Cycle length distributions in random permutations with diverging cycle weights

by

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Summary. We study the model of random permutations with diverging cycle weights, which was recently considered by Ercolani and Ueltschi, and others. Assuming only regular variation of the cycle weights we obtain a very precise local limit theorem for the size of a typical cycle, and use this to show that the empirical distribution of properly rescaled cycle lengths converges in probability to a gamma distribution.

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1 Introduction

We study the empirical cycle distributions in models of random permutations with weights depending on the length of the cycles. In this model, for any cycle of length j the weight of the permutation gets multiplied with a factor proportional to θ_j. More precisely, the probability of a permutation π of n elements is defined as

$$\mathbb{P}_n(\pi) = \frac{1}{h_n n!} \prod_{j\geq 1} \theta_j^{r_j(\pi)}, \tag{1}$$

where $r_j(\pi)$ is the number of cycles in the permutation π of length j, and h_n is a normalisation. The case of constant cycle weights $\theta_j = \theta$ corresponds to the Ewens measure from population biology and is well studied. In this paper our focus is on cycle weights (θ_j) which form a diverging sequence of regular variation. Studying random permutation with cycle weights described by their asymptotic behaviour was considered in [BG05] and is also motivated by the study of the quantum Bose gas [BU09, BU11]. The case of convergent sequences (θ_j) has also been studied, see e.g. [BG05, BUV11, Lug09].

The case of diverging cycle weights was treated by Betz et al. [BUV11], Ercolani and Ueltschi [EU13], Nikeghbali and Zeindler [NZ13] and by Maples et al. [MNZ12]. If the growth of the cycle weights is of polynomial order the length of a typical cycle goes to zero. Moreover, Ercolani and Ueltschi [EU13] show for a particular choice of the sequence (θ_j) that the length L_1 of the cycle containing one, behaves like

$$L_1 \sim n^{\frac{1}{\gamma+1}} X,$$

where $\gamma := \lim \frac{\log \theta_j}{\log j} > 0$ and X is gamma distributed with shape parameter $\gamma + 1$.

The aim of this paper is to generalise this result in several ways. First we allow for completely general sequences (θ_j) of regular variation with positive index, going well beyond the setting of [EU13]. See [BGT87] for definitions and general results on this class of sequences. Second, we considerably refine the asymptotics and obtain a full local limit theorem. And third, building on this result, we extend the convergence to full convergence of the empirical cycle length distribution to a gamma distribution. The latter fact brings this result in line with similar results obtained in the study of condensation phenomena recently obtained by the authors in [DM13] and [Der13].

While the studies carried out for this model so far rely on the (often quite heavy) machinery of analytic combinatorics, like saddle-point analysis [EU13], singularity analysis [NZ13] or further generating function methods [MNZ12], our proofs rely on a direct analysis of the renewal-type equations relating the normalisation factors h_n to the cycle weights. The flexibility of this method is due to the fact that no inversion of generating functions has to be performed. One can expect that this method can also be used to extend further results from [EU13] and other papers in this area.

2 Statement of the main results

Recall the definition (1) of the random partitions and impose the following assumptions on the sequence (θ_j) of cycle weights:

(A1) (θ_j) is regularly varying, i.e. there exists an index $\gamma > 0$ and a slowly varying function ℓ such that $\theta_j = j^\gamma \ell(j)$ for all $j \in \mathbb{N}$.

(A2) (θ_j) is nondecreasing.

We let $\beta_0 = 0$ and $\beta_n := \sum_{j=1}^n \theta_j$ for integers $n \geq 1$. By Lemma 3.4 below we have

$$\beta_n \sim \tfrac{1}{\gamma+1} n^{\gamma+1} \, \ell(n)$$

and, denoting by

$$\beta^{\leftarrow}(t) := \min\{n \geq 0 : \beta_n \geqslant t\}, \qquad \text{for } t \in [0, \infty),$$

its generalised inverse, there is a slowly varying function $\ell^{\leftarrow} \colon [0,\infty) \to [0,\infty)$ such that

$$\beta^{\leftarrow}(t) = t^{\frac{1}{\gamma+1}} \ell^{\leftarrow}(t),$$

or in other words that $(\beta^{\leftarrow}(t))$ is regularly varying with index $\frac{1}{\gamma+1}$. Finally, define

$$d_\gamma := \Gamma(\gamma+2)^{\frac{1}{\gamma+1}}$$

and recall that $(\gamma+1)\, x^\gamma \, e^{-d_\gamma x}, x \geqslant 0$ is the probability density of a gamma distribution with shape parameter $\gamma+1$. We denote by $L_k = L_k(\sigma)$, the length of the cycle containing the symbol $k \in \{1, \dots, n\}$. The following local limit theorem is the first main result of this paper.

Theorem 2.1 (Local limit theorem)**.** *For every $M > 0$ we have*

$$\sup_{j \leqslant M\beta^{\leftarrow}(n)} \left| \frac{n}{\theta_j} \mathbb{P}_n\{L_1 = j\} - e^{-d_\gamma j/\beta^{\leftarrow}(n)} \right| \to 0 \quad as \quad n \to \infty.$$

Moreover, for every $\varepsilon > 0$ there exist $M > 0$ with

$$\limsup_{n\to\infty} \mathbb{P}_n\big\{L_1 \geqslant M\beta^{\leftarrow}(n)\big\} < \varepsilon.$$

Theorem 2.1 implies that a typical cycle under $\mathbb{P}_n$ has length of order $\beta^{\leftarrow}(n)$. The following corollary is a slightly weaker version of Theorem 2.1, which is more illuminating in the case that j is of the order of a typical cycle length, and readily implies a global limit theorem.

Corollary 2.2. *For every $M>0$ we have*

$$\sup_{j\,\leqslant\, M\beta^{\leftarrow}(n)}\Big|\beta^{\leftarrow}(n)\,\mathbb{P}_n\{L_1=j\}-(\gamma+1)\big(\tfrac{j}{\beta^{\leftarrow}(n)}\big)^{\gamma}\,e^{-d_\gamma j/\beta^{\leftarrow}(n)}\Big|\to 0\quad as\quad n\to\infty,$$

and therefore we have the global limit theorem

$$\frac{L_1}{\beta^{\leftarrow}(n)}\ \overset{\mathbb{P}_n}{\Longrightarrow}\ X,$$

where X is gamma distributed with shape parameter $\gamma+1$.

