Ageing in the parabolic Anderson model

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Abstract: The parabolic Anderson model is the Cauchy problem for the heat equation with a random potential. We consider this model in a setting which is continuous in time and discrete in space, and focus on time-constant, independent and identically distributed potentials with polynomial tails at infinity. We are concerned with the long-term temporal dynamics of this system. Our main result is that the periods, in which the profile of the solutions remains nearly constant, are increasing linearly over time, a phenomenon known as ageing. We describe this phenomenon in the weak sense, by looking at the asymptotic probability of a change in a given time window, and in the strong sense, by identifying the almost sure upper envelope for the process of the time remaining until the next change of profile. We also prove functional scaling limit theorems for profile and growth rate of the solution of the parabolic Anderson model.

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1 Introduction

1.1 Motivation and overview

The long term dynamics of disordered complex systems out of equilibrium have been the subject of great interest in the past decade. A key paradigm in this research programme is the notion of ageing. Roughly speaking, in an ageing system the probability that there is no essential change of the state between time t and time $t + s(t)$ is of constant order for a period $s(t)$ which depends increasingly, and often linearly, on the time t. Hence, as time goes on, in an ageing system changes become less likely and the typical time scales of the system are increasing. Therefore, as pointed out in [BF05], ageing can be associated to the existence of infinitely many time-scales that are inherently relevant to the system. In that respect, ageing systems are distinct from metastable systems, which are characterized by a finite number of well separated time-scales, corresponding to the lifetimes of different metastable states.

Ageing systems are typically rather difficult to analyse analytically. Most results to date concern either the Langevin dynamics of relatively simple mean field spin glasses, see e.g. [BADG01], or phenomenological models like the class of *trap models*, see e.g. [Bou92, Čer06, GMW09]. The idea behind the latter is to represent a physical system as a particle moving in a random energy landscape with infinitely many valleys, or traps. Given the landscape, the particle moves according to a continuous time random walk remaining at each trap for an exponential time with a rate proportional to its depth. While there is good experimental evidence for the claim that trap models capture the dynamical behaviour of many more complex systems, a rigorous mathematical derivation of this fact exists only in very few cases.

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Two recent papers, Dembo and Deuschel [DD07] and Aurzada and Doering [AD09], investigate weaker forms of ageing based on correlations. Both deal with a class of models which includes as a special case a parabolic Anderson model with time-variable potential and show absence of correlation-based ageing in this case. While this approach is probably the only way to deal rigorously with complicated models, it is not established that the effect picked up by these studies is actually really due to the existence or absence of ageing in our sense, or whether other moment effects are accountable.

In the present work we show that the parabolic Anderson model exhibits ageing behaviour, at least if the underlying random potential is sufficiently heavy-tailed. As a lattice model with random disorder the parabolic Anderson model is a model of significant complexity, but its linearity and strong localization features make it considerably easier to study than, for example, the dynamics of most non-mean field spin glass models.

Our work has led to three main results. The first one, Theorem 1.1, shows that the probability that during the time window $[t, t+\theta t]$ the profiles of the solution of the parabolic Anderson problem remain within distance $\varepsilon > 0$ of each other converges to a constant $I(\theta)$, which is strictly between zero and one. This shows that ageing holds on a linear time scale. Our second main result, Theorem 1.3, is an almost sure ageing result. We define a function $R(t)$ which characterizes the waiting time starting from time t until the profile changes again. We determine the precise almost sure upper envelope of $R(t)$ in terms of an integral test. The third main result, Theorem 1.6, is a functional scaling limit theorem for the location of the peak, which determines the profile, and for the growth rate of the solution. We give the precise statements of the results in Section 1.2, and in Section 1.3 we provide a rough guide to the proofs.

1.2 Statement of the main results

The parabolic Anderson model is given by the heat equation on the lattice $\mathbb{Z}^d$ with a random potential, i.e. we consider the solution $u\colon (0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z)\,, && \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d\,,\\ \lim_{t\downarrow 0} u(t,z) &= \mathbb{1}_0(z)\,, && \text{for } z\in\mathbb{Z}^d\,. \end{aligned}$$

Here Δ is the discrete Laplacian

$$\Delta f(x) = \sum_{\substack{y\in\mathbb{Z}^d\\ y\sim x}} \big(f(y)-f(x)\big)\,,$$

and $y\sim x$ means that y is a nearest-neighbour of site x. The potential $\xi = (\xi(z)\colon z\in\mathbb{Z}^d)$ is a collection of independent, identically distributed random variables, which we assume to be Pareto-distributed for some $\alpha > d$, i.e.

$$\mathrm{Prob}\{\xi(z)\le x\} = 1-x^{-\alpha}\,,\quad \text{for } x\ge 1\,.$$

The condition $\alpha > d$ is necessary and sufficient for the Cauchy problem to have a unique, nonnegative solution, see [GM90]. We write

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z) \qquad \text{for } t\ge 0,$$

for the *total mass* of the solution (which is finite at all times) and

$$v(t,z) = \frac{u(t,z)}{U(t)} \qquad \text{for } t\ge 0, z\in\mathbb{Z}^d,$$

for its *profile*. It is not hard to see that the total mass grows superexponentially in time. Our interest is therefore focused on the changes in the profile of the solution.

1.2.1 Ageing: a weak limit theorem

Our first ageing result is a weak limit result. We show that for an observation window whose size is growing linearly in time, the probability of seeing no change during the window converges to a nontrivial value. The same limit is obtained when only the states at the endpoints of the observation window are considered.

Theorem 1.1. *For any $\theta > 0$ there exists $I(\theta) > 0$ such that, for all sufficiently small $\varepsilon > 0$,*

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{R}^d} \sup_{s\in[t,t+t\theta]} \big|v(t,z) - v(s,z)\big| < \varepsilon \Big\} \\ = \lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{R}^d} \big|v(t,z) - v(t+t\theta,z)\big| < \varepsilon \Big\} \\ = I(\theta).\end{aligned}$$

Remark 1.2.

- Note that we only have one ageing regime, which is contrast to the behaviour of the unsymmetric trap models described in [BAČ05]
- An integral representation of $I(\theta)$ will be given in Proposition 2.4, which shows that the limit is not derived from the generalized arcsine law as in the universal scheme for trap models described in [BAČ08]. In Proposition 2.5, we show that there are positive constants C_0, C_1 such that

$$\lim_{\theta\downarrow 0} \theta^{-1}\big(1 - I(\theta)\big) = C_0 \qquad \text{and} \qquad \lim_{\theta\uparrow\infty} \theta^d\, I(\theta) = C_1\,.$$

1.2.2 Ageing: an almost-sure limit theorem

The crucial ingredient in our ageing result is the fact that in the case of Pareto distributed potentials the profile of the solution of the parabolic Anderson problem can be essentially described by one parameter, the location of its *peak*. This is due to the one-point localization theorem [KLMS09, Theorem 1.2] which states that, for any $\mathbb{Z}^d$-valued process $(X_t\colon t \geq 0)$ with the property that $v(t, X_t)$ is the maximum value of the profile at time t, we have

$$v(t, X_t) \to 1 \text{ in probability.} \tag{1}$$

In other words, asymptotically the profile becomes completely localized in its peak. Assume for definiteness that $t \mapsto X_t$ is right-continuous and define the *residual lifetime* function by $R(t) = \sup\{s \geq 0\colon X_t = X_{t+s}\}$, for $t \geq 0$. Roughly speaking, $R(t)$ is the waiting time, at time t, until the next change of peak, see the schematic picture in Figure 1. We have shown in Theorem 1.1 that the law of $R(t)/t$ converges to the law given by the distribution function $1 - I$. In the following theorem, we describe the smallest asymptotic upper envelope for the process $(R(t)\colon t \geq 0)$.

Theorem 1.3 (Almost sure ageing)**.** *For any nondecreasing function $h\colon (0,\infty) \to (0,\infty)$ we have, almost surely,*

$$\limsup_{t\to\infty} \frac{R(t)}{th(t)} = \begin{cases} 0 & \text{if } \displaystyle\int_1^\infty \frac{\mathrm{d}t}{th(t)^d} < \infty, \\ \infty & \text{if } \displaystyle\int_1^\infty \frac{\mathrm{d}t}{th(t)^d} = \infty. \end{cases}$$

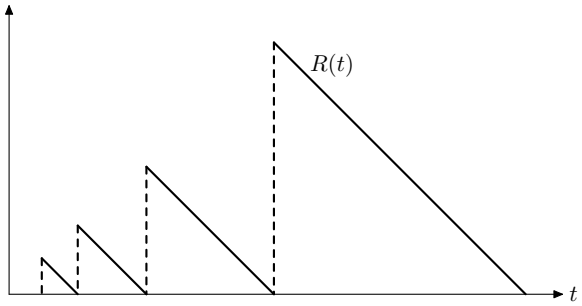


Figure 1: A schematic representation of the residual lifetime function R.

1.2.3 A functional scaling limit theorem

To complete the discussion of the temporal behaviour of the solution it is natural to look for a functional limit theorem under suitable space-time scaling of the solution. From [HMS08, Theorem 1.2] we know that there are heavy fluctuations even in the *logarithm* of the total mass, as we have for $t \uparrow \infty$,

$$\frac{(\log t)^{\frac{d}{\alpha-d}}}{t^{\frac{\alpha}{\alpha-d}}} \log U(t) \Rightarrow Y\,, \tag{2}$$

where Y is a random variable of extremal Fréchet type with shape parameter $\alpha - d$. We therefore focus on the profile of the solution and extend it to $(0,\infty)\times\mathbb{R}^d$ by taking the integer parts of the second coordinate, letting $v(t,x) = v(t,\lfloor x\rfloor)$. Taking nonnegative measurable functions on $\mathbb{R}^d$ as densities with respect to the Lebesgue measure, we can interpret $a^d v(t,ax)$ for any $a,t>0$ as an element of the space $\mathcal{M}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$. By $\delta(y)\in\mathcal{M}(\mathbb{R}^d)$ we denote the Dirac point mass located in $y\in\mathbb{R}^d$.

Proposition 1.4 (Convergence of the scaled profile to a wandering point mass)**.** *There exists a nondegenerate stochastic process $(Y_t\colon t>0)$ such that, as $T\uparrow\infty$, the following functional scaling limit holds,*

$$\Big(\big(\tfrac{T}{\log T}\big)^{\frac{\alpha d}{\alpha-d}}\, v\big(tT, \big(\tfrac{T}{\log T}\big)^{\frac{\alpha}{\alpha-d}} x\big)\colon t>0\Big) \Rightarrow \big(\delta(Y_t)\colon t>0\big)\,, \tag{3}$$

in the sense of convergence of finite dimensional distributions on the space $\mathcal{M}(\mathbb{R}^d)$ equipped with the weak topology.

Remark 1.5. The process $(Y_t\colon t>0)$ will be described explicitly in and after Remark 1.7 (iii).

In this formulation of a scaling limit theorem the mode of convergence is not optimal. Also, under the given scaling, islands of diameter $o((\frac{t}{\log t})^{\frac{\alpha}{\alpha-d}})$ at time t would still be mapped onto single points, and hence the spatial scaling is not sensitive to the one-point localization described in the previous section. We now state an optimal result in the form of a functional scaling limit theorem in the Skorokhod topology for the localization point itself. Additionally, we prove joint convergence of the localization point together with the value of the potential there. This leads to a Markovian limit process which is easier to describe, and from which the non-Markovian process $(Y_t\colon t>0)$ can be derived by projection. This approach also yields an extension of (2) to a functional limit theorem. Here and in the following we denote by $|x|$ the ℓ^1-norm of $x\in\mathbb{R}^d$.

Theorem 1.6 (Functional scaling limit theorem)**.**
There exists a time-inhomogeneous Markov process $((Y_t^{(1)},Y_t^{(2)})\,:\,t>0)$ on $\mathbb{R}^d\times\mathbb{R}$ such that,

(a) as $T\to\infty$, we have

$$\Big(\big(\big(\tfrac{\log T}{T}\big)^{\frac{\alpha}{\alpha-d}} X_{tT}, \big(\tfrac{\log T}{T}\big)^{\frac{d}{\alpha-d}}\xi(X_{tT})\big)\,:\,t>0\Big) \Rightarrow \Big(\big(Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha-d}|Y_t^{(1)}|\big)\,:\,t>0\Big)\,,$$

in distribution on the space $D(0,\infty)$ of càdlàg functions $f\colon (0,\infty) \to \mathbb{R}^d \times \mathbb{R}$ with respect to the Skorokhod topology on compact subintervals;

(b) as $T \to \infty$, we have

$$\Big(\big(\tfrac{\log T}{T}\big)^{\frac{d}{\alpha-d}} \tfrac{\log U(tT)}{tT} \,:\, t>0 \Big) \Rightarrow \big(Y_t^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{t}\big)|Y_t^{(1)}| \,:\, t>0 \big)\,,$$

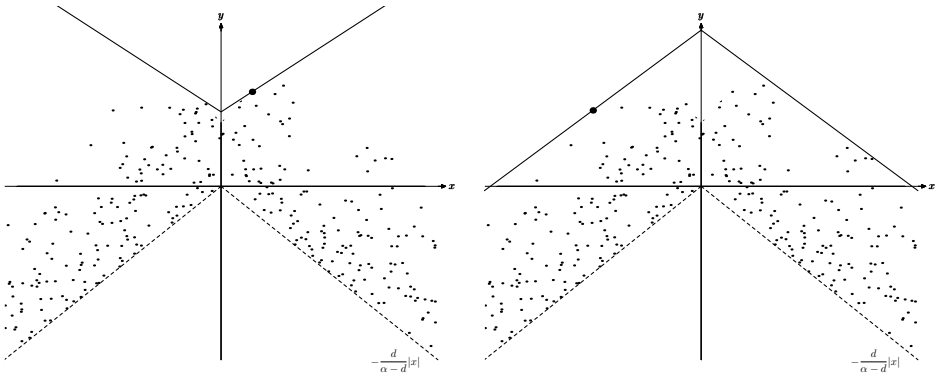
in distribution on the space $C(0,\infty)$ of continuous functions $f\colon (0,\infty) \to \mathbb{R}$ with respect to the uniform topology on compact subintervals.

Remark 1.7.

(i) Projecting the process onto the first component at time $t = 1$ we recover the result of [KLMS09, Theorem 1.3]. This result shows in particular that the peak X_t of the profile escapes with superlinear speed.

(ii) From the proof of this result it is easy to see that the convergence in both parts of Theorem 1.6 also holds simultaneously on the space of càdlàg functions $f\colon (0,\infty) \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ with respect to the Skorokhod topology on compact subintervals.

(iii) The process $(Y_t\colon t>0)$ in Proposition 1.4 is is equal to the projected process $(Y_t^{(1)}\colon t>0)$.



(a) $t < 1$. (b) $t > 1$.

Figure 2: The definition of the process $(Y_t^{(1)}, Y_t^{(2)})$ in terms of the point process Π. Note that t parametrizes the opening angle of the cone, see (a) for $t < 1$ and (b) for $t > 1$.

In order to describe the limit process we need to introduce some notation. Denote by Π a Poisson point process on $H^0 = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}\colon y > -\frac{d}{\alpha-d}|x|\}$ with intensity measure

$$\nu(\mathrm{d}x\,\mathrm{d}y) = \mathrm{d}x \otimes \frac{\alpha \mathrm{d}y}{(y + \frac{d}{\alpha-d}|x|)^{\alpha+1}}.$$

Given the point process, we can define an $\mathbb{R}^d$-valued process $Y_t^{(1)}$ and an $\mathbb{R}$-valued process $Y_t^{(2)}$ in the following way. Fix $t > 0$ and define the open cone with tip $(0,z)$

$$\mathcal{C}_t(z) = \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y + \tfrac{d}{\alpha-d}(1-\tfrac{1}{t})|x| > z\big\}\,,$$

and let

$$\mathcal{C}_t = \mathrm{cl}\Big(\bigcup_{\substack{z>0 \\ \Pi(\mathcal{C}_t(z))=0}} \mathcal{C}_t(z) \Big)\,.$$

Informally, $\mathcal{C}_t$ is the closure of the first cone $\mathcal{C}_t(z)$ that 'touches' the point process as we decrease z from infinity. Since $\mathcal{C}_t \cap \Pi$ contains at most two points, we can define $(Y_t^{(1)}, Y_t^{(2)})$ as the point in

this intersection whose projection on the first component has the largest ℓ^1-norm, see Figures 2(a) and 2(b) for illustration. The resulting process $((Y_t^{(1)}, Y_t^{(2)})\colon t > 0)$ is an element of $D(0, \infty)$.

The derived processes in Theorem 1.6 can be described as follows:

- $((Y_t^{(2)} + \frac{d}{\alpha - d}|Y_t^{(1)}|)\colon t > 0)$ corresponds to the *vertical distance* of the point $(Y_t^{(1)}, Y_t^{(2)})$ to the boundary of the domain given by the curve $y = -\frac{d}{\alpha-d}|x|$;
- $((Y_t^{(2)} + (1 - \frac{1}{t})|Y_t^{(1)}|)\colon t > 0)$ corresponds to the *y-coordinate of the tip* of the cone $\mathcal{C}_t$.

Remark 1.8. *Time evolution of the process.*
$(Y_1^{(1)}, Y_1^{(2)})$ is the 'highest' point of the Poisson point process Π. Given $(Y_t^{(1)}, Y_t^{(2)})$ and $s \geq t$ we consider the surface given by all $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ such that

$$y = Y_t^{(2)} - \tfrac{d}{\alpha-d}\big(1 - \tfrac{1}{s}\big)(|x| - |Y_t^{(1)}|)\,.$$

For $s = t$ there are no points of Π above this surface, while $(Y_t^{(1)}, Y_t^{(2)})$ (and possibly one further point) is lying on it. We now increase the parameter s until the surface hits a further point of Π. At this time $s > t$ the process jumps to this new point $(Y_s^{(1)}, Y_s^{(2)})$. Geometrically, increasing s means opening the cone further, while keeping the point $(Y_t^{(1)}, Y_t^{(2)})$ on the boundary and moving the tip upwards on the y-axis. Similarly, given the point $(Y_t^{(1)}, Y_t^{(2)})$ one can go backwards in time by decreasing s, or equivalently closing the cone and moving the tip downwards on the y-axis. The general independence properties of Poisson processes ensure that this procedure yields a process $((Y_t^{(1)}, Y_t^{(2)})\colon t > 0)$ which is Markovian in both the forward and backward direction. The process $(Y_t^{(2)} + \frac{d}{\alpha-d}(1 - \frac{1}{t})|Y_t^{(1)}|\colon t > 0)$ is continuous, which can be seen directly from its interpretation as the y-coordinate of the tip of the cone. An animation of the process $((Y_t^{(1)}, Y_t^{(2)})\colon t > 0)$ can be found on the second author's homepage at http://people.bath.ac.uk/maspm/animation_ageing.pdf.

1.3 Strategy of the proofs and overview

Let us first collect some of the key ingredients common to the proofs of our three main results. It is shown in [KLMS09] that, almost surely, for all large t the total mass $U(t)$ can be approximated by a variational problem. More precisely,

$$\frac{1}{t}\log U(t) \sim \max_{z \in \mathbb{Z}^d} \Phi_t(z)\,, \tag{4}$$

where, for any $t \geq 0$, the functional Φ_t is defined as

$$\Phi_t(z) = \xi(z) - \frac{|z|}{t}\log \xi(z) + \frac{\eta(z)}{t},$$

for $z \in \mathbb{Z}^d$ with $t\xi(z) \geq |z|$, and $\Phi_t(z) = 0$ for other values of z. Here $\eta(z)$ is the logarithm of the number of paths of length $|z|$ leading from 0 to z.

