline{T_0}$. It becomes evident that the quadratic potential in the transformed equation does not matter in the first case while in the second it will be of "good" sign.

The existence of the DM-soliton depends strongly on the solvability of the nonlinear TM-equations. Recapitulating the results it is now clear that the different regimes, where the DM-soliton does exist or not, could be explained by investigating the nonlinear TM-equations (see Remark 3.3 below). This is another surprising similarity between the above ODE (nonlinear TM-equations) and the PDE (DM-NLS) model.

Remark 3.3.

- *For $\langle D\rangle > 0$ it is known from numerical simulations that the DM-soliton exists for all values of the map strength. Furthermore it is known that the shape of the DM-soliton varies with decreasing map strength. In the next section we will derive a new class of equations and will explain how our method can be used to understand this fundamental property of the DM-soliton.*
- *For $\langle D\rangle < 0$ there is no solution of the nonlinear TM-equations in the case of small map strength, whereas there exist two solutions in the opposite case. This seems to be a new explanation of what was obtained by Pelinovsky [12], i.e. two pulses in the case of negative residual dispersion.*
- *For $\langle D\rangle = 0$ there occurs a bifurcation at the critical map strength $\overline{y}$. In particular this shows the existence of a "nontrivial" periodic solution of the nonlinear TM-equations. Of course, this corresponds to the situation which leads to the existence of the DM-soliton in the case of vanishing residual dispersion. From [10] it is known that the DM-soliton exists if the map strength is above a critical value.*

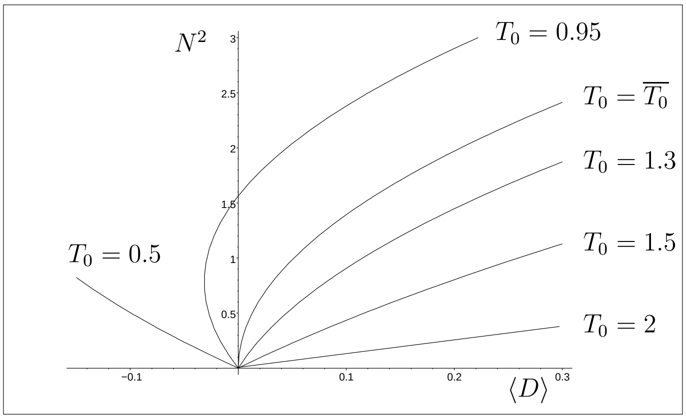


Figure 4: The value of N^2 as a function of $\langle D\rangle$ for $d = 8$ ($\Rightarrow \overline{T_0} \sim 1.09767$) and different values of T_0. Analytical values.

The dependence of N^2 on T_0 and $\langle D\rangle$ is shown in Figure 4. It becomes evident how the scales are transformed under lens transformation. In particular, it is clarified that the dependence of N^2 on $\langle D \rangle$ is not necessary linear and, consequently, residual dispersion and nonlinear effects can be of different order. However, it is essential to observe that the behavior substantially changes at $\overline{T_0}$. Note that $N^2(\langle D \rangle) = \mathcal{O}(\sqrt{\langle D \rangle})$ for $T_0 = \overline{T_0}$.

In Figure 5, the $(\langle D \rangle, T_0 - \overline{T_0})$-plane is divided into four regions:

- $A := \{(\langle D \rangle, T_0 - \overline{T_0}) | \langle D \rangle > 0, T_0 > \overline{T_0}\}$, where $N^2(\langle D \rangle) = \mathcal{O}(\langle D \rangle)$
- $B := \{(\langle D \rangle, T_0 - \overline{T_0}) | \langle D \rangle \geq 0, T_0 < \overline{T_0}\}$, where $N^2(0) \neq 0$.
- $C := \{(\langle D \rangle, T_0 - \overline{T_0}) | \langle D \rangle < 0, T_0 < T_0^*\}$, where there are two solutions.
- $D := \{(\langle D \rangle, T_0 - \overline{T_0}) | \langle D \rangle \leq 0, T_0 > T_0^*\}$, where no solution does exist.

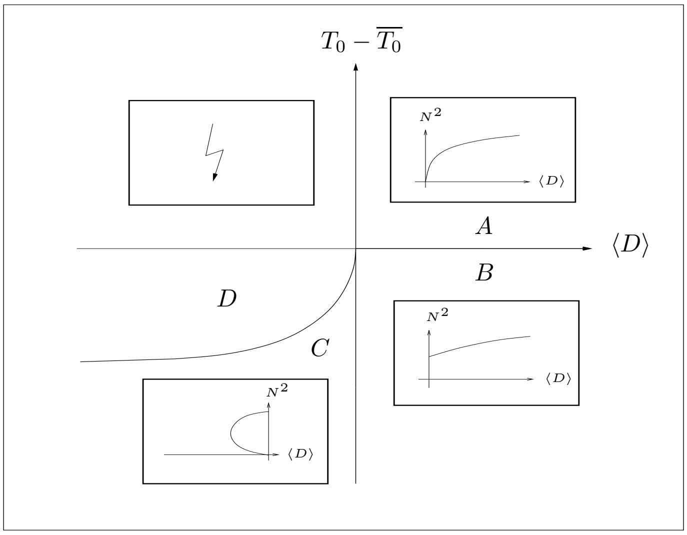


Figure 5: Bifurcation diagram in the $(\langle D \rangle, T_0 - \overline{T_0})$-plane.

It becomes evident that in region B the parameter N^2 is no longer of the same order as the residual dispersion, a corresponding equation has not been studied in this context. The recent results for positive residual dispersion [7, 23] are only valid in region A.

In the next section we will present an averaging procedure for the lens-transformed DM-NLS and derive a new class of model equations involving a quadratic potential. We will show that in region A the potential is not repelling in contrast to some former discussions of the problem. Moreover, we will derive a equation in region B by a modified averaging method which can be regarded as a model equation for DM-solitons with large energy which are of practical interest due to their reduced signal-to-noise ratios.

4 Averaging of the lens-transformed DM-NLS

In this section we describe how the lens transformed DM-NLS can be averaged using the method by Zharnitsky et al. [23]. At first we explain how the averaging procedure can be applied to the DM-NLS after lens transformation. Thereby, we will first examine the case $T_0 >> \overline{T_0}$, where analytical formulas for the mean values are at hand. Adapting the averaging procedure and using numerical simulations we then verify that the quadratic potential is not repelling for $T_0 > \overline{T_0}$, that is region A in Figure 5. Due to the quadratic potential the existence of ground states of the averaged equations is assured. In the remainder of this section we present a modified method for the other regions and briefly discuss the averaged equations.

Recall that the resulting equation after lens transformation is given by (10):

$$iQ_z + \frac{D(z)}{T^2(z)}(Q_{xx} - x^2Q) + \frac{N^2}{T(z)}(x^2Q + |Q|^2Q) = 0.$$

There is similarity between (10) and (1): In both equations the dispersion coefficient D resp. D/T^2 has high local dispersion compared to average dispersion. Therefore it is natural to use the same method as Zharnitsky et al. [23], i.e. removing strong variations of D/T^2 by variation of constants and average the resulting equation. It was shown in the last section how the scales change under lens transformation.