We now define the *empirical cycle length distribution* as the random probability measure on $[0,1]$ given by

$$\mu_n=\frac{1}{n}\sum_{k=1}^{n}\delta_{\frac{L_k}{\beta^{\leftarrow}(n)}}=\frac{1}{n}\sum_{i\geq 1}\lambda_i\,\delta_{\frac{\lambda_i}{\beta^{\leftarrow}(n)}},$$

where the integers $\lambda_1\geqslant\lambda_2\geqslant\cdots$ are the ordered cycle lengths of a permutation chosen randomly according to $\mathbb{P}_n$. We derive a limit theorem for the empirical cycle length distribution, showing that it converges in probability to a deterministic limit given by a gamma distribution.

Theorem 2.3 (Asymptotic shape of the cycle length distribution)**.** *For every $x\geqslant 0$,*

$$\lim_{n\to\infty}\mu_n\big[0,x\big]=(\gamma+1)\int_0^x y^{\gamma}e^{-d_\gamma\,y}\,dy,\qquad in\ probability.$$

Our interest in Theorem 2.3 stems mostly from the analogy to results on the emergence of condensation, which also exhibit an incomplete gamma function describing the empirical distribution of a condensing quantity prior to condensation, see [DM13] for a speculative treatment of this universal phenomenon and results in the case of Kingman's model of selection and mutation and [Der13] for results on random networks.

3 Proofs

3.1 Some first observations

The following two lemmas hold without any assumptions on (θ_j). Crucial in the analysis of the model is the sequence $(h_n)_{n\geq 0}$ of normalisations.

Lemma 3.1.

(a) The sequence of normalisations is determined by the recurrence equation

$$h_0=1\quad and\quad h_n=\frac{1}{n}\sum_{j=1}^{n}\theta_j\,h_{n-j}\quad for\ n\in\mathbb{N}. \tag{2}$$

(b) The law of L_1 is given by

$$\mathbb{P}_n\big\{L_1(\sigma) = j\big\} = \frac{\theta_j h_{n-j}}{n h_n} \qquad \text{for } j \geqslant 1.$$

Proof. See also Proposition 2.1 in [EU13]. We have that (a) follows from (b) by summing over all $j \in \{1, \ldots, n\}$. For (b) we first sum over all the possible elements of the cycle containing one, in order, and then look at all the permutations of the remaining indices. This yields

$$\mathbb{P}_n\{L_1 = j\} = \frac{\theta_j}{n! h_n}\,(n-1)(n-2)\cdots(n-j+1)\,(n-j)!\,h_{n-j} = \frac{\theta_j h_{n-j}}{n h_n}. \qquad \square$$

Lemma 3.2. *Given the cycle containing one, the conditional distribution of the permutation on the remaining indices is given by $\mathbb{P}_{n-L_1}$.*

Proof. Note that the number of possible cycles of length l containing one is $(n-1)(n-2)\cdots(n-l+1)$, and by Lemma 3.1 (b) the law of L_1 is given as

$$\mathbb{P}_n\{L_1 = l\} = \frac{\theta_l h_{n-l}}{n h_n}.$$

Hence the conditional weight of any permutation σ containing the given cycle is

$$\frac{\mathbb{P}_n(\sigma)}{\frac{\theta_l h_{n-l}}{h_n\, n(n-1)\cdots(n-l+1)}} = \frac{\prod_{j\geq 1} \theta_j^{r_j(\sigma)}}{(n-l)!\theta_l h_{n-l}} = \mathbb{P}_{n-l}(\tilde{\sigma}),$$

where $\tilde{\sigma}$ is obtained from σ by removing the cycle containing one and relabelling the remaining indices as $\{1, \ldots, n-l\}$. $\square$

The next lemma is a simple consequence of assumption (A2).

Lemma 3.3. *The sequence $(nh_n)_{n\geq 0}$ is nondecreasing.*

Proof. Let $n \in \mathbb{N}$ and observe that

$$n\, h_n = \sum_{j=1}^{n} \theta_j\, h_{n-j} \leqslant \sum_{j=0}^{n} \theta_{j+1}\, h_{n-j} = (n+1)\, h_{n+1}$$

by the nonnegativity of (h_n) and assumption (A2). Further, $0h_0 = 0 \leqslant 1h_1$. $\square$

We collect relevant asymptotic estimates in the following lemma.

Lemma 3.4 (Asymptotic estimates)**.**

(i) $\beta_n = \sum_{j=1}^{n} \theta_j \sim \frac{1}{\gamma+1} n^{\gamma+1}\, \ell(n)$

(ii) There exists a slowly varying function $\ell^{\leftarrow} \colon (0,\infty) \to [0,\infty)$ such that

$$\beta^{\leftarrow}(t) := \min\big\{n \in \mathbb{N} \cup \{0\} \colon \beta_n \geqslant t\big\} = t^{\frac{1}{\gamma+1}}\, \ell^{\leftarrow}(t)$$

(iii) $\displaystyle\lim_{n\to\infty} \frac{\beta^{\leftarrow}(n)\, \theta_{\beta^{\leftarrow}(n)}}{n} = 1 + \gamma.$

Proof. **(i)** This follows immediately from Karamata's theorem (direct half), see [BGT87, Proposition 1.5.8]. **(ii)** This follows immediately from the asymptotic inversion principle for regularly varying functions, see [BGT87, Theorem 1.5.12]. **(iii)** One has $\beta_N \sim (1+\gamma)^{-1} N\,\theta_N$. Replacing N by $\beta^{\leftarrow}(n)$ and letting n tend to infinity, one gets $n \sim \frac{1}{1+\gamma}\,\beta^{\leftarrow}(n)\,\theta_{\beta^{\leftarrow}(n)}$, which immediately implies (iii). $\square$

3.2 Proof of Theorem 2.1

The key to our analysis is to study the asymptotic behaviour of the normalising sequence (h_n) using the recurrence relation (2). Our main technical step, Proposition 3.9, shows that defining

$$g_t^{(N)} := h_{N+\lfloor t\beta^{\leftarrow}(N)\rfloor} \qquad \text{for } t \in \mathbb{R}, \tag{3}$$

with the convention that $h_n = 0$ for $n \in -\mathbb{N}$, we have

$$\lim_{N\to\infty} \frac{g_b^{(N)}}{g_a^{(N)}} = e^{d_\gamma\,(b-a)} \tag{4}$$

uniformly in the values a, b taken from a compact interval.