Furthermore, [KLMS09] show that the peaks X_t agree for most times t with the maximizer Z_t of the functional Φ_t. This maximizer is uniquely defined, if we impose the condition that $t \mapsto Z_t$ is right-continuous. Defining the two scaling functions

$$r_t = \big(\tfrac{t}{\log t}\big)^{\frac{\alpha}{\alpha-d}} \quad \text{and} \quad a_t = \big(\tfrac{t}{\log t}\big)^{\frac{d}{\alpha-d}},$$

it is shown in [KLMS09], refining the argument of [HMS08], that, as $t \to \infty$, the point process

$$\Pi_t = \sum_{\substack{z \in \mathbb{Z}^d \\ t\xi(z) \geq |z|}} \delta_{(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t})} \tag{5}$$

converges (in a suitable sense) to the Poisson point process Π on H_0 defined above.

Section 2 is devoted to the proof of the 'annealed' ageing result, Theorem 1.1. We show in Section 2.2, see Lemma 2.9, that

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{R}^d} \sup_{s\in[t,t+t\theta]} \big|v(t,z)-v(s,z)\big| < \varepsilon \Big\} = \lim_{t\to\infty} \mathrm{Prob}\big\{ Z_t = Z_{t+t\theta} \big\}.$$

Therefore we begin this proof, in Section 2.1, by discussing the limit on the right hand side. To this end we approximate the probability in terms of the point process Π_t. We are able to write

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\Phi_t(z)}{a_t} + \frac{\theta}{1+\theta}\,\frac{d}{\alpha-d}\,\frac{|z|}{r_t} + \text{error}, \tag{6}$$

where the error can be suitably controlled, see Lemma 2.3. Hence (in symbolic notation)

$$\begin{aligned}&\mathrm{Prob}\big\{ Z_t = Z_{t+t\theta} \big\} \\ &\quad\approx \iint \mathrm{Prob}\Big\{ \Pi_t(\mathrm{d}x\,\mathrm{d}y) > 0, \Pi_t\{(\bar x,\bar y)\colon \bar y > y\} = 0, \\ &\qquad\qquad \Pi_t\{(\bar x,\bar y)\colon |\bar x| > |x| \text{ and } \bar y > y - \tfrac{d}{\alpha-d}\tfrac{\theta}{1+\theta}(|\bar x|-|x|)\} = 0 \Big\},\end{aligned}$$

where the first line of conditions on the right means that x is a maximizer of Φ_t with maximum y, and the second line means that x is also a maximizer of $\Phi_{t+\theta t}$. As $t\uparrow\infty$ the point process Π_t is replaced by Π and we can evaluate the probability.

Section 3 is devoted to the 'quenched' ageing result, Theorem 1.3. This proof is technically more involved, because we cannot exploit the point process approach and have to do significant parts of the argument from first principles. We now have to consider events

$$\mathrm{Prob}\Big\{ \frac{R(t)}{t} \ge \theta_t \Big\} \approx \mathrm{Prob}\big\{ Z_t = Z_{t+t\theta_t} \big\},$$

for $\theta_t\uparrow\infty$. We have to significantly refine the argument above and replace the convergence of $\mathrm{Prob}\{Z_t = Z_{t+t\theta}\}$ by a moderate deviation statement, see Section 3.1. Indeed, for $\theta_t\uparrow\infty$ not too fast we show that

$$\mathrm{Prob}\big\{ Z_t = Z_{t+t\theta_t} \big\} \sim C\,\theta_t^{-d},$$

for a suitable constant $C>0$, see Proposition 3.1. Then, if $\varphi(t) = th(t)$, this allows us to show in Sections 3.2 and 3.3 that, for any $\varepsilon>0$, the series $\sum_n \mathrm{Prob}\{R(e^n)\ge\varepsilon\varphi(e^n)\}$ converges if $\sum_n h(e^n)^{-d}$ converges, which is essentially equivalent to $\int h(t)^{-d}\mathrm{d}t/t<\infty$. By Borel-Cantelli we get that

$$\limsup_{n\to\infty} \frac{R(e^n)}{\varphi(e^n)} = 0,$$

which implies the upper bound in Theorem 1.3, and the lower bound follows similarly using a slightly more delicate second moment estimate, see Lemma 3.5.

The proofs of the scaling limit theorems, Proposition 1.4 and Theorem 1.6 are given in Section 4. By (6) we can describe Z_{tT} approximately as the maximizer of

$$\frac{\Phi_T(z)}{a_T} + \frac{d}{\alpha-d}\Big(1-\frac{1}{t}\Big)\frac{|z|}{r_T}.$$

Instead of attacking the proof of Theorem 1.6 directly, we first show in Sections 4.1 and 4.2 a limit theorem for

$$\Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T}\big) \,:\, t>0 \Big), \tag{7}$$

see Proposition 4.1. Informally, we obtain

$$\begin{aligned}&P\big\{ \tfrac{Z_{tT}}{r_T}\in A,\ \tfrac{\Phi_{tT}(Z_{tT})}{a_T}\in B \big\} \\ &\quad\approx \iint\limits_{\substack{x\in A,\\ y+q(1-\frac1t)|x|\in B}} \mathrm{Prob}\Big\{ \Pi_T(\mathrm{d}x\,\mathrm{d}y)>0,\ \Pi_T\big\{(\bar x,\bar y)\colon \bar y - y > \tfrac{d}{\alpha-d}\big(1-\tfrac1t\big)(|x|-|\bar x|)\big\}=0 \Big\},\end{aligned}$$

where the first line of conditions on the right means that there is a site $z \in \mathbb{Z}^d$ such that $x = z/r_T \in A$ and $y = \Phi_T(z)/a_T \in B - q(1-\frac{1}{t})|x|$, and the second line means that $\Phi_{tT}(z)$ is not surpassed by $\Phi_{tT}(\bar{z})$ for any other site $\bar{z} \in \mathbb{Z}^d$ with $\bar{x} = \bar{z}/r_T$. We can then use the convergence of Π_T to Π inside the formula to give a limit theorem for the one-dimensional distributions of (7). A minor strengthening of this argument given in Section 4.1 shows convergence of the finite dimensional distributions, see Lemma 4.2. In Section 4.2 we check a tightness criterion in Skorokhod space, see Lemma 4.5, and thus complete the proof of the convergence

$$\Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T} \big) \,:\, t > 0 \Big) \Rightarrow \Big(\big(Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha - d}(1 - \tfrac{1}{t})|Y_t^{(1)}| \big) \,:\, t > 0 \Big) .$$

Based on this result we complete the proof of the scaling limit results in Section 4.3. Theorem 1.6 (b) follows using (4) and projecting on the second component. Observe that the convergence in (b) automatically holds in the uniform sense, as all involved processes are continuous. We note further that

$$\frac{\xi(z)}{a_T} = \frac{\Phi_T(z)}{a_T} + \frac{d}{\alpha - d} \frac{|z|}{r_T} + \text{error},$$

see Lemma 4.6. This allows us to deduce Theorem 1.6 (a), and Proposition 1.4 is an easy consequence of this.

2 Ageing: a weak limit theorem

This section is devoted to the proof of Theorem 1.1. In Section 2.1 we show ageing for the two point function of the process $(Z_t \colon t \geq 0)$ of maximizers of the variational problem Φ_t, using the point process approach which was developed in [HMS08] and extended in [KLMS09]. In Section 2.2 we use this and the localization of the profile in Z_t to complete the proof.

2.1 Ageing for the maximizer of Φ_t

In this section, we prove ageing for the two point function of the process $(Z_t \colon t \geq 0)$, which from now on is chosen to be left-continuous. The value $I(\theta)$ will be given by the formula in Proposition 2.4 below.

Proposition 2.1. *Let $\theta > 0$, then* $\lim_{t\to\infty} \mathrm{Prob}\big\{ Z_t = Z_{t+\theta t} \big\} = I(\theta) \in (0,1)$.

Throughout the proofs we use the abbreviation

$$q = \frac{d}{\alpha - d} .$$

For any $t > 0$ consider the point process Π_t on $\mathbb{R}^d \times \mathbb{R}$ defined in (5). Define a locally compact Borel set

$$\widehat{H} = \dot{\mathbb{R}}^{d+1} \setminus \big(\{ (x,y) \in \mathbb{R}^d \times \mathbb{R} \colon y < -q(1-\varepsilon)|x| \} \cup \{0\} \big) ,$$

where $0 < \varepsilon < \frac{1}{1+\theta}$ and $\dot{\mathbb{R}}^{d+1}$ is the one-point compactification of $\mathbb{R}^{d+1}$. As in Lemma 6.1 of [KLMS09] one can show that the point process Π_t restricted to the domain $\widehat{H}$ converges in law to a Poisson process Π on $\widehat{H}$ with intensity measure

$$\nu(\mathrm{d}x\, \mathrm{d}y) = \frac{\alpha\, \mathrm{d}x\, \mathrm{d}y}{(y + q|x|)^{\alpha+1}} . \tag{8}$$

Here, Π_t and Π are random elements of the set of point measures on $\widehat{H}$, which is given the topology of vague convergence. For more background on point processes and similar arguments, see [HMS08].

Our strategy is to express the condition $Z_t = Z_{t+\theta t}$ in terms of the point process Π_t. In order to be able to bound error functions that appear in our calculations, we have to restrict our attention to the point process Π on a large box. To this end, define the two boxes

$$B_N = \{(x,y) \in \mathbb{R}^d \times [0,\infty) \,:\, |x| \le N,\, \tfrac{1}{N} \le y \le N\}\,,$$
$$\widehat{B}_N = \{(x,y) \in \widehat{H} \,:\, |x| \le N, y \le N\}\,.$$

Now note that the condition $Z_t = Z_{t+\theta t}$ means that

$$\Phi_{t+\theta t}(z) \le \Phi_{t+\theta t}(Z_t)\,, \tag{9}$$

for all $z \in \mathbb{Z}^d$. We now show that it suffices to guarantee that this condition holds for all z in a sufficiently large bounded box.

Lemma 2.2. *Define the event*

$$A(N,t) = \Big\{ \big(\tfrac{Z_t}{r_t}, \tfrac{\Phi_t(Z_t)}{a_t}\big) \in B_N\,, \Phi_{t+\theta t}(z) \le \Phi_{t+\theta t}(Z_t)\, \forall z \in \mathbb{Z}^d \text{ s.t. } \big(\tfrac{|z|}{r_t}, \tfrac{\Phi_t(z)}{a_t}\big) \in \widehat{B}_N \Big\}\,.$$

Then, provided the limit on the right-hand side exists, we find that

$$\lim_{t\to\infty} \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} = \lim_{N\to\infty} \lim_{t\to\infty} \mathrm{Prob}(A(N,t))\,.$$

Proof. We have the lower bound,

$$\begin{aligned}\mathrm{Prob}\{Z_t = Z_{t+\theta t}\} &\ge \mathrm{Prob}\big\{Z_t = Z_{t+\theta t}\,,\, \big(\tfrac{Z_t}{r_t}, \tfrac{\Phi_t(Z_t)}{a_t}\big) \in B_N\big\} \\ &\ge \mathrm{Prob}(A(N,t)) - \mathrm{Prob}\big\{\tfrac{|Z_{t+\theta t}|}{r_t} > N\big\}\,.\end{aligned}$$

Recall that, by [KLMS09, Lemma 6.2], we have that

$$\big(\tfrac{Z_t}{r_t}, \tfrac{\Phi_t(Z_t)}{a_t}\big) \Rightarrow (Y^{(1)}, Y^{(2)})\,, \tag{10}$$

where $(Y^{(1)}, Y^{(2)})$ is a random variable on $\mathbb{R}^d \times [0,\infty)$ with an explicit density. In particular, we find that since $r_{t+\theta t} = (1+\theta)^{q+1} r_t (1+o(1))$

$$\lim_{t\to\infty} \mathrm{Prob}\big\{\tfrac{|Z_{t+\theta t}|}{r_t} > N\big\} = \mathrm{Prob}\big\{|Y^{(1)}| > \tfrac{N}{(1+\theta)^{q+1}}\big\}\,,$$

which converges to zero as $N \to \infty$.

Now, for an upper bound on $\mathrm{Prob}\{Z_t = Z_{t(1+\theta)}\}$ we find that

$$\mathrm{Prob}\{Z_t = Z_{t(1+\theta)}\} \le \mathrm{Prob}(A(N,t)) + \mathrm{Prob}\big\{\tfrac{|Z_t|}{r_t} \ge N\big\} + \Big(1 - \mathrm{Prob}\big\{\tfrac{1}{N} \le \tfrac{\Phi_t(Z_t)}{a_t} \le N\big\}\Big)\,.$$

As above, using the convergence (10) one can show that the limit of the last two summands is zero when taking first $t \to \infty$ and then $N \to \infty$, which completes the proof of the lemma. □

We would like to translate the condition (9) into a condition on the point process Π_t. Therefore, we have to express $\Phi_{t+\theta t}(z)$ in terms of $\Phi_t(z)$.

Lemma 2.3. *For any $z \in \mathbb{Z}^d$ such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) \ge |z|$,*

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\Phi_t(z)}{a_t} + \frac{q\theta}{1+\theta}\frac{|z|}{r_t} + \delta_\theta\big(t, \tfrac{|z|}{r_t}, \tfrac{\Phi_t(z)}{a_t}\big)\,,$$

where the error δ_θ converges to zero as $t \to \infty$ uniformly. Moreover, almost surely, eventually for all large enough t, for all $z \in \mathbb{Z}^d$ such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) < |z|$, we have that $\Phi_{t+\theta t}(z) \le 0$, and such a $z \in \mathbb{Z}^d$ will automatically satisfy (9).

Proof. Consider any z such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) \geq |z|$. Then, using that $r_t = \frac{t}{\log t}\, a_t$ we obtain

$$\begin{aligned}\frac{\Phi_{t+\theta t}(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{1}{a_t t + \theta t}\big(|z| \log \xi(z) - \eta(z)\big) \\ &= \frac{\Phi_t(z)}{a_t} + \frac{\theta}{1+\theta}\frac{|z|}{r_t \log t} \log a_t + \frac{\theta}{1+\theta}\Big(\frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} - \frac{\eta(z)}{t a_t}\Big) \\ &= \frac{\Phi_t(z)}{a_t} + \frac{\theta q}{1+\theta}\frac{|z|}{r_t} + \delta'_\theta\big(t, \tfrac{z}{r_t}, \tfrac{\xi(z)}{a_t}\big)\,,\end{aligned} \tag{11}$$

where using that $\log a_t = (q + o(1)) \log t$ and $0 \leq \eta(z) \leq |z| \log d$, we can write

$$\delta'_\theta\big(t, \tfrac{z}{r_t}, \tfrac{\xi(z)}{a_t}\big) = \frac{\theta}{1+\theta}\Big(\frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + o(1)\frac{|z|}{r_t}\Big)\,. \tag{12}$$

First of all, we have to show that this expression is of the form $\delta_\theta(t, z/r_t, \Phi_t(z)/a_t)$ for some suitable error function. With this in mind, using that $a_t t = r_t \log t$, we obtain for z such that $t\xi(z) \geq |z|$

$$\begin{aligned}\frac{\Phi_t(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{|z|}{r_t \log t} \log \xi(z) + \frac{\eta(z)}{a_t t} \\ &= \frac{\xi(z)}{a_t} - (q + o(1))\frac{|z|}{r_t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + \frac{\eta(z)}{a_t t}\,, \\ &= \chi_\rho\big(\tfrac{\xi(z)}{a_t}\big) - (q + o(1))\frac{|z|}{r_t},\end{aligned}$$

where $\chi_\rho(x) = x - \rho \log x$ and $\rho = \frac{|z|}{r_t \log t}$. Note that χ_ρ is strictly increasing on $[\rho, \infty)$ and also that $\xi(z)/a_t > \rho$ is equivalent to $t\xi(z) > |z|$ which is satisfied by assumption. Therefore, we can write

$$\frac{\xi(z)}{a_t} = \chi_\rho^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\tfrac{|z|}{r_t}\big)\,,$$

and obtain that the error in (12) is of the required form

$$\begin{aligned}\delta'_\theta\big(t, \tfrac{z}{r_t}, \tfrac{\xi(z)}{a_t}\big) &= \frac{\theta}{1+\theta}\Big(\frac{|z|}{r_t \log t} \log \chi_\rho^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\tfrac{|z|}{r_t}\big) + o(1)\frac{|z|}{r_t}\Big) \\ &=: \delta_\theta\big(t, \tfrac{z}{r_t}, \tfrac{\Phi_t(z)}{a_t}\big)\,.\end{aligned} \tag{13}$$

We now show that this error tends to zero uniformly for all z satisfying $t\xi(z) > |z|$ and $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$. For a lower bound we first use that $x \log x \geq -e^{-1}$ to obtain

$$\begin{aligned}&\frac{|z|}{r_t \log t} \log \chi_\rho^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\tfrac{|z|}{r_t}\big) \\ &\qquad \geq \frac{|z|}{r_t \log t} \log \frac{|z|}{r_t \log t} \geq -\frac{1}{\log t}\, e^{-1} - \frac{\log\log t}{\log t}\frac{|z|}{r_t} \geq -\frac{1}{\log t}\, e^{-1} - \frac{\log\log t}{\log t} N\,.\end{aligned}$$

To bound the expression in (13) from above note that $\rho = \frac{|z|}{r_t \log t} \leq \frac{N}{\log t}$ and we can thus assume that $\rho < 1$, which implies that for $x > 1$ we find $\chi_1(x) \leq \chi_\rho(x)$. Hence, either

$$\chi_\rho^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\, \tfrac{|z|}{r_t}\big) \leq 1\,,$$

or we can estimate

$$\chi_\rho^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\, \tfrac{|z|}{r_t}\big) \leq \chi_1^{-1}\big(\tfrac{\Phi_t(z)}{a_t} + (q + o(1))\, \tfrac{|z|}{r_t}\big) \leq \chi_1^{-1}\big((N(1 + 2q)\big)\,.$$

which completes the proof of the first part of the lemma.