As mentioned in the last section we have to distinguish between different regimes depending on the value of N^2 resp. T_0 as indicated in Figure 5. Here we have chosen to distinguish the different regimes by the value of T_0 which corresponds to initial pulse width. But this could also be done by using the ratio of d and some d_{crit} for a given T_0. Since the map strength is defined as $y = d/(2T_0^2)$ everything can also be expressed in terms of the map strength. However, in the next section we will get rid of the artificial parameter T_0 and formulate the results in terms of N^2 which corresponds to initial pulse energy in order to make the results more applicable.

4.1 The case $\langle D \rangle > 0$

In this subsection we will analyse the case of positive residual dispersion. It will turn out that the quadratic potential is attractive for a large class of system parameters. Note that previous results [7, 23] are only valid for $T_0 > \overline{T_0}$ since they consider nonlinear effects and residual dispersion to be of the same order.

4.1.1 The region of small map strength

In the region of small map strength we have $T_0 >> \overline{T_0}$. From (11) it follows with $\langle D \rangle = \epsilon\alpha$

$$N^2(\epsilon, T_0) = \frac{\epsilon\alpha/T_0}{F(d/(2T_0^2))} =: \epsilon C$$

where $F(y) := 2/\sqrt{1+y^2} - y^{-1}\ln(y + \sqrt{1+y^2})$. In contrast to (1) it is not obvious how the unperturbed ($\epsilon = 0$) part of equation (10) looks like. Note that T is ϵ-depending with $\lim_{\epsilon\to 0} T(z;\epsilon) = T_{\text{lin}}(z)$ because $\lim_{\epsilon\to 0} N^2(\epsilon) = 0$. Letting now $\epsilon \to 0$ in (10) gives

$$iQ_z(x,z) + \frac{D_{\text{loc}}(z)}{T_{\text{lin}}^2(z)}(Q_{xx}(x,z) - x^2Q(x,z)) = 0. \tag{14}$$

Let $U(t)$ be the unitary group generated by $\triangle - x^2$ and define the accumulated effective dispersion as

$$R^{\text{eff}}(z) := \int_0^z D_{\text{loc}}(z')/T_{\text{lin}}^2(z')dz'. \tag{15}$$

Then $S(z) := U(R^{\text{eff}}(z))$ is the solution operator of the unperturbed equation (14) which is the harmonic oscillator with an additional z-dependent factor $D_{\text{loc}}(z)/T_{\text{lin}}^2(z)$. Since

$\langle D_{\rm loc}/T_{\rm lin}^2\rangle = 0$ the operator S is 1-periodic. Moreover it is unitary and conserves momentum and energy [11]. Splitting (10) into unperturbed and $\epsilon-$dependent part we obtain

$$iQ_z + \frac{D_{\rm loc}}{T_{\rm lin}^2}(Q_{xx} - x^2 Q) + \left((\frac{D}{T^2} - \frac{D_{\rm loc}}{T_{\rm lin}^2})(Q_{xx} - x^2 Q) + \frac{N^2}{T}(x^2 Q + |Q|^2 Q) \right) = 0.$$

Using the solution of the unperturbed equation we introduce a canonical transformation

$$Q(z,x) = S(z)v(z,x) \tag{16}$$

which yields after applying $S^{-1}(z)$ and some straight-forward simplifications using the fact that a semigroup and its generator commute

$$iv_z + (\frac{D}{T^2} - \frac{D_{\rm loc}}{T_{\rm lin}^2})(v_{xx} - x^2 v) + S^{-1}(z)\left(\frac{N^2}{T}(x^2 S(z)v + |S(z)v|^2 S(z)v) \right) = 0. \tag{17}$$

It is essential that both $D/T^2 - D_{\rm loc}/T_{\rm lin}^2$ and N^2/T are now of order ϵ. Therefore, averaging of the above equation makes sense. Equation (17) possesses a Hamiltonian similar to that of the cubic NLS with additional quadratic potential:

$$H = (\frac{D}{T^2} - \frac{D_{\rm loc}}{T_{\rm lin}^2})\int_{-\infty}^{\infty} \left(|v_x|^2 + x^2|v|^2\right)dx - \frac{N^2}{T}\int_{-\infty}^{\infty}\left(x^2|S(z)v|^2 + \frac{1}{2}|S(z)v|^4 \right) dx.$$

Using $\langle D_{\rm loc}/T_{\rm lin}^2\rangle = 0$, formal averaging now yields

$$\langle H\rangle = \langle\frac{D}{T^2}\rangle\int_{-\infty}^{\infty}(|u_x|^2 + x^2|u|^2)dx - \int_0^1 \frac{N^2}{T(z)}\int_{-\infty}^{\infty}\left(x^2|S(z)u|^2 + \frac{1}{2}|S(z)u|^4 \right) dxdz$$

with corresponding Euler-Lagrange equation

$$\langle\frac{D}{T^2}\rangle(u_{xx} - x^2 u) + \int_0^1 S^{-1}(z')\left(\frac{N^2}{T(z')}(x^2 S(z')u + |S(z')u|^2 S(z')u) \right) dz' = \lambda u. \tag{18}$$

Solutions of (18) are standing wave solutions of the averaged DM-NLS after lens transformation

$$iu_z + \langle\frac{D}{T^2}\rangle(u_{xx} - x^2 u) + \int_0^1 S^{-1}(z')\left(\frac{N^2}{T(z')}(x^2 S(z')u + |S(z')u|^2 S(z')u) \right) dz' = 0. \tag{19}$$

Remark 4.1. *For a jusitification of the averaging procedure we refer to [23]. Translating their proof to our situation would require a lengthy and cumbersome expansion of all coefficients up to second order and is omitted here.*

The following Lemma is essential to simplify the averaged equation:

Lemma 4.2. *For all* $u \in X := \{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} x^2|u|^2 dx < \infty\}$ *we have:*

$$S^{-1}(z)\left(\frac{N^2}{T(z)} x^2 S(z) u \right) = \frac{N^2}{T(z)}\Big(\frac{\cos(4R^{\rm eff}(z)) - 1}{2} u_{xx} + \frac{\cos(4R^{\rm eff}(z)) + 1}{2} x^2 u + i\,\frac{\sin(4R^{\rm eff}(z))}{2}(u + 2xu_x) \Big).$$

Proof. see Appendix B. ☐

Note that R^{eff} is defined in (15). Due to symmetry the last term averages out, i.e. $\langle \sin(4R^{\text{eff}})/T \rangle = 0$. This together with the Lemma implies the following result: The averaged lens-transformed DM-NLS reads as

$$iu_z + \quad au_{xx} - bx^2u + \int_0^1 S^{-1}(z)\left(\tfrac{N^2}{T(z)}|S(z)u|^2 S(z)u\right) dz = 0, \tag{20}$$

where

$$a: \quad = \langle \tfrac{D}{T^2} - \tfrac{N^2}{T}\tfrac{1-\cos(4R^{\text{eff}})}{2}\rangle, \; b: \quad = \langle \frac{D}{T^2} - \frac{N^2}{T}\frac{1+\cos(4R^{\text{eff}})}{2}\rangle. \tag{21}$$

Of course the existence and behavior of the solutions strongly depends on the sign of a and b.

