Robustness of spatial preferential attachment networks

Emmanuel Jacob and Peter Mörters

[1] École Normale Supérieure de Lyon
[2] University of Bath

Abstract. We study robustness under random attack for a class of networks, in which new nodes are given a spatial position and connect to existing vertices with a probability favouring short spatial distances and high degrees. In this model of a scale-free network with clustering one can independently tune the power law exponent $\tau > 2$ of the degree distribution and a parameter $\delta > 1$ determining the decay rate of the probability of long edges. We argue that the network is robust if $\tau < 2 + \frac{1}{\delta}$, but fails to be robust if $\tau > 2 + \frac{1}{\delta-1}$. Hence robustness depends not only on the power-law exponent but also on the clustering features of the network.

Keywords: Scale-free network, Barabasi-Albert model, preferential attachment, geometric random graph, power law, clustering, robustness, giant component, resilience.

1 Introduction

Scientific, technological or social systems can often be described as complex networks of interacting components. Many of these networks have been empirically found to have strikingly similar topologies, shared features being that they are *scale-free*, i.e. the degree distribution follows a power law, *small worlds*, i.e. the typical distance of nodes is logarithmic or doubly logarithmic in the network size, or *robust*, i.e. the network topology is qualitatively unchanged if an arbitrarily large proportion of nodes chosen at random is removed from the network. Barabási and Albert [2] therefore concluded fifteen years ago 'that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of the individual systems.' They suggested a model of a growing family of graphs, in which new vertices are added successively and connected to vertices in the existing graph with a probability proportional to their degree, and a few years later these features were rigorously verified in the work of Bollobás and Riordan, see [8],[5],[6].

A characteristic feature present in most real networks that is not picked up by preferential attachment is that of *clustering*, the formation of clusters of nodes with an edge density significantly higher than in the overall network. A natural way to integrate this feature in the model is by giving every node an individual feature and implementing a preference for edges connecting vertices with similar features. This is usually done by spatial positioning of nodes and rewarding short edges, see for example [17], [1], [21]. Here we investigate a model, introduced in [19], which is a generalisation of the model of Aiello et al. [1]. It is defined as a growing family of graphs in which a new vertex gets a randomly allocated spatial position on the torus. This vertex then connects to every vertex in the existing graph independently, with a probability which is a decreasing function of the spatial distance of the vertices, the time, and the inverse of the degree of the vertex. The relevance of this *spatial preferential attachment model* lies in the fact that, while it is still a scale-free network governed by a simple rule of self-organisation, it has been shown to exhibit clustering. The present paper investigates the problem of robustness.

In mathematical terms, we call a growing family of graphs *robust* if the critical parameter for vertex percolation is zero, which means that whenever vertices are deleted independently at random from the graph with a positive retention probability, a connected component comprising an asymptotically positive proportion of vertices remains. For several scale-free models, including non-spatial preferential attachment networks, it has been shown that the transition between robust and non-robust behaviour occurs when the power law exponent τ crosses the value three, see for example [5], [14]. Robustness in scale-free networks relies on the presence of a hierarchically organised core of vertices with extremely high degrees, such that every vertex is connected to the next higher layer by a small number of edges, see for example [22]. Our analysis of the spatial model shows that, if $\tau < 3$, whether vertices in the core are sufficiently close in the graph distance to the next higher layer depends critically on the speed at which the connection probability decreases with spatial distance, and hence depending on this speed robustness may hold or fail. The phase transition between robustness and non-robustness therefore occurs at value of τ strictly smaller than three.

The main structural difference between the spatial and classical model of preferential attachment is that the former exhibits *clustering*. Mathematically this is measured in terms of a positive clustering coefficient, meaning that, starting from a randomly chosen vertex, and following two different edges, the probability that the two end vertices of these edges are connected remains positive as the graph size is growing. This implies in particular that local neighbourhoods of typical vertices in the spatial network do not look like trees. However, the main ingredient in almost every mathematical analysis of scale-free networks so far has been the approximation of these neighbourhoods by suitable random trees, see [7], [13], [4], [16]. As a result, the analysis of spatial preferential attachment models requires a range of entirely new methods, which allow to study the robustness of networks without relying on the local tree structure that turned out to be so useful in the past.

