The universality classes in the parabolic Anderson model

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We consider the unique continuous nonnegative solution $v\colon [0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ to the Cauchy problem with random coefficients and localised initial datum,

$$\begin{aligned} \tfrac{\partial}{\partial t}\, v(t,z) &= \Delta^{\mathrm{d}} v(t,z) + \xi(z)\, v(t,z), \qquad && \text{for } (t,z) \in (0,\infty)\times\mathbb{Z}^d, \\ v(0,z) &= \mathbb{1}_0(z), && \text{for } z \in \mathbb{Z}^d. \end{aligned}$$

Here $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty,\infty)$, and Δ^{d} is the discrete Laplacian. This parabolic problem is called the *parabolic Anderson model*, it is well-studied in the mathematics and mathematical physics literature because it exhibits an *intermittency effect*. This means, loosely speaking, that most of the total mass

$$U(t) = \sum_{z\in\mathbb{Z}^d} v(t,z), \qquad \text{for } t > 0,$$

of the solution is concentrated on a small number of spatially well separated localised islands.

This talk is mainly concerned with the *annealed* behaviour, i.e. with the behaviour of the expectations $\langle v(t,z)\rangle$ with respect to the random potential. If the potential is truly random this behaviour is markedly different from the behaviour of the heat equation in the mean potential. This is due to the fact that exceptionally favourable potentials ξ dominate the average solutions. Important questions in the annealed case are:

- How fast (if at all) does the expected mass spread into space?
- What is the long-term behaviour of the expected total mass $\langle U(t)\rangle$?
- What is the shape of the favourable potentials ξ dominating the average?

These questions have been studied for *special classes* of potentials, for example in [GM98] for the double-exponential distribution, [A95, BK01] for potentials bounded from above, and [S98, GK00] for variants of the model in continuous space. See also [CM94, GK05] for surveys.

In [HKM05] we initiate the study of the precise dependence of the answers to these questions on the distribution of the potential ξ. We assume that the logarithmic moment generating function,

$$H(t) = \log \left\langle e^{t\xi(0)} \right\rangle$$

is finite for all $t > 0$, so that all moments $\langle U(t)^p\rangle$ exist at all times. To simplify the presentation, we make the assumption that if ξ is bounded, then $\operatorname{esssup}\xi(0) = 0$, so that $\lim_{t\to\infty} H(t)/t \in \{0,\infty\}$. This is no loss of generality, as additive constants in the potential appear as additive constants in the logarithmic asymptotics of $\langle U(t)^p\rangle$. To avoid 'mixed behaviour' we also make the mild technical assumptions that $H(t)/t$ is in the de Haan class, which implies that H is regularly varying with some index $\gamma \geq 0$, and that the auxiliary function has a limit $\kappa^* \in [0,\infty]$.

Under these assumptions we can identify precisely *four* qualitatively different classes of behaviour that can arise in the parabolic Anderson model. In all cases there is a scale function $\alpha(t)$ such that, in a suitable sense, the mass of the solution at time t is confined to the ball of radius of order $\alpha(t)$ about the origin. Moreover the first two terms in an expansion of $\langle U(t)^p\rangle$ are

$$\frac{1}{pt}\log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\,\big(\chi + o(1)\big), \qquad \text{as } t\uparrow\infty,$$

where χ is a real number given in terms of a variational problem encoding (at least on a heuristical level) the optimal shape of the potential fields and associated solutions that make the dominant contribution to the average. The four classes are given in terms of the parameters γ and κ^* as follows:

(1) $\boxed{\gamma > 1, \text{ or } \gamma = 1 \text{ and } \kappa^* = \infty.}$

This case is included in the discussion of [GM90, GM98]. Here $\chi = 2d$, $\alpha(t) = 1$ and the first term of the expansion dominates the sum, which diverges to infinity. The expected mass remains localised in the origin in the sense that

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \frac{\langle v(t,0)\rangle}{\langle \sum_{z\in\mathbb{Z}^d} v(t,z)\rangle} = 0\,.$$

We therefore call this case the *single-peak case*, it comprises, for example, the case of Gaussian potentials.