The averaged Hamiltonian can be written as

$$\langle H \rangle \quad = a\int_{-\infty}^{\infty} |u_x|^2 dx + b\int_{-\infty}^{\infty} x^2|u|^2 dx - \int_0^1 \tfrac{N^2}{2T}\int_{-\infty}^{\infty} |S(z)u|^4 \; dx \; dz. \tag{22}$$

At first sight we would require $a > 0$ and $b \geq 0$ to obtain ground states.

Remark 4.3.

- *The structure of the lens-transformed DM-NLS is always of the above type. Only the value (and the sign) of a and b changes in the different regions.*
- *Note that naive averaging of* (10) *would give [20]*

$$iu_z + \langle \frac{D}{T^2}\rangle u_{xx} - (\langle \frac{D}{T^2}\rangle - \langle \frac{N^2}{T}\rangle)x^2u + \langle \frac{N^2}{T}\rangle |u|^2 u = 0 \tag{23}$$

  in contrast to (20)*. As $\langle D/T^2\rangle > 0$ and $\langle D/T^2\rangle - \langle N^2/T\rangle < 0$ the potential in* (23) *is of non-trapping type and there exist no square-integrable ground states of the corresponding Hamiltonian and accordingly no pulse-like solutions.*

Remember that in the situation discussed in this section formula (11) is valid. The following approximations hold with $y = d/(2T_0^2)$ [14]:

$$\langle \frac{D}{T^2}\rangle \quad \sim \quad \frac{N^2}{T_0\sqrt{1+y^2}} \quad \text{and} \quad \langle \frac{N^2}{T}\rangle \sim \frac{N^2}{T_0} y^{-1}\ln(y+\sqrt{1+y^2}).$$

In a similar way we can show that [8]

$$\langle \frac{N^2\cos(4R^{\text{eff}})}{T}\rangle \sim \langle \frac{N^2\cos(4R^{\text{eff}})}{T_{\text{lin}}}\rangle = \frac{N^2}{T_0}(\frac{2}{\sqrt{1+y^2}} - y^{-1}\ln(y+\sqrt{1+y^2})).$$

Combining the above results we arrive at

$$\begin{aligned} a \quad &= \quad (\langle \frac{D}{T^2}\rangle - N^2\langle \frac{1-\cos(4R^{\text{eff}})}{2T}\rangle) \\ &\sim \quad \frac{N^2}{T_0}(\frac{2}{\sqrt{1+y^2}} - y^{-1}\ln(y+\sqrt{1+y^2})) = \frac{N^2}{T_0}F(y), \end{aligned} \tag{24}$$

$$b \quad = \quad (\langle \frac{D}{T^2}\rangle - N^2\langle \frac{1+\cos(4R^{\text{eff}})}{2T}\rangle) \sim 0. \tag{25}$$

It is interesting that $F(y)$ does again play a role. For large T_0 as in this section we have $y < \overline{y}$ implying $F(y) > 0$ and accordingly $a > 0$. Using the above expression for b the quadratic potential does not matter in first order approximation. Note that numerical simulations show that b is small but positive. The averaged Hamiltonian is now

$$\langle H \rangle \ = \epsilon C \left(\tfrac{F(y)}{T_0} \int_{-\infty}^{\infty} |u_x|^2 dx - \int_0^1 \tfrac{1}{2T(z)} \int_{-\infty}^{\infty} |Su|^4 \ dx \ dz \right) + \mathcal{O}(\epsilon^2),$$

where we have set $N^2 = \epsilon C$ according to (11). As we only consider first order averaging we should neglect terms of higher order and minimize

$$\langle H \rangle \ = \epsilon C \left(\tfrac{F(y)}{T_0} \int_{-\infty}^{\infty} |u_x|^2 dx - \int_0^1 \tfrac{1}{2T} \int_{-\infty}^{\infty} |Su|^4 \ dx \ dz \right)$$

with respect to some constraint in order to obtain stable ground state solutions of the averaged equation.

The above averaged Hamiltonian is similar to the one obtained by Zharnitsky et al. [23]. The only difference is that the nonlinear term consists now of $S(z)$ instead of $U(z) = \exp(iR(z)\triangle)$ and the nonlinear coefficient is now z-dependent (but it could be assumed to be constant without loss of generality).

The constrained minimization procedure works similar to the proof of Zharnitsky et al. Note that the crucial lemma in [23] was the bound on localization in the linear Schrödinger equation which is also valid for the case of the harmonic oscillator. Following the proof in [23] which is nearly identical to the one by Yew et al. [21] one can show the existence of standing wave solutions by applying a mountain pass argument. We will omit the proof since we are mainly interested in the case where the potential does matter, see the following section. It should be noted that there is one advantage in our approach: It seems to be more suitable for numerical investigations because the equations become much more simple if we use the basis of Gauss-Hermite eigenfunctions. Moreover the corresponding Euler-Lagrange equations are similar to those investigated by Turitsyn et al. [14, 16] which admit a very good and efficient approximation of the DM-soliton.

4.1.2 The region of medium map strength

In the region of medium map strength ($T_0 \sim \overline{T_0}$) we have to treat the cases $T_0 > \overline{T_0}$ and $T_0 < \overline{T_0}$ separately since the behavior completely changes.

The case $T_0 > \overline{T_0}$. This case is much more exciting than the previous one: The quadratic potential will have "good" sign and therefore we will find a ground state u with $xu \in L^2(\mathbb{R})$. Bearing in mind the last section, it would be natural to require

$$\langle \frac{N^2 \cos(4R^{\text{eff}})}{T_{\text{lin}}} \rangle \sim 0 \tag{26}$$

in order to obtain positive sign for both $\int_{-\infty}^{\infty} |u_x|^2 dx$ and $\int_{-\infty}^{\infty} x^2 |u|^2 dx$. Using

$$\langle \frac{N^2 \cos(4R^{\text{eff}})}{T_{\text{lin}}} \rangle = \frac{N^2}{T_0} \left(\frac{2}{\sqrt{1+y^2}} - y^{-1} \ln(y + \sqrt{1+y^2}) \right) = \frac{N^2}{T_0} F(y)$$

we can conclude that condition (26) is valid for $y \sim \overline{y}$ implying $T_0 \sim \overline{T_0}$. In what follows we will explain that this "naive" idea will work.

