Class Number Type Relations for Fourier Coefficients of Mock Modular Forms

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The Hurwitz class number

Recall: Class number of discriminant d

$$h(d) = \#\mathcal{Q}_d / \operatorname{SL}_2(\mathbb{Z})$$

with

$$\mathcal{Q}_d = \left\{ \left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix} \right) \in \mathbb{Z}^{2\times 2} \;\middle|\; a > 0, \gcd(a,b,c) = 1, b^2 - 4ac = d \right\}$$

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. Hurwitz class number of discriminant $-d$

$$H(d) = \begin{cases} -\frac{1}{12} & \text{if } d = 0 \\ \sum\limits_{f^2 \mid d} \frac{h(-d/f^2)}{w(-d/f^2)} & \text{if } d \in \mathbb{N} \\ 0 & \text{if } d \notin \mathbb{N}_0 \end{cases},$$

$$w(n) = \begin{cases} 3 & \text{if } n = -3 \\ 2 & \text{if } n = -4 \\ 1 & \text{otherwise} \end{cases}$$

Class number relations

- L. Kronecker (1860), A. Hurwitz (1884):

$$\sum_{s\in\mathbb{Z}} H(4n-s^2) + 2\lambda_1(n) = 2\sigma_1(n)$$

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- M. Eichler, A. Selberg (1955/56):

$$\sum_{s\in\mathbb{Z}}(s^2-n)H(4n-s^2)+2\lambda_3(n)=0$$

$$\sum_{s\in\mathbb{Z}}(s^4-3ns^2+n^2)H(4n-s^2)+2\lambda_5(n)=0$$

$$\ldots$$

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Harmonic Maaß forms

Definition

A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maaß form of weight $k \in \frac{1}{2}\mathbb{Z}$, level $N \in \mathbb{N}$, and character χ modulo N (with $4 \mid N$ if $k \notin \mathbb{Z}$) if it fulfills the following properties:

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The vector space of harmonic weak Maaß forms of weight k, level N, and character χ is denoted by $\mathcal{H}_k(N, \chi)$.

Splitting

Lemma

Let f be a harmonic weak Maaß form of weight $k \neq 1$. Then f has a canonical splitting into

$$f(\tau) = f^{+}(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^{-}(0)} + f^{-}(\tau),$$

where for some $m_0, n_0 \in \mathbb{Z}$ we have the Fourier expansions

$$f^{+}(\tau) = \sum_{m=m_0}^{\infty} c_f^{+}(m) q^m$$

and

$$f^{-}(\tau) = \sum_{\substack{n=n_0 \\ n \neq 0}}^{\infty} \overline{c_f^{-}(n)} n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n}.$$

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- Let f be a mock modular form of weight k and g a modular form of weight ℓ. Then $f \cdot g$ is called a mixed mock modular form of weight (k, ℓ).

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- Let f be a mock modular form of weight k and g a modular form of weight ℓ. Then $f \cdot g$ is called a mixed mock modular form of weight (k, ℓ).
- The νth Rankin-Cohen bracket of f and g is called a mixed mock modular form of degree ν.

The shadow

Proposition

For $k \neq 1$, the mapping

$$\xi_k : \mathcal{H}_k(N,\chi) \to M^!_{2-k}(N,\overline{\chi}),\ f \mapsto 2iy^k \overline{\frac{\partial}{\partial \overline{\tau}} f}$$

is welldefined and surjective with kernel $M^!_k(N,\chi)$. Moreover, for $f \in \mathcal{H}_k(N,\chi)$, we have that

$$(\xi_k f)(\tau) = (4\pi)^{1-k} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.$$

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- $$\mathcal{M}_k(\Gamma,\chi) = \xi_k^{-1}(M_{2-k}(\Gamma,\overline{\chi}))$$
  $$\mathcal{S}_k(\Gamma,\chi) = \xi_k^{-1}(S_{2-k}(\Gamma,\overline{\chi})).$$

An example

Theorem (D. Zagier, 1976)