2 The model

While spatial preferential attachment models may be defined in a variety of metric spaces, we focus here on homogeneous space represented by a one-dimensional torus of unit volume, given as $\mathbb{T}_1 = (-1/2, 1/2]$ with the endpoints identified. We use d_1 to denote the torus metric. Let $\mathcal{X}$ denote a homogeneous Poisson point process of finite intensity $\lambda > 0$ on $\mathbb{T}_1 \times (0, \infty)$. A point $\mathbf{x} = (x, s)$ in $\mathcal{X}$ is a vertex $\mathbf{x}$, born at time s and placed at position x. Observe that, almost surely, two points of $\mathcal{X}$ neither have the same birth time nor the same position. We say that (x, s) is *older* than (y, t) if $s < t$. For $t > 0$, write $\mathcal{X}_t$ for $\mathcal{X} \cap (\mathbb{T}_1 \times (0, t])$, the set of vertices already born at time t.

We construct a growing sequence of graphs $(G_t)_{t>0}$, starting from the empty graph, and adding successively the vertices in $\mathcal{X}$ when they are born, so that the vertex set of G_t equals $\mathcal{X}_t$. Given the graph G_{t-} at the time of birth of a vertex $\mathbf{y} = (y, t)$, we connect $\mathbf{y}$, independently of everything else, to each vertex $\mathbf{x} = (x, s) \in G_{t-}$, with probability

$$\varphi\left(\frac{t}{f(Z(\mathbf{x}, t-))}\mathrm{d}_1(x, y)\right), \tag{1}$$

where $Z(\mathbf{x}, t-)$ is the *indegree* of vertex $\mathbf{x}$, defined as the total number of edges between $\mathbf{x}$ and younger vertices, at time $t-$. The model parameters in (1) are the *attachment rule* $f\colon \mathbb{N} \cup \{0\} \to (0, \infty)$, which is a nondecreasing function regulating the strength of the preferential attachment, and the *profile function* $\varphi\colon [0, \infty) \to (0, 1)$, which is an integrable nonincreasing function regulating the decay of the connection probability in terms of the interpoint distance. The connection probabilities in (1) may look arcane at a first glance, but are in fact completely natural. To ensure that the probability of a new vertex connecting to its nearest neighbour does not degenerate, as $t \uparrow \infty$, it is necessary to scale $\mathrm{d}_1(x, y)$ by $1/t$, which is the order of the distance of a point to its nearest neighbour at time t. The linear dependence of the argument of φ on time ensures that the expected number of edges connecting a new vertex to vertices of bounded degree remains bounded from zero and infinity, as $t \uparrow \infty$, as long as $x \mapsto \varphi(|x|)$ is integrable.

The model parameters λ, f and φ are not independent. If $\int \varphi(|x|)\, dx = \mu > 0$, we can modify φ to $\varphi \circ (\mu\, \mathrm{Id})$ and f to μf, so that the connection probabilities remain unchanged and

$$\int \varphi(|x|)\, dx = 1. \tag{2}$$

Similarly, if the intensity of the Poisson point process $\mathcal{X}$ is $\lambda > 0$, we can replace $\mathcal{X}$ by $\{(x, \lambda s)\colon (x, s) \in \mathcal{X}\}$ and f by λf, so that again the connection probabilities are unchanged and we get a Poisson point process of unit intensity. From now on we will assume that both of these normalisation conventions are in place. Under these assumptions the regime for the attachment rule f which leads to power law degree distributions is characterised by asymptotic linearity, i.e.

$$\lim_{k \uparrow \infty} \frac{f(k)}{k} = \gamma,$$

for some $\gamma > 0$. We henceforth assume asymptotic linearity with the additional constraint that $\gamma < 1$, which excludes cases with infinite mean degrees.

We finally assume that the profile function φ is either regularly varying at infinity with index $-\delta$, for some $\delta > 1$, or φ decays quicker than any regularly varying function. In the latter case we set $\delta = \infty$. Intuitively, the bigger δ, the stronger the clustering in the network. See Figures 1 and 2 for simulations of the spatial preferential attachment network indicative of the parameter dependence.

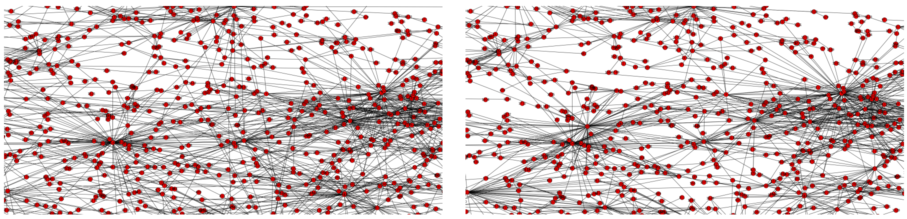

Fig. 1. Simulations of the network for the two-dimensional torus, based on the same realisation of the Poisson process, with parameters $\gamma = 0.75$ and $\delta = 2.5$ (left) and $\delta = 5$ (right). Both networks have the same edge density, but the one with larger δ shows more pronounced clustering. The pictures zoom into a typical part of the torus.