Letting $\langle D\rangle$ tend to zero in the nonlinear TM-equations we conclude using (13)

$$N^2(T_0, \langle D\rangle) \to N^2(T_0, 0) = 0$$

and hence $T \to T_{\text{lin}}$. According to the above explanations the unperturbed part of (10) is again given by (14) with $T_{\text{lin}}(0)$ now close to $\overline{T_0}$. Using again $Q(z, x) = S(z)v(z, x)$ and the same calculations as in the previous case we again obtain the averaged equation (20) with a and b defined as above, see (21). Unfortunately it is very unpleasant to calculate approximations for the mean values since we have to expand all quantities up to the second order. By the choice of T_0 we expect $N^2\langle\cos(4R^{\text{eff}})/T\rangle$ to be of high order. Instead of doing all this calculations we refer to numerical simulations which show

$$\langle D/T^2\rangle - N^2\langle(1 \pm \cos(4R^{\text{eff}}))/2T\rangle > 0.$$

In Figure 6 the dependence of a and b on the initial pulse width is presented. Starting with arbitrary big initial pulse width T_0 we see that b, the coefficient of the quadratic potential, is close to zero in analogy to the analytical result. Making T_0 smaller the parameter b becomes positive and therefore the quadratic potential comes into play. Now we are in

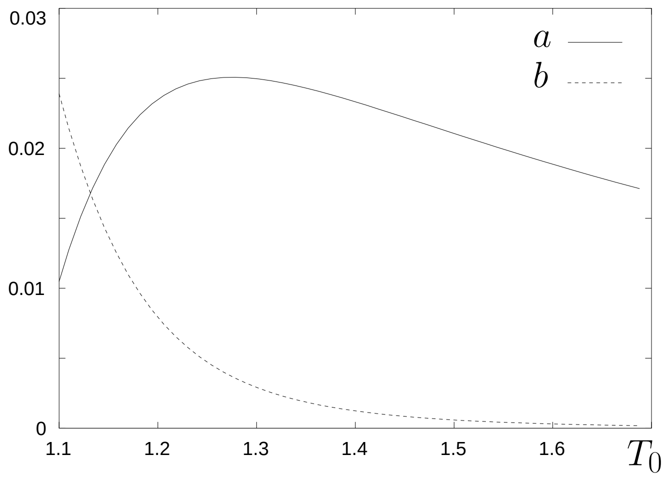


Figure 6: Numerical values for a (solid line) and b (dashed line) in the case $d = 8 (\Rightarrow T_0 \sim 1.09767)$ and $\langle D\rangle = 0.05$

position to define the constrained minimization problem which gives us weak standing wave solutions of the averaged DM-NLS after lens transformation. Let

$$I(u) := \langle H\rangle(u) = a\int_{-\infty}^{\infty}|u_x|^2dx + b\int_{-\infty}^{\infty}x^2|u|^2dx - \int_0^1\frac{N^2}{2T}\int_{-\infty}^{\infty}|S(z)u|^4\ dx\ dz \quad (27)$$

It is natural to look for minimizers of I in the following well known Hilbert space:

$$X := \left\{u \in H^1(\mathbb{R})\ \middle|\ xu \in L^2(\mathbb{R})\right\}.$$

For $u, v \in X$ we can define the inner product

$$< u, v >_X := < u_x, v_x >_{L^2} + < u, v >_{L^2} + < xu, xv >_{L^2}$$

and energy norm $\| u \|_X^2 := \| xu \|_{L^2}^2 + \| u \|_{H^1}^2$. Note that X equipped with the above norm is compactly embedded into $L^2(\mathbb{R})$ [22]. Using this fact it is easy to obtain ground states:

Theorem 4.4 (Existence of ground states). *The constrained minimization problem* $I_\omega = \min\{I(u)|u \in X, \| u \|_2^2 = \omega\}$ *has at least one nontrivial solution* $u \in X$ *for all* $\omega > 0$*. This minimizer satisfies the corresponding Euler-Lagrange equation with Lagrange-multiplier* $\lambda = \lambda^\omega$.

Proof. see Appendix A. ☐

The parameter λ is the quasi-impulse, i.e. we have shown that there exists a solution of the NLS after lens transformation (10) which is close to $S(z)u(x)\exp(i\lambda z)$, recall that $S(z)$ is 1-periodic. Note that the ground state u is shown to have the decay property $xu \in L^2(\mathbb{R})$ which is a new theoretical decay result. The fast decay results in suppression of soliton interaction, and consequently, in the possibility of denser information packing.

Remark 4.5.

- *Following the argument of Zhang [22] the ground states are orbital stable.*
- *From the type of the potential one would expect that the DM-soliton has Gaussian tails. For example this was shown by Kavian and Weissler [3] for a simpler nonlinearity. However, from numerical simulations it is known that the tails have oscillating structure and decay exponentially, cf. [10, 19]. The Gaussian decay is lost due the nonlocal properties of the nonlinearity derived in this paper.*
- *For a similar equation a detailed bifurcation analysis is known [5]. A complete description of the solutions and the bifurcation behavior will be given in a continuation of this paper [9]*
- *Note that expanding u using the Gauss-Hermite eigenfunctions we obtain equations similar to those derived by Turitsyn et al. [16] which describe the true DM-soliton very well [14, 16], see the next section for details.*

Thus, we have shown the following theorem which is our main result:

Theorem 4.6 (Existence of standing waves of arbitrary energy). *For positive residual dispersion the averaged lens transformed DM-NLS in the case of medium map strength and* $T_0 > \overline{T_0}$ *reads as*

$$iu_z + au_{xx} - bx^2u + \int_0^1 S^{-1}(z')\left(\frac{N^2}{T(z')}|S(z')u|^2 S(z')u\right) dz' = 0 \text{ with } a, b > 0.$$

It possesses standing wave solutions $u(x,t) = v(x)\exp(i\lambda t)$ *of arbitrary energy* $\| v \|_2^2 = \omega$*, where* $v \in X$.

Note that the above theorem includes the case $T_0 >> \overline{T_0}$ as limit because b vanishes for large T_0. Since the aim of this paper is to explain how the lens transformation can be used to verify various numerically and experimentally well-known properties of the DM-soliton we do not give a more detailed investigation of Equation (20).

Recapitulating the derived results we have explained the parameter-dependence of the DM-soliton, partly analytical and partly by numerical calculations. The fact that the quadratic potential is attractive is essential, nevertheless we can use our method to describe the behavior of the DM-solitons in other regions. This will be done briefly in the remainder of this section.

The case $T_0 < \overline{T_0}$. In this case N^2 is no longer of the same order as $\langle D \rangle$, i.e. the unperturbed equation is nonlinear and the method used in the previous cases does not work. To overcome this problem one has to treat both, residual dispersion $\langle D \rangle$ and $\overline{T_0} - T_0$, as perturbations, then

$$\lim_{\epsilon \to 0} N^2(T_0, < D >) = N^2(\overline{T_0}, 0) = 0.$$

In this case the unperturbed equation reads as

$$iQ_z + \frac{D_{\text{loc}}}{T_{\text{lin}}^2}(Q_{xx} - x^2 Q) = 0 \text{ with } T_{\text{lin}}(0) = \overline{T_0},$$

where the difference to the other regimes is the value of $T_{\text{lin}}(0)$.