For the second part, recall that for all $t > 0$ we have $\Phi_t(Z_t) > 0$, since $\Phi_t(0) > 0$. Suppose $t\xi(z) < |z|$, then $\Phi_t(z) = 0$ and hence $z \neq Z_t$. We want to show that $\Phi_{t+\theta t}(z) \leq 0$ which ensures that z satisfies (9). Indeed, if $(t+\theta t)\xi(z) < |z|$, then this is true as $\Phi_{t+\theta t}(z) = 0$, and otherwise we can estimate as above that

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\xi(z)}{a_t} - \frac{q\theta}{1+\theta}\frac{|z|}{r_t} + \tilde{\delta}_\theta\big(t, \tfrac{|z|}{a_t}, \tfrac{\Phi_t(z)}{a_t}\big)\,,$$

where $\tilde{\delta}_\theta(t,x,y)$ converges to zero uniformly in $(x,y) \in \widehat{B}_N$. In particular, it follows that

$$\frac{\Phi_{t+\theta t}(z)}{a_t} \leq \Big(-\frac{q\theta}{1+\theta} + \frac{1}{\log t}\Big)\frac{|z|}{r_t} + \tilde{\delta}_\theta\big(t, \tfrac{|z|}{a_t}, \tfrac{\Phi_t(z)}{a_t}\big)\,,$$

which is negative for all t large enough, uniformly for all z such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in B_N$. □

We now calculate $\mathrm{Prob}(A(t,N))$ in the limit as $t \to \infty$, i.e. we are interested in

$$\iint\limits_{(x,y)\in B_N} \mathrm{Prob}\Big\{ \tfrac{Z_t}{r_t} \in \mathrm{d}x, \tfrac{\Phi_t(Z_t)}{a_t} \in \mathrm{d}y, \Phi_{t+\theta t}(z) \leq \Phi_{t+\theta t}(Z_t)\,\forall z \in \mathbb{Z}^d \text{ s.t. } \big(\tfrac{|z|}{r_t}, \tfrac{\Phi_t(z)}{a_t}\big) \in \widehat{B}_N \Big\}\,.$$

First, we express the probability under the integral for fixed $(x,y) \in B_N$ in terms of the point process Π_t. Given that Π_t contains the point (x,y) we require that there are no points in the set $\mathbb{R}^d \times (y,\infty)$, and requiring (9) for all points z with $(|z|/r_t, \Phi_t(z)/a_t) \in \widehat{B}_N$ is, by Lemma 2.3, equivalent to the requirement that Π_t should have no points in the set

$$\big\{(\bar{x},\bar{y}) \in \widehat{B}_N \colon \bar{y} + \tfrac{q\theta}{1+\theta}|\bar{x}| > y + \tfrac{q\theta}{1+\theta}|x|\big\}.$$

Hence, defining the set

$$D_\theta^N(r,y) = \big\{(\overline{x},\overline{y}) \in \mathbb{R}^d \times \mathbb{R} \,:\, \overline{y} > y\big\} \cup \big\{(\overline{x},\overline{y}) \in \widehat{B}_N \,:\, |\overline{x}| > r, \overline{y} > y - \tfrac{q\theta}{1+\theta}(|\overline{x}| - r)\big\}\,,$$

we see that, as $t \to \infty$,

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Prob}(A(N,t)) &= \iint\limits_{(x,y)\in B_N} \mathrm{Prob}\big\{\Pi(\mathrm{d}x\,\mathrm{d}y) = 1, \Pi(D_\theta^N(|x|,y)) = 0\big\} \\ &= \iint\limits_{(x,y)\in B_N} e^{-\nu(D_\theta^N(|x|,y))}\nu(\mathrm{d}x\,\mathrm{d}y)\,.\end{aligned}$$

Taking the limit in this way is justified as $D_\theta^N(|x|,y)$ is relatively compact in $\widehat{H}$ and (x,y) ranges only over elements in B_N. Finally, if we similarly define (see also Figure 3)

$$D_\theta(r,y) = \big\{(\overline{x},\overline{y}) \in \mathbb{R}^d \times \mathbb{R} \,:\, |\overline{x}| \leq r, \overline{y} > y \text{ or } |\overline{x}| > r, \overline{y} > y - \tfrac{q\theta}{1+\theta}(|\overline{x}| - r)\big\}\,.$$

we can invoke Lemma 2.2 to see that

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} &= \lim_{N\to\infty}\lim_{t\to\infty} \mathrm{Prob}(A(N,t)) \\ &= \lim_{N\to\infty} \iint\limits_{(x,y)\in B_N} e^{-\nu(D_\theta^N(|x|,y))}\nu(\mathrm{d}x\,\mathrm{d}y) \\ &= \int_{y\geq 0}\int_{x\in\mathbb{R}^d} e^{-\nu(D_\theta(|x|,y))}\nu(\mathrm{d}x\,\mathrm{d}y)\,,\end{aligned}$$

where the last equality follows by dominated convergence, as the integrand is dominated by $e^{-\nu(D_0(|x|,y))}$ which is integrable with respect to ν by the direct calculation in the next proposition.

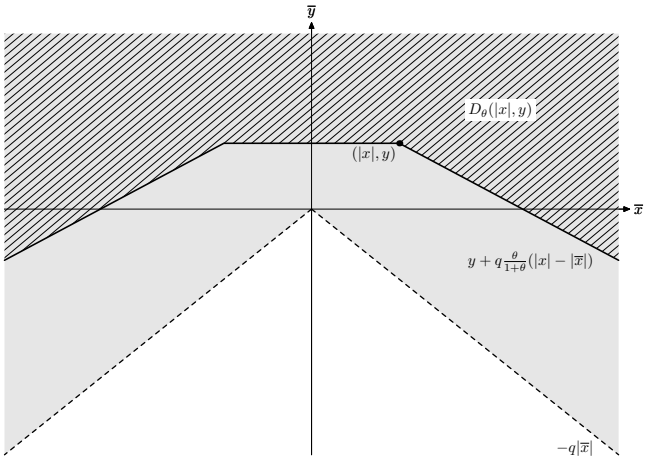


Figure 3: The point process Π is defined on the set $\widehat{H}$ indicated in grey. If we fix $Z_t/r_t = x, \Phi_t(Z_t)/a_t = y$, the condition that $Z_t = Z_{t+\theta t}$ corresponds to the requirement that the point process Π has no points in the shaded region $D_\theta(|x|, y)$.

We now simplify the expression that arises from the point process calculation. We denote by $B(a, b)$ the Beta function with parameters a, b and define the normalized incomplete Beta function

$$\tilde{B}(x, a, b) = \frac{1}{B(a, b)} \int_0^x v^{a-1}(1-v)^{b-1}\,\mathrm{d}v\,.$$

Proposition 2.4 (Explicit form of $I(\theta)$)**.** *For any $\theta \geq 0$, we have*

$$\int_{y\geq 0}\int_{x\in\mathbb{R}^d} e^{-\nu(D_\theta(|x|,y))}\nu(\mathrm{d}x\,\mathrm{d}y) = I(\theta) := \frac{1}{B(\alpha-d+1,d)}\int_0^1 v^{\alpha-d}(1-v)^{d-1}\,\varphi_\theta(v)\,\mathrm{d}v\,,$$

where the weight $\varphi_\theta(v)$ is defined by

$$\tfrac{1}{\varphi_\theta(v)} = 1 - \tilde{B}(v, \alpha-d, d) + (1+\theta)^\alpha\left(\tfrac{\theta}{v}+1\right)^{d-\alpha}\tilde{B}\big(\tfrac{v+\theta}{1+\theta}, \alpha-d, d\big)\,. \tag{14}$$

Proof. First of all, we compute $\nu(D_\theta(r, y))$ for some $r > 0$,

$$\begin{aligned}\nu(D_c(r,y)) &= \int_{|\overline{x}|\leq r}\int_y^\infty \frac{\alpha\,\mathrm{d}\overline{x}\,\mathrm{d}\overline{y}}{(\overline{y}+q|\overline{x}|)^{\alpha+1}} + \int_{|\overline{x}|>r}\int_{y-\frac{q\theta}{1+\theta}(|\overline{x}|-r)}^\infty \frac{\alpha\,\mathrm{d}\overline{x}\,\mathrm{d}\overline{y}}{(\overline{y}+q|\overline{x}|)^{\alpha+1}}\\ &= \int_{|\overline{x}|\leq r}\frac{\mathrm{d}\overline{x}}{(y+q|\overline{x}|)^\alpha} + \int_{|\overline{x}|>r}\frac{\mathrm{d}\overline{x}}{(y+\frac{q\theta}{1+\theta}r+\frac{q}{1+\theta}|\overline{x}|)^\alpha}\,.\end{aligned}$$

Next, we can rewrite the two last summands. We exploit the invariance of the integrand under reflections at the axes, then for $x_i \geq 0$ we use the substitution $u_1 = \overline{x}_1 + \cdots + \overline{x}_d$, $u_i = \overline{x}_i$ for $i \geq 2$ and then the substitution $y + qu_1 = y/v$, so that

$$\begin{aligned}\int_{|\overline{x}|\leq r}\frac{\mathrm{d}\overline{x}}{(y+q|\overline{x}|)^\alpha} &= 2^d\int_0^r \frac{u_1^{d-1}}{(y+qu_1)^\alpha}\Big(\int_{\substack{u_2+\cdots+u_d\leq 1\\ u_i\geq 0}}\mathrm{d}u_2\ldots\mathrm{d}u_d\Big)\mathrm{d}u_1\\ &= \tfrac{2^d}{(d-1)!}\int_0^r\frac{u_1^{d-1}}{(y+qu_1)^\alpha}\,\mathrm{d}u_1 = \tfrac{2^d y^{d-\alpha}}{q^d(d-1)!}\int_{\frac{y}{y+qr}}^1 v^{\alpha-d-1}(1-v)^{d-1}\mathrm{d}v\\ &= \vartheta y^{d-\alpha}\Big(1-\tilde{B}\big(\tfrac{y}{y+qr},\alpha-d,d\big)\Big)\,,\end{aligned} \tag{15}$$

where $\vartheta = \frac{2^d B(\alpha-d,d)}{q^d(d-1)!}$. A similar calculation shows that

$$\int_{|\overline{x}|>r} \frac{\mathrm{d}\overline{x}}{(y+\frac{q\theta}{1+\theta}r+\frac{q}{1+\theta}|\overline{x}|)^\alpha} = \vartheta(1+\theta)^d\big(y+\tfrac{q\theta}{1+\theta}r\big)^{d-\alpha}\tilde{B}\big(\tfrac{y+\frac{q\theta}{1+\theta}r}{y+qr},\alpha-d,d\big)\,.$$

Combining the previous displays, and using the substitution $y+qr = y/v$ yields

$$\begin{aligned}\nu\big(D_\theta(r,y)\big) &= \vartheta y^{d-\alpha}\Big(1-\tilde{B}\big(\tfrac{y}{y+qr},\alpha-d,d\big)+(1+\theta)^d\big(1+\tfrac{q\theta}{1+\theta}\tfrac{r}{y}\big)^{d-\alpha}\tilde{B}\Big(\tfrac{y+\frac{q\theta}{1+\theta}r}{y+qr},\alpha-d,d\Big)\Big)\\ &= \vartheta y^{d-\alpha}\Big(1-\tilde{B}(v,\alpha-d,d)+(1+\theta)^\alpha\big(1+\tfrac{\theta}{v}\big)^{d-\alpha}\tilde{B}\big(\tfrac{v+\theta}{1+\theta},\alpha-d,d\big)\Big)\\ &= \vartheta y^{d-\alpha}\varphi_\theta(v)^{-1}\,. \end{aligned} \tag{16}$$

To calculate the integral over $x\in\mathbb{R}^d$ we substitute $r = x_1+\ldots+x_d$ and $u_i = x_i$ for $i\geq 2$,

$$\int_{\mathbb{R}^d} e^{-\nu(D_\theta(|x|,y))}\frac{\alpha}{(y+q|x|)^{\alpha+1}}\,\mathrm{d}x = \tfrac{2^d}{(d-1)!}\int_0^\infty e^{-\nu(D_\theta(r,y))}\frac{\alpha r^{d-1}}{(y+qr)^{\alpha+1}}\,\mathrm{d}r.$$

Finally, we integrate over $y\geq 0$ and use the above formula for $\nu(D_\theta(r,y))$ together with the substitution $y+qr = y/v$ and $w = \vartheta y^{d-\alpha}$ to obtain

$$\begin{aligned}\int_{y\geq 0}\int_{x\in\mathbb{R}^d} e^{-\nu(D_\theta(|x|,y))}\,\nu(\mathrm{d}x\,\mathrm{d}y) &= \tfrac{2^d}{(d-1)!}\int_0^\infty\int_0^\infty e^{-\nu(D_\theta(r,y))}\frac{\alpha r^{d-1}\mathrm{d}r}{(y+qr)^{\alpha+1}}\,\mathrm{d}y\\ &= \tfrac{2^d}{q^d(d-1)!}\int_0^1 \alpha v^{\alpha-d}(1-v)^{d-1}\int_0^\infty \exp\{-\vartheta y^{d-\alpha}\varphi_\theta(v)^{-1}\}\,y^{d-\alpha-1}\,\mathrm{d}y\,\mathrm{d}v\\ &= \tfrac{\alpha}{B(\alpha-d,d)(\alpha-d)}\int_0^1 v^{\alpha-d}(1-v)^{d-1}\int_0^\infty e^{-w\varphi_\theta(v)^{-1}}\,\mathrm{d}w\,\mathrm{d}v\\ &= \tfrac{1}{B(\alpha-d+1,d)}\int_0^1 v^{\alpha-d}(1-v)^{d-1}\varphi_\theta(v)\,\mathrm{d}v\,,\end{aligned}$$

where we used the identity $B(x+1,y)\,(x+y) = B(x,y)\,x$ for $x,y>0$ in the last step. □

Proposition 2.5 (Tails of I)**.**

(a) $\lim\limits_{\theta\to\infty}\theta^d I(\theta) = \frac{1}{d\,B(\alpha-d+1,d)}$.

(b) $\lim\limits_{\theta\downarrow 0}\theta^{-1}(1-I(\theta)) = C_0$, *where the constant* C_0 *is given by*

$$C_0 = \tfrac{1}{B(\alpha-d+1,d)}\Big(\int_0^1 \alpha v^{\alpha-d}(1-v)^{d-1}\tilde{B}(v;\alpha-d,d)\mathrm{d}v + B\big(2(\alpha-d),2d-1\big)\Big)\,.$$

Proof. (a) As $\tilde{B}(v,\alpha-d,d)\leq 1$ and $v\mapsto\tilde{B}(v,\alpha-d,d)$ is nondecreasing we get, for $0\leq v\leq 1$,

$$\begin{aligned}\tfrac{1}{\varphi_\theta(v)} &= 1-\tilde{B}(v,\alpha-d,d)+(1+\theta)^\alpha\big(\tfrac{\theta}{v}+1\big)^{d-\alpha}\tilde{B}\big(\tfrac{v+\theta}{1+\theta},\alpha-d,d\big)\\ &\geq (1+\theta)^d\,v^{\alpha-d}\,\big(\tfrac{1+\theta}{v+\theta}\big)^{\alpha-d}\tilde{B}\big(\tfrac{\theta}{1+\theta},\alpha-d,d\big)\geq \tfrac{1}{2}(1+\theta)^d v^{\alpha-d}\,,\end{aligned}$$

where we chose θ large enough such that $\tilde{B}(\frac{\theta}{1+\theta},\alpha-d,d)\geq\frac{1}{2}$. Hence, for θ large enough,

$$\begin{aligned}\theta^d I(\theta) &= \tfrac{1}{B(\alpha-d+1,d)}\int_0^1 v^{\alpha-d}(1-v)^{d-1}\theta^d\varphi_\theta(v)\,\mathrm{d}v\\ &\leq \tfrac{2}{B(\alpha-d+1,d)}\int_0^1(1-v)^{d-1}\,\mathrm{d}v = \tfrac{2}{dB(\alpha-d+1,d)}\,.\end{aligned}$$

Therefore, since $\theta^d\varphi_\theta(v)\to v^{d-\alpha}$ pointwise for every $v\in(0,1)$ as $\theta\to\infty$, we can invoke the dominated convergence theorem to complete the proof of the lemma.

(b) We can write

$$1 - I(\theta) = \tfrac{1}{B(\alpha-d+1,d)} \int_0^1 v^{\alpha-d}(1-v)^{d-1}\varphi_\theta(v)(\varphi_\theta^{-1}(v)-1)\,.$$

Set $\tilde{B}(v) = \tilde{B}(v;\alpha-d,d)$. Then, since $\varphi_\theta(v) \to 1$ for every v as $\theta \downarrow 0$, we can concentrate on

$$\begin{aligned}\varphi_\theta(v)^{-1} - 1 &= (1+\theta)^d v^{\alpha-d}\left(\tfrac{1+\theta}{v+\theta}\right)^{\alpha-d}\tilde{B}\big(\tfrac{\theta+v}{1+v}\big) - \tilde{B}(v)\\ &= \tilde{B}\big(\tfrac{\theta+v}{1+\theta}\big)\left(\tfrac{(1+\theta)^\alpha v^{\alpha-d}}{(v+\theta)^{\alpha-d}} - 1\right) + \tilde{B}\big(\tfrac{\theta+v}{1+\theta}\big) - \tilde{B}(v).\end{aligned}$$

The first summand can be bounded by $(1+\theta)^\alpha - 1 \le 2\alpha\theta$, eventually for all θ. For the second term, we have that

$$\tilde{B}\big(\tfrac{\theta+v}{1+\theta}\big) - \tilde{B}(v) = \int_v^{\frac{v+\theta}{1+\theta}} u^{\alpha-d-1}(1-u)^{d-1}\mathrm{d}u \le \theta(1-v)^d \max\{v^{\alpha-d-1},1\}\,.$$

Combining the two estimates we obtain that $\theta^{-1}(1-I(\theta))$ is bounded, so that by the dominated convergence theorem, we may take the limit of $\theta^{-1}(\varphi_\theta^{-1}(v)-1)$ as $\theta \downarrow 0$ under the integral. □

2.2 Ageing for the solution profile

In this section, we prove Theorem 1.1 by combining the results about ageing for the maximizer Z_t from the previous section with the localization results in [KLMS09]. We start with a preliminary calculation that will be used several times in the remainder.

Lemma 2.6. *If $\Phi_t(x) = \Phi_t(y)$ for some $t > 0$ and $x, y \in \mathbb{Z}^d$ such that $t\xi(x) > |x|$ and $t\xi(y) > |y|$, then for all $s > 0$ such that $s\xi(x) > |x|$ and $s\xi(y) > |y|$, we have that*

$$\Phi_s(x) - \Phi_s(y) = (\xi(x) - \xi(y))\big(1 - \tfrac{t}{s}\big)\,.$$

Proof. By the assumptions on t, x, y, we find that

$$\Phi_t(x) - \Phi_t(y) = (\xi(x)-\xi(y)) - \tfrac{1}{t}\big(|x|\log\xi(x) - |y|\log\xi(y) - \eta(x) + \eta(y)\big) = 0\,.$$

Rearranging, we can substitute into

$$\begin{aligned}\Phi_s(x) - \Phi_s(y) &= (\xi(x)-\xi(y)) - \tfrac{1}{s}\big(|x|\log\xi(x) - |y|\log\xi(y) - \eta(x) + \eta(y)\big)\\ &= (\xi(x)-\xi(y))\left(1 - \tfrac{t}{s}\right),\end{aligned}$$

which completes the proof. □

Remark 2.7. Let $Z_t^{(1)}, Z_t^{(2)}, \ldots \in \mathbb{Z}^d$ be sites in $\mathbb{Z}^d$ producing the largest values of Φ_t in descending order (choosing the site with largest ℓ^1-norm in case of a tie), and recall that $Z_t = Z_t^{(1)}$. It is then easy to see that $t\xi(Z_t^{(i)}) > |Z_t^{(i)}|$ for $i = 1, 2$ and all $t \ge 1$. Hence, if $\tau > 1$ is a jump time of the process $(Z_t \colon t > 0)$, then $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$, so that we can apply Lemma 2.6 with $x = Z_\tau^{(1)}$ and $y = Z_\tau^{(2)}$ and the conclusion holds for all $s \ge \tau$.