Let us first see how Theorem 2.1 follows from this. By Lemma 3.1 (b)

$$\frac{n}{\theta_j}\,\mathbb{P}_n\{L_1 = j\} = \frac{h_{n-j}}{h_n} = \frac{g^{(N)}_{-\frac{j}{\beta^{\leftarrow}(n)}}}{g_0^{(N)}},$$

so that by (4), for every $M > 0$, we have

$$\sup_{j\,\leqslant\, M\beta^{\leftarrow}(n)} \left| \frac{n}{\theta_j}\,\mathbb{P}_n\{L_1 = j\} - e^{-d_\gamma j/\beta^{\leftarrow}(n)} \right| \to 0 \;\text{ as }\; n \to \infty.$$

The additional statement of Theorem 2.1 will be proved in Subsection 3.2.1. It constitutes the first step in the proof of (4), which will be carried out in four steps in Subsections 3.2.1 to 3.2.4. Subsection 3.2.5 is devoted to the proof of Corollary 2.2.

3.2.1 The recurrence equation

In this section we show that for every $\varepsilon > 0$ there exist $M > 0$ with

$$\limsup_{n\to\infty} \mathbb{P}_n\big\{L_1 \geqslant M\beta^{\leftarrow}(n)\big\} < \varepsilon. \tag{5}$$

This is a direct consequence of the following lemma.

Lemma 3.5. *For every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,*

$$\frac{1}{n} \sum_{j=\lceil M\beta^{\leftarrow}(n)\rceil}^{n} \theta_j h_{n-j} \leqslant \varepsilon h_{n-1}.$$

Indeed, using Lemma 3.1(b), Lemma 3.3, and Lemma 3.5 we get for every $\varepsilon > 0$ some M and n_0 such that, for all $n \geqslant n_0$,

$$\mathbb{P}_n\big\{L_1 \geqslant M\beta^{\leftarrow}(n)\big\} = \sum_{j=\lceil M\beta^{\leftarrow}(n)\rceil}^{n} \frac{\theta_j h_{n-j}}{n h_n}$$
$$\leqslant \frac{1}{n-1} \sum_{j=\lceil M\beta^{\leftarrow}(n)\rceil}^{n} \frac{\theta_j h_{n-j}}{h_{n-1}} < \varepsilon.$$

Proof. We analyse the sequence (h_n) at a large reference time $N \in \mathbb{N}$. By Lemma 3.4, there exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$

$$\beta^{\leftarrow}(2n) < n/2 \text{ and } \beta^{\leftarrow}(2(n+1)) - \beta^{\leftarrow}(2n) \in \{0, 1\};$$

and we define a sequence $(\alpha_k^{(N)})$ inductively by letting $\alpha_0^{(N)} := N$ and $\alpha_{k+1}^{(N)} = \alpha_k^{(N)} - \beta^{\leftarrow}(2\alpha_k^{(N)})$ as long as $\alpha_k^{(N)} \geqslant n_0$. We denote by $K = K(N)$ the largest index for which $\alpha_K^{(N)} \geqslant n_0$ so that we end up with a sequence $\alpha_0^{(N)}, \ldots, \alpha_{K+1}^{(N)}$ of positive integers. The sequence is used to partition $\{0, \ldots, N-1\}$ into sets

$$\mathbb{I}_k^{(N)} = \{\alpha_{k+1}^{(N)}, \ldots, \alpha_k^{(N)} - 1\}$$

for $k = 0, \ldots, K$, and the remainder $\mathbb{I}_{K+1}^{(N)} := \{0, \ldots, \alpha_{K+1}^{(N)} - 1\}$. For $k = 0, \ldots, K$, we consider

$$M_k^{(N)} := \min\{h_n : n \in \mathbb{I}_k^{(N)}\}.$$

First we prove that, for $k = 1, \ldots, K$,

$$M_{k-1}^{(N)} \geqslant 2M_k^{(N)}.$$

Let $n = \alpha_k^{(N)}$ which is the smallest index in $\mathbb{I}_{k-1}^{(N)}$. Since $n - \beta^{\leftarrow}(2n) = \alpha_{k+1}^{(N)}$ and $\beta_{\beta^{\leftarrow}(2n)} \geqslant 2n$ by definition, we get (conveniently dropping the round-off symbols in the summation)

$$h_n \geqslant \frac{1}{n} \sum_{j=1}^{\beta^{\leftarrow}(2n)} \theta_j h_{n-j} \geqslant M_k^{(N)} \frac{1}{n} \sum_{j=1}^{\beta^{\leftarrow}(2n)} \theta_j \geqslant 2M_k^{(N)}.$$

Next, let $n = \alpha_k^{(N)} + 1$. By assumption $n - 1 \geqslant n_0$ so that $n - \beta^{\leftarrow}(2n) \geqslant \alpha_{k+1}^{(N)}$ and as above

$$h_n \geqslant \frac{1}{n}\Big(\theta_1 \underbrace{h_{n-1}}_{\geqslant 2M_k^{(N)}} + \sum_{j=2}^{\beta^{\leftarrow}(2n)} \theta_j h_{n-j}\Big) \geqslant 2M_k^{(N)}.$$

Similarly, it follows by induction over n, that

$$M_{k-1}^{(N)} = \min\{h_n : n = \alpha_k^{(N)}, \ldots, \alpha_{k-1}^{(N)} - 1\} \geqslant 2M_k^{(N)}. \tag{6}$$

Second, we provide an estimate for h_n where $n \in \mathbb{I}_k^{(N)}$ and $k \in \{0, \dots, K+1\}$. To begin with, let $k \in \{1, \dots, K\}$ and let $m \in \mathbb{I}_{k-1}^{(N)}$ be the index where (h_n) takes its mimimum on the set $\mathbb{I}_{k-1}^{(N)}$. Then, by Lemma 3.3 and (6), one has

$$h_n \leqslant \frac{m}{n}\, h_m = \frac{m}{n}\, M_{k-1}^{(N)} \leqslant 4\, 2^{-(k-1)}\, h_{N-1},$$

where we used that $\alpha_k^{(N)}/\alpha_{k+1}^{(N)} \leqslant 2$ for $k \in \{0, \dots, K\}$, by construction. The estimate remains true for $n \in \mathbb{I}_0^{(N)}$ and, for $n \in \mathbb{I}_{K+1}^{(N)} \backslash \{0\}$, one has

$$h_n \leqslant \alpha_{K+1}^{(N)} h_{\alpha_{K+1}^{(N)}} \leqslant 4\, n_0\, 2^{-K}\, h_{N-1}. \tag{7}$$