This can be seen in a more analytical way as follows: With $N^2 = \mathcal{O}(\epsilon)$ and $\langle D \rangle = \mathcal{O}(\epsilon^2)$ one can see that $\delta = T_0 - \overline{T_0} < 0$ is of order ϵ using the formula

$$N^2 = -\frac{b\delta}{2a} + \frac{\sqrt{b^2\delta^2 - 4ac\langle D \rangle}}{2a},$$

which implies

$$\delta = -\frac{b^2}{2a}\frac{\langle D \rangle}{N^2} + \frac{N^2}{2a}.$$

In our situation we have $\langle D \rangle / N^2 = \mathcal{O}(\epsilon)$ and $N^2 = \mathcal{O}(\epsilon)$. Therefore it follows $\delta = \mathcal{O}(\epsilon)$ and accordingly $T_0 \to \overline{T_0}$ for $\epsilon \to 0$. It should be mentioned that this behavior is supported by numerical simulations, see Figure 7 (a).

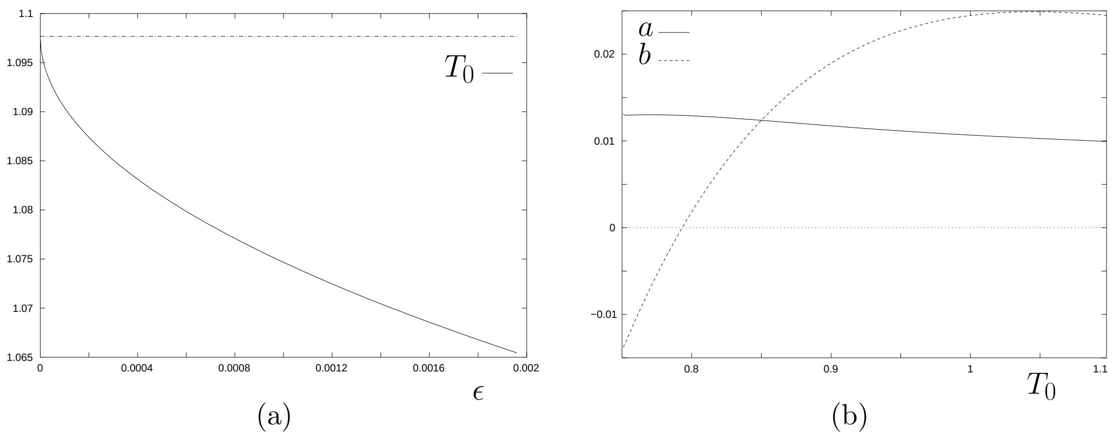


Figure 7: (a) The parameter $T_0(N^2, \langle D \rangle)$ as a function of ϵ. Here $N^2 = \epsilon C$ and $\langle D \rangle = \epsilon^2 \alpha$ with $C = 4$ and $\alpha = 0.2$. It is $\overline{T_0} = 1.09767$. (b) The values of a and b for $\langle D \rangle = 0.05$ between $T_0 = 0.75$ and $T_0 = \overline{T_0}$. Here $\widetilde{T_0} \sim 0.793$

With the above rescaling of the problem the unperturbed part is again (14), but now $T_{\text{lin}}(0) = \overline{T_0}$ and the method developed in the last section can be applied. Numerical simulations then show the existence of a critical parameter $\widetilde{T_0}$ depending on d and $\langle D \rangle$ where the parameter b changes its sign, see Figure 7 (b). Thus, we have established

Theorem 4.7 (Averaged Equation for $T_0 < \overline{T_0}$). *If $T_0 < \overline{T_0}$ and $\langle D\rangle = \epsilon^2\alpha$ with $\alpha > 0$ the averaged equation after lens-transformation reads as*

$$iu_z + au_{xx} - bx^2u + \int_0^1 S^{-1}(z)\left(\frac{N^2}{T(z)}|S(z)u|^2S(z)u\right)dz = 0$$

where $a > 0$ and

- *$b > 0$ for $T_0 > \widetilde{T_0}$*
- *$b < 0$ for $T_0 < \widetilde{T_0}$*

Thereby, $a, b = \mathcal{O}(\langle D\rangle) = \mathcal{O}(\epsilon^2)$ and $N^2 = \mathcal{O}(\epsilon)$.

Remark 4.8.

- *One could ask why one should include the terms of higher order in the averaged equation. Applying an averaging procedure with normal transformations it turns out that second order correction terms vanish due to symmetry, see [13] for details. However, a detailed analysis of this situation is beyond the scope of this paper.*
- *The switching of the potential from attracting to repelling corresponds to the transition to a flatter profile which is well-known for solutions in the regime of vey large map strength. In the vicinity of the critical parameter $\widetilde{T_0}$ the DM-soliton concept breaks down; in the next section we will reformulate this result in terms of input pulse energy to make it more meaningful for applications.*

4.2 The case $\langle D\rangle < 0$

In this section we give an overview on the case of negative residual dispersion. In order to prevent confusion we will write the Hamiltonians in different situations only in a qualitative sense, i.e. we are only interested in the sign of a and b without regarding its absolut value. The nonlinear part is always the same, we write $N^2 = \epsilon C$ and

$$N(u) := \int_0^1 \frac{C}{2T}\int_{-\infty}^{\infty} |S(z)u|^4 \, dx \, dz.$$

4.2.1 The lower branch.

The lower branch corresponds to the solutions with smaller energy. Applying the averaging procedure together with numerical calculations shows that the corresponding Hamiltonian is of the following type

$$\langle H\rangle = -\epsilon\int_{-\infty}^{\infty} |u_x|^2 + \epsilon\int_{-\infty}^{\infty} x^2|u|^2 \, dx - \epsilon N(u).$$

It can be seen easily that the Hamiltonian is unbounded.