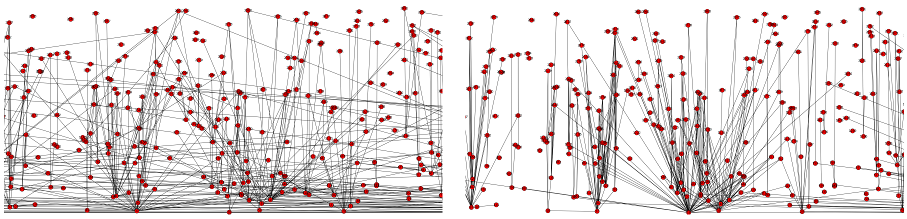

Fig. 2. Simulations of the network for the one-dimensional torus, the vertical axis indicating birth time of the nodes. Parameters are $\gamma = 0.75$ and $\delta = 2$ (left), resp. $\delta = 5$ (right) and both networks have the same edge density and power law exponent. Our results show that the network on the left is robust, the one on the right is not.

A similar spatial preferential attachment model was introduced in [1] and studied further in [20], [10]. There it is assumed that the profile functions has bounded support, more precisely $\varphi = p\mathbb{1}_{[0,r]}$, for $p \in (0, 1]$ and r satisfying (2). This choice, roughly corresponding to the boundary case $\delta \uparrow \infty$, is too restrictive for the problems we study in this paper, as it turns out that robustness does not hold for any value of τ. Other spatial models with a phase transition between a robust and a non-robust phase are the scale-free percolation model of Deijfen et al. [11], and the Chung-Lu model in hyperbolic space, discussed in Candellero and Fountoulakis [9]. In both cases the transition happens when the power law exponent of the degree distribution crosses the value 3.

Local properties of the spatial preferential attachment model were studied in [19], where this model was first introduced. It is shown there that

– The *empirical degree distribution* of G_t converges in probability to a deterministic limit μ. The probability measure μ on $\{0\} \cup \mathbb{N}$ satisfies

$$\mu(k) = k^{-(1+\frac{1}{\gamma})+o(1)} \qquad \text{as} \quad k \uparrow \infty.$$

The network $(G_t)_{t>0}$ is *scale-free* with power-law exponent $\tau = 1 + \frac{1}{\gamma}$, which can be tuned to take any value $\tau > 2$. See [19, Theorem 1 and 2].

– The average over all vertices $v \in G_t$ of the empirical local clustering coefficient at v, defined as the proportion of pairs of neighbours of v which are themselves connected by an edge in G_t, converges in probability to a positive constant $c_\infty^{\text{av}} > 0$, called the *average clustering coefficient.* In other words the network $(G_t)_{t>0}$ exhibits *clustering.* See [19, Theorem 3].

3 Statement of the result

Recall that the number of vertices of the graphs G_t, $t > 0$, form a Poisson process of unit intensity, and is therefore almost surely equivalent to t as $t \uparrow \infty$. Let $C_t \subset G_t$ be the largest connected component in G_t and denote by $|C_t|$ its size. We say that the network has a *giant component* if C_t is of linear size or, more precisely, if

$$\lim_{\varepsilon \downarrow 0} \limsup_{t\to\infty} \mathbb{P}\left(\frac{|C_t|}{t} \le \varepsilon\right) = 0;$$

and it has *no giant component* if C_t has sublinear size or, more precisely, if

$$\liminf_{t\to\infty} \mathbb{P}\left(\frac{|C_t|}{t} \le \varepsilon\right) = 1 \text{ for any } \varepsilon > 0.$$

If G is a graph with vertex set $\mathcal{X}$, and $p \in (0,1)$, we write ${}^pG$ for the random subgraph of G obtained by Bernoulli percolation with retention parameter p on the vertices of G. We also use ${}^p\mathcal{X}$ for set of vertices surviving percolation. The network $(G_t)_{t>0}$ is said to be *robust* if, for any fixed $p \in (0,1]$, the network $({}^pG_t)_{t>0}$ has a giant component and *non-robust* if there exists $p \in (0,1]$ so that $({}^pG_t)_{t>0}$ has no giant component.