Since $\beta^{\leftarrow}(2n_0) \leqslant n_0/2$, one has $\beta_{\lfloor n_0/2 \rfloor} = \sum_{j=1}^{\lfloor n_0/2 \rfloor} \theta_j \geqslant 2n_0$. Hence, there exists $n \in \{1, \dots, \lfloor n_0/2 \rfloor\} \subset \mathbb{I}_{K+1}^{(N)}$ with $\theta_n \geqslant 4$ and one obtains $h_n \geqslant \frac{\theta_n}{n} \geqslant \frac{4}{n_0}$. Consequently,

$$h_0 = 1 \leqslant \frac{n_0}{4} h_n \leqslant n_0^2\, 2^{-K} h_{N-1}.$$

by (7). Altogether, we get that there is a constant c only depending on n_0 such that, for $k \in \{0, \dots, K+1\}$ and $n \in \mathbb{I}_k^{(N)}$,

$$h_n \leqslant c\, 2^{-k}\, h_{N-1}. \tag{8}$$

Fix a constant $M \in 2\mathbb{N}$ and analyse

$$\xi_N := \frac{1}{N} \sum_{j = M\beta^{\leftarrow}(2N)/2+1}^{N} \theta_j\, h_{N-j}$$

For $j \in \mathbb{N}$, we set

$$i^{(N)}(j) := \max\{l \in \{0, \dots, K+1\} : N - \alpha_l^{(N)} + 1 \leqslant j\}$$

which is the unique index l for which one has $N - j \in \mathbb{I}_l^{(N)}$. By (8), one has

$$\xi_N \leqslant c\, \frac{1}{N} \sum_{j = M\beta^{\leftarrow}(2N)/2+1}^{N} \theta_j\, 2^{-i^{(N)}(j)}\, h_{N-1}.$$

Since, for $k = 0, \dots, K+1$,

$$N - \alpha_k^{(N)} = \sum_{l=1}^{k} \beta^{\leftarrow}(2\alpha_{l-1}^{(N)}) \leqslant k\, \beta^{\leftarrow}(2N),$$

one has

$$i^{(N)}(j) \geqslant \max\{l \in \{0, \dots, K+1\} : 1 + l\beta^{\leftarrow}(2N) \leqslant j\} = \Big\lfloor \frac{j-1}{\beta^{\leftarrow}(2N)} \Big\rfloor \wedge (K+1).$$

Therefore, as long as $\beta^{\leftarrow}(2N) \geqslant n_0$, one has

$$\xi_N \leqslant c\, h_{N-1} \sum_{k=M/2}^{\infty} 2^{-k} \frac{1}{N} \sum_{j=k\beta^{\leftarrow}(2N)+1}^{(k+1)\beta^{\leftarrow}(2N)} \theta_j \leqslant c\, h_{N-1} \sum_{k=M/2}^{\infty} 2^{-k} \frac{\beta((k+1)\beta^{\leftarrow}(2N))}{N}.$$

Clearly, one has $\beta(\beta^{\leftarrow}(2N)) \sim 2N$ as $N \to \infty$. Further, the Potter bound [BGT87, Theorem 1.5.6] implies that for sufficiently large n and any $m \geqslant n$ one has

$$\beta(m) \leqslant 2\Big(\frac{m}{n}\Big)^{\gamma+2} \beta(n).$$

Consequently, one gets that, for sufficiently large N,

$$\xi_N \leqslant 5c\, h_{N-1} \sum_{k=M/2}^{\infty} 2^{-k} (k+1)^{\gamma+2}.$$

Since $M\beta^{\leftarrow}(2N)/2 + 1 \sim M 2^{-\gamma/(1+\gamma)} \beta^{\leftarrow}(N)$ and $2^{-\gamma/(1+\gamma)} < 1$, we have for sufficiently large N that

$$\frac{1}{N} \sum_{j=M\beta^{\leftarrow}(N)}^{N} \theta_j h_{N-j} \leqslant \xi_N \leqslant 5c\, h_{N-1} \sum_{k=M/2}^{\infty} 2^{-k} (k+1)^{\gamma+2}.$$

The statement follows by choosing M sufficiently large. □

3.2.2 Estimates against the Volterra equation

Our aim is to show that $g^{(N)}$, as defined in (3), is close to the solution of an integral equation on an interval $[-L, L]$ with $L > 0$ being fixed, but arbitrarily large.

Lemma 3.6. *For any $\varepsilon > 0$, there exists $\kappa > 0$ such that for any $L > 0$ one has, for all sufficiently large $N \in \mathbb{N}$ and all $t \in [-L, L]$,*

$$g_t^{(N)} \leqslant e^{\varepsilon}(1+\gamma) \int_0^{\kappa} s^{\gamma}\, g_{t-s}^{(N)}\, ds$$

Conversely, for every $\varepsilon, \kappa > 0$ and $L > 0$ one has, for all sufficiently large $N \in \mathbb{N}$ and all $t \in [-L, L]$,

$$g_t^{(N)} \geqslant e^{-\varepsilon}(1+\gamma) \int_0^{\kappa} s^{\gamma}\, g_{t-s}^{(N)}\, ds.$$

Proof. We only prove the first statement, as the second can be proved analogously. Fix $\varepsilon \in (0, 1/2)$ and choose $M > 0$ according to Lemma 3.5. In the following, we denote by $0 < \iota_1 < \iota_2 < \dots$ constants that can be chosen arbitrarily small and that do not depend on N and t. The following estimates are valid for sufficiently large N and all $t \in [-L, L]$.