Lemma 2.8. *Almost surely, the function $u \mapsto \xi(Z_u)$ is nondecreasing on $(1, \infty)$.*

Proof. Let $\{\tau_n\}$ be the successive jump times of the process $(Z_t \colon t \ge 1)$. By definition,

$$\Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(1)}) = \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(2)})$$

and by right-continuity of $t \mapsto Z_t^{(1)}$, we have that $Z_{\tau_{n+1}}^{(2)} = Z_{\tau_n}^{(1)}$. Now, consider $\tau_{n+1} < t < \tau_{n+2}$ such that $Z_t^{(i)} = Z_{\tau_{n+1}}^{(i)}$ for $i = 1, 2$, then by Lemma 2.6 and Remark 2.7 we know that

$$\begin{aligned} \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) &= \Phi_t(Z_{\tau_{n+1}}^{(1)}) - \Phi_t(Z_{\tau_{n+1}}^{(2)}) = (\xi(Z_{\tau_{n+1}}^{(1)}) - \xi(Z_{\tau_{n+1}}^{(2)}))(1 - \tfrac{\tau_{n+1}}{t}) \\ &= (\xi(Z_{\tau_{n+1}}^{(1)}) - \xi(Z_{\tau_n}^{(1)}))(1 - \tfrac{\tau_{n+1}}{t})\,. \end{aligned} \tag{17}$$

As $t < \tau_{n+2}$, and $t \mapsto \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is not constant, the left hand side of (17) is strictly positive, which implies that $\xi(Z_{\tau_{n+1}}) - \xi(Z_{\tau_n}) > 0$, thus completing the proof. □

As an immediate consequence of this lemma, we get that $(Z_t \colon t > 1)$ never returns to the same point in $\mathbb{Z}^d$. We now prove the first part of Theorem 1.1.

Lemma 2.9. *For any sufficiently small $\varepsilon > 0$,*

$$\lim_{t\to\infty} \mathrm{Prob}\big\{ \sup_{z\in\mathbb{R}^d} \big|v(t,z) - v(t+\theta t, z)\big| < \varepsilon \big\} = \lim_{t\to\infty} \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} = I(\theta)\,.$$

Proof. Suppose $0 < \varepsilon < \frac{1}{2}$ and let us throughout this proof argue on the event

$$A_t = \big\{\, v(t, Z_t) > 1 - \tfrac{\varepsilon}{2},\, v(t+\theta t, Z_{t+\theta t}) > 1 - \tfrac{\varepsilon}{2} \big\}\,.$$

Now, if $z \neq Z_t$, then

$$u(t,z) \le \sum_{x \neq Z_t} u(t,x) = U(t) - u(t, Z_t) < \tfrac{\varepsilon}{2}\, U(t)\,,$$

and similarly if $z \neq Z_{t+\theta t}$, then $u(t+\theta t, z) \le \frac{\varepsilon}{2} U(t+\theta t)$. In particular, if $z \neq Z_t$ and $z \neq Z_{t+\theta t}$, then $|v(t,z) - v(t+\theta t, z)| < \varepsilon$. Now, if $Z_t = Z_{t+\theta t}$, then by assumption A_t we have $|v(t,z) - v(t+\theta t, z)| < \varepsilon$ for any $z \in \mathbb{Z}^d$. Conversely, suppose that $Z_t \neq Z_{t+\theta t}$. From above we then get $u(t+\theta t, Z_t) < \frac{\varepsilon}{2} U(t+\theta t)$ and since we argue on the event A_t, we find that $v(t, Z_t) - v(t+\theta t, Z_t) > 1 - \varepsilon > \varepsilon$, so that

$$\sup_{z\in\mathbb{Z}^d} \big|v(t,z) - v(t+\theta t, z)\big| \ge \big|v(t, Z_t) - v(t+\theta t, Z_t)\big| > \varepsilon\,.$$

To complete the proof, it remains to notice that since $v(t, Z_t)$ converges weakly to one, we have that $\mathrm{Prob}(A_t) \to 1$ as $t \to \infty$. □

Before we can prove the remaining part of Theorem 1.1, we need to collect the following fact about the maximizers $Z^{(1)}$ and $Z^{(2)}$.

Lemma 2.10. *Let $\lambda_t = (\log t)^{-\beta}$ for some $\beta > 1 + \frac{1}{\alpha - d}$. If $t_1 \le t_2$ are sufficiently large, satisfy $Z_{t_1}^{(1)} = Z_{t_2}^{(1)}$ and*

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \ge \tfrac{1}{2}\, a_t \lambda_t\,, \tag{18}$$

holds for $t = t_1$ and $t = t_2$, then (18) holds for all $t \in [t_1, t_2]$.

Proof. First, we additionally assume that $Z_t^{(2)} = Z_{t_1}^{(2)}$ for all $t \in [t_1, t_2)$. By Lemma 2.8 we have that $Z_t^{(1)} = Z_{t_1}^{(1)}$ for all $t \in [t_1, t_2]$. Using also the continuity of $t \mapsto \Phi_t(Z_t^{(i)})$, $i = 1, 2$, we get

$$\begin{aligned} \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) &= \Phi_t(Z_{t_1}^{(1)}) - \Phi_t(Z_{t_1}^{(2)}) \\ &= \xi(Z_{t_1}^{(1)}) - \xi(Z_{t_1}^{(2)}) - \tfrac{1}{t}\big(|Z_{t_1}^{(1)}| \log \xi(Z_{t_1}^{(1)}) - |Z_{t_1}^{(2)}| \log \xi(Z_{t_1}^{(2)}) - \eta(Z_{t_1}^{(1)}) + \eta(Z_{t_1}^{(2)})\big) \\ &= A - \tfrac{1}{t} B \quad \text{for all } t \in [t_1, t_2]\,, \end{aligned}$$

for some constants $A, B \in \mathbb{R}$ depending only on t_1. Now, defining

$$f(t) = A - \tfrac{1}{t} B - \tfrac{1}{2}\, a_t \lambda_t\,,$$

we get that $f(t_1) \geq 0$ and $f(t_2) \geq 0$ by our assumption. Moreover,

$$f'(t) = \frac{1}{t^2}\Big(B - \tfrac{1}{2}\,\frac{t^{q+1}}{(\log t)^{q+\beta}}\big(q - \tfrac{q+\beta}{\log t}\big)\Big),$$

which is negative for t larger than some threshold depending on t_1. Also, if t_1 is large enough, the function $t \mapsto \frac{t^{q+1}}{(\log t)^{q+\beta}}\big(q - \frac{q+\beta}{\log t}\big)$ is strictly increasing for $t \geq t_1$, hence f' has at most one zero for $t \geq t_1$. Therefore, if f' has a zero $t' \geq t_1$, then f' is negative for all $t > t'$, implying that f does not have a minimum at $t' \in (t_1, t_2)$. If f' does not have a zero for $t \geq t_1$, then it follows that $f'(t) < 0$ for all $t \geq t_1$. In either case, $f(t_1) \geq 0$ and $f(t_2) \geq 0$ imply that $f(t) \geq 0$ for all $t \in [t_1, t_2]$, in other words (18) holds for all $t \in [t_1, t_2]$.

Now we drop the extra assumption on $Z_t^{(2)}$ and define the jump times

$$\tau^- = \sup\big\{t < t_1 \,:\, Z_t^{(1)} \neq Z_{t_1}^{(1)}\big\} \quad \text{and} \quad \tau^+ = \inf\big\{t > t_2 \,:\, Z_t^{(1)} \neq Z_{t_1}^{(1)}\big\}\,.$$

Furthermore, define a sequence $s^{(i)}$ by setting $s^{(0)} = \tau^-$ and for $i \geq 1$ setting

$$s^{(i)} = \inf\{s > s^{(i-1)} \,:\, \Phi_s(Z_s^{(2)}) = \Phi_s(Z_s^{(3)})\}\,.$$

Then, there exists $N \geq 1$ such that $s^{(N)} < \tau^+ < s^{(N+1)}$, where $N \geq 1$ since, by Lemma 2.8,

$$Z_{\tau^-}^{(2)} = \lim_{t \uparrow \tau^-} Z_t^{(1)} \neq Z_{\tau^+}^{(1)}.$$

Using that $\Phi_{s^{(i)}}(Z_{s^{(i)}}^{(2)}) = \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(3)})$ and Proposition 3.4 in [KLMS09],

$$\Phi_{s^{(i)}}(Z_{s^{(i)}}^{(1)}) - \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(2)}) = \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(1)}) - \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(3)}) \geq\, a_{s^{(i)}}\lambda_{s^{(i)}}\,.$$

Therefore, (18) holds for $t = s^{(i)}$, $i = 1, \ldots, N$. Hence, the additional assumption that we made in the first part of the proof holds for each of the intervals $[t_1, s^{(1)})$, $[s^{(i)}, s^{(i+1)})$, for $i = 1, \ldots, N-1$ and $[s^{(N)}, t_2)$. Thus, we can deduce that (18) holds for all t in the union of these intervals, which completes the proof. □

Finally, we can now show the stronger form of ageing for the profile v and thereby complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.1, it suffices to show that

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{\sup_{\substack{z\in\mathbb{R}^d\\ s\in[t,t+\theta t]}} \big|v(t,z) - v(s,z)\big| < \varepsilon\Big\} = \lim_{t\to\infty} \mathrm{Prob}\{Z_t^{(1)} = Z_{t+\theta t}^{(1)}\}\,.$$

First of all, note that by Lemma 2.8 we know that $Z_t^{(1)} = Z_{t+\theta t}^{(1)}$ if and only if $Z_t^{(1)} = Z_s^{(1)}$ for all $s \in [t, t+\theta t]$. We will work on the event

$$A_t = \big\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq a_t\lambda_t/2\big\} \cap \big\{\Phi_{t+\theta t}(Z_{t+\theta t}^{(1)}) - \Phi_{t+\theta t}(Z_{t+\theta t}^{(2)}) \geq a_{t+\theta t}\lambda_{t+\theta t}/2\big\}\,.$$

Recall from Proposition 5.3 in [KLMS09] that if $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(2)})$ are sufficiently far apart, then the profile is localized in $Z_t^{(1)}$. More precisely, almost surely,

$$\lim_{t\to\infty} \sum_{z\in\mathbb{Z}^d\setminus\{Z_t^{(1)}\}} v(t,z)\,\mathbb{1}\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq a_t\lambda_t/2\} = 0\,.$$

In particular, for given $\varepsilon < \frac{1}{2}$, we can assume that t is sufficiently large, so that for all $s \geq t$,

$$\sum_{z\in\mathbb{Z}^d\setminus\{Z_s^{(1)}\}} v(s,z)\,\mathbb{1}\{\Phi_s(Z_s^{(1)}) - \Phi_s(Z_s^{(2)}) \geq a_s\lambda_s/2\} < \tfrac{\varepsilon}{2}\,. \tag{19}$$

Now, if $Z_t^{(1)} \neq Z_{t+\theta t}^{(1)}$, then on A_t, we know by (19) that $v(t+\theta t, Z_t^{(1)}) \leq \frac{\varepsilon}{2}$. Combining this with the fact that $v(t, Z_t^{(1)}) > 1 - \frac{\varepsilon}{2}$, we have that

$$\sup_{\substack{z\in\mathbb{Z}^d \\ s\in[t,t+\theta t]}} \big|v(t,z) - v(s,z)\big| \geq \big|v(t, Z_t^{(1)}) - v(t+\theta t, Z_t^{(1)})\big| > 1 - \varepsilon > \varepsilon\,.$$

Conversely, assume that $Z_t^{(1)} = Z_{t+\theta t}^{(1)}$, then by Lemma 2.8, $Z_t^{(1)} = Z_s^{(1)}$ for all $s \in [t, t+\theta t]$. Now, on the event A_t we know by Lemma 2.10 that for all $s \in [t, t+\theta t]$,

$$\Phi_s(Z_s^{(1)}) - \Phi_s(Z_s^{(2)}) \geq a_s \lambda_s / 2\,. \tag{20}$$

This implies by (19) that

$$\sum_{z\in\mathbb{Z}^d\setminus\{Z_s^{(1)}\}} v(s,z) < \varepsilon/2 \qquad \text{for all } s \in [t, t+\theta t].$$

As in the proof of Lemma 2.9, this yields that

$$\sup_{\substack{z\in\mathbb{R}^d \\ s\in[t,t+\theta t]}} \big|v(t,z) - v(s,z)\big| < \varepsilon\,.$$

Hence, to complete the proof, it remains to notice that by [KLMS09, Lemma 6.2] the pair $(\Phi_t(Z_t^{(1)})/a_t, \Phi_t(Z_t^{(2)})/a_t)$ converges weakly to a limit random variable with a density, from which we conclude that $\mathrm{Prob}(A_t) \to 1$ as $t \to \infty$. □

3 Ageing: an almost-sure limit theorem

In this section, we prove Theorem 1.3. As in the previous section, we first concentrate on an analogous theorem for the maximizer of the variational problem Φ_t. In particular, in Section 3.1, we extend Proposition 2.1 to a moderate deviations principle. This estimate allows us to prove the equivalent of the almost sure ageing Theorem 1.3 in the setting of the variational problem in Section 3.2. Finally, in Section 3.3, we transfer this result to the maximizer of v.

3.1 Moderate deviations

Recall from Proposition 2.5 that

$$\lim_{t\to\infty} \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} = I(\theta) \sim \frac{1}{dB(\alpha - d + 1, d)}\, \theta^{-d}\,,$$

where the latter asymptotic equivalence holds for θ tending to infinity. We now show that we obtain the same asymptotic for $\mathrm{Prob}\{Z_t = Z_{t+\theta t}\}$ if we allow θ to grow slowly with t.

Proposition 3.1 (Moderate deviations)**.** *For any positive function θ_t such that $\theta_t \to \infty$ and $\theta_t \leq (\log t)^\delta$ for some $\delta > 0$, we have that*

$$\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} = \big(\tfrac{1}{dB(\alpha-d+1,d)} + o(1)\big)\, \theta_t^{-d}\,.$$

Unlike in the proof of Proposition 2.1, we cannot directly use the point process techniques, as the weak convergence only applies to compact sets, whereas here we deal with sets that increase slowly with t to a set that has infinite mass under the intensity measure ν. We start by expressing $\Phi_t(z)$ in terms of $\xi(z)$ and $|z|$, while carefully controlling the errors.

Lemma 3.2. *There exist $C_1, C_2 > 0$ and $t_0 > 0$ such that, for all $z \in \mathbb{Z}^d, t > t_0$ with $t\xi(z) > |z|$,*

$$\frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}\big(1 + 2\tfrac{\log(N_t+qg_t)}{\log t}\big) \le \frac{\Phi_t(z)}{a_t} \le \frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}\big(1 - C_1\tfrac{\log\log t}{\log t}\big) + C_2\frac{1}{\log t}\,,$$

where the lower bound holds uniformly for all functions N_t, g_t such that $\Phi_t(z) \le a_t N_t, |z| \le r_t g_t$ and $N_t, g_t \to \infty$ as $t \to \infty$. Similarly, for $\theta \ge 0$ and $z \in \mathbb{Z}^d$ such that $(1+\theta)t\xi(z) > |z|$, we have

$$\frac{\xi(z)}{a_t} - \frac{q}{1+\theta}\frac{|z|}{r_t}\big(1 + 2\tfrac{\log(N_t+qg_t)}{\log t}\big) \le \frac{\Phi_{t+\theta t}(z)}{a_t} \le \frac{\xi(z)}{a_t} - \frac{q}{1+\theta}\frac{|z|}{r_t}\big(1 - C_1\tfrac{\log\log t}{\log t}\big) + C_2\frac{1}{\log t}\,,$$

again with the restriction that for the lower bound we assume that $\Phi_t(z) \le a_t N_t$ and $|z| \le r_t g_t$.

Proof. Using that $r_t = \frac{t}{\log t} a_t$, we have, for $t\xi(z) > |z|$, that

$$\frac{\Phi_t(z)}{a_t} = \frac{\xi(z)}{a_t} - \frac{1}{ta_t}(|z|\log\xi(z) - \eta(z)) = \frac{\xi(z)}{a_t} - q\frac{|z|}{r_t} + \mathrm{error}(t,z),$$

where

$$\mathrm{error}(t,z) = q\frac{|z|}{r_t}\frac{\log\log t}{\log t} - \frac{|z|}{r_t\log t}\log\frac{\xi(z)}{a_t} + \frac{\eta(z)}{r_t\log t}\,.$$

It thus suffices to find suitable upper and lower bounds for the last two terms. For the upper bound, we use that $\eta(z) \le |z|\log d$ and also that $x\log x \ge -e^{-1}$ for any $x > 0$, to get

$$-\frac{|z|}{r_t\log t}\log\frac{\xi(z)}{a_t} + \frac{\eta(z)}{a_t t} \le -\frac{|z|}{r_t\log t}\log\frac{|z|}{r_t\log t} + \frac{|z|}{r_t}\frac{\log d}{\log t} \le \frac{1}{\log t}e^{-1} + \frac{|z|}{r_t}\frac{\log\log t + \log d}{\log t}\,,$$

so that the upper bound holds for $C_1 \ge \frac{1}{q}(1+\log d) + q$ and $C_2 \ge e^{-1}$. For the lower bound we note that either $\xi(z)/a_t < (1 + \frac{g_t}{\log t})^2$, or we can use $\log x \le x^{1/2}$, for all $x > 0$, to estimate

$$\begin{aligned}\frac{\Phi_t(z)}{a_t} + q\frac{|z|}{r_t} &\ge \frac{\xi(z)}{a_t} - \frac{|z|}{r_t\log t}\log\frac{\xi(z)}{a_t} \ge \frac{\xi(z)}{a_t} - \frac{|z|}{r_t\log t}\Big(\frac{\xi(z)}{a_t}\Big)^{1/2} \\ &\ge \frac{\xi(z)}{a_t} - \frac{g_t}{\log t}\Big(\frac{\xi(z)}{a_t}\Big)^{1/2} \ge \Big(\frac{\xi(z)}{a_t}\Big)^{1/2}.\end{aligned}$$

Therefore, we have $\xi(z)/a_t \le \max\{(1+\frac{g_t}{\log t})^2, (\frac{\Phi_t(z)}{a_t} + q\frac{|z|}{r_t})^2\} \le (N_t + qg_t)^2$. Hence, we can conclude that

$$\mathrm{error}(t,z) \ge -q\frac{|z|}{r_t}\,\frac{2\log(N_t+qg_t)}{\log t}\,.$$

For the bound on $\Phi_{t+\theta t}(z)$ it suffices to note that

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\xi(z)}{a_t} - \frac{q}{1+\theta}\frac{|z|}{r_t} + \frac{1}{1+\theta}\,\mathrm{error}(t,z)\,,$$

where $\mathrm{error}(t,z)$ is precisely the same error term as in the first part of the lemma. □

In analogy to the proof of Proposition 2.1, we will have to restrict $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes in $\mathbb{R}^d \times \mathbb{R}$. The first step is therefore to estimate the probability that $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ lies outside a large box.