Theorem 1. *The spatial preferential attachment network* $(G_t)_{t>0}$ *is*

(a) robust *if* $\gamma > \frac{\delta}{1+\delta}$ *or, equivalently, if* $\tau < 2 + \frac{1}{\delta}$*;*
(b) non-robust *if* $\gamma < \frac{\delta-1}{\delta}$ *or, equivalently, if* $\tau > 2 + \frac{1}{\delta-1}$*.*

Remark 1 The network is also non-robust if $\gamma < \frac{1}{2}$ or, equivalently, if $\tau > 3$. But the surprising result here is that for $\delta > 2$ the transition between the two phases occurs at a value strictly below 3. This phenomenon is new and due to the clustering structure in the network. It offers a new perspective on the 'classical' results on network models without clustering.

Remark 2

- We conjecture that the result in (a) is sharp, i.e. nonrobustness occurs if $\gamma < \frac{\delta}{1+\delta}$. If this holds, the critical value for τ equals $2 + \frac{1}{\delta}$. Our proof techniques currently do not allow to prove this.
- Our approach also provides heuristics indicating that in the robust phase $\delta(\tau - 2) < 1$ the typical distances in the robust giant component are asymptotically
$$(4 + o(1)) \frac{\log \log t}{-\log(\delta(\tau - 2))},$$
namely doubly logarithmic, just as in some nonspatial preferential attachment models. The constant coincides with that of the nonspatial models in the limiting case $\delta \downarrow 1$, see [15],[12], and goes to infinity as $\delta(\tau - 2) \to 1$. It is an interesting open problem to confirm these heuristics rigorously.

4 Proof ideas and strategies

Before describing the strategies of our proofs, we briefly summarise the techniques developed in [19] in order to describe the local neighbourhoods of typical vertices by a limit model.

Canonical representation We first describe a canonical representation of our network $(G_t)_{t>0}$. To this end, let $\mathcal{X}$ be a Poisson process of unit intensity on $\mathbb{T}_1 \times (0, \infty)$, and endow the point process $\mathcal{X} \times \mathcal{X}$ with independent marks which are uniformly distributed on $[0, 1]$. We denote these marks by $\mathcal{V}_{\mathbf{x},\mathbf{y}}$ or $\mathcal{V}(\mathbf{x}, \mathbf{y})$, for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. If $\mathcal{Y} \subset \mathbb{T}_1 \times (0, \infty)$ is a finite set and $\mathcal{W}\colon \mathcal{Y} \times \mathcal{Y} \to [0, 1]$ a map, we define a graph $G^1(\mathcal{Y}, \mathcal{W})$ with vertex set $\mathcal{Y}$ by establishing edges in order of age of the younger endvertex. An edge between $\mathbf{x} = (x, t)$ and $\mathbf{y} = (y, s)$, $t < s$, is present if and only if
$$\mathcal{W}(\mathbf{x}, \mathbf{y}) \le \varphi\left(\frac{s\, \mathrm{d}_1(x, y)}{f(Z(\mathbf{x}, s-))}\right), \tag{3}$$
where $Z(\mathbf{x}, s-)$ is the indegree of $\mathbf{x}$ at time $s-$. A realization of $\mathcal{X}$ and $\mathcal{V}$ then gives rise to the family of graphs $(G_t)_{t>0}$ with vertex sets $\mathcal{X}_t = \mathcal{X} \cap (\mathbb{T}_1 \times (0, t])$, given by $G_t = G^1(\mathcal{X}_t, \mathcal{V})$, which has the distribution of the spatial preferential attachment network.

Space-time rescaling The construction above can be generalised in a straightforward manner from $\mathbb{T}_1$ to the torus of volume t, namely $\mathbb{T}_t = (-\frac{1}{2}\, t, \frac{1}{2}\, t]$, equipped with its canonical torus metric d_t. The resulting functional, mapping a finite subset $\mathcal{Y} \subset \mathbb{T}_t \times (0, \infty)$ and a map from $\mathcal{Y} \times \mathcal{Y} \to [0, 1]$ onto a graph, is now denoted by G^t. We introduce the *rescaling mapping*
$$\begin{aligned} h_t : \mathbb{T}_1 \times (0, t] &\to \mathbb{T}_t \times (0, 1], \\ (x, s) &\mapsto (tx, s/t) \end{aligned}$$

which expands the space by a factor t, the time by a factor $1/t$. The mapping h_t operates on the set $\mathcal{X}$, but also on $\mathcal{V}$, by $h_t(\mathcal{V})_{h_t(\mathbf{x}),h_t(\mathbf{y})} := \mathcal{V}_{\mathbf{x},\mathbf{y}}$. The operation of h_t preserves the rule (3), and it is therefore simple to verify that we have

$$G^t(h_t(\mathcal{X}_t), h_t(\mathcal{V})) = h_t(G^1(\mathcal{X}_t, \mathcal{V})) = h_t(G_t),$$

that is, it is the same to construct the graph and then rescale the picture, or to first rescale the picture, then construct the graph on this rescaled picture. Observe also that $h_t(\mathcal{X}_t)$ is a Poisson point process of intensity 1 on $\mathbb{T}_t \times (0,1]$, while $h_t(\mathcal{V})$ are independent marks attached to the points of $h_t(\mathcal{X}_t) \times h_t(\mathcal{X}_t)$ which are uniformly distributed on $[0,1]$.