(2) $\boxed{\gamma = 1 \text{ and } 0 < \kappa^* < \infty.}$

This case, covering distributions in the vicinity of the double-exponential distribution, is the main objective of [GM98]. Here $\alpha(t) \to 1/\sqrt{\kappa^*} \in (0,\infty)$, so that the expected solution does not spread into space, but remains essentially confined to a finite ball. The first term in the expansion dominates the sum, which goes to infinity. Moreover,

$$\chi = \min_{\substack{g\,:\,\mathbb{Z}^d\to\mathbb{R}\\ \sum g^2(z)=1}} \Big\{ \tfrac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \big(g(x)-g(y)\big)^2 - \rho \sum_{x\in\mathbb{Z}^d} g^2(x)\log g^2(x) \Big\},$$

where $x \sim y$ means that the points are neighbours, and ρ is an additional parameter. This variational problem is difficult to analyse. It has a solution, which is unique for sufficiently large values of ρ, and heuristically this minimizer represents the shape of the solution.

(3) $\boxed{\gamma = 1 \text{ and } \kappa^* = 0.}$

Potentials in this class are called *almost bounded* in [GM98]. The class contains both bounded and unbounded potentials with very light upper tails, and is discussed in detail for the first time in our paper [HKM05]. We show that $\alpha(t)$ is going to infinity, but is slowly varying. In particular $\alpha(t) \uparrow \infty$ more slowly than any polynomial in t. The first term of the expansion dominates the sum, which may go to infinity or zero. Moreover,

$$\chi = \min_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \|\nabla g\|_2^2 - \rho \int g^2(x)\log g^2(x)\,dx \Big\}\,.$$

This problem is the continuous variant of the problem in (2) and is much easier to solve. By the logarithmic Sobolev inequality $\chi = d\rho\big(1 - \frac{1}{2}\log\frac{\rho}{\pi}\big)$ and there is a unique minimiser,

$$g_*(x) = \big(\frac{\rho}{\pi}\big)^{d/4} \exp\big(-\frac{\rho}{2}|x|^2\big),$$

which heuristically represents the shape of the solution in the scale $\alpha(t)$.

(4) $\boxed{\gamma < 1.}$

This is the case of *completely bounded potentials*, which is treated in [BK01]. Here $\alpha(t) \uparrow \infty$ and $t \mapsto \alpha(t)$ is regularly varying of index $\frac{1-\gamma}{d+2-d\gamma} < \frac{1}{2}$. Moreover,

$$\chi = \inf_{\substack{g \in H^1(\mathbb{R}^d) \\ \|g\|_2 = 1}} \left\{ \|\nabla g\|_2^2 - \rho \int_{\mathbb{R}^d} \frac{g^{2\gamma}(x) - g^2(x)}{\gamma - 1} \, dx \right\}.$$

The two terms of the expansion are of the *same order*, and $\langle U(t)^p \rangle$ converges to zero.

Besides this classification, our main contribution is the detailed investigation of the almost bounded case (3). We use the Feynman-Kac formula to express $v(t,z)$ in terms of exponential functionals of the local times of a continuous-time random walk. Three major technical ingredients enter in our proofs:

- a compactification argument of Biskup and König [BK01] based on estimating Dirichlet eigenvalues in large boxes against maximal Dirichlet eigenvalues in small subboxes;
- a powerful inequality of Brydges, van der Hofstad and König [BHK05] which refines the classical Donsker-Varadhan large deviation upper bound and allows to study highly discontinuous functionals of the local time field of continuous time random walks;
- a local large deviation upper bound for q-norms of the normalised local time field of continuous time random walks based on the combinatorial approach of [KM02].

Full details of the proofs and additional results on the *quenched* problem, i.e. the behaviour of the *random* solutions $v(t,z)$, can be found in our preprint [HKM05].

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