Lemma 3.3. *There exist constants $C, C' > 0$ such that for all $t > 0$ large enough, uniformly for all $N \ge 1$,*

(a) $\mathrm{Prob}\big\{\frac{|Z_t|}{r_t} \ge N\big\} \le C\,N^{d-\alpha}$,

(b) $\mathrm{Prob}\big\{\frac{\Phi_t(Z_t)}{a_t} \ge N\big\} \le C\,N^{d-\alpha}$,

(c) for any positive function $\eta_t \le 1$ such that $\eta_t a_t \to \infty$ we have

$$\mathrm{Prob}\big\{\tfrac{\Phi_t(Z_t)}{a_t} \le \eta_t\big\} \le Ce^{-C'\eta_t^{d-\alpha}}\,.$$

Proof. (a) Using Lemma 3.2, we can estimate

$$\begin{aligned}\mathrm{Prob}\{|Z_t| \ge Nr_t\} &\le \mathrm{Prob}\{\exists z \in \mathbb{Z}^d \text{ with } |z| \ge Nr_t, \tfrac{\Phi_t(z)}{a_t} \ge 0\}\\ &\le \sum_{\substack{z\in\mathbb{Z}^d\\ |z|\ge Nr_t}} \mathrm{Prob}\Big\{\tfrac{\xi(z)}{a_t} \ge q\tfrac{|z|}{r_t}\big(1 - C_1\tfrac{\log\log t}{\log t}\big) - C_2\tfrac{1}{\log t}\Big\}\\ &= (1+o(1))\sum_{\substack{z\in\mathbb{Z}^d\\ |z|\ge Nr_t}} a_t^{-\alpha}\big(q\tfrac{|z|}{r_t}\big)^{-\alpha} = (1+o(1))q^{-\alpha}r_t^{\alpha-d}\sum_{\substack{z\in\mathbb{Z}^d\\ |z|\ge Nr_t}} |z|^{-\alpha}\,,\end{aligned}$$

where we used that $r_t^d = a_t^\alpha$ and $o(1)$ tends to 0 as $t\to\infty$ uniformly in $N \ge 1$. We obtain the required bound by noting that the sum is bounded by a constant multiple of $(Nr_t)^{d-\alpha}$.

(b) For the second estimate, we use again Lemma 3.2 to obtain

$$\begin{aligned}\mathrm{Prob}\{\Phi_t(Z_t) \ge Na_t\} &\le \sum_{z\in\mathbb{Z}^d} \mathrm{Prob}\{\Phi_t(z) \ge Na_t\}\\ &\le \sum_{z\in\mathbb{Z}^d} \mathrm{Prob}\Big\{\tfrac{\xi(z)}{a_t} \ge N + q\tfrac{|z|}{r_t}\Big(1 - C_1\tfrac{\log\log t}{\log t}\Big) - C_2\tfrac{1}{\log t}\Big\}\\ &\le (1+o(1))\sum_{z\in\mathbb{Z}^d} a_t^{-\alpha}\big(N + q\tfrac{|z|}{r_t}\big)^{-\alpha}\,.\end{aligned}$$

Similarly as before, observe that the sum is bounded by a constant multiple of $\int_0^\infty r^{d-1}(N+qr)^{-\alpha}$, which itself is bounded by a constant multiple of $N^{d-\alpha}$.

(c) For the last bound, note first that by Lemma 3.2, that if $t\xi(z) > |z|$ and $|z|/r_t < g_t := \log t$ and $\Phi_t(z)/a_t < 1$, then there exists $C > 0$ such that

$$\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}\big(1 + C\tfrac{\log\log t}{\log t}\big) < \tfrac{\Phi_t(z)}{a_t}\,.$$

Hence, we can estimate

$$\begin{aligned}\mathrm{Prob}\big\{\tfrac{\Phi_t(Z_t)}{a_t} \le \eta_t\big\} &\le \mathrm{Prob}\big\{\tfrac{\Phi_t(z)}{a_t} \le \eta_t \text{ for all } z \text{ with } t\xi(z) > |z| \text{ and } |z| < r_t(\log t)\big\}\\ &\le \prod_{\substack{z\in\mathbb{Z}^d\\ |z|\le r_t g_t}} \mathrm{Prob}\Big\{t\xi(z) \le |z| \text{ or } \tfrac{\xi(z)}{a_t} \le \eta_t + q\tfrac{|z|}{r_t}\big(1 + C\tfrac{\log\log t}{\log t}\big)\Big\}\,.\end{aligned}$$

Now, if $t\xi(z) \le |z|$ and t is large enough, the second inequality must hold as well. Hence we obtain

$$\begin{aligned}\mathrm{Prob}\big\{\tfrac{\Phi_t(Z_t)}{a_t} \le \eta_t\big\} &\le \prod_{\substack{z\in\mathbb{Z}^d\\ |z|\le r_t g_t}} \mathrm{Prob}\Big\{\tfrac{\xi(z)}{a_t} \le \eta_t + q\tfrac{|z|}{r_t}\big(1 + C\tfrac{\log\log t}{\log t}\big)\Big\}\\ &= \exp\Big\{\sum_{\substack{z\in\mathbb{Z}^d\\ |z|\le r_t g_t}} \log\Big(1 - a_t^{-\alpha}\big(\eta_t + q\tfrac{|z|}{r_t}\big(1 + C\tfrac{\log\log t}{\log t}\big)\big)^{-\alpha}\Big)\Big\}\\ &\le \exp\Big\{-(1+o(1))\sum_{\substack{z\in\mathbb{Z}^d\\ |z|\le r_t g_t}} a_t^{-\alpha}\big(\eta_t + q\tfrac{|z|}{r_t}\big)^{-\alpha}\Big\}\,,\end{aligned}$$

using that $a_t\eta_t \to \infty$ and $\log(1-x) \le -x$ for $x < 1$. The sum can be bounded from below by a constant multiple of

$$r_t^{-d}\int_0^{r_t g_t - \frac12} r^{d-1}\big(\eta_t + q\tfrac{r+\frac12}{r_t}\big)^{-\alpha}\mathrm{d}r = (1+o(1))\int_0^{g_t - \frac12 r_t^{-1}} r^{d-1}(\eta_t + qr)^{-\alpha}\mathrm{d}r,$$

and the latter integral can be seen to be bounded from below by a constant multiple of $\eta_t^{d-\alpha}$. □

Proof of Proposition 3.1. The *main idea* is again to restrict $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes to be able to control the error when approximating Φ_t. To set up the notation, we introduce functions $\eta_t = (\log t)^{-\beta'}$, $N_t = (\log t)^{\beta}$, $g_t = (\log t)^{\gamma}$ for some parameters $\beta, \beta', \gamma > 0$, which we will choose later on depending on the function θ_t such that

$$\mathrm{Prob}\big\{Z_t = Z_{t(1+\theta_t)}\big\} = \mathrm{Prob}\big\{Z_t = Z_{t(1+\theta_t)},\, |Z_t| \le r_t g_t,\, \tfrac{\Phi_t(Z_t)}{a_t} \in [\eta_t, N_t]\big\} + o(\theta_t^{-d})\,.$$

Once these growing boxes are defined, we can find by Lemma 3.2 a constant $C > 0$ such that the function $\delta_t = C\frac{\log\log t}{\log t}$ satisfies

$$\frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}(1+\delta_t) \le \frac{\Phi_t(z)}{a_t} \le \frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}(1-\delta_t) + \delta_t\,,$$

where the upper bound holds for all $z \in \mathbb{Z}^d$ and the lower bound for all $z \in \mathbb{Z}^d$ such that $|z| \le r_t g_t$ and $\Phi_t(z) \le a_t N_t$.

Upper bound. We use a slight variation on the general idea, and consider

$$\begin{aligned}\mathrm{Prob}\big\{Z_t = &Z_{t(1+\theta_t)}\big\} \le \mathrm{Prob}\big\{Z_t = Z_{t(1+\theta_t)},\, \eta_t \le \tfrac{\xi(Z_t)}{a_t} - q\tfrac{|Z_t|}{r_t}(1-\delta_t) + \delta_t < N_t\big\}\\ &+ \mathrm{Prob}\big\{\Phi_t(Z_t) < \eta_t a_t\big\} + \sum_{z\in\mathbb{Z}^d} \mathrm{Prob}\big\{\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t \ge N_t\big\}\,.\end{aligned} \tag{21}$$

By Lemma 3.3(c) and the proof of (b), we have that

$$\mathrm{Prob}\{\Phi_t(Z_t) < \eta_t a_t\} + \sum_{z\in\mathbb{Z}^d} \mathrm{Prob}\Big\{\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t \ge N_t\Big\} \le C_1\big(e^{-C_2\eta_t^{d-\alpha}} + N_t^{d-\alpha}\big)\,,$$

so that this error term is of order $o(\theta_t^{-d})$ if $\beta > 0$ is large enough.

Now, we can unravel the definition of Z_t being the maximizer of Φ_t (in particular we know $t\xi(Z_t) > |Z_t|$ and $\Phi_t(Z_t)$ is positive) and write

$$\begin{aligned}&\mathrm{Prob}\big\{Z_t = Z_{t+\theta_t},\, \eta_t' a_t \le \Phi_t(Z_t) \le \eta_t a_t,\, |Z_t| \le g_t r_t\big\}\\ &\quad= \int_{\eta_t}^{N_t} \sum_{z\in\mathbb{Z}^d} \mathrm{Prob}\left\{\begin{array}{cc} \Phi_t(\overline{z}) \le \Phi_t(z) & \text{for } \overline{z} \text{ with } t\xi(\overline{z}) > |\overline{z}|;\\ \Phi_{t(1+\theta_t)}(\overline{z}) \le \Phi_{t(1+\theta_t)}(z) & \text{for } \overline{z} \text{ with } t(1+\theta_t)\xi(\overline{z}) > |\overline{z}|;\\ t\xi(z) > |z| & \text{for } \frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}(1-\delta_t) + \delta_t \in \mathrm{d}y\end{array}\right\}\,.\end{aligned} \tag{22}$$

Let z be such that $|z| < g_t r_t$, and $\frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}(1-\delta_t) + \delta_t = y < N_t$. For any $\overline{z}$ with $|\overline{z}| < g_t r_t$ and

$$\begin{array}{cc}\Phi_t(\overline{z}) \le \Phi_t(z) & \text{if } t\xi(\overline{z}) > |\overline{z}|,\\ \Phi_{t(1+\theta_t)}(\overline{z}) \le \Phi_{t(1+\theta_t)}(z) & \text{if } t(1+\theta_t)\xi(\overline{z}) > |\overline{z}|,\end{array}$$

we can deduce from Lemma 3.2 that

$$\begin{array}{cc}\frac{\xi(\overline{z})}{a_t} - q\frac{|\overline{z}|}{r_t}(1+\delta_t) \le y & \text{if } t\xi(\overline{z}) > |\overline{z}|,\\ \frac{\xi(\overline{z})}{a_t} - \frac{q}{1+\theta_t}\frac{|\overline{z}|}{r_t}(1+\delta_t) \le y + \frac{q\theta_t}{1+\theta_t}\frac{|z|}{r_t}(1-\delta_t) & \text{if } t(1+\theta_t)\xi(\overline{z}) > |\overline{z}|.\end{array}$$

Recalling that $r_t \log t = t a_t$ it is easy to see that the inequalities on the left hold automatically for sufficiently large t, if the conditions on the right are violated. Therefore, using the independence of the $\xi(z)$, we get an upper bound on the expression in (22),

$$\begin{aligned}\int_{\eta_t}^{N_t} \sum_{z\in\mathbb{Z}^d} &\mathrm{Prob}\big\{\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t \in \mathrm{d}y\big\} \prod_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} \mathrm{Prob}\big\{\tfrac{\xi(\overline{z})}{a_t} \le y + q\tfrac{|\overline{z}|}{r_t}(1+\delta_t)\big\}\\ &\times \prod_{\substack{\overline{z}\in\mathbb{Z}^d\\ |z|<|\overline{z}|<r_t g_t}} \mathrm{Prob}\big\{\tfrac{\xi(\overline{z})}{a_t} - \tfrac{q}{1+\theta_t}\tfrac{|\overline{z}|}{r_t}(1+\delta_t) \le y + \tfrac{q\theta_t}{1+\theta_t}\tfrac{|z|}{r_t}(1-\delta_t)\big\}\,.\end{aligned} \tag{23}$$

We now require that $\beta' < 1$, so that $\delta_t \eta_t^{-1} \to 0$. In the following steps, we treat each of the products in the above expression separately. First of all, as $\xi(0)$ is Pareto-distributed,

$$\begin{aligned}\frac{1}{\mathrm{d}y}\,\mathrm{Prob}\big\{\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t \in \mathrm{d}y\big\} &= \alpha\, a_t^{-\alpha}\big(y + q\tfrac{|z|}{r_t}(1-\delta_t) - \delta_t\big)^{-(\alpha+1)} \\ &\le (1-\delta_t\eta_t^{-1})^{-(\alpha+1)}\alpha\, a_t^{-\alpha}\big(y + q\tfrac{|z|}{r_t}\big)^{-(\alpha+1)}\,.\end{aligned}$$

For the second expression in (23), we find that for all $y > \eta_t$, we know that $a_t y > a_t\eta_t > 1$, assuming that t is large enough. In particular, we can use the approximation $\log(1-x) < -x$ for $x < 1$ to obtain uniformly for all $y > \eta_t$ and all z,

$$\begin{aligned}\prod_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} \mathrm{Prob}\big\{\tfrac{\xi(\overline{z})}{a_t} \le y + q\tfrac{|\overline{z}|}{r_t}(1+\delta_t)\big\} &\le \exp\Big\{\sum_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} \log\big(1 - a_t^{-\alpha}\big(y + q\tfrac{|\overline{z}|}{r_t}(1+\delta_t)\big)^{-\alpha}\big)\Big\} \\ &\le \exp\Big\{-\sum_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} a_t^{-\alpha}\big(y + q\tfrac{|\overline{z}|}{r_t}(1+\delta_t)\big)^{-\alpha}\Big\} \\ &\le \exp\Big\{-(1+\delta_t)^{-\alpha}\int_{|\overline{z}|<|z|-\frac12} r_t^{-d}\big(y + q\tfrac{|\overline{z}|+\frac12}{r_t}\big)^{-\alpha}\mathrm{d}\overline{z}\Big\} \\ &\le (1+o(1))\exp\Big\{-(1+o(1))\int_{|\overline{x}|<\frac{|z|}{r_t}} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\}\,,\end{aligned}$$

where our assumptions on η_t guarantee that all the error terms are of order $o(1)$. Finally, we consider the last product in (23), and a similar calculation to above shows that uniformly in $y \ge \eta_t$ and for all $z \in \mathbb{Z}^d$,

$$\begin{aligned}&\prod_{\substack{\overline{z}\in\mathbb{Z}^d\\ |z|<|\overline{z}|<r_t g_t}} \mathrm{Prob}\big\{\tfrac{\xi(\overline{z})}{a_t} - \tfrac{q}{1+\theta_t}\tfrac{|\overline{z}|}{r_t}(1+\delta_t) \le y + \tfrac{q\theta_t}{1+\theta_t}\tfrac{|z|}{r_t}(1-\delta_t)\big\} \\ &\qquad\le (1+o(1))\exp\Big\{-(1+o(1))\int_{\frac{|z|}{r_t}\le|\overline{x}|\le g_t}\big(y + \tfrac{q\theta_t}{1+\theta_t}\tfrac{|z|}{r_t} + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\}\,.\end{aligned}$$

Combining these estimates to bound (23) and thus (22), we obtain

$$\begin{aligned}&\mathrm{Prob}\big\{Z_t = Z_{t+\theta_t},\, \eta_t' a_t \le \tfrac{\xi(Z_t)}{a_t} - q\tfrac{|Z_t|}{r_t}(1-\delta_t) + \delta_t \le \eta_t a_t,\, |Z_t| \le g_t r_t\big\} \\ &\quad\le (\alpha+o(1))\int_{\eta_t}^{N_t}\sum_{z\in\mathbb{Z}^d} r_t^{-d}\exp\Big\{-(1+o(1))\int_{|\overline{x}|<\frac{|z|}{r_t}}(y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\ &\qquad\times\exp\Big\{-(1+o(1))\int_{\frac{|z|}{r_t}\le|\overline{x}|\le g_t}\big(y + \tfrac{q\theta_t}{1+\theta_t}\tfrac{|z|}{r_t} + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\}\big(y+q\tfrac{|z|}{r_t}\big)^{-(\alpha+1)}\mathrm{d}y \\ &\quad\le (1+o(1))\int_{\eta_t}^{N_t}\int_{x\in\mathbb{R}^d}\exp\Big\{-(1+o(1))\int_{|\overline{x}|<|x|}(y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\ &\qquad\times\exp\Big\{-(1+o(1))\int_{|x|\le|\overline{x}|\le g_t}\big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\}\frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}}\,,\end{aligned}$$

where, as before, the approximation of the sum by an integral works because $\eta_t a_t \to \infty$. Note also that, uniformly in x and y,

$$\int_{|\overline{x}|\ge g_t}\big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x} \le (1+\theta_t)^{\alpha}q^{-\alpha}\int_{|\overline{x}|\ge g_t}|\overline{x}|^{-\alpha} \le C'\theta_t^{\alpha}g_t^{d-\alpha}\,,$$

where $C' > 0$ is some universal constant. Choosing $\gamma > 0$ large enough ensures that this term

tends to 0. Hence, together with (21) we have shown that

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&\le (1+o(1)) \int_{y>0} \int_{x\in\mathbb{R}^d} \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha} \mathrm{d}\overline{x} \Big\} \\
&\qquad \exp\Big\{ -(1+o(1)) \int_{|x|\le|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha} \mathrm{d}\overline{x} \Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} + o(\theta_t^{-d})\,.
\end{aligned}$$

Lower bound. Before we simplify the expression for the upper bound, we derive a similar expression for the lower bound. As in the upper bound, we follow the main idea and restrict our attention to large boxes and estimate

$$\begin{aligned}
\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} &\ge \sum_{\substack{z\in\mathbb{Z}^d \\ |z|\le r_t g_t}} \mathrm{Prob}\big\{ Z_t = z = Z_{t+\theta t},\ \tfrac{\xi(z)}{a_t} - 2q\tfrac{|z|}{r_t} \le N_t \big\} \\
&= \sum_{\substack{z\in\mathbb{Z}^d \\ |z|\le r_t g_t}} \mathrm{Prob} \left\{ \begin{array}{cc} \Phi_t(\overline{z}) \le \Phi_t(z) & \text{for } \overline{z} \text{ with } t\xi(\overline{z}) > |\overline{z}|; \\ \Phi_{t(1+\theta_t)}(\overline{z}) \le \Phi_{t(1+\theta_t)}(z) & \text{for } \overline{z} \text{ with } t(1+\theta_t)\xi(\overline{z}) > |\overline{z}|; \\ t\xi(z) > |z|; & \tfrac{\xi(z)}{a_t} - 2q\tfrac{|z|}{r_t} \le N_t \end{array} \right\}.
\end{aligned} \tag{24}$$

The proof of Lemma 3.2 shows that if z is such that $|z| \le g_t r_t$ and $\frac{\xi(z)}{a_t} - 2q\frac{|z|}{r_t} \le N_t$, then we can find $C > 0$ such that with $\delta_t = C\frac{\log\log t}{\log t}$ we have that

$$\begin{aligned}
\tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1+\delta_t) \le \tfrac{\Phi_t(z)}{a_t} \le \tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t \text{ and} \\
\tfrac{\xi(z)}{a_t} - \tfrac{q}{1+\theta}\tfrac{|z|}{r_t}(1+\delta_t) \le \tfrac{\Phi_{t+\theta t}(z)}{a_t} \le \tfrac{\xi(z)}{a_t} - \tfrac{q}{1+\theta}\tfrac{|z|}{r_t}(1-\delta_t) + \delta_t.
\end{aligned}$$