Convergence to the limit model We now denote by $\mathcal{X}$ a Poisson point process with unit intensity on $\mathbb{R} \times (0,1]$, and endow the points of $\mathcal{X} \times \mathcal{X}$ with independent marks $\mathcal{V}$, which are uniformly distributed on $[0,1]$. For each $t > 0$, identify $(-\frac{1}{2}\,t, \frac{1}{2}\,t]$ and $\mathbb{T}_t$, and write $\mathcal{X}^t$ for the restriction of $\mathcal{X}$ to $\mathbb{T}_t \times (0,1]$, and $\mathcal{V}^t$ for the restriction of $\mathcal{V}$ to $\mathcal{X}^t \times \mathcal{X}^t$. In the following, we write G^t or $G^t(\mathcal{X},\mathcal{V})$ for $G^t(\mathcal{X}^t,\mathcal{V}^t)$. We have seen that for fixed $t \in (0,\infty)$, the graphs G^t and $h_t(G_t)$ have the same law. Thus any results of robustness we prove for the network $(G^t)_{t>0}$ also hold for the network $(G_t)_{t>0}$. It was shown in [19, Proposition 5] that, almost surely, the graphs G^t converge to a locally finite graph $G^\infty = G^\infty(\mathcal{X},\mathcal{V})$, in the sense that the neighbours of any given vertex $\mathbf{x} \in \mathcal{X}$ coincide in G^t and in G^∞, if t is large enough. It is important to note the fundamentally different behaviour of the processes $(G^t)_{t>0}$ and $(G_t)_{t>0}$. While in the former the degree of any fixed vertex stabilizes, in the latter the degree of any fixed vertex goes to ∞, as $t \uparrow \infty$. We will exploit the convergence of G^t to G^∞ in order to decide the robustness of the finite graphs G^t, and ultimately G_t, from properties of the limit model G^∞.

Law of large numbers We now state a limit theorem for the graphs ${}^pG^t$ centred in a randomly chosen point. To this end we denote by ${}^p\mathbb{P}$ the law of $\mathcal{X},\mathcal{V}$ together with independent Bernoulli percolation with retention parameter p on the points of $\mathcal{X}$. For any $\mathbf{x} \in \mathbb{R} \times (0,1]$ we denote by ${}^p\mathbb{P}_\mathbf{x}$ the *Palm measure*, i.e. the law ${}^p\mathbb{P}$ conditioned on the event $\{\mathbf{x} \in {}^p\mathcal{X}\}$. Note that by elementary properties of the Poisson process this conditioning simply adds the point $\mathbf{x}$ to ${}^p\mathcal{X}$ and independent marks $\mathcal{V}_{\mathbf{x},\mathbf{y}}$ and $\mathcal{V}_{\mathbf{y},\mathbf{x}}$, for all $\mathbf{y} \in \mathcal{X}$, to $\mathcal{V}$. We also write ${}^p\mathbb{E}_\mathbf{x}$ for the expectation under ${}^p\mathbb{P}_\mathbf{x}$. Let $\xi = \xi\,(\mathbf{x}, G)$ be a bounded functional of a locally-finite graph G with vertices in $\mathbb{R} \times (0,1]$ and a vertex $\mathbf{x} \in G$, which is invariant under translations of $\mathbb{R}$. Also, let $\xi_t = \xi_t\,(\mathbf{x}, G)$ be a bounded family of functionals of a graph G with vertices in $\mathbb{T}_t \times (0,1]$ and a vertex $\mathbf{x} \in G$, invariant under translations of the torus. We assume that, for U an independent uniform random variable on $(0,1]$, we have that $\xi_t((0,U), {}^pG^t)$ converges to $\xi((0,U), {}^pG^\infty)$ in ${}^p\mathbb{P}_{(0,U)}$-probability. By [19, Theorem 7], in ${}^p\mathbb{P}$-probability,

$$\frac{1}{t} \sum_{\mathbf{x} \in {}^p\mathcal{X}^t} \xi_t\left(\mathbf{x}, {}^pG^t\right) \underset{t\to\infty}{\longrightarrow} p \int_0^1 {}^p\mathbb{E}_{(0,u)}[\xi((0,u), {}^pG^\infty)]\, du. \tag{4}$$