4.2.2 The upper branch.

In this case N^2 is no longer $\mathcal{O}(\epsilon)$ and the modified method should be applied. Numerical calculations show that b is always negative and there is a critical parameter $\widetilde{T_0}$ where a changes its sign. Thus, we have the following qualitative Hamiltonians

$$\begin{aligned}\langle H\rangle &= -\epsilon^2\textstyle\int_{-\infty}^{\infty}|u_x|^2dx - \epsilon^2\int_{-\infty}^{\infty}x^2|u|^2\,dx - \epsilon N(u), \quad T_0 < \widetilde{T_0},\\ \langle H\rangle &= \epsilon^2\textstyle\int_{-\infty}^{\infty}|u_x|^2dx - \epsilon^2\int_{-\infty}^{\infty}x^2|u|^2\,dx - \epsilon N(u), \quad T_0 > \widetilde{T_0}.\end{aligned}$$

This situation corresponds to the case $T_0 > \overline{T_0}$ for positive residual dispersion. In analogy, a and b are of order $\langle D \rangle = \mathcal{O}(\epsilon^2)$ which means they are of higher order. Note that for $T_0 < \widetilde{T_0}$ the only critical points of the averaged Hamiltonian are saddles and maxima; we omit the proof here. This is a an analogon to the result of Zharnitsky et al. [23]

Remark 4.9. *The averaged Hamiltonian of the DM-NLS for negative residual dispersion has also been investigated by Jackson, Jones and Zharnitsky [2] who have shown the non-existence of a minimizer. In our case we have explained that there is no solution of the nonlinear TM-equations in the case of small map strength, whereas there exist two solutions in the opposite case. This corresponds to the result of Pelinovsky [12]. Jackson et al. [2] have shown by a numerical averaging procedure that in the case of negative residual dispersion close to zero a local minimum of a reduced Hamiltonian persists. It vanishes when residual dispersion is decreased further.*

4.3 The case $\langle D \rangle = 0$

In the case $\langle D \rangle = 0$ we have shown in Section 3 that nontrivial periodic solutions of the nonlinear TM-equations exist only for $T_0 < \overline{T_0}$. Using Equation (13) we have

$$N^2 \sim -9.33\ (T_0 - \overline{T_0}).$$

Regarding N^2 as a small quantity yields that $T_0 - \overline{T_0}$ is of the same order, say ϵ. Hence in the unperturbed problem we have $T_{\rm lin}(0) = \overline{T_0}$ and the modified averaging method should be employed. Numerical simulations then show that both, a and b, are of order ϵ^3. Therefore the correct Hamiltonian in this case is of type

$$H = -\epsilon N(u).$$

Remark 4.10. *If residual dispersion is vanishing numerical simulations show the existence of a stable periodic pulse. Using the approach of Zharnitsky one has no control on the derivative in the averaged equation, i.e. there is no term including u_{xx} in the resulting equation. The existence of a minimizer was recently shown by Kunze [6]. Note that in contrast to the result of Zharnitsky [23] now the harmonic oscillator semigroup is involved. This could make it easier to obtain a ground state in some L^p-space.*

5 Summary of the results in terms of the originial equation

In this section we translate our results to the DM-NLS (1). Since the energy

$$E = \int_{\mathbb{R}} |A(z,t)|^2 dt$$

is a conserved quantity of the DM-NLS we will use E to parametrize the solutions instead of the more artificial parameter T_0. We have

$$\begin{aligned} E = \int_{\mathbb{R}} |A(z,t)|^2 dt &= \frac{N^2}{T(z)} \int_{\mathbb{R}} \left| Q(t/T(z), z) \exp\left(it^2 \frac{M(z)}{T(z)} \right) \right|^2 dt \\ &= N^2 \int_{\mathbb{R}} |Q(x,z)|^2\ dx,\ \text{with } x = t/T(z). \end{aligned}$$

After transformation $Q(x,z) = S(z)v(x,z)$ and ansatz $v(x,z) = \exp(i\lambda z)u(x)$ we end up at

$$E = N^2 \int_{\mathbb{R}} |u(x)|^2\ dx. \tag{28}$$

Thus the parameter N^2 in the lens transformation can be regarded as scaling of energy. For prescribed energy $E = N^2$ we determine the parameter T_0 such that the corresponding solution of the nonlinear TM-equations is 1-periodic and search for solutions with unit-norm of the DM-NLS after lens transformation. Note that the parameter T_0 is uniquely determined for given values of N^2 and $\langle D \rangle$. Figure 8 shows the curves $N^2(T_0, \langle D \rangle)$ for different values of T_0, these curves cover the union of the regions $A, \dots, E$.

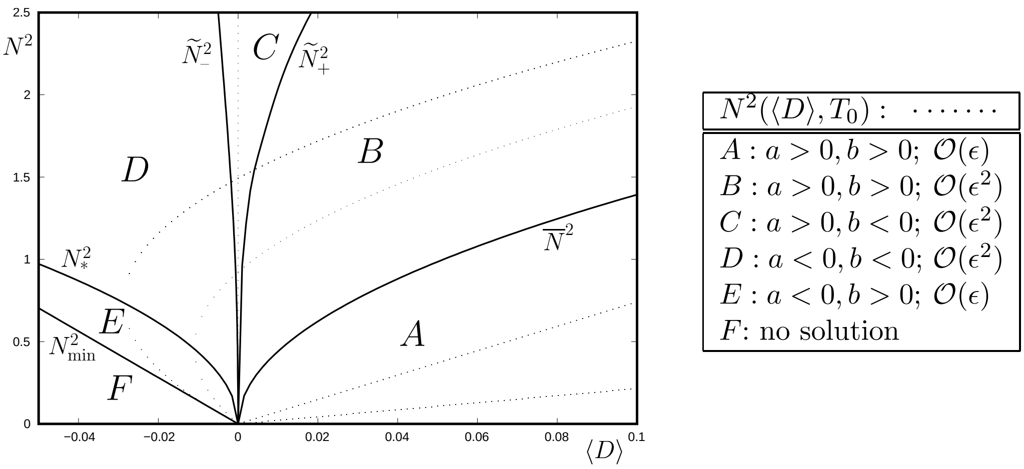


Figure 8: Partition of the $(N^2, \langle D \rangle)$ in regions with constant sign of a and b.

Remark 5.1. *The recent publications deal with the situation where $\langle D \rangle$ and N^2 are of same order ϵ. This is exactly true in region A resp. E, where the original averaging procedure is applicable. In this regions both, a and b, are $\mathcal{O}(\epsilon)$. The modified averaging procedure is valid in the other regions, where in terms of the characteristic lengths no longer $Z_{\rm rd} \sim Z_{\rm nl}$, but rather $Z_{\rm disp} << Z_{\rm nl} << Z_{\rm rd}$. To the best of our knowledge such a situation has not been studied in the literature. Therefore the derived equations could be regarded as a first step towards the analysis of DM-solitons with large energy which are of practical interest due to their small signal-to-noise ratio.*

The main result of this paper then reads as follows

Theorem 5.2 (Existence of the DM-soliton). *For prescribed energy $E = N^2$ there exists a solution of the DM-NLS which is close (in the sense of the averaging procedure) to the 1-periodic function*

$$A(t,z) = N \frac{U(R^{\rm eff}(z))\,\{\exp(i\lambda z) u(t/T(z))\}}{\sqrt{T(z)}} \exp\left(i t^2 \frac{M(z}{T(z)} \right),$$

where T and M are 1-periodic solutions of the nonlinear TM-equations and the profile $u \in H^1(\mathbb{R})$ fulfils $xu \in L^2(\mathbb{R})$. Moreover, u is orbital stable as a ground state of the averaged Hamiltonian.