Therefore, we can approximate (24) further by

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&\ge \sum_{\substack{z\in\mathbb{Z}^d \\ |z|\le r_t g_t}} \int_{\eta_t}^{N_t} \mathrm{Prob} \left\{ \begin{array}{cc} \frac{\xi(\overline{z})}{a_t} - q\frac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y & \text{for } \overline{z} \ne z; \\ \frac{\xi(\overline{z})}{a_t} - \frac{q}{1+\theta_t}\frac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y + \frac{q\theta_t}{1+\theta_t}(1+\delta_t) & \text{for } \overline{z} \ne z; \\ \frac{\xi(z)}{a_t} - q\frac{|z|}{r_t}(1+\delta_t) \in \mathrm{d}y \end{array} \right\}
\end{aligned} \tag{25}$$

We now show that, depending on whether $|z| \le |\overline{z}|$ or $|z| > |\overline{z}|$ one of the two conditions in the bracket above is superfluous. Indeed, if $|\overline{z}| \le |z|$ and the first condition holds we can deduce that

$$\frac{\xi(\overline{z})}{a_t} - \frac{q}{1+\theta_t}\frac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y + \frac{q\theta_t}{1+\theta_t}\frac{|\overline{z}|}{r_t}(1-\delta_t) \le y + \frac{q\theta_t}{1+\theta_t}\frac{|z|}{r_t}(1+\delta_t)\,.$$

Conversely, if $|\overline{z}| > |z|$ and we assume the second condition it follows that

$$\frac{\xi(\overline{z})}{a_t} - q\frac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y - \frac{q\theta_t}{1+\theta_t}\frac{|\overline{z}|}{r_t}(1-\delta_t) + \frac{q\theta_t}{1+\theta_t}\frac{|z|}{r_t}(1-\delta_t) \le y\,.$$

Hence, we have found a lower bound which can be expressed using the independence of the ξ as

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&\ge \sum_{\substack{z\in\mathbb{Z}^d \\ |z|\le r_t g_t}} \int_{\eta_t}^{N_t} \mathrm{Prob}\big\{ \tfrac{\xi(z)}{a_t} - q\tfrac{|z|}{r_t}(1+\delta) \in \mathrm{d}y \big\} \prod_{\substack{\overline{z}\in\mathbb{Z}^d \\ |\overline{z}|<|z|}} \mathrm{Prob}\big\{ \tfrac{\xi(\overline{z})}{a_t} - q\tfrac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y \big\} \\
&\qquad \times \prod_{\substack{\overline{z}\in\mathbb{Z}^d \\ |\overline{z}|>|z|}} \mathrm{Prob}\Big\{ \tfrac{\xi(\overline{z})}{a_t} - \tfrac{q}{1+\theta_t}\tfrac{|\overline{z}|}{r_t}(1-\delta_t) + \delta_t \le y + \tfrac{q\theta_t}{1+\theta_t}\tfrac{|z|}{r_t}(1-\delta_t) \Big\}\,.
\end{aligned} \tag{26}$$

We use that $\log(1-x) \geq -x(1+x)$ for $0 < x < 1/2$ to see that

$$\begin{aligned}
&\prod_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} \mathrm{Prob}\big\{\tfrac{\xi(\overline{z})}{a_t} - q\tfrac{|\overline{z}|}{r_t}(1-\delta_t)+\delta_t \leq y\big\} \\
&\qquad = \exp\Big\{ \sum_{\substack{\overline{z}\in\mathbb{Z}^d\\ |\overline{z}|<|z|}} \log\big(1 - a_t^{-\alpha}\big(y + q\tfrac{|\overline{z}|}{r_t}(1-\delta_t) - \delta_t\big)^{-\alpha}\big)\Big\} \\
&\qquad \geq (1+o(1))\exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<\frac{|z|}{r_t}} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\},
\end{aligned}$$

where $o(1)$ tends to 0 uniformly in $y \geq \eta_t$ and all $x \in \mathbb{Z}^d$. Similarly as in the upper bound, we can deal with the other products in (26) and approximate the sums by integrals to obtain

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&\quad \geq (1+o(1)) \int_{|x|\leq g_t} \int_{\eta_t}^{N_t} \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\
&\qquad \times \exp\Big\{ -(1+o(1)) \int_{|x|\leq|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}},
\end{aligned}$$

which is almost the same expression as for the upper bound. In order to control the difference, we first estimate

$$\begin{aligned}
&\int_{|x|\geq g_t} \int_{\eta_t}^{N_t} \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\
&\qquad\qquad \times \exp\Big\{ -(1+o(1)) \int_{|x|\leq|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} \\
&\quad \leq \int_{|x|\geq g_t} \int_{\eta_t}^{N_t} \exp\Big\{ -(1+o(1)) \int (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} \\
&\quad = \tfrac{\alpha 2^d}{(d-1)!} \int_{\eta_t}^{N_t} \int_{r\geq g_t} e^{-(1+o(1))\vartheta y^{d-\alpha}} \frac{r^{d-1}\mathrm{d}r\,\mathrm{d}y}{(y+qr)^{\alpha+1}},
\end{aligned}$$

where we used the same simplification as in Proposition 2.4 and $\vartheta = \frac{2^d B(\alpha-d,d)}{q^d(d-1)!}$. Similarly, we get an upper bound of

$$\begin{aligned}
&\leq \tfrac{\alpha 2^d}{(d-1)!q^d} \int_0^\infty e^{-(1+o(1))\vartheta y^{d-\alpha}} y^{d-\alpha-1} \int_0^{N_t/(qg_t)} v^{\alpha-d}(1-v)^{d-1}\mathrm{d}v\,\mathrm{d}y \\
&\leq (1+o(1))\tfrac{\alpha}{B(\alpha-d,d)(\alpha-d)} \big(\tfrac{N_t}{qg_t}\big)^{\alpha-d+1}.
\end{aligned}$$

Making $\beta > 0$ larger depending on θ_t, and then choosing $\gamma > 0$ large depending on β and θ_t, we can ensure that this term is of order $o(\theta_t^{-d})$. Similar calculations yield

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d} \int_0^{\eta_t} \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\
&\qquad\qquad \times \exp\Big\{ -(1+o(1)) \int_{|x|\leq|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} \\
&\quad \leq \tfrac{\alpha 2^d B(\alpha-d+1,d)}{(d-1)!q^d}\, e^{-(1+o(1))\vartheta\eta_t^{d-\alpha}} \eta_t^{d-\alpha},
\end{aligned}$$

which is of order $o(\theta_t^{-d})$, and

$$\begin{aligned}
&\int_{x\in\mathbb{R}^d} \int_{N_t}^\infty \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha}\mathrm{d}\overline{x}\Big\} \\
&\qquad \times \exp\Big\{ -(1+o(1)) \int_{|x|\leq|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} \leq C N_t^{\alpha-d},
\end{aligned}$$

for some constant $C > 0$, which by choice of $\beta > 0$ is also of order $o(\theta_t^{-d})$.

Final step. Combining the upper and lower bound we have shown that

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&= (1+o(1)) \int_{y>0} \int_{x\in\mathbb{R}^d} \exp\Big\{ -(1+o(1)) \int_{|\overline{x}|<|x|} (y+q|\overline{x}|)^{-\alpha} \mathrm{d}\overline{x} \Big\} \\
&\qquad \exp\Big\{ -(1+o(1)) \int_{|x|\le|\overline{x}|} \big(y + \tfrac{q\theta_t}{1+\theta_t}|x| + \tfrac{q}{1+\theta_t}|\overline{x}|\big)^{-\alpha} \mathrm{d}\overline{x} \Big\} \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y+q|x|)^{\alpha+1}} + o(\theta_t^{-d})\,.
\end{aligned}$$

Simplifying the integrals as in Proposition 2.4, we obtain that $\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)}\} = (1+o(1))I(\theta_t) + o(\theta_t^{-d})$, and an appeal to Proposition 2.5 completes the proof. □

Remark 3.4. In fact, the proof of Proposition 3.1 even shows a slightly stronger statement. Namely, let $\gamma > 0$ and suppose ℓ_t is a function such that $\ell_t \to \infty$ as $t \to \infty$. Then for any $\varepsilon > 0$, there exists $T > 0$ such that for all $t \ge T$ and all $\ell_t \le \theta \le (\log t)^\gamma$, we have that

$$(1-\varepsilon)\tfrac{1}{d\,B(\alpha-d-1)}\,\theta^{-d} \le \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} \le (1+\varepsilon)\tfrac{1}{d\,B(\alpha-d-1)}\,\theta^{-d}\,.$$

As indicated in Section 1.3 the previous proposition suffices to prove the upper bound in Theorem 1.3. For the lower bound we also need to control the decay of correlations.

Lemma 3.5. *Let θ_t be a positive, nondecreasing function such that $\theta_t \to \infty$ as $t \to \infty$ and for some $\delta > 0$, $\theta_t \le (\log t)^\delta$ for all $t > 0$. Then, for any $t > 0$ and $s \ge (1+\theta_t)t$,*

$$\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)} \ne Z_s = Z_{s(1+\theta_s)}\} \le (1+o(1))\frac{1}{d^2 B(\alpha-d+1,d)^2}\theta_t^{-d}\theta_s^{-d}\,,$$

where $o(1)$ is an error term that vanishes as $t \to \infty$.

Proof. We use a similar notation as in the proof of Proposition 3.1. In particular, we will choose functions g_t, η_t, N_t depending on θ_t. Also, let $\delta_t = C\frac{\log\log t}{\log t}$, where C is the constant implied in the error bounds in Lemma 3.2. A lengthy routine calculation similar to Lemma 3.3 shows that

$$\begin{aligned}
&\mathrm{Prob}\{Z_t = Z_{t(1+\theta_t)} \ne Z_s = Z_{s(1+\theta_s)}\} \\
&\qquad = \mathrm{Prob}\left\{ \begin{array}{c} Z_t = Z_{t(1+\theta_t)} \ne Z_s = Z_{s(1+\theta_s)}; \\ \frac{\xi(Z_t)}{a_t} - q\frac{|Z_t|}{r_t}(1-\delta_t) + \delta_t \in [\eta_t, N_t]; \\ \frac{\xi(Z_s)}{a_s} - q\frac{|Z_s|}{r_s}(1-\delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\} + \mathrm{error}(s,t)\,,
\end{aligned} \tag{27}$$

where, for some constants $C_1, C_2 > 0$,

$$\mathrm{error}(t,s) \le C_1(e^{-C_2\eta_t^{d-\alpha}} + N_t^{d-\alpha})(e^{-C_2\eta_s^{d-\alpha}} + \theta_s^{-d} + N_s^{d-\alpha}) + C_1\theta_t^{-d}(e^{-C_2\eta_s^{d-\alpha}} + N_s^{d-\alpha})\,.$$

Taking $N_t = \theta_t^{q+3/2}$ and $\eta_t = \theta_t^{-\beta'}$ for $\beta' > 0$ ensures that the error is of order $o(\theta_t^{-d}\theta_s^{-d})$. We can therefore focus the probability on the right hand side of (27). Using Lemma 3.2, we find the following upper bound

$$\begin{aligned}
&\mathrm{Prob}\left\{ \begin{array}{c} Z_t = Z_{t(1+\theta_t)} \ne Z_s = Z_{s(1+\theta_s)}\,; \\ \frac{\xi(Z_t)}{a_t} - q\frac{|Z_t|}{r_t}(1-\delta_t) + \delta_t \in [\eta_t, N_t]\,; \\ \frac{\xi(Z_s)}{a_s} - q\frac{|Z_s|}{r_s}(1-\delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\} \\
&\le \sum_{z_1\in\mathbb{Z}^d} \sum_{z_2\in\mathbb{Z}^d\setminus\{z_1\}} \mathrm{Prob}\left\{ \begin{array}{c} \Phi_{t(1+\theta_t)}(\overline{z}) \le \Phi_{t(1+\theta_t)}(z_1) \quad \forall|\overline{z}| \le r_t g_t \text{ with } \overline{z} \ne z_1, z_2\,; \\ \Phi_{s(1+\theta_s)}(\overline{z}) \le \Phi_{s(1+\theta_s)}(z_2) \quad \forall r_t g_t < |\overline{z}| \le r_s g_s \text{ with } \overline{z} \ne z_1, z_2\,; \\ \frac{\xi(z_1)}{a_t} - q\frac{|z_1|}{r_t}(1-\delta_t) + \delta_t \in [\eta_t, N_t]\,; \\ \frac{\xi(z_2)}{a_s} - q\frac{|z_2|}{r_s}(1-\delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\},
\end{aligned}$$

which, taking $g_t = \theta_t^{q+3/2}$ and using the independence, we can estimate as

$$\leq \sum_{z_1\in\mathbb{Z}^d}\sum_{z_2\in\mathbb{Z}^d\setminus\{z_1\}}\int_{\eta_t}^{N_t}\int_{\eta_s}^{N_s}\prod_{\substack{|\overline{z}|<g_t r_t\\ \overline{z}\neq z_1,z_2}}\mathrm{Prob}\Big\{\tfrac{\xi(\overline{z})}{a_t}-q\tfrac{|\overline{z}|}{r_t}(1+\delta_t)\leq y_1+\tfrac{q\theta_t}{1+\theta_t}\tfrac{|z_1|}{r_t}(1-\delta_t)\Big\}$$
$$\times\prod_{\substack{g_t r_t<|\overline{z}|<g_s r_s\\ \overline{z}\neq z_1,z_2}}\mathrm{Prob}\Big\{\tfrac{\xi(\overline{z})}{a_s}-q\tfrac{|\overline{z}|}{r_s}(1+\delta_s)\leq y_2+\tfrac{q\theta_s}{1+\theta_s}\tfrac{|z_2|}{r_s}(1-\delta_s)\Big\}$$
$$\times\mathrm{Prob}\Big\{\tfrac{\xi(z_1)}{a_t}-q\tfrac{|z_1|}{r_t}(1-\delta_t)+\delta_t\in\mathrm{d}y_1\Big\}\mathrm{Prob}\Big\{\tfrac{\xi(z_2)}{a_s}-q\tfrac{|z_2|}{r_s}(1-\delta_s)+\delta_s\in\mathrm{d}y_2\Big\}.$$

As before, we can work out the probabilities, and approximate the sums by integrals to finally obtain $(1+o(1))$ times

$$\begin{aligned}&\int\int_{\eta_t}^{N_t}\exp\Big\{-(1+o(1))\int_{|\overline{x}|<g_t}\big(y_1+\tfrac{q}{1+\theta_t}|\overline{x}|+\tfrac{q\theta_t}{1+\theta_t}|x_1|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\}\frac{\alpha\,\mathrm{d}x_1\,\mathrm{d}y_1}{(y_1+q|x_1|)^{\alpha+1}}\\ &\times\int\int_{\eta_s}^{N_s}\exp\Big\{-(1+o(1))\int_{g_t r_t/r_s<|\overline{x}|<g_s}\big(y_2+\tfrac{q}{1+\theta_s}|\overline{x}|+\tfrac{q\theta_s}{1+\theta_s}|x_2|\big)^{-\alpha}\mathrm{d}\overline{x}\Big\}\frac{\alpha\,\mathrm{d}x_2\,\mathrm{d}y_2}{(y_2+q|x_2|)^{\alpha+1}}\,.\end{aligned}\tag{28}$$

In the remainder of the proof, we have to show that the first term is of order θ_t^{-d}, whereas the second is of order θ_s^{-d}. The integral in the first factor equals in polar coordinates

$$\begin{aligned}&\tfrac{2^d}{(d-1)!}\int_{0<r<g_t}\big(y_1+\tfrac{q}{1+\theta_t}r+\tfrac{q\theta_t}{1+\theta_t}|x_1|\big)^{-\alpha}r^{d-1}\mathrm{d}r\\ &\quad\geq\tfrac{2^d}{(d-1)!}(1+\theta_t)^d\int_{0<r<\frac{g_t}{1+\theta_t}}(y_1+qr+q|x_1|\big)^{-\alpha}r^{d-1}\mathrm{d}r\\ &\quad=\tfrac{2^d}{(d-1)!}(1+\theta_t)^d\Big\{q^{-\alpha}(y_1+q|x_1|)^{d-\alpha}B(\alpha-d,d)-\int_{r>\frac{g_t}{(1+\theta_t)}}(y_1+qr+q|x_1|\big)^{-\alpha}r^{d-1}\mathrm{d}r\Big\}.\end{aligned}$$

The subtracted integral is bounded from above by $q^{-\alpha}g_t^{d-\alpha}(1+\theta_t)^{\alpha-d}$ and therefore, by our assumptions, together with the $(1+\theta_t)^d$ factor tends to zero. Hence we can conclude that, with ϑ as before, the first factor in (28) is bounded from above by

$$\begin{aligned}&(1+o(1))\int\int_{\eta_t}^{N_t}e^{-(1+o(1))\vartheta(1+\theta_t)^d(y_1+q|x_1|)^{d-\alpha}}\frac{\alpha\,\mathrm{d}y_1\,\mathrm{d}x_1}{(y_1+q|x_1|)^{\alpha+1}}\\ &\leq(1+o(1))\,\tfrac{2^d}{(d-1)!}\int_0^\infty\int_0^\infty e^{-(1+o(1))\vartheta(1+\theta_t)^d(y+qr)^{d-\alpha}}\frac{\alpha\,r^{d-1}\mathrm{d}y_1\,\mathrm{d}r}{(y_1+qr)^{\alpha+1}}\\ &\leq(1+o(1))\,\tfrac{\alpha 2^d}{(d-1)!q^d}\int_0^\infty\int_0^1 e^{-(1+o(1))\vartheta(1+\theta_t)^d y^{d-\alpha}v^{\alpha-d}}y^{\alpha-d-1}v^{\alpha-d}(1-v)^{d-1}\mathrm{d}v\,\mathrm{d}y\\ &=(1+o(1))\,(1+\theta_t)^{-d}\tfrac{\alpha}{(\alpha-d)B(\alpha-d,d)}\int_0^1(1-v)^{d-1}\mathrm{d}v=(1+o(1))\,\theta_t^{-d}\tfrac{1}{dB(\alpha-d+1,d)}\,.\end{aligned}$$

For the second factor in (28), we almost get the same expression, and it suffices to consider the following term and, using similar arguments as above, we can estimate uniformly in $y_2\geq\eta_s$,

$$\begin{aligned}&\int_{|\overline{x}|<\frac{g_t r_t}{r_s}}(y_2+\tfrac{q}{1+\theta_s}|\overline{x}|+q|x_2|)^{-\alpha}\mathrm{d}\overline{x}\\ &\qquad\leq(1+\theta_s)^d\tfrac{2^d}{q^d(d-1)!}(y_2+q|x_2|)^{d-\alpha}\int_{1-\frac{qg_t r_t}{r_s(1+\theta_s)\eta_s}}^1 u^{\alpha-d-1}(1-u)^{d-1}\mathrm{d}u\,.\end{aligned}$$

Using that $s/t \geq (1+\theta_t)$ and recalling that $\eta_t = \theta_t^{-\beta'}$, where we can assume $0 < \beta' < 1$ and $g_t = \theta_t^{q+3/2}$, we obtain

$$\frac{g_t r_t}{r_s(1+\theta_s)\eta_s} \leq \frac{g_t(\log t + \log(1+\theta_t))^{q+1}}{(\log t)^{q+1}\theta_t^{q+2-\beta'}} \leq (1+o(1))\theta_t^{\beta'-\frac{1}{2}},$$

so that, by choosing $\beta' < \frac{1}{2}$, this term tends to 0. Now, we can simplify the second factor in (28) in the same way as the first one to show that it is of the required form. □

3.2 Almost sure asymptotics for the maximizer of Φ_t

In analogy with the residual lifetime function R for the process X_t, we can also define the residual lifetime function R^V for the maximizer Z_t of the variational problem, by setting

$$R^V(t) = \sup\{s \geq 0 \,:\, Z_t = Z_{t+s}\}.$$

Using the moderate deviation principle, Proposition 3.1, developed in the previous section together with the Borel-Cantelli lemma, we aim to prove the following analogue of Theorem 1.3.