4.1 Robustness: strategy of proof

Existence of an infinite component in the limit model We first show that, under the assumptions that $\gamma > \frac{\delta}{1+\delta}$, or equivalently $\frac{\gamma}{\delta(1-\gamma)} > 1$, the percolated limit model ${}^pG^\infty$ has an infinite connected component. This uses the established strategy of the hierarchical core. Young vertices, born after time $\frac{1}{2}$, are called *connectors*. We find $\alpha > 1$ such that, starting from a sufficiently old vertex $\mathbf{x}_0 \in {}^pG^\infty$, we establish an infinite chain $(\mathbf{x}_k)_{k\geq 1}$ of vertices $\mathbf{x}_k = (x_k, s_k)$ such that $s_k < s_{k-1}^\alpha$, i.e. we move to increasingly older vertices, and $\mathbf{x}_{k-1}$ and $\mathbf{x}_k$ are connected by a path of length two, using a connector as a stepping stone. The following lemma is the key. Roughly speaking, we call a vertex born at time s *good* if its indegree at time $\frac{1}{2}$ is close to its expectation, i.e. of order $s^{-\gamma}$.

Lemma 1. *Choose first* $\alpha \in (1, \frac{\gamma}{\delta(1-\gamma)})$ *then* $\beta \in (\alpha, \frac{\gamma}{\delta}(1+\alpha\delta))$. *If x is a good vertex born at time s, then with very high probability there exists a good vertex y born before time s^α with $|x-y| < s^{-\beta}$ such that x and y are connected through a connector.*

Proof (Sketch).

- The existence of a good vertex y is easy because it just needs to be located in a box of sidelengths s^α and $2s^{-\beta}$, and $s^\alpha s^{-\beta} \to \infty$.
- At time $\frac{1}{2}$ the good vertex x has indegree of order $s^{-\gamma}$. The number of connectors at distance $\leq s^{-\gamma}$, which are connected to x is therefore stochastically bounded from below by a Poisson variable with intensity $s^{-\gamma}$.
- For each of these connectors the probability that they connect to a good y is at least
$$\varphi\Big(\frac{\frac{1}{2}d(x,y)}{s^{-\alpha\gamma}}\Big) \leq \text{cst.} s^{-\delta(\alpha\gamma-\beta)}.$$
We succeed because $-\gamma - \delta(\alpha\gamma-\beta) < 0$.

Transfer to finite graphs using the law of large numbers To infer robustness of the network $(G^t)_{t>0}$ from the behaviour of the limit model we use (4) on the functional $\xi_t(\mathbf{x}, G)$ defined as the indicator of the event that there is a path in G connecting $\mathbf{x}$ to the oldest vertex of G. We denote by $\xi(\mathbf{x}, G)$ the indicator of the event that the connected component of $\mathbf{x}$ is infinite and let

$${}^p\theta := \int_0^1 {}^p\mathbb{P}_{(0,u)}\big\{ \text{ the component of } (0,u) \text{ in } {}^pG^\infty \text{ is infinite } \big\}\, du. \tag{5}$$

If $\lim \xi_t((0,U), {}^pG^t) = \xi((0,U), {}^pG^\infty)$ in probability, then the law of large numbers (4) implies that $\lim(1/t)\sum_{\mathbf{x}\in {}^p\mathcal{X}^t} \xi_t(\mathbf{x}, {}^pG^t) = p\, {}^p\theta$. The sum is the number of vertices in ${}^pG^t$ connected to the oldest vertex, and we infer that this number grows linearly in t so that a giant component exists in $({}^pG^t)_{t>0}$. This implies that $(G^t)_{t>0}$ and hence $(G_t)_{t>0}$ is a robust network. However, while it is easy to see that $\limsup_{t\uparrow\infty} \xi_t((0,U), {}^pG^t) \leq \xi((0,U), {}^pG^\infty)$, checking that

$$\liminf_{t\uparrow\infty} \xi_t((0,U), {}^pG^t) \geq \xi((0,U), {}^pG^\infty), \tag{6}$$

is the difficult part of the argument.