Figure 8 explains why the shape of the DM-soliton varies with increasing pulse energy supposed the fiber parameters d and $\langle D \rangle > 0$ are given. For small values of the energy the parameter b is of higher order. Thus, the quadratic potential has little influence and the DM-soliton is close to the traditional NLS-soliton. For larger values of E the significance of the quadratic potential increases, the corresponding solution is the well-known energy-enhanced DM-soliton with Gaussian core. Increasing E above a critical value $\widetilde{N}^2$, the parameter b becomes negative and the solution of the averaged equation is no longer in $L^2(\mathbb{R})$, this corresponds to numerically observed flatter profile.

Remark 5.3. *Using the basis $\{u_n\}_{n\in\mathbb{N}_0}$ of Gauss-Hermite eigenfunctions of the harmonic oscillator the ground state u can be represented as*

$$u(x) = \sum_{m=0}^{\infty} F_m u_m(x).$$

Inserting this ansatz in the averaged DM-NLS after lens transformation and taking the L^2-scalar product of the resulting equation with u_n we end up at the infinite-dimensional algebraic system

$$\begin{aligned} - \quad & \lambda F_n + \langle \frac{D(z)}{T^2(z)} - \frac{N^2}{2T(z)} \rangle \lambda_n F_n \\ + \quad & N^2 \langle \frac{\cos(4R^{\text{eff}}(z))}{2T(z)} \rangle \left(\sqrt{(n+1)(n+2)} F_{n+2} + \sqrt{n(n-1)} F_{n-2} \right) \\ + \quad & \sum_{k,l,m=0}^{\infty} \int_0^1 \frac{N^2}{T(z)} \exp\left(2i(k-l+m-n)R^{\text{eff}}(z)\right) dz F_k \overline{F_l} F_m V_{n,m,l,k} = 0, \end{aligned} \tag{29}$$

where

$$V_{n,m,l,k} := \int_{\mathbb{R}} u_m(x) u_k(x) u_l(x) u_n(x) dx.$$

and $\lambda_n = -2n - 1$ denote the eigenvalues corresponding to u_n. A system similar to the above one has been previously derived, cf. [16], where the averaging procedure has been performed by Lie-transform technique. The only difference is that R^{eff} in (29) *is replaced by $R(z)$ defined through $R'(z) = D(z)/T^2(z) - \langle D/T^2 \rangle$. Strictly speaking, R still depends on the smallness parameter $\langle D \rangle = \mathcal{O}(\epsilon)$ and hence R and R^{eff} differ by an ϵ-dependent term. For an analysis of system* (29) *we refer to [16].*

As mentioned before the quadratic potential helps to establish further properties of the DM-soliton, we have restricted ourselves to the existence of ground states of the averaged variational principle in the present paper.

Acknowledgements

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A Existence of ground states of the averaged variational principle

In this section we will give the proof of Theorem 4.4. In order to simplify notation we assume I to be of the following form

$$I(u) := \int_{-\infty}^{\infty} |u_x|^2 dx + \int_{-\infty}^{\infty} x^2 |u|^2 dx - \int_0^1 \int_{-\infty}^{\infty} |S(z)u|^4 \, dx \, dz. \tag{30}$$

Note that there exist $T_*, T^* > 0$ such that $T_* < T(z) < T^*$ which means that we can omit T without loss of generality. Moreover all constants are normalized to 1.

Now we are going to prove

Theorem 4.4 *The constrained minimization problem*

$$I_\omega = \min\{I(u) | u \in X, \| u \|_2^2 = \omega\}$$

has at least one nontrivial solution $u \in X$ for all $\omega > 0$.

At first we show that I is bounded from below.

Lemma A.1. $\forall\, \omega \in (0, \infty)$ $I_\omega > -\infty$

Proof. Let $u \in X$ with $\| u \|_2^2 = \omega$ and define $v(x, z) := S(z)u(x)$. In this proof we use the following equivalent norm on X:

$$\| u \|_{X'}^2 := \| xu \|_{L^2}^2 + \| u_x \|_{L_2}^2 . \tag{31}$$

Due to the Gagliardo-Nirenberg-Sobolev inequality we can estimate as follows

$$\| v \|_4^4 \leq C \, \| v_x \|_2 \| v \|_2^3 = C \, \| v_x \|_2 \| u \|_2^{3/2} = C \, \| v_x \|_2 \, \omega^{3/2}$$

Because of the conservation of energy in the linear Schrödinger equation we have

$$\| v_x \|_2^2 \leq \| v_x \|_2^2 + \| xv \|_2^2 = \| v \|_{X'}^2 = \| u \|_{X'}^2 .$$

Therefore

$$\int_0^1 \int_{\mathbb{R}} |S(z)u(x)|^4 \, dx \, dz = \int_0^1 \| v \|_4^4 \, dz \leq C \, \| u \|_{X'} \, \omega^{3/2}$$

This implies

$$I(u) \geq \| u \|_{X'}^2 - C \, \| u \|_{X'} \, \omega^{3/2},$$

which is a quadratic polynomial in $\| u \|_{X'}$ and hence bounded from below. □ □

Next we show that every minimizing sequence is bounded in X:

Lemma A.2. *Let $\{u_n\}$ be a sequence in X with $\| u_n \|_2^2 = \omega$ and $I(u_n) \to I_\omega$. Then there exists $M > 0$ with $\| u_n \|_X \leq M$.*

Proof. Because $I(u_n)$ is a converging sequence in $\mathbb{R}$ we can find a positive constant M, independent of n, with $I(u_n) \leq M$. Using the inequality obtained in the proof of Lemma A.1 we have $M \geq I(u_n) \geq \| u_n \|_{X'}^2 - C \, \| u_n \|_{X'} \, \omega^{3/2}$. But this can only be true if $\| u_n \|_{X'}$ and therefore $\| u_n \|_X$ is bounded. □ □

Now we are able to show the existence of a minimizer, i.e.

$$\exists u \in X : \| u \|_2^2 = \omega, I(u) = I_\omega.$$

Let $\{u_n\}$ be a minimizing sequence. Since $\{u_n\}$ is a bounded sequence in the Hilbert space X we can extract a weakly converging subsequence $u_{n_k} \rightharpoonup u$ in X for some $u \in X$. Without loss of generality we assume $u_{n_k} = u_n$. Because the embedding $X \subset\subset L^2(\mathbb{R})$ is compact, we have strong convergence $u_n \to u$ in L^2. Thus, $\| u \|_2^2 = \omega$. We can write

$$I(u) = \| u \|_X^2 - J(u), \text{ where } J(u) = \int_0^1 \int_{\mathbb{R}} |S(z)u|^4 \, dx \, dz.$$