We let $K \in \mathbb{N}$ and set $\delta = M/K$. Applying Lemma 3.5, we get

$$g_t^{(N)} = \frac{1}{N + \lfloor t\beta^{\leftarrow}(N)\rfloor} \sum_{j=1}^{N+\lfloor t\beta^{\leftarrow}(N)\rfloor} \theta_j \, g^{(N)}_{t-\frac{j}{\beta^{\leftarrow}(N)}}$$
$$\leqslant \frac{1}{N + \lfloor t\beta^{\leftarrow}(N)\rfloor} \sum_{k=1}^{K} \sum_{j=\lfloor (k-1)\delta\beta^{\leftarrow}(N)\rfloor+1}^{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor} \theta_j \, g^{(N)}_{t-\frac{j}{\beta^{\leftarrow}(N)}} + \varepsilon g_t^{(N)}.$$

For large N, one has $N/(N + \lfloor -L\beta^{\leftarrow}(N)\rfloor) \leqslant e^{\iota_1}$ so that

$$g_t^{(N)} \leqslant e^{\iota_1} \sum_{k=1}^{K} \frac{\beta^{\leftarrow}(N)\,\theta_{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor}}{N} \sum_{j=\lfloor (k-1)\delta\beta^{\leftarrow}(N)\rfloor+1}^{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor} \frac{g^{(N)}_{t-\frac{j}{\beta^{\leftarrow}(N)}}}{\beta^{\leftarrow}(N)} + \varepsilon g_t^{(N)}.$$

By definition of $g_t^{(N)}$, one has

$$\sum_{j=\lfloor (k-1)\delta\beta^{\leftarrow}(N)\rfloor+1}^{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor} \frac{g^{(N)}_{t-\frac{j}{\beta^{\leftarrow}(N)}}}{\beta^{\leftarrow}(N)} = \int_{a_{k-1}}^{a_k} g^{(N)}_{t-s}\, ds$$

for $a_k := a_k^{(N,t)} := t - \frac{\lfloor t\beta^{\leftarrow}(N)\rfloor - \lfloor k\delta\beta^{\leftarrow}(N)\rfloor}{\beta^{\leftarrow}(N)}$. Here we used that $g^{(N)}_{t-\cdot}$ is constant on intervals of length $\beta^{\leftarrow}(N)$. Hence,

$$g_t^{(N)} \leqslant e^{\iota_1} \sum_{k=1}^{K} \frac{\beta^{\leftarrow}(N)\,\theta_{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor}}{N} \int_{a_{k-1}}^{a_k} g^{(N)}_{t-s}\, ds + \varepsilon g_t^{(N)}.$$

We note that, for each $k = 1, \ldots, K$,

$$\theta_{\lfloor k\delta\beta^{\leftarrow}(N)\rfloor} \sim (k\delta)^{\gamma}\, \theta_{\beta^{\leftarrow}(N)}$$

Further, by Lemma 3.4, we have $\frac{\beta^{\leftarrow}(N)\,\theta_{\beta^{\leftarrow}(N)}}{N} \to 1 + \gamma$. Consequently,

$$g_t^{(N)} \leqslant e^{\iota_2}(1+\gamma) \sum_{k=1}^{K} (k\delta)^{\gamma} \int_{a_{k-1}}^{a_k} g^{(N)}_{t-s}\, ds + \varepsilon g_t^{(N)}. \tag{9}$$

So far we have not imposed any assumptions on the positive constants ε and δ. We now assume that K is sufficiently large (or, equivalently, $\delta = M/K$ is sufficiently small) in order to guarantee existence of a nonnegative integer $K_0 < K$ with

$$(1+\gamma)(K_0\delta)^{\gamma+1} \leqslant \varepsilon/2 \quad \text{and} \quad e^{\iota_2}\frac{(K_0+1)^{\gamma}}{K_0^{\gamma}} \leqslant e^{\iota_3}.$$

One has

$$e^{\iota_2}(1+\gamma) \sum_{k=1}^{K_0} (k\delta)^{\gamma} \int_{a_{k-1}}^{a_k} g^{(N)}_{t-s}\, ds \leqslant e^{\iota_2}(1+\gamma)(K_0\delta)^{\gamma} \int_0^{a_{K_0}} g^{(N)}_{t-s}\, ds.$$

From Lemma 3.3 we infer that

$$\sup_{\substack{u,v\in[-L,L]\\ u\leqslant v}} \frac{g_u^{(N)}}{g_v^{(N)}} \to 1, \quad \text{as } N\to\infty.$$

Further, $a_{K_0} \to K_0\delta$ uniformly in t as $N\to\infty$ and assuming that $e^{\iota_2} < 2$, we conclude with the definition of K_0, that, for N sufficiently large,

$$e^{\iota_2}(1+\gamma)(K_0\delta)^\gamma \int_0^{a_{K_0}} g_{t-s}^{(N)}\, ds \leqslant \varepsilon g_t^{(N)}.$$

Combining this with (9) and the estimate $e^{\iota_2}k^\gamma/(k-1)^\gamma \leqslant e^{\iota_3}$ for $k > K_0$, yields that

$$g_t^{(N)} \leqslant e^{\iota_4}(1+\gamma)\int_{a_{K_0}}^{a_K} s^\gamma\, g_{t-s}^{(N)}\, ds + 2\varepsilon g_t^{(N)},$$

where we used that

$$\sup_{k=K_0+1,\dots,K}\ \sup_{s\in[a_{k-1},a_k]} \frac{s^\gamma}{(k\delta)^\gamma} \to 1,$$

which is a consequence of the uniform convergence $a_k \to k\delta$ as $N\to\infty$.

Finally, we subtract $2\varepsilon g_t^{(N)}$, divide by $1-2\varepsilon$ to deduce that for all sufficiently large N and all $t\in[-L,L]$

$$g_t^{(N)} \leqslant \frac{1}{1-2\varepsilon} e^{\iota_3}(1+\gamma)\int_{a_{K_0}}^{a_K} s^\gamma\, g_{t-s}^{(N)}\, ds \leqslant \frac{1}{1-2\varepsilon} e^{\iota_3}(1+\gamma)\int_0^{M+1} s^\gamma\, g_{t-s}^{(N)}\, ds$$

which proves the statement since ε and ι_3 can be chosen arbitrarily small. □

3.2.3 Analysis of the Volterra equation

Lemma 3.6 relates our problem to the Volterra equation

$$g_\varepsilon(t) = \int_0^t k_\varepsilon(t-s) g_\varepsilon(s)\, ds + f(t), \qquad \text{for } t\geqslant T, \tag{10}$$

where $\varepsilon\in\mathbb{R}$, $T\in\mathbb{R}$, $k_\varepsilon(u) = e^\varepsilon(1+\gamma)u^\gamma$, for $u\geqslant 0$, and $f\colon [T,\infty)\to\mathbb{R}$ denotes a locally integrable function, see Remark 3.8 below for more details on this relation. We now collect some facts about this equation taken from [GLS90, Chapter 2]. We only consider the case $T=0$, since the general case can be easily obtained from the particular case by applying a time change. Further we write $g=g_0$ and $k=k_0$.