The geometric argument The proof of (6) is the most technical part of the proof. We first look at the finite graph ${}^pG^t$ and establish the existence of a core of old and well-connected vertices, which includes the oldest vertex. Any pair of vertices in the core are connected by a path with a bounded number of edges, in particular all vertices of the core are in the same connected component. This part of the argument is similar to the construction in the limit model. We then use a simple continuity argument to establish that if the vertex $(0, U)$ is in an infinite component in the limit model, then it is also in an infinite component for the limit model based on a Poisson process $\mathcal{X}$ with a slightly reduced intensity. In the main step we show that under this assumption the vertex $(0, U)$ is connected in ${}^pG^t$ with reduced intensity to a moderately old vertex. In this step we have to rule out explicitly the possibilities that the infinite component of ${}^pG^\infty$ either avoids the set of eligible moderately old vertices, or connects to them only by a path which moves very far away from the origin. The latter argument requires good control over the length of edges in the component of $(0, U)$ in ${}^pG^\infty$. Once the main step is established, we can finally use the still unused vertices, which form a Poisson process with small but positive intensity, to connect the moderately old vertex we have found to the core by means of a classical sprinkling argument.

4.2 Non-robustness: strategy of proof

Using the limit model If $\gamma < \frac{1}{2}$ it is very plausible that the spatial preferential attachment network is non-robust, as the classical models with the same power-law exponents are non-robust [5], [14] and it is difficult to see how the spatial structure could help robustness. We have not been able to use this argument for a proof, though, as our model cannot be easily dominated by a non-spatial model with the same power-law exponent. Instead we use a direct approach, which turns out to yield non-robustness also in some cases where $\gamma > \frac{1}{2}$. The key is again the use of the limit model, and in particular the law of large numbers. We apply this now to the functionals $\xi^{(k)}(\mathbf{x}, G)$ defined as the indicator of the event that the connected component of $\mathbf{x}$ has no more than k vertices. By the law of large numbers (4) the proportion of vertices in ${}^pG^t$ which are in components no bigger than k converge, as first $t \uparrow \infty$ and then $k \uparrow \infty$ to $1 - {}^p\theta$. Hence if ${}^p\theta = 0$ for some $p > 0$, then $(G^t)_{t>0}$ and hence $(G_t)_{t>0}$ is non-robust. It is therefore sufficient to show that, for some sufficiently small $p > 0$, there is no infinite component in the percolated limit model ${}^pG^\infty$.

Positive correlation between edges We first explain why a naïve first moment calculation fails. If $(0, U)$ has positive probability of belonging to an infinite component of ${}^pG^\infty$ then, with positive probability, we could find an infinite self-avoiding path in ${}^pG^\infty$ starting from $\mathbf{x}_0 = (0, U)$. A direct first moment calculation would require to give a bound on the probability of the event $\{\mathbf{x}_0 \leftrightarrow \mathbf{x}_1 \leftrightarrow \cdots \leftrightarrow \mathbf{x}_n\}$ that a sequence $(\mathbf{x}_0, \ldots, \mathbf{x}_n)$ of distinct points $\mathbf{x}_i = (x_i, s_i)$ conditioned to be in $\mathcal{X}$ forms a path in G^∞. If this estimate allows us to bound the expected number of paths of length n in G^∞ starting in $\mathbf{x}_0 = (0, U)$ by C^n, for some constant C, we can infer with Borel-Cantelli that, if $p < 1/C$, almost surely there is no arbitrarily long self-avoiding paths in ${}^pG^\infty$. The problem here is that the events $\{\mathbf{x}_j \leftrightarrow \mathbf{x}_{j+1}\}$ and $\{\mathbf{x}_k \leftrightarrow \mathbf{x}_{k+1}\}$ are positively correlated if the interval $I = (s_j, s_{j+1}) \cap (s_k, s_{k+1})$ is nonempty, because the existence of a vertex in $\mathcal{X} \cap (\mathbb{R} \times I)$ may make their indegrees grow simultaneously. Because the positive correlations play against us, it seems not possible to give an effective upper bound on the probability of a long sequence to be a path, therefore making this first moment calculation impossible.

Quick paths, disjoint occurrence, and the BK inequality As a solution to this problem we develop the concept of *quick paths*. If ${}^pG^\infty$ contains an infinite path, then there is an infinite quick path in G^∞ with at least half of its points lying in ${}^pG^\infty$. The expected number of quick paths of length n can be bounded by C^n, for some $C > 0$, and the naïve argument above can be carried through.