Let $\mathscr{H}(\tau) = \sum_{n=0}^{\infty} H(n) q^n$ and

$$\mathscr{R}(\tau) := \frac{1+i}{16\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\vartheta(z)}{(z+\tau)^{\frac{3}{2}}} dz = \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}; 4\pi n^2 y\right) q^{-n^2}.$$

Then the function $\widehat{\mathscr{H}} = \mathscr{H} + \mathscr{R}$ is a harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0(4)$ with shadow $\xi\widehat{\mathscr{H}} = \frac{1}{8\sqrt{\pi}}\vartheta$. In particular, $\mathscr{H}$ is a mock modular form of weight $\frac{3}{2}$.

Appell-Lerch sums

Definition

Let $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z} \oplus \mathbb{Z}\tau)$.
The Appell-Lerch sum of level 1 is then the expression

$$A_1(u,v;\tau) = e^{\pi i u} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.$$

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- Zweger's thesis: Appell-Lerch sums are one of three ways to realize Ramanujan's mock theta functions
- the others: indefinite theta functions and Fourier coefficients of meromorphic Jacobi forms

Appell-Lerch sums

Definition

The completion of the Appell-Lerch sum $A_1(u,v;\tau)$ is given by

$$\widehat{A}_1(u,v;\tau) = A_1(u,v;\tau) + \frac{i}{2}\Theta(v;\tau)R(u-v;\tau),$$

with

$$R(u;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{ \operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im} u}{y}\right)\sqrt{2y}\right)\right\} \times (-1)^{\nu-\frac{1}{2}} q^{-\frac{\nu^2}{2}} e^{-2\pi i\nu u}$$

$$E(t) := 2\int_0^t e^{-\pi u^2}\,du, \qquad \Theta(v;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\frac{\nu^2}{2}} e^{2\pi i\nu(v+\frac{1}{2})}$$

A real-analytic Jacobi form

Theorem (S. Zwegers, 2002)

$\widehat{A}_1$ transforms like a Jacobi form of weight 1 and index $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$:

1. $\widehat{A}_1(-u,-v) = -\widehat{A}_1(u,v)$.

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2. For $\lambda_i, \mu_i \in \mathbb{Z}$:

$$\widehat{A}_1(u+\lambda_1\tau+\mu_1, v+\lambda_2\tau+\mu_2) = (-1)^{(\lambda_1+\mu_1)} e^{2\pi i((\lambda_1-\lambda_2)u-\lambda_1 v)} q^{\frac{\lambda_1^2}{2}-\lambda_1\lambda_2} \widehat{A}_1(u,v)$$

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3. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

$$\widehat{A}_1\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)e^{\pi i c\frac{-u^2+2uv}{c\tau+d}} \widehat{A}_1(u,v;\tau).$$

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A generalized function

From Dirichlet's class number formula we generalize for $r \in \mathbb{N}$

$$h_r(n) = \begin{cases} (-1)^{\lfloor r/2 \rfloor} \frac{(r-1)! n^{r-1/2}}{2^{r-1}\pi^r} L(r, \chi_{(-1)^r n}) & \text{if } n > 0 \text{ and} \\ & (-1)^r n \equiv 0, 1\ (4) \\ 0 & \text{otherwise} \end{cases}$$

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N.B.: $h_1(n) = h(-n)$, $H_1(n) = H(n)$ and $\mathscr{H}_1(n) = \mathscr{H}(n)$

Theorem (H. Cohen, 1975)

For $r \geq 2$, the function $\mathscr{H}_r$ is a (holomorphic) modular form of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$.

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- Results into many interesting identities for special values of L-functions.
- For $r = 1$ the Eisenstein series don't converge $\rightsquigarrow$ Hecke's trick and analytic continuation yield proof af Zagier's theorem.

Conjecture

Conjecture (H. Cohen, 1975

The coefficient of X^ν in

$$S_4^1(\tau, X) := \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \left[\sum_{s\in\mathbb{Z}} \frac{H(n-s^2)}{1-2sX+nX^2} + \sum_{k=0}^{\infty} \lambda_{2k+1}(n) X^{2k} \right] q^n$$

is a (holomorphic) modular form of weight $2\nu + 2$ on $\Gamma_0(4)$.