Proposition 3.6. *For any nondecreasing function $h : (0,\infty) \to (0,\infty)$ we have, almost surely,*

$$\limsup_{t\to\infty} \frac{R^V(t)}{th(t)} = \begin{cases} 0 & \text{if } \int_1^\infty \frac{\mathrm{d}t}{th(t)^d} < \infty, \\ \infty & \text{if } \int_1^\infty \frac{\mathrm{d}t}{th(t)^d} = \infty. \end{cases}$$

Proof of the first part of Proposition 3.6. Consider $h\colon (0,\infty) \to (0,\infty)$ such that $\int_1^\infty \frac{\mathrm{d}t}{th(t)^d} < \infty$ which is equivalent to $\int_{t>1} h(\frac{1}{3}e^t)^{-d}\mathrm{d}t < \infty$, so that

$$\sum_{n=1}^{\infty} h(\tfrac{1}{3}e^n)^{-d} < \infty. \tag{29}$$

It is not hard to see that $h(t) \to \infty$ and that we can assume, without loss of generality, that $h(t) \leq (\log t)^\gamma$ for some $\gamma > 1$, replacing $h(t)$ by $\tilde{h}(t) = h(t) \wedge (\log t)^\gamma$ if necessary.

Fix $\varepsilon > 0$ and an increasing sequence $t_n \to \infty$. It suffices to show that almost surely,

$$\limsup_{n\to\infty} \frac{R^V(t_n)}{t_n h(t_n)} \leq \varepsilon.$$

To this end, we now show that for all but finitely many n,

$$\tfrac{R^V(t_n)}{t_n} > \varepsilon h(t_n) \quad \text{implies} \quad \tfrac{R^V(t)}{t} > \tfrac{1}{4}\varepsilon h(t_n) \text{ for all } t \in [t_n, 3t_n]. \tag{30}$$

By definition, $R^V(t_n) > \varepsilon\, t_n h(t_n)$ implies that Z_t does not jump during the interval $[t_n, t_n(1+\varepsilon h(t_n))]$. As R^V is affine with slope -1 on this interval

$$\frac{R^V(t)}{t} = \frac{R^V(t_n)+t_n-t}{t} > \frac{(1+\varepsilon h(t_n))t_n - t}{t} \geq \frac{\varepsilon h(t_n)}{4} \quad \text{for } t \in [t_n, t_n\tfrac{(1+\varepsilon h(t_n))}{(1+\frac{1}{4}\varepsilon h(t_n))}]$$

Recall that $h(t) \to \infty$, and hence we have, for all but finitely many n, that $(1+\varepsilon h(t_n)) \geq 3(1+\frac{1}{4}\varepsilon h(t_n))$, completing the proof of (30).

Now, define $k(n) = \inf\{k \,:\, e^k \geq t_n\}$, so that in particular $t_n \leq e^{k(n)} < 3t_n$. Then, by (30) and monotonicity of φ, we can deduce that for n large enough

$$\tfrac{R^V(t_n)}{t_n} \geq \varepsilon h(t_n) \quad \text{implies} \quad \tfrac{R^V(e^{k(n)})}{e^{k(n)}} \geq \tfrac{\varepsilon}{4} h(t_n) \geq \tfrac{\varepsilon}{12} h(\tfrac{1}{3}e^{k(n)}).$$

This shows in particular that

$$\mathrm{Prob}\big\{\tfrac{R^V(t_n)}{t_n} \geq \varepsilon h(t_n) \text{ infinitely often}\big\} \leq \mathrm{Prob}\big\{\tfrac{R^V(e^n)}{e^n} \geq \tfrac{\varepsilon}{12} h(\tfrac{1}{3}e^n) \text{ infinitely often}\big\}\,.$$

By Proposition 3.1 we can deduce that exists a constant $\tilde{C}$ such that for all n large enough

$$\mathrm{Prob}\big\{\tfrac{R^V(e^n)}{e^n} \geq \tfrac{\varepsilon}{12} h(\tfrac{1}{3}e^n)\big\} \leq \tilde{C} h(\tfrac{1}{3}e^n)^{-d}\,.$$

By (29) these probabilities are summable, so that Borel-Cantelli completes the proof. □

For the second part of Proposition 3.6, we need to prove a lower bound on the limit superior, so our strategy is to use the fine control over the decay of correlations that we developed in the previous section and combine it with the Kochen-Stone lemma.

Proof of second part of Proposition 3.6. Let $h\colon (0,\infty) \to (0,\infty)$ be such that $\int_1^\infty \frac{\mathrm{d}t}{t h(t)^d} = \infty$. Then, we can deduce that

$$\sum_{n=1}^{\infty} h(e^n)^{-d} = \infty. \tag{31}$$

Without loss of generality, we can assume that $h(t) \to \infty$ and also additionally that $h(t) \leq (\log t)^{2/d}$ for all t. Indeed, if necessary, we may replace $h(t)$ by $\tilde{h}(t) = h(t) \wedge (\log t)^{2/d}$. For fixed $\kappa > 0$, define the event $E_n = \{\frac{R(e^n)}{e^n} \geq \kappa h(e^n)\}$. By Proposition 3.1,

$$\mathrm{Prob}(E_n) = \tfrac{1}{d\,B(\alpha-d+1,d)}(1+o(1))\kappa^{-d} h(e^n)^{-d}\,,$$

so that by (31) we have $\sum_{n=1}^\infty \mathrm{Prob}(E_n) = \infty$. By the Kochen-Stone lemma, see for instance [FG97], we then have that

$$\mathrm{Prob}\{E_n \text{ infinitely often }\} \geq \limsup_{k\to\infty} \frac{\big(\sum_{n=1}^k \mathrm{Prob}(E_n)\big)^2}{\sum_{n=1}^k \sum_{m=1}^k \mathrm{Prob}(E_m \cap E_n)}\,. \tag{32}$$

Fix $\varepsilon > 0$. By Proposition 3.1 and Remark 3.4 we can deduce that we can choose N large enough such that for all $t \geq N$ and all $(\log t)^{\frac{1}{2d}} \wedge h(t) \leq \theta \leq (\log t)^6$, we have that

$$(1-\varepsilon)\tfrac{1}{d\,B(\alpha-d+1,d)}\,\theta^{-d} \leq \mathrm{Prob}\{Z_t = Z_{t+\theta t}\} \leq (1+\varepsilon)\tfrac{1}{d\,B(\alpha-d+1,d)}\,\theta^{-d}\,. \tag{33}$$

Also, by Lemma 3.5, we know that we can assume N is large enough such that such that for all $n \geq N$ and $m \geq n + \log(1+\kappa h(e^n))$, we have that

$$\begin{aligned}\mathrm{Prob}\{Z_{e^n} = Z_{e^n(1+\kappa h(e^n))} \neq Z_{e^m} = Z_{e^m(1+\kappa h(e^m))}\}& \\ \leq (1+\varepsilon)\big(\tfrac{1}{dB(\alpha-d+1,1)}\big)^2 \kappa^{-2d} h(e^n)^{-d} h(e^m)^{-d}& \\ \leq \tfrac{1+\varepsilon}{1-\varepsilon}\,\mathrm{Prob}(E_n)\mathrm{Prob}(E_m)\,.&\end{aligned} \tag{34}$$

Note that by Lemma 2.8, we know that Z_t never returns to the same point, therefore we have

$$\begin{aligned}&\mathrm{Prob}(E_n \cap E_m)\\ &= \mathrm{Prob}\big\{Z_{e^n} = Z_{e^m(1+\kappa h(e^n))}\big\} + \mathrm{Prob}\big\{Z_{e^n} = Z_{e^n(1+\kappa h(e^n))} \neq Z_{e^m} = Z_{e^m(1+\kappa h(e^m))}\big\}.\end{aligned}$$

In particular, notice that the second probability is zero if $n \leq m \leq n + \log(1+\kappa h(e^n))$. Hence,

we can estimate for $n > N$ and for k large enough, using (33) and (34),

$$\begin{aligned}
&\sum_{m=n}^{k} \mathrm{Prob}(E_n \cap E_m) \\
&\quad \le \sum_{m=n}^{n+2\log n} \mathrm{Prob}\{Z_{e^n} = Z_{e^m(1+\kappa h(e^n))}\} + \sum_{m=n+2\log n}^{k} \mathrm{Prob}\{Z_{e^m n^{-2}} = Z_{e^m(1+\kappa h(e^m))}\} \\
&\qquad + \tfrac{1+\varepsilon}{1-\varepsilon} \sum_{m=n+\log(1+\kappa h(e^n))}^{k} \mathrm{Prob}(E_n)\mathrm{Prob}(E_m) \\
&\quad \le \tilde{C}\,\mathrm{Prob}(E_n) \sum_{m=n}^{k} e^{d(n-m)} + \tilde{C}\, n^{-2d} \sum_{m=n}^{k} \mathrm{Prob}(E_m) + \tfrac{1+\varepsilon}{1-\varepsilon} \sum_{m=n}^{k} \mathrm{Prob}(E_n)\mathrm{Prob}(E_m)\,,
\end{aligned}$$

where $\tilde{C}$ is some suitable constant. Finally, in order to bound the right hand side of (32), we can estimate for $k > N$,

$$\begin{aligned}
\sum_{n=1}^{k}\sum_{m=1}^{k} \mathrm{Prob}(E_n \cap E_m) &\le 2N \sum_{n=1}^{k} \mathrm{Prob}(E_n) + \sum_{n=N}^{k}\sum_{m=N}^{k} \mathrm{Prob}(E_n \cap E_m) \\
&\le 2\sum_{n=1}^{k} \Big(N + \sum_{m=1}^{k} m^{-2d} + \sum_{m=n}^{k} \tilde{C} e^{d(n-m)}\Big)\mathrm{Prob}(E_n) \\
&\qquad + 2\tfrac{1+\varepsilon}{1-\varepsilon} \sum_{n=N}^{k}\sum_{m=n}^{k} \mathrm{Prob}(E_n)\mathrm{Prob}(E_m) \\
&\le C' \sum_{n=1}^{k} \mathrm{Prob}(E_n) + \tfrac{1+\varepsilon}{1-\varepsilon} \sum_{n=1}^{k}\sum_{k=1}^{k} \mathrm{Prob}(E_n)\mathrm{Prob}(E_m)\,,
\end{aligned}$$

where $C' > 0$. Therefore, we can conclude from (32) that $\mathrm{Prob}\{E_n \text{ infinitely often }\} \ge \frac{1-\varepsilon}{1+\varepsilon}$, and since $\varepsilon > 0$ and $\kappa > 0$ were arbitrary, the second statement of Proposition 3.6 follows. □

3.3 Almost sure asymptotics for the maximizer of the solution profile

In this section, we prove Theorem 1.3. Thus, we have to transfer the almost sure ageing result of Proposition 3.6, which was formulated on the level of the variational problem, to the residual lifetime function of the maximizer X_t of the profile v. The underlying idea is that most of the time X_t and the maximizer of the variational problem Z_t agree and we only have to control the length of the intervals when they can disagree. The latter scenario corresponds to those times during which the processes relocate to another point. Therefore, our strategy is to look at the jump times and show that both processes jump at almost the same times.

The period when the maximizers relocates correspond exactly to those times when $Z_t^{(1)}$ and $Z_t^{(2)}$ produce a comparable value of Φ. With this in mind, define for $\lambda_t = (\log t)^{-\beta}$ with $\beta > 1 + \frac{1}{\alpha - d}$, the set of exceptional transition times

$$\mathcal{E} = \mathcal{E}(\beta) = \big\{t > t_0 \,:\, \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \le \tfrac{1}{2} a_t \lambda_t\big\}\,, \tag{35}$$

where t_0 is chosen sufficiently large and, to avoid trivialities, such that $t_0 \neq \inf \mathcal{E}$. By [KLMS09, Lemma 3.4] we can choose t_0 large enough such that for all $t > t_0$,

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) > a_t \lambda_t\,. \tag{36}$$

Lemma 3.7. *The process $(Z_t^{(1)} \colon t \ge t_0)$ jumps only at times contained in the set $\mathcal{E}$. Moreover, each connected component of $\mathcal{E}$ contains exactly one such jump time.*

Proof. The first part of the statement is trivial, since at each jump time $\tau \geq t_0$ of $Z_t^{(1)}$ we have that $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$ so that $\tau \in \mathcal{E}$. For the second statement, let $[b^-, b^+]$ be a connected component of $\mathcal{E}$, then

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) = \tfrac{1}{2} a_t \lambda_t \,,$$

for $t = b^-, b^+$ (here we use that $b^- \geq \inf \mathcal{E} \neq t_0$). Now, since $t \mapsto \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is never constant, if $Z_{b^-}^{(1)} = Z_{b^+}^{(1)}$ then by Lemma 2.10 there is $t \in (b^-, b^+)$ such that $t \notin \mathcal{E}$ contradicting the connectedness of $[b^-, b^+]$. Thus, we can conclude that $Z_t^{(1)}$ jumps at least once in $[b^-, b^+]$. Finally, the fact that, by Lemma 2.8, $Z_t^{(1)}$ never returns to the same point combined with (36) guarantees that $Z_t^{(1)}$ only jumps once in $[b^-, b^+]$ (namely from $Z_{b^-}^{(1)}$ to $Z_{b^-}^{(2)}$). □

Denote by (τ_n) the jump times of the maximizer process $(Z_t^{(1)} \colon t \geq t_0)$ in increasing order. In the next lemma we have collected some of their basic properties.

Lemma 3.8. *(i) Fix $\beta > 1 + \frac{1}{\alpha - d}$, then, almost surely, for all but finitely many n,*

$$(\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\tfrac{\tau_{n+1} - \tau_n}{\tau_n}\right) \geq a_{\tau_n} (\log \tau_n)^{-\beta} \,.$$

(ii) Fix $\gamma > 1 + \frac{2}{\alpha - d}$, then, almost surely, for all but finitely many n,

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} \geq (\log \tau_n)^{-\gamma} \,.$$

(iii) Fix $\delta > 1 + \frac{1}{\alpha - d} + \frac{1}{d}$, then, almost surely, for all but finitely many n,

$$\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)}) \geq a_{\tau_n} (\log \tau_n)^{-\delta} \,.$$

Proof. (i) By Lemma 2.6 and Remark 2.7 we find that

$$\begin{aligned} \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(1)}) - \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(2)}) &= (\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\tfrac{\tau_{n+1} - \tau_n}{\tau_{n+1}}\right) \\ &\leq (\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\tfrac{\tau_{n+1} - \tau_n}{\tau_n}\right) . \end{aligned} \tag{37}$$

Now, we can estimate the difference on the left-hand side from below by using that $Z_{\tau_n}^{(2)}$ cannot produce more than the third largest value of Φ at time τ_{n+1}. Indeed, Lemma 2.8 ensures that $Z^{(1)}$ never visits the same point again, so that $Z_{\tau_n}^{(2)} = Z_{\tau_n - 1}^{(1)} \neq Z_{\tau_{n+1}}^{(i)}$ for $i = 1, 2$ since $Z_{\tau_{n+1}}^{(2)} = Z_{\tau_n}^{(1)}$. Hence, using [KLMS09, Proposition 3.4] for the second inequality,

$$\begin{aligned} \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(2)}) &\leq \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(3)}) \leq \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(1)}) - a_{\tau_{n+1}} (\log \tau_{n+1})^{-\beta} \\ &\leq \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(1)}) - a_{\tau_n} (\log \tau_n)^{-\beta} \,, \end{aligned}$$

where in the last step we again used that $Z_{\tau_n}^{(1)} = Z_{\tau_{n+1}}^{(2)}$ and that $t \mapsto a_t (\log t)^{-\beta}$ is increasing for all sufficiently large t. Substituting this inequality into (37) completes the proof of part (i).

(ii) By the first part, we need to get an upper bound on $\xi(Z_{\tau_n}^{(1)})$. Therefore, our first claim is that for any $\delta > \frac{1}{\alpha - d}$, and all t sufficiently large

$$\xi(Z_t^{(1)}) \leq a_t (\log t)^{\delta} \,. \tag{38}$$

Indeed, by [HMS08, Lemma 3.5], for $\varepsilon = \frac{1}{3}(\delta - \frac{1}{\alpha - d})$ we have for all sufficiently large r,

$$\max_{|z| \leq r} \xi(z) \leq r^{\frac{d}{\alpha}} (\log r)^{\frac{1}{\alpha} + \varepsilon} \,.$$

Moreover, by [KLMS09, Lemma 3.2], for $\varepsilon' = \frac{\alpha}{d}(\frac{1}{3}(\delta - \frac{1}{\alpha - d}))$, we find that $|Z_t^{(1)}| < r_t (\log t)^{\frac{1}{\alpha - d} + \varepsilon'}$ and therefore

$$\begin{aligned} \xi(Z_t^{(1)}) &\leq a_t (\log t)^{(\frac{1}{\alpha - d} + \varepsilon') \frac{d}{\alpha}} (\log(r_t (\log t)^{\frac{1}{\alpha - d} + \varepsilon'}))^{\frac{1}{\alpha} + \varepsilon} \\ &= a_t (q+1)^{\frac{1}{\alpha} + \varepsilon} (\log t)^{\frac{1}{\alpha - d} + \varepsilon' \frac{d}{\alpha} + \varepsilon} (1 + o(1)) < a_t (\log t)^{\delta} \,, \end{aligned}$$

eventually for all t sufficiently large.

Now, if we combine part (i) for $\beta = \frac{1}{2}(\gamma+1) > 1 + \frac{1}{\alpha - d}$ with (38) for $\delta = \frac{1}{2}(\gamma - 1)$, we get

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} \geq \frac{a_{\tau_n}(\log \tau_n)^{-\beta}}{\xi(Z^{(1)}_{\tau_n})} \geq \frac{a_{\tau_n}(\log \tau_n)^{-\beta}}{a_{\tau_n}(\log \tau_n)^{\delta}} = (\log \tau_n)^{-(\delta+\beta)} = (\log \tau_n)^{-\gamma},$$

which completes the proof of the lemma.