Starting with a geodesic path $x_0 \leftrightarrow \cdots \leftrightarrow x_\ell$ in ${}^pG_0^\infty$ we first construct a subsequence $y_n = x_{\varphi(n)}$ by letting $\varphi(0) = 0$ and $\varphi(n+1)$ be the maximal $k > \varphi(n)$ such that there is $y \in G^\infty$ younger than $x_{\varphi(n)}$ and x_k with $x_{\varphi(n)} \leftrightarrow y \leftrightarrow x_k$. We emphasise that y need not be in ${}^pG^\infty$ but only in G^∞. The vertex y is called a *common child* of the vertices $x_{\varphi(n)}$ and $x_{\varphi(n+1)}$, and if there is no common child we let $\varphi(n+1) = \varphi(n) + 1$. The *quick path* $z_0 \leftrightarrow \cdots \leftrightarrow z_m$ associated with the geodesic path $x_0 \leftrightarrow \cdots \leftrightarrow x_\ell$ is obtained by inserting between y_n and y_{n+1}, if they are not connected by an edge, their oldest common child $y \in G^\infty$. Quick paths are characterised by the properties;

(i) A vertex which is not a local maximum (i.e. younger than its two neighbours in the chain) cannot be connected by an edge to a younger vertex of the path, except possibly its neighbours.

(ii) Two vertices z_n and z_{n+j}, with $j \geq 2$, which are not local maxima, can have common children only if $j = 2$ and z_{n+1} is a local maximum. In that case, z_{n+1} is their oldest common child.

Introduce a *splitting at index i* if either z_i is younger than both z_{i-1} and z_{i-2}, or younger than both z_{i+1} and z_{i+2}. We write $n_0 = 0 < n_1 < \cdots < n_k = m$ for the splitting indices in increasing order. Let

$$A_j = \{z_{n_{j-1}} \leftrightarrow \cdots \leftrightarrow z_{n_j}\}.$$

Then if $z_0 \leftrightarrow \cdots \leftrightarrow z_m$ is a path in G^∞ that satisfies (i) and (ii), then $A_1, \ldots, A_k$ occur disjointly. The concept of disjoint occurrence is due to van den Berg and Kesten. Two increasing events A and B *occur disjointly* if there exists disjoint subsets of the domain of the Poisson process such that A occurs if the points falling in the first subset are present, and B occurs if the points falling in the second subset are present. The famous *BK-inequality*, see [3] for the variant most useful in our context, states that the probability of events occurring disjointly is bounded by the probability of their product. The events A_j involve five or fewer consecutive vertices and Figure 3 shows the six possible types, up to symmetry. The probability of these types can be estimated by a direct calculation.

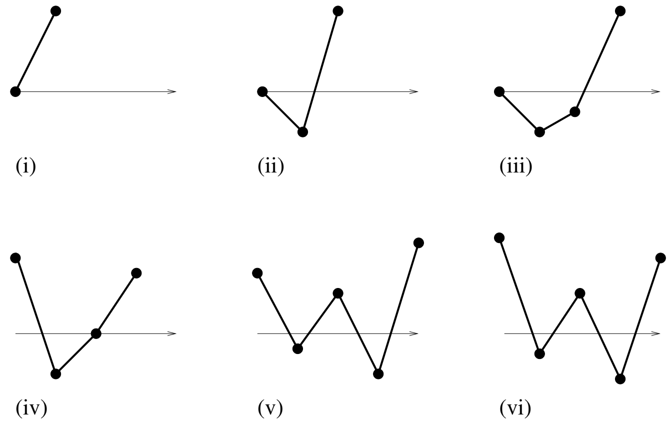

Fig. 3. Up to symmetry there are six types of small parts after the splitting. Illustrated, with the index of a point on the abscissa and time on the ordinate, these are (i) one single edge, (ii) a V shape with two edges, (iii) a V shape with three edges and the end vertex of the short leg between the two vertices of the long leg, (iv) a V shape with three edges and both vertices of the long leg below the end vertex of the short leg, (v) a W shape with the higher end vertex on the side of the deeper valley, (vi) a W shape with the lower end vertex on the side of the deeper valley.

An refinement of the method The method described so far, allows to show non-robustness only in the case $\tau > 3$. To show non-robustness in the case $\tau > 2 + \frac{1}{\delta - 1}$ a refinement is needed, which we now briefly describe.

A vertex z born at time u has typically of order $u^{-\gamma}$ younger neighbours, which may be a lot. As most of these neighbours are close to z, namely within distance u^{-1}, and their local neighbourhoods are therefore strongly correlated, our bounds are far from sharp. No matter how many vertices within distance u^{-1} of z belong to the component of z, it will not help much to connect z to vertices far away. Indeed, defining the *region around* z as

$$C_z = \{z' \text{ born at } u' \geq u, |z' - z| \leq 2u^{-1} - u'^{-1}\},$$

we show that the typical number of vertices outside C_z that are connected to z, or any other vertex in C_z, is only of order $\log(u^{-1})$. To estimate the probability of a path it therefore makes sense to take all the points within C_z for granted and consider only those edges of a quick path straddling a suitably defined boundary of C_z. This improves our bounds because few edges straddle the boundary, and the boundary remains small as u becomes small.

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