The unique solution to (10) can be expressed in terms of a fundamental solution. It is the unique solution to

$$r_\varepsilon(t) = \int_0^t k_\varepsilon(t-s)\, r_\varepsilon(s)\, ds + k_\varepsilon(t), \qquad \text{for } t\geqslant 0. \tag{11}$$

Again we abbreviate $r = r_0$. With the fundamental solution we can represent the unique solution g_ε to (10) as

$$g_\varepsilon(t) = \int_0^t r_\varepsilon(t-s)\, f(s)\, ds + f(t).$$

We will make use of the following properties.

Lemma 3.7.

(1) We have $r_\varepsilon(t) = e^{\varepsilon/(\gamma+1)} r(e^{\varepsilon/(\gamma+1)} t)$.

(2) We have $r(t) \sim \mu^{-1} e^{d_\gamma t}$ as $t \to \infty$ where $\mu := (1+\gamma) \int_0^\infty e^{-d_\gamma u} u^{\gamma+1}\, du$.

Proof. (1) is easy to verify. For (2) we multiply (11) (with $\varepsilon = 0$) by $e^{-d_\gamma t}$ and observe that the structure of the equation is retained with a new kernel $\bar{k}(u) := e^{-d_\gamma u} k(u)$, which is directly Riemann integrable and defines a probability density on the positive halfline. Hence, by the renewal theorem for densities (see for instance the 'alternative form' of the renewal theorem in [Fel71, XI.1]), one has for the corresponding fundamental solution $\bar{r}(t) = e^{-d_\gamma t} r(t)$ that $\lim_{t\to\infty} \bar{r}(t) = \mu^{-1}$, as required. □

Remark 3.8. Lemma 3.6 allows to compare $g^{(N)}$ with a solution to the Volterra equation on an arbitrarily fixed window $[-L, L]$. Fix $\varepsilon > 0$ and choose $\kappa \geqslant 2L$ as in the lemma. For sufficiently large N, one has

$$g_t^{(N)} \leqslant \int_0^{t+L} k_\varepsilon(s)\, g_{t-s}^{(N)}\, ds + \underbrace{\int_{t+L}^{\kappa} k_\varepsilon(s)\, g_{t-s}^{(N)}\, ds}_{=: F_\varepsilon^{(N,L,\kappa)}(t)},$$

for $t \in [-L, L]$, where we used that κ exceeds the length of the window $[-L, L]$. This is dominated by the unique solution $G_\varepsilon^{(N,L,\kappa)} : [-L, L] \to [0, \infty)$ of the equation

$$G_\varepsilon^{(N,L,\kappa)}(t) = \int_0^{t+L} k_\varepsilon(s)\, G_\varepsilon^{(N,L,\kappa)}(t-s)\, ds + F_\varepsilon^{(N,L,\kappa)}(t).$$

As this is a Volterra equation we use the above representation of its solution to get that, for sufficiently large N,

$$g_t^{(N)} \leqslant G_\varepsilon^{(N,L,\kappa)}(t) = \int_{-L}^t r_\varepsilon(t-s)\, F_\varepsilon^{(N,L,\kappa)}(s)\, ds + F_\varepsilon^{(N,L,\kappa)}(t), \tag{12}$$

for $t \in [-L, L]$. Analogously, we obtain that, for sufficiently large N and $t \in [-L, L]$,

$$g_t^{(N)} \geqslant G_{-\varepsilon}^{(N,L,\kappa)}(t) = \int_{-L}^t r_{-\varepsilon}(t-s)\, F_{-\varepsilon}^{(N,L,\kappa)}(s)\, ds + F_{-\varepsilon}^{(N,L,\kappa)}(t).$$

3.2.4 Exponential behaviour of $g_t^{(N)}$

In this section we finish the proof of (4) and hence of Theorem 2.1. We achieve this by combining the approximation and the results on the Volterra equation.

Proposition 3.9. *Let $L, \delta > 0$. One has, for sufficiently large $N \in \mathbb{N}$, that*

$$e^{-\delta} e^{d_\gamma (b-a)} \leqslant \frac{g_b^{(N)}}{g_a^{(N)}} \leqslant e^{\delta} e^{d_\gamma (b-a)}$$

for $-L \leqslant a \leqslant b \leqslant L$.

Proof. Given $\varepsilon > 0$ and $\kappa \geqslant 2L$ we defineF_ε as in Remark 3.8 and note that $F_\varepsilon^{(N,L,\kappa)}(t) = e^{\varepsilon} F^{(N,L,\kappa)}(t)$ with $F := F_0$. We use the properties of the fundamental solution provided by Lemma 3.7 to rephrase (12) as follows

$$g_t^{(N)} \leqslant e^{\varepsilon + \varepsilon/(\gamma+1)} \int_{-L}^{t} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds + e^{\varepsilon}\, F^{(N,L,\kappa)}(t).$$

We start with the derivation of an upper bound. Let $\delta \in (0,1]$ be arbitrary. We will suppose that $\varepsilon \in (0, \delta]$ is a sufficiently small parameter, the actual value of which will be chosen later in the discussion. This choice may depend on L and κ but not on N or t. Assuming that $\varepsilon \leqslant \delta$ we get that, for sufficiently large N,

$$g_t^{(N)} \leqslant e^{2\delta} \int_{-L}^{t} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds + e^{\delta}\, F^{(N,L,\kappa)}(t). \tag{13}$$

By Lemma 3.7, there exists $T > 0$ only depending on δ such that

$$r(t) \leqslant e^{\delta}\, \mu^{-1} e^{d_\gamma t} \quad \text{for} \quad t \geqslant T.$$

We restrict attention to $t \in [-L+T, L]$. We split the integral in (13) into two parts. The dominant part is

$$\int_{-L}^{t-T} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds \leqslant e^{\delta}\, \mu^{-1} \int_{-L}^{t-T} \exp\{d_\gamma e^{\varepsilon/(\gamma+1)}(t-s)\} F^{(N,L,\kappa)}(s)\, ds.$$