Since $\|\cdot\|_X$ is weakly lower semi-continuous as a norm on X, we obtain the inequality

$$\| u \|_X \leq \lim_{n\to\infty} \inf \| u_n \|_X \ .$$

To complete the proof it suffices to show that $J(u_n) \to J(u)$. To see this, we can show similar as in the proof of Lemma A.1 that the following estimate is true:

$$\int_0^1 \int_{\mathbb{R}} |S(z)(u_n(x) - u(x))|^4 \, dx \, dt \leq C \, \| u_n - u \|_X \left(\int_{\mathbb{R}} |u_n - u|^2 \, dx \right)^{3/2} .$$

The first factor in the right hand side of the above equation is bounded because of the weak convergence $u_n \rightharpoonup u$ in X, the second factor tends to zero due to strong convergence in L^2. Thus we have shown

$$I(u) \leq \lim_{n\to\infty} I(u_n)$$

which, together with $\| u \|_2^2 = \omega$, completes the proof. Note, that $\| u_n \|_X \to \| u \|_X$ and weak convergence $u_n \rightharpoonup u$ in X give $u_n \to u$ strongly in X. □

B Proof of Lemma 4.2

It remains to show

Lemma 4.2 *For all $u \in X := \{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} x^2 |u|^2 dx < \infty\}$ we have:*

$$S^{-1}(z) \left(\frac{N^2}{T(z)} x^2 S(z) u \right) = \frac{N^2}{T(z)} \Big(\frac{\cos(4R^{\text{eff}}(z)) - 1}{2} \, u_{xx} + \frac{\cos(4R^{\text{eff}}(z)) + 1}{2} \, x^2 u + i \, \frac{\sin(4R^{\text{eff}}(z))}{2} \, (u + 2x u_x) \Big).$$

Proof. Using the basis of Gauss-Hermite eigenfunctions we can write

$$u(x) = \sum_{n=0}^{\infty} a_n u_n(x), \ \text{with } a_n \in \mathbb{C}.$$

Taking advantage of the fact that $S(z)u_n$ is explicitly known we have

$$S(z)u(x) = \sum_{n=0}^{\infty} a_n S(z) u_n(x) = \sum_{n=0}^{\infty} a_n \exp(i\lambda_n R^{\text{eff}}(z)) u_n().$$

Thereby, $\lambda_n = 2n + 1$ is the eigenvalue corresponding to u_n. Further it is known that [15]:

$$x u_n(x) = n u_{n-1}(x) + \frac{1}{2} u_{n+1}(x).$$

Applying this formula twice yields

$$x^2 u_n(x) = n(n-1) u_{n-2}(x) + \frac{1}{2}(2n+1) u_n(x) + \frac{1}{4} u_{n+2}(x),$$

hence

$$x^2 S(z)u(x) = \sum_{n=0}^{\infty} a_n \exp(i\lambda_n R^{\text{eff}}(z)) x^2 u_n(x)$$

$$= \sum_{n=0}^{\infty} a_n \exp(i\lambda_n R^{\text{eff}}(z)) \left(n(n-1)u_{n-2}(x) + \frac{1}{2}(2n+1)u_n(x) + \frac{1}{4}u_{n+2}(x) \right).$$

Now we are in position to calculate as follows

$$\begin{aligned} &S^{-1}(z)\left(x^2 S(z)u(x)\right) \\ &= \sum_{n=0}^{\infty} a_n e^{i\lambda_n R^{\text{eff}}(z)} S^{-1}(z) \left(n(n-1)u_{n-2}(x) + \frac{2n+1}{2}u_n(x) + \frac{1}{4}u_{n+2}(x) \right) \\ &= \sum_{n=0}^{\infty} a_n \left(n(n-1)e^{4iR^{\text{eff}}(z)} u_{n-2}(x) + \frac{2n+1}{2}u_n(x) + \frac{1}{4}e^{-4iR^{\text{eff}}(z)} u_{n+2}(x) \right), \end{aligned}$$

where in the last equation we have used $\lambda_n - \lambda_{n\pm 2} = \mp 4$. Next we turn to the right hand side. Using the expansion of u we obtain

$$\begin{aligned} S^{-1}(z)\big(x^2 S(z)u(x)\big) = \sum_{n=0}^{\infty} a_n \Big(& \frac{\cos(4R^{\text{eff}}(z))-1}{2} u_n''(x) + \frac{\cos(4R^{\text{eff}}(z))+1}{2} x^2 u_n(x) \\ & + i\, \frac{\sin(4R^{\text{eff}}(z))}{2} \left(u_n(x) + 2x u_n'(x)\right) \Big). \end{aligned}$$

Due to $-u_n''(x) + x^2 u_n(x) = \lambda_n u_n(x)$ it follows

$$\begin{aligned} &\frac{\cos(4R^{\text{eff}}(z)) - 1}{2} u_n''(x) + \frac{\cos(4R^{\text{eff}}(z)) + 1}{2} x^2 u_n(x) \\ &= \frac{\cos(4R^{\text{eff}}(z)) - 1}{2} \left(x^2 u_n(x) - \lambda_n u_n(x)\right) + \frac{\cos(4R^{\text{eff}}(z)) + 1}{2} x^2 u_n(x) \\ &= \cos(4R^{\text{eff}}(z)) x^2 u_n(x) - \frac{\cos(4R^{\text{eff}}(z)) - 1}{2} \lambda_n u_n(x) \\ &= \cos(4R^{\text{eff}}(z)) \left(n(n-1)u_{n-2}(x) + \frac{2n+1}{2} u_n(x) + \frac{u_{n+2}(x)}{4} \right) \\ &\quad - \frac{\cos(4R^{\text{eff}}(z))-1}{2} \lambda_n u_n(x) \\ &= n(n-1)\cos(4R^{\text{eff}}(z)))u_{n-2}(x) + \frac{2n+1}{2} u_n(x) + \frac{1}{4}\cos(4R^{\text{eff}}(z)) u_{n+2}(x). \end{aligned}$$

To get control on the term $xu_n'(x)$ we use the relation $u_n' = nu_{n-1} - u_{n+1}/2$, see [15] and conclude

$$x u_n'(x) = n(n-1)u_{n-2}(x) - \frac{1}{2}u_n(x) - \frac{1}{4}u_{n+2}(x)$$

Hence we have

$$\frac{\sin(4R^{\text{eff}}(z))}{2} \left(u_n(x) + 2x u_n'(x)\right) = \sin(4R^{\text{eff}}(z)) \left(n(n-1)u_{n-2}(x) - \frac{u_{n+2}(x)}{4} \right).$$

Due to $\exp(4iR^{\text{eff}}(z))) = \cos(4R^{\text{eff}}(z)) + i\sin(4R^{\text{eff}}(z))$ the right-hand side can be written as

$$\sum_{n=0}^{\infty} a_n \left(n(n-1)e^{4iR^{\text{eff}}(z)} u_{n-2}(x) + \frac{2n+1}{2} u_n(x) + \frac{e^{-4iR^{\text{eff}}(z)}}{4} u_{n+2}(x) \right),$$

which gives the assertion of the lemma. □ □

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