(iii) Note that for any $\delta' > \frac{1}{d}$, Proposition 3.6, shows that for all but finitely many n,

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} = \frac{R^V(\tau_n)}{\tau_n} \leq (\log \tau_n)^{\delta'}.$$

This observation together with part (i), immediately implies the statement of part (iii). □

A similar statement to Lemma 3.7 also holds for the process $X_t = \operatorname{argmax}\{u(t,z) : z \in \mathbb{Z}^d\}$. Fix $0 < \varepsilon < \frac{1}{3}$, then by [KLMS09, Proposition 5.3] we can assume additionally that t_0 in the definition (35) of $\mathcal{E}$ is chosen large enough such that for all $t > t_0$

$$\Big[U(t)^{-1} \sum_{\substack{z \in \mathbb{Z}^d \\ z \neq Z^{(1)}_t}} u(t,z)\Big] \mathbb{1}\{\Phi_t(Z^{(1)}_t) - \Phi_t(Z^{(2)}_t) \geq \tfrac{1}{2} a_t \lambda_t\} < \varepsilon. \tag{39}$$

Furthermore, by the 'two cities theorem' [KLMS09, Theorem 1.1], we may assume that for all $t \geq t_0$,

$$\frac{u(t, Z^{(1)}_t) + u(t, Z^{(2)}_t)}{U(t)} > 1 - \varepsilon. \tag{40}$$

Lemma 3.9. *The process $(X_t : t \geq t_0)$ only jumps at times contained in $\mathcal{E}$ and each connected component of $\mathcal{E}$ contains exactly one such jump time. Furthermore, it never returns to the same point in $\mathbb{Z}^d$.*

Proof. By (39), for any $t \in [t_0, \infty) \setminus \mathcal{E}$, we have $X_t = Z_t$ so that, in particular, X_t jumps only at times in $\mathcal{E}$. Now, let $[b^-, b^+]$ be a connected component of $\mathcal{E}$. Note that the proof of Lemma 3.7 shows that for all $t \in [b^-, b^+]$, the set $\{Z^{(1)}_t, Z^{(2)}_t\}$ consists of exactly two points, $z^{(1)} := Z^{(1)}_{b^+}$ and $z^{(2)} := Z^{(2)}_{b^+} = Z^{(1)}_{b^-}$. Hence, by (39) we find that $X_{b^-} = z^{(2)}$ and $X_{b^+} = z^{(1)}$. Also, the two-point localization (40) implies that $\{X_t : t \in [b^-, b^+]\} = \{z^{(1)}, z^{(2)}\}$. Hence, it remains to show that $(X_t : t > 0)$ jumps only once (from $z^{(2)}$ to $z^{(1)}$) in the interval $[b^-, b^+]$.

Define the function

$$g(t) = \frac{u(t, z^{(1)})}{u(t, z^{(2)})}.$$

Then, note that since u solves the heat equation, for $z \in \{z^{(1)}, z^{(2)}\}$,

$$\tfrac{\partial}{\partial t} u(t,z) = \Delta u(t,z) + \xi(z)\, u(t,z) = \sum_{y \sim z} (u(t,y) - u(t,z)) + \xi(z)\, u(t,z).$$

Furthermore, by [KLMS09, Lemmas 2.2, 3.2], we have $z^{(1)} \not\sim z^{(2)}$ so that using (40) we get

$$\begin{aligned}(-2d + \xi(z))\, u(t,z) < \tfrac{\partial}{\partial t} u(t,z) &< 2d\varepsilon U(t) - 2d u(t,z) + \xi(z)\, u(t,z) \\ &< 2d \tfrac{\varepsilon}{1-\varepsilon} (u(t, z^{(1)}) + u(t, z^{(2)})) + (\xi(z) - 2d)\, u(t,z).\end{aligned}$$

Therefore,

$$\begin{aligned} g'(t) &= \frac{\frac{\partial}{\partial t} u(t, z^{(1)}) u(t, z^{(2)}) - u(t, z^{(1)}) \frac{\partial}{\partial t} u(t, z^{(2)})}{u(t, z^{(2)})^2} \\ &> \frac{1}{u(t, z^{(2)})^2} \Big[\big(\xi(z^{(1)}) - \xi(z^{(2)}) - 2d \tfrac{\varepsilon}{1-\varepsilon}\big) u(t, z^{(1)}) u(t, z^{(2)}) - 2d \tfrac{\varepsilon}{1-\varepsilon} u(t, z^{(1)})^2 \Big] \\ &= g(t) \Big(\xi(z^{(1)}) - \xi(z^{(2)}) - 2d \tfrac{\varepsilon}{1-\varepsilon} (1 + g(t)) \Big). \end{aligned}$$

Now, since $z^{(1)} = Z^{(1)}_{b^+}$ and $z^{(2)} = Z^{(1)}_{b^-}$, Lemma 3.8 shows (again assuming that t_0 is large enough) that, for any $\delta > 1 + \frac{1}{\alpha-d} + \frac{1}{d}$, if τ is the jump time of $Z^{(1)}$ in the interval $[b^-, b^+]$, then

$$\xi(z^{(1)}) - \xi(z^{(2)}) \geq a_\tau (\log \tau)^{-\delta} .$$

Hence, we can deduce that if there exists t' such that $g(t') = 1$, then $g'(t') > 0$. Using the continuity of u we see that first there can be at most one such t' and $g(t) < 1$ if $t < t'$ and $g(t) > 1$ if $t > t'$, and second that there exists $t' \in [b^-, b^+]$ such that $g(t') = 1$. Therefore it has to be unique and $u(t, z^{(1)}) < u(t, z^{(2)})$ if $t < t'$ and $u(t, z^{(1)}) > u(t, z^{(2)})$ if $t > t'$. Thus, we can see that X_t jumps exactly once in the interval $[b^-, b^+]$. □

In order to be able to deduce the asymptotics of the jump times of $(X_t \colon t > 0)$ from those of $(Z_t \colon t > 0)$, we find bounds for the length of a connected component of $\mathcal{E}$.

Lemma 3.10. *Suppose in the definition (35) we choose $\beta > 1 + \frac{q+2}{d} + \frac{1}{\alpha-d}$. Then, for any $0 < \varepsilon < \frac{1}{2}(\beta - (1 + \frac{q+2}{d} + \frac{1}{\alpha-d}))$, almost surely for any connected component $[b^-, b^+]$ of $\mathcal{E}$ with b^- large enough, we find that*

$$\frac{b^+ - b^-}{\tau} \leq (\log \tau)^{-\varepsilon} ,$$

where τ is the jump time of the process $(Z_t \colon t > 0)$ in the interval $[b^-, b^+]$.

Proof. We start by expressing the distances $b^+ - \tau$ and $\tau - b^-$ in terms of the potential values at the sites $Z^{(1)}_\tau$ and $Z^{(2)}_\tau$. As we have seen in the proof of Lemma 3.7, $Z^{(i)}_\tau = Z^{(i)}_{b^+}$ for $i = 1, 2$. Hence, we obtain that

$$\Phi_{b^+}(Z^{(1)}_\tau) - \Phi_{b^+}(Z^{(2)}_\tau) = \Phi_{b^+}(Z^{(1)}_{b^+}) - \Phi_{b^+}(Z^{(2)}_{b^+}) = \tfrac{1}{2} a_{b^+} \lambda_{b^+} .$$

Moreover, by Lemma 2.6 we get that

$$\Phi_{b^+}(Z^{(1)}_\tau) - \Phi_{b^+}(Z^{(2)}_\tau) = (\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau))(1 - \tfrac{\tau}{b^+}) .$$

Combining the previous two displayed equations and rearranging yields

$$b^+ - \tau = \frac{\frac{1}{2} b^+ a_{b^+} \lambda_{b^+}}{\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau)} . \tag{41}$$

Similarly, we know that $Z^{(1)}_{b^-} = Z^{(2)}_\tau$ and $Z^{(2)}_{b^-} = Z^{(1)}_\tau$ and deduce in the same way that

$$\tau - b^- = \frac{\frac{1}{2} b^- a_{b^-} \lambda_{b^-}}{\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau)} . \tag{42}$$

Define τ^+ as the next jump of $Z^{(1)}_t$ after τ, then $b^+ \leq \tau^+$. We use (41) and (42) to get

$$\begin{aligned} \frac{b^+ - b^-}{\tau} &= \frac{b^+ - \tau}{\tau} + \frac{\tau - b^-}{\tau} = \frac{1}{2} \frac{1}{\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau)} \Big(a_{b^+} \lambda_{b^+} \frac{b^+}{\tau} + a_{b^-} \lambda_{b^-} \frac{b^-}{\tau} \Big) \\ &\leq \frac{1}{2} \frac{1}{\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau)} \Big(a_{b^+} \lambda_{b^+} \frac{\tau^+}{\tau} + a_\tau \lambda_\tau \Big) , \end{aligned} \tag{43}$$

where we used in the last step that $\beta^- \leq \tau$ and that $t \mapsto a_t (\log t)^{-\beta} = \frac{t^q}{(\log t)^{q+\beta}}$ is increasing for all t large enough. Next, by the definition of a_t and λ_t, we obtain that

$$a_{b^+} \lambda_{b^+} = \tfrac{(b^+)^q}{(\log b^+)^{q+\beta}} \leq \tfrac{\tau^q}{(\log \tau)^{q+\beta}} \big(\tfrac{\tau^+}{\tau} \big)^q = a_\tau \lambda_\tau \big(\tfrac{\tau^+}{\tau} \big)^q , \tag{44}$$

where we used that $b^+ \leq \tau^+$ for the inequality. Using Lemma 3.8(i), if τ is large enough, for $\beta' = 1 + \frac{1}{\alpha-d} + \frac{\varepsilon}{2}$, we get

$$\xi(Z^{(1)}_\tau) - \xi(Z^{(2)}_\tau) \geq \frac{\tau}{\tau^+ - \tau} a_\tau (\log \tau)^{-\beta'} .$$

Hence, substituting this estimate into (43) together with the previous estimate (44) yields

$$\frac{b^+ - b^-}{\tau} \le \frac{\tau^+ - \tau}{\tau} (\log \tau)^{\beta' - \beta} \Big(\big(\tfrac{\tau^+}{\tau}\big)^{q+1} + 1 \Big) \le 2 \big(\tfrac{\tau^+}{\tau}\big)^{q+2} (\log \tau)^{\beta' - \beta} .$$

It remains to bound the term τ^+/τ. By Proposition 3.6, for $\delta = \frac{1}{d} + \frac{\varepsilon}{2(q+2)}$, we get

$$\frac{\tau_+}{\tau} = 1 + \frac{\tau^+ - \tau}{\tau} \le (\log \tau)^{\delta} .$$

Finally, we have shown that if b^- is large enough $\frac{b^+ - b^-}{\tau} \le 2 (\log \tau)^{\beta' - \beta + (q+2)\delta} < (\log \tau)^{-\varepsilon}$, which completes the proof. □

We are now in the position to translate the results from Section 3.2 from the setting of the variational problem to the setting of the residual lifetime function of the maximizer of the solution.

Proof of Theorem 1.3. Suppose $t \mapsto h(t)$ is a nondecreasing function such that

$$\int_1^\infty \frac{\mathrm{d}t}{t h(t)^d} < \infty.$$

Without loss of generality, we can assume that there exists $\gamma' > 0$ such that $h(t) \le (\log t)^{\gamma'}$ for all $t > 0$. Also, let $\gamma > 1 + \frac{2}{\alpha - d}$. Fix $\varepsilon > 0$ and choose $\beta > 1 + \frac{q+2}{d} + \frac{1}{\alpha - d}$ large enough such that

$$\delta := \tfrac{1}{4} \big(\beta - \big(1 + \tfrac{q+2}{d} + \tfrac{1}{\alpha - d} \big) \big) > \gamma' + \gamma .$$

Define $\mathcal{E} = \mathcal{E}(\beta)$ as in (35) and denote by $[b_n^-, b_n^+], n \ge 1$, the connected components of $\mathcal{E}$. By Lemmas 3.7 and 3.9 each of the processes $(X_t \colon t \ge t_0)$ and $(Z_t \colon t \ge t_0)$ jumps only at times in $\mathcal{E}$ and each interval $[b_n^-, b_n^+]$ contains exactly one jump time, which we denote by σ_n for X_t and τ_n for Z_t. By Lemma 3.8 and Proposition 3.6, for all n sufficiently large,

$$2 (\log \tau_n)^{-\gamma} \le \frac{\tau_{n+1} - \tau_n}{\tau_n} \le \tfrac{\varepsilon}{3} h(\tfrac{1}{2}\tau_n) \le \tfrac{1}{2} (\log \tau_n)^{\gamma'} . \tag{45}$$

We now want to translate the upper bound to the jump times (σ_n). For this purpose, we can invoke Lemma 3.10 to find that by our choice of β and δ we have that for all n sufficiently large

$$\frac{b_n^+ - b_n^-}{\tau_n} \le (\log \tau_n)^{-\delta} . \tag{46}$$

Now, we first use that $|\sigma_n - \tau_n| \le b_n^+ - b_n^-$ and then the estimates (45) and (46) to obtain

$$\begin{aligned} \frac{R(\sigma_n)}{\sigma_n h(\sigma_n)} = \frac{\sigma_{n+1} - \sigma_n}{\sigma_n h(\sigma_n)} &\le \Big(\frac{\tau_{n+1} - \tau_n}{\tau_n} + \frac{b_{n+1}^+ - b_{n+1}^-}{\tau_{n+1}} \frac{\tau_{n+1}}{\tau_n} + \frac{b_n^+ - b_n^-}{\tau_n} \Big) \\ &\qquad \times \big((1 - (\log \tau_n)^{-\delta}) h(\tau_n (1 - (\log \tau_n)^{-\delta})) \big)^{-1} \\ &\le \big(\tfrac{\tau_{n+1} - \tau_n}{\tau_n} + (\log \tau_{n+1})^{-\delta + \gamma'} + (\log \tau_n)^{-\delta} \big) \big(\tfrac{1}{2} h(\tfrac{1}{2}\tau_n) \big)^{-1} \\ &\le 3 \, \tfrac{\tau_{n+1} - \tau_n}{h(\frac{1}{2}\tau_n)\tau_n} \le \varepsilon , \end{aligned}$$

for all but finitely many n. In particular, this shows that, almost surely,

$$\limsup_{n \to \infty} \frac{R(\sigma_n)}{\sigma_n h(\sigma_n)} = 0 .$$

However, since R jumps only at the points σ_n and decreases on $[\sigma_n, \sigma_{n+1})$, this immediately implies the first part of Theorem 1.3, see also Figure 1.

For the *second part* of the proof, suppose $t \mapsto h(t)$ is a nondecreasing function such that

$$\int_1^\infty \frac{\mathrm{d}t}{th(t)^d} = \infty\,.$$

Fix $\kappa > 0$, then by Proposition 3.6, we know that there exists a sequence $(t_n)_{n\geq 1}$ such that $R^V(t_n) \geq 3\kappa t_n h(2t_n)$. Define a subsequence of the jump times (τ_n) by choosing n_k such that for some index j we have that $t_j \in [\tau_{n_k}, \tau_{n_k+1})$. In particular, since R^V is decreasing on the interval $[\tau_{n_k}, \tau_{n_k+1})$, we can deduce that for k large enough

$$\frac{\tau_{n_k+1}-\tau_{n_k}}{\tau_{n_k}h(2\tau_{n_k})} = \frac{R^V(\tau_{n_k})}{\tau_{n_k}h(2\tau_{n_k})} \geq \frac{R^V(t_j)}{t_j h(2t_j)} \geq 3\kappa\,,$$

Similarly as for the upper bound, we can estimate

$$\begin{aligned}
\frac{R(\sigma_{n_k})}{\sigma_{n_k}h(\sigma_{n_k})} = \frac{\sigma_{n_k+1}-\sigma_{n_k}}{\sigma_{n_k}h(\sigma_{n_k})} &\geq \frac{\tau_{n_k+1}-\tau_{n_k}-(b^+_{n_k+1}-b^-_{n_k+1})-(b^+_{n_k}-b^-_{n_k})}{(\tau_{n_k}+(b^+_{n_k}-b^-_{n_k}))h(\tau_{n_k}+b^+_{n_k}-b^-_{n_k})} \\
&\geq \Big(1-\frac{b^+_{n_k+1}-b^-_{n_k+1}}{\tau_{n_k+1}}\frac{\tau_{n_k+1}}{\tau_{n_k}}\frac{\tau_{n_k}}{\tau_{n_k+1}-\tau_{n_k}} - \frac{b^+_{n_k}-b^-_{n_k}}{\tau_{n_k}}\frac{\tau_{n_k}}{\tau_{n_k+1}-\tau_{n_k}}\Big) \\
&\quad \times \frac{\tau_{n_k+1}-\tau_{n_k}}{\tau_{n_k}}\Big((1+(\log\tau_{n_k})^{-\delta})h(\tau_{n_k}(1+(\log\tau_{n_k})^{-\delta}))\Big)^{-1} \\
&\geq \frac{\tau_{n_k+1}-\tau_{n_k}}{\tau_{n_k}}(1-(\log\tau_{n_k+1})^{\gamma+\gamma'-\delta}-(\log\tau_{n_k})^{\gamma-\delta})(2h(2\tau_{n_k}))^{-1} \\
&\geq \tfrac{1}{3}\frac{\tau_{n_k+1}-\tau_{n_k}}{\tau_{n_k}h(2\tau_{n_k})} \geq \kappa\,,
\end{aligned}$$

eventually for all k large enough. This implies that $\limsup_{t\to\infty}\frac{R(t)}{th(t)} = \limsup_{t\to\infty}\frac{R(t)}{\varphi(t)} \geq \kappa$, thus completing the proof of Theorem 1.3. □

4 A functional scaling limit theorem

The aim of this section is to prove Theorem 1.6. As in previous sections, we start by dealing with the maximizer of the variational problem formulating a limit theorem for the process

$$\Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T}\big) \,:\, t>0\Big). \tag{47}$$

Convergence will take place in the Polish space $D(0,\infty) := D((0,\infty),\mathbb{R}^{d+1})$ of all càdlàg processes defined on $(0,\infty)$ taking values in $\mathbb{R}^{d+1}$ equipped with the Skorokhod topology on compact subintervals. This means that $f_n \to f$ if, for every $0<a<b<\infty$ we can find a continuous and strictly increasing time-changes $\lambda_n\colon [a,b]\to[a,b]$ such that

$$\sup_{t\in[a,b]}|\lambda_n(t)-t| \to 0 \quad \text{and} \quad \sup_{t\in[a,b]}|f(t)-f_n(\lambda_n(t))| \to 0,$$

for more details see [Bil99]. The main part of this section is devoted to the proof of the following proposition stated in terms of the maximizer of the variational problem.

Proposition 4.1. *As $T\to\infty$*

$$\Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T}\big) \,:\, t>0\Big) \Rightarrow \Big(\big(Y^{(1)}_t, Y^{(2)}_t + q\big(1-\tfrac{1}{t}\big)|Y^{(1)}_t|\big) \,:\, t>0\Big)\,,$$

in the sense of weak convergence on $D(0,\infty)$.

We will prove this result by first showing convergence of the finite-dimensional distributions in Section 4.1 and then tightness in Section 4.2. In Section 4.3, we transfer the results to the maximizer of the profile and the potential value at that site, hence showing Theorem 1.6 and, by a slight variation, also Proposition 1.4.

4.1 Finite-dimensional distributions

The next lemma shows that the finite-dimensional distributions of the process (47) converge weakly to those of the limiting process defined in terms of $Y = (Y^{(1)}, Y^{(2)})$.

Lemma 4.2. *Fix $0 < t_1 < \ldots < t_k < \infty$. Then as $T \to \infty$,*

$$\Big(\big(\tfrac{Z_{t_1 T}}{r_T}, \tfrac{\Phi_{t_1 T}(Z_{t_1 T})}{a_T}\big), \ldots, \big