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- $\nu = 0$ yields Eichler's class number relation.

New class number relations

Corollary

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$$=-\frac{1}{12}\sum_{n=x^2+y^2+z^2+t^2}\left(x^4-6x^2y^2+y^4\right)$$

$$\sum_{s\in\mathbb{Z}}\left(64s^6-80s^4n+24s^2n^2-n^3\right)H\left(n-s^2\right)+\lambda_7\left(n\right)$$

$$=-\frac{1}{3}\sum_{n=x^2+y^2+z^2+t^2}\left(x^6-5x^4y^2-10x^4z^2+30x^2y^2z^2+5x^2z^4-5y^2z^4\right)$$

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$$= -\frac{1}{3} \sum_{n=x^2+y^2+z^2+t^2} \left(x^6 - 5x^4y^2 - 10x^4z^2 + 30x^2y^2z^2 + 5x^2z^4 - 5y^2z^4\right)$$

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Solution

Theorem 1 (M., 2013)

Cohen's conjecture is true. Moreover, $\operatorname{coeff}_{X^{2\nu}}(S_4^1(\tau; X))$ is a cusp form for $\nu > 0$.

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Recall Rankin-Cohen brackets: f, g modular of weights k, ℓ

$$[f,g]_\nu := \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu} D^\mu f D^{\nu-\mu} g.$$

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$$\operatorname{coeff}_{X^{2\nu}}(S_4^1(\tau;X)) = \frac{c_\nu}{2} \left([\mathscr{H}, \vartheta]_\nu(\tau) - [\mathscr{H}, \vartheta]_\nu \left(\tau + \tfrac{1}{2}\right)\right) + \Lambda_{2\nu+1,odd}(\tau),$$

$$c_\nu = \nu! \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})}$$

Some Lemmas

Lemma

$$\Lambda_{2\nu+1,odd} = \frac{1}{2}\left(D_v^{2\nu+1} A_1^{odd}\right)\left(0, \tau + \tfrac{1}{2}; 2\tau\right),$$

where

$$A_1^{odd}(u, v; \tau) = \frac{1}{2}\left(A_1(u, v; \tau) - A_1\left(u, v + \frac{1}{2}; \tau\right)\right).$$

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Lemma

The R-function completing the Appell-Lerch sums is in the heat kernel, i.e.

$$D_u^2 R = -2D_\tau R.$$

Outline of the proof

- Complete $[\mathscr{H}, \vartheta]_\nu$ and $\Lambda_{2\nu+1,odd}$ to transform like modular forms

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  - $\nu \mapsto \nu + 1$: Use heat kernel property of R

Remark

Let $T_n^{(k)}$ be the nth Hecke operator on $S_k(\Gamma_0(4))$. Then

$$\operatorname{coeff}_{X^{2\nu}}(S_4^1(\tau; X)) = -3 \sum_{n\ odd} \operatorname{trace}\left(T_n^{(2\nu+2)}\right) q^n$$

(not proved by this method)

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where $\delta, \varepsilon > 0$, and $2 < k \in \frac{1}{2}\mathbb{Z}$.
Then we define the holomorphic projection of f by

$$(\pi_{hol} f)(\tau) := (\pi_{hol}^k f)(\tau) := \sum_{n=0}^{\infty} c(n) q^n,$$

with

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_f(n, y) e^{-4\pi n y} y^{k-2} dy, \qquad n > 0.$$

Properties

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- The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).
- If f is modular of weight $k > 2$ on $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ then we have

$$\langle f, g \rangle = \langle \pi_{hol}(f), g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson scalar product, for every cusp form $g \in S_k(\Gamma)$.

Rankin-Cohen brackets

From now on: $f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau) \in \mathcal{M}_k(\Gamma)$, $g \in M_\ell(\Gamma)$, $\nu > 0$ s.t. $[f,g]_\nu$ satisfies conditions in definition.