Assuming that $(e^{\varepsilon/(\gamma+1)} - 1) 2L d_\gamma \leqslant \delta$ we arrive at

$$\int_{-L}^{t-T} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds \leqslant \mu^{-1} e^{2\delta} \int_{-L}^{t-T} e^{d_\gamma (t-s)} F^{(N,L,\kappa)}(s)\, ds. \tag{14}$$

In order to show that the remaining part of the integral is asymptotically negligible, we first derive an estimate for $F^{(N,L,\kappa)}(s)$ for $s \in [-L+1, L]$. One has

$$F^{(N,L,\kappa)}(s) = (1+\gamma) \int_{s-\kappa}^{-L} (s-u)^{\gamma} g_u^{(N)}\, du$$

and we observe that for the relevant values of u we have

$$(s-u)^\gamma = (s+L-(L+u))^\gamma = (s+L)^\gamma\big(1+\tfrac{-(L+u)}{s+L}\big)^\gamma \leqslant (s+L)^\gamma(1-L-u)^\gamma,$$

where we have used that $s+L\geqslant 1$ and that the numerator is nonnegative. Hence,

$$F^{(N,L,\kappa)}(s) \leqslant (1+\gamma)(L+s)^\gamma \int_{s-\kappa}^{-L} (-L+1-u)^\gamma g_u^{(N)}\, du \leqslant (L+s)^\gamma\, F^{(N,L,\kappa)}(-L+1).$$

Consider now the remaining part of the integral in (13) for $t\in[-L+T+1,L]$. One has

$$\int_{t-T}^{t} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds \leqslant \mu^{-1}e^{\delta}\exp\{d_\gamma e^{\varepsilon/(\gamma+1)}T\}\int_{t-T}^{t} F^{(N,L,\kappa)}(s)\, ds$$

and using the above estimate for $F^{(N,L,\kappa)}(s)$ we arrive at

$$\int_{t-T}^{t} r(e^{\varepsilon/(\gamma+1)}(t-s))\, F^{(N,L,\kappa)}(s)\, ds \leqslant e^{-2\delta}C_T(L+t)^{\gamma+1}F^{(N,L,\kappa)}(-L+1), \qquad (15)$$

where $C_T\geqslant 1$ is a constant only depending on T but not on the choice of L,κ,δ and ε. Combining (13) with (14) and (15) we get

$$g_t^{(N)} \leqslant \mu^{-1}e^{4\delta}\int_{-L}^{t-T} e^{d_\gamma(t-s)}\, F^{(N,L,\kappa)}(s)\, ds + 2C_T(L+t)^{\gamma+1}\, F^{(N,L,\kappa)}(-L+1).$$

Next, we compare the negligible with the dominant term. For $s\in[-L+\frac12,-L+1]$ we find

$$F^{(N,L,\kappa)}(s) \geqslant (1+\gamma)\int_{-L+1-\kappa}^{-L} (s-u)^\gamma g_u^{(N)}\, du \geqslant 2^{-\gamma}F^{(N,L,\kappa)}(-L+1),$$

where we have used that $s-u\geqslant \frac12(-L+1-u)$ on the domain of integration. Hence, for $t\in[-L+T+1,L]$,

$$\int_{-L}^{t-T} e^{d_\gamma(t-s)}\, F^{(N,L,\kappa)}(s)\, ds \geqslant \int_{-L+\frac12}^{-L+1} e^{d_\gamma(t-s)}\, F^{(N,L,\kappa)}(s)\, ds \geqslant \tfrac{1}{2^{\gamma+1}}e^{d_\gamma(t+L-1)}\, F^{(N,L,\kappa)}(-L+1).$$

Consequently, there exists $T'\geqslant T+2$ only depending on C_T (and thus on T) but not on L and δ so that, for sufficiently large N and $t\in[-L+T',L]$,

$$g_t^{(N)} \leqslant \mu^{-1}e^{5\delta}\int_{-L}^{t-T} e^{d_\gamma(t-s)}\, F^{(N,L,\kappa)}(s)\, ds.$$

An analogous lower bound can be proved similarly. By switching variables we get that, for $\delta\in(0,1]$ arbitrary, there exist $T,T'>0$ such that for any $L>0$ and sufficiently large $\kappa\geqslant 2L$ one has, for N sufficiently large and $t\in[-L+T',L]$,

$$\mu^{-1}e^{-\delta}\int_{-L}^{t-T} e^{d_\gamma(t-s)}F^{(N,L,\kappa)}(s)\, ds \leqslant g_t^{(N)} \leqslant \mu^{-1}e^{\delta}\int_{-L}^{t-T} e^{d_\gamma(t-s)}F^{(N,L,\kappa)}(s)\, ds.$$

This implies that, for $-L+T'\leqslant a<b\leqslant L$,

$$e^{-2\delta}e^{d_\gamma(b-a)} \leqslant \frac{g_b^{(N)}}{g_a^{(N)}} \leqslant e^{2\delta}e^{d_\gamma(b-a)},$$

finishing the proof. □

3.2.5 Proof of Corollary 2.2

Using regular variation of (θ_j) and Lemma 3.4(iii), for any $\varkappa > 0$,

$$\sup_{j \leqslant \varkappa\beta^{\leftarrow}(n)} \theta_j = \theta_{\lfloor \varkappa\beta^{\leftarrow}(n)\rfloor} \sim \varkappa^{\gamma}(1+\gamma)\,\frac{n}{\beta^{\leftarrow}(n)}. \tag{16}$$

Plugging this into Theorem 2.1 with M in the role of $\varkappa$ gives

$$\frac{\beta^{\leftarrow}(n)}{n} \sup_{j \leqslant M\beta^{\leftarrow}(n)} \Big| n\,\mathbb{P}_n\{L_1 = j\} - \theta_j\, e^{-d_\gamma j/\beta^{\leftarrow}(n)}\Big| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence we are done with the local result once we show that

$$\sup_{j \leqslant M\beta^{\leftarrow}(n)} \Big| \theta_j \frac{\beta^{\leftarrow}(n)}{n} - (\gamma+1)\Big(\frac{j}{\beta^{\leftarrow}(n)}\Big)^{\gamma}\Big| \to 0 \quad \text{as} \quad n \to \infty.$$