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Lemma

$$\frac{(4\pi)^{1-k}}{k-1}\overline{c_f^-(0)}\pi_{hol}([y^{1-k},g]_\nu) = \kappa\overline{c_f^-(0)}\sum_{n=0}^{\infty} n^{k+\nu-1}a_g(n)q^n,$$

with

$$\kappa = \kappa(k,\ell,\nu) = \frac{1}{(k+\ell+2\nu-2)!(k-1)}\sum_{\mu=0}^{\nu}\left[\frac{\Gamma(2-k)\Gamma(\ell+2\nu-\mu)}{\Gamma(2-k-\mu)} \times\binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu}\right].$$

Rankin-Cohen brackets

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$$P_{a,b}(X,Y) := \sum_{j=0}^{a-2} \binom{j+b-2}{j} X^j (X+Y)^{a-j-2}.$$

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Theorem

$$\pi_{hol}([f^-, g]_\nu) = \sum_{r=1}^{\infty} b(r) q^r,$$

where

$$b(r) = -\Gamma(1-k) \sum_{m-n=r} \sum_{\mu=0}^{\nu} \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} m^{\nu-\mu} a_g(m) \overline{c_f^-(n)}$$
$$\times \left(m^{\mu-2\nu-\ell+1} P_{k+\ell+2\nu, 2-k-\mu}(r,n) - n^{k+\mu-1} \right)$$

Properities of $P_{a,b}(X,Y)$

Lemma

For $b \neq 1, 2$, the polynomial $P_{a,b}(X,Y)$ satisfies

$$P_{a,b}(X,Y) = \sum_{j=0}^{a-2} \binom{a+b-3}{j} X^j Y^{a-2-j}$$
$$= \sum_{j=0}^{a-2} \binom{a+b-3}{a-2-j} \binom{j+b-2}{j} (X+Y)^{a-2-j} (-Y)^j.$$

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The Theorem of Serre-Stark

Theorem (J.-P. Serre, H. Stark, 1977)

Let φ be a modular form of weight $\frac{1}{2}$ on $\Gamma_1(N)$. Then φ is a unique linear combination of the theta series

$$\vartheta_{\chi,t}(\tau) = \sum_{n\in\mathbb{Z}} \chi(n) q^{tn^2}$$

with χ a primitive even character with conductor $F(\chi)$ and $t \in \mathbb{N}$ such that $4F(\chi)^2 t \mid N$.

Proposition

Let $r = m - n$. Then it holds that

$$\sum_{\mu=0}^{\nu} \binom{\nu+\frac{1}{2}}{\nu-\mu} \binom{\nu-\frac{1}{2}}{\mu} \left(m^{\frac{1}{2}-\nu} P_{2\nu+2,\frac{1}{2}-\mu}(r,n) - n^{\frac{1}{2}+\mu} m^{\nu-\mu} \right)$$

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$$= 2^{-2\nu} \binom{2\nu}{\nu} (m^{\frac{1}{2}} - n^{\frac{1}{2}})^{2\nu+1}.$$

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$$\sum_{\mu=0}^{\nu} \binom{\nu-\frac{1}{2}}{\nu-\mu} \binom{\nu+\frac{1}{2}}{\mu} \left(m^{-\nu-\frac{1}{2}} P_{2\nu+2,\frac{3}{2}-\mu}(r,n) - n^{\mu-\frac{1}{2}} m^{\nu-\mu} \right)$$

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$$= - 2^{-2\nu} \binom{2\nu}{\nu} (mn)^{-\frac{1}{2}} \left(m^{\frac{1}{2}} - n^{\frac{1}{2}} \right)^{2\nu+1}.$$

Result in weight $(\frac{3}{2}, \frac{1}{2})$

Theorem 2 (M., 2013)

Let $f \in \mathcal{M}_{\frac{3}{2}}(\Gamma)$ and $g \in M_{\frac{1}{2}}(\Gamma)$ with $\Gamma = \Gamma_1(4N)$ for some $N \in \mathbb{N}$ and fix $\nu \in \mathbb{N}$. Then there is a finite linear combination $L_\nu^{f,g}$ of functions of the form

$$\Lambda_{s,t}^{\chi,\psi}(\tau;\nu) = \sum_{r=1}^{\infty} \left(2 \sum_{\substack{sm^2 - tn^2 = r \\ m,n \geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m - \sqrt{t}n)^{2\nu+1} \right) q^r + \overline{\psi(0)} \sum_{r=1}^{\infty} \chi(r)(\sqrt{s}r)^{2\nu+1} q^{sr^2}$$

with $s, t \in \mathbb{N}$ and χ, ψ are even characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that $[f,g]_\nu + L_\nu^{f,g}$ is a holomorphic cusp form of weight $2\nu + 2$.

Result for mock theta functions

Theorem 3 (M., 2013)

Let $f \in \mathcal{S}_{\frac{1}{2}}(\Gamma)$ be a completed mock theta function and $g \in S^{\theta}_{\frac{3}{2}}(\Gamma)$, where $\Gamma = \Gamma_1(4N)$ for some $N \in \mathbb{N}$ and let ν be a fixed non-negative integer. Then there is a finite linear combination $D^{f,g}_{\nu}$ of functions of the form

$$\Delta^{\chi,\psi}_{s,t}(\tau;\nu) = 2\sum_{r=1}^{\infty}\left(\sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m-\sqrt{t}n)^{2\nu+1}\right)q^r,$$

where $s,t \in \mathbb{N}$ and χ,ψ are odd characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that $[f,g]_\nu + D^{f,g}_\nu$ is a holomorphic cusp form of weight $2\nu+2$.

Generating systems

Corollary

With the notation from the theorem the following is true.
The equivalence classes $\Lambda_{s,t}^{\chi,\psi} + M_{2\nu+2}^!(\Gamma_1(N))$ generate the $\mathbb{C}$-vector space

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- in general, minimal divisor power sums in a real-(bi)quadratic number field ($\rightsquigarrow$ generalized Appell-Lerch sums?)

Example: Eichler-Selberg and Cohen's conjecture

- Recall: $\mathscr{H}(\tau)=\sum_{n=0}^{\infty}H(n)q^n\in\mathcal{M}_{\frac{3}{2}}^{mock}(\Gamma_0(4))$ with shadow $\frac{1}{8\sqrt{\pi}}\vartheta$.

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- From Theorem 2: For $\nu > 0$,

$$\pi_{hol}([\widehat{\mathscr{H}}, \vartheta]_\nu)(\tau) = [\mathscr{H}, \vartheta]_\nu(\tau) + 2^{-2\nu-1}\binom{2\nu}{\nu}\Lambda'(\tau) \in S_{2\nu+2}(\Gamma_0(4)),$$

where $\Lambda' = \Lambda_{1,1}^{1,1}$.

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where $\Lambda' = \Lambda_{1,1}^{1,1}$.

- easy to see

$$(\Lambda|U(4))(\tau;\nu) = 2^{2\nu+1}\sum_{n=1}^{\infty} 2\lambda_{2\nu+1}(n)q^n$$

$$(\Lambda|S_{2,1})(\tau;\nu) = 2\sum_{n \text{ odd}} \lambda_{2\nu+1}(n)q^n.$$

Example: Eichler-Selberg and Cohen's conjecture

- Rewrite:

$$\sum_{n=1}^{\infty}\left(\sum_{s\in\mathbb{Z}} g_\nu^{(1)}(s,n)H(4n-s^2)\right)q^n + 2\sum_{n=0}^{\infty}\lambda_{2\nu+1}(n)q^n \in S_{2\nu+2}(\Gamma_0(1))$$

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- Rankin-Selberg unfolding trick (f normalized Hecke eigenform):

$$\langle[\widehat{\mathscr{H}},\vartheta]_\nu, f\rangle = \langle\pi_{hol}([\widehat{\mathscr{H}},\vartheta]_\nu), f\rangle \doteq \langle f, f\rangle$$

$\Rightarrow$ trace formulae for $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(4)$.

Thank you for your attention.