Note that if $j/\beta^{\leftarrow}(n)$goes to zero, the second term inside the supremum vanishes asymptotically, and so does the first term by an application of (16) with an arbitrarily small value of $\varkappa > 0$. Hence we can assume that the supremum is over $\varepsilon\beta^{\leftarrow}(n) \leqslant j \leqslant M\beta^{\leftarrow}(n)$, for some fixed $\varepsilon > 0$. But on this domain we can exploit again that (θ_j) is regularly varying and Lemma 3.4 (iii) to obtain

$$\theta_j \frac{\beta^{\leftarrow}(n)}{n} \sim \Big(\frac{j}{\beta^{\leftarrow}(n)}\Big)^{\gamma} \theta_{\beta^{\leftarrow}(n)} \frac{\beta^{\leftarrow}(n)}{n} \sim (\gamma+1)\Big(\frac{j}{\beta^{\leftarrow}(n)}\Big)^{\gamma}$$

uniformly on the domain, which completes the proof of the local result in Corollary 2.2. To infer that this implies the global limit theorem we observe that

$$\begin{aligned}
\mathbb{P}_n\{L_1(\sigma) \leqslant x\beta^{\leftarrow}(n)\} &= \frac{1}{\beta^{\leftarrow}(n)} \sum_{j=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor} \beta^{\leftarrow}(n)\mathbb{P}_n\{L_1(\sigma) = j\} \\
&= \Bigg((\gamma+1)\,\frac{1}{\beta^{\leftarrow}(n)} \sum_{j=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor} \Big(\frac{j}{\beta^{\leftarrow}(n)}\Big)^{\gamma} e^{-d_\gamma \frac{j}{\beta^{\leftarrow}(n)}} \Bigg) + o(1).
\end{aligned}$$

The term in brackets is a Riemann sum and therefore asymptotically equal to

$$(\gamma+1)\int_0^x y^{\gamma}\, e^{-d_\gamma y}\, dy,$$

which is the distribution function of a gamma distribution with shape parameter $\gamma + 1$.

3.3 Proof of Theorem 2.3

We now derive Theorem 2.3 from Corollary 2.2 using the first two moments of $\mu_n[0, x]$, for fixed $x > 0$. The first moment is

$$\begin{aligned}
\mathbb{E}\mu_n[0, x] &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}_n\{L_k(\sigma) \leqslant x\beta^{\leftarrow}(n)\} = \mathbb{P}_n\{L_1(\sigma) \leqslant x\beta^{\leftarrow}(n)\} \\
&\sim (\gamma+1)\int_0^x y^{\gamma}\, e^{-d_\gamma y}\, dy.
\end{aligned}$$

Now let $L^{(1)} := L_1$ and $L^{(2)}$ be the length of the cycle containing the smallest index not in the cycle of one. The second moment is

$$\begin{aligned}\mathbb{E}\mu_n\big[0,x]^2 &= \frac{1}{n^2}\sum_{k=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}_n\big\{L_i = k, L_j \leqslant x\beta^{\leftarrow}(n)\big\}\\ &= \sum_{k=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor}\Big(\tfrac{n-k}{n}\,\mathbb{P}_n\big\{L^{(1)} = k, L^{(2)} \leqslant x\beta^{\leftarrow}(n)\big\} + \tfrac{k}{n}\mathbb{P}_n\{L^{(1)} = k\}\Big).\end{aligned}$$

By Corollary 2.2 we have,

$$\sum_{k=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor}\tfrac{k}{n}\mathbb{P}_n\{L^{(1)} = k\} \leqslant \frac{x^2\beta^{\leftarrow}(n)}{n}\big((\gamma+1)x^{\gamma} + o(1)\big) \longrightarrow 0.$$

To estimate the main term we use Lemma 3.2 to see that

$$\mathbb{P}_n\big\{L^{(1)} = k, L^{(2)} \leqslant x\beta^{\leftarrow}(n)\big\} = \sum_{l=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor}\mathbb{P}_n\{L_1 = k\}\mathbb{P}_{n-k}\{L_1 = l\}.$$

Using this together with Corollary 2.2 we get

$$\sum_{k=1}^{\lfloor x\beta^{\leftarrow}(n)\rfloor}\tfrac{n-k}{n}\,\mathbb{P}_n\big\{L^{(1)} = k, L^{(2)} \leqslant x\beta^{\leftarrow}(n)\big\} \sim (1+\gamma)^2\Big(\int_0^x y^{\gamma}\, e^{-d_{\gamma}y}\, dy\Big)^2,$$

which implies that the variance of $\mu_n\big[0,x]$ goes to zero. Hence the convergence in Theorem 2.3 holds in the L^2 sense, completing its proof.

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References

[BG05] A. D. Barbour and Boris L. Granovsky. Random combinatorial structures: the convergent case. *J. Combin. Theory Ser. A*, 109:203–220, 2005.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[BU09] Volker Betz and Daniel Ueltschi. Spatial random permutations and infinite cycles. *Comm. Math. Phys.*, 285:469–501, 2009.

[BU11] Volker Betz and Daniel Ueltschi. Spatial random permutations with small cycle weights. *Probab. Theory Related Fields*, 149:191–222, 2011.

[BUV11] Volker Betz, Daniel Ueltschi, and Yvan Velenik. Random permutations with cycle weights. *Ann. Appl. Probab.*, 21:312–331, 2011.

[Der13] Steffen Dereich. Random networks with preferential attachment: unfolding Bose-Einstein condensation. *In preparation*, 2013.

[DM13] Steffen Dereich and Peter Mörters. Emergence of condensation in Kingman's model of selection and mutation. *Acta Appl. Math.*, to appear, 2013.

[EU13] N.M. Ercolani and D. Ueltschi. Cycle structure of random permutations with cycle weights. *Random Structures & Algorithms*, to appear, 2013.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[GLS90] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[Lug09] Michael Lugo. Profiles of permutations. *Electron. J. Combin.*, 16:Research Paper 99, 20, 2009.

[MNZ12] Kenneth Maples, Ashkan Nikeghbali, and Dirk Zeindler. On the number of cycles in a random permutation. *Elect. Comm. in Probab.*, 17:Article 20, 2012.

[NZ13] Ashkan Nikeghbali and Dirk Zeindler. The generalized weighted probability measure on the symmetric group and the asymptotic behavior of the cycles. *Annales de l'Institut Poincaré: Probab. et Stat.*, to appear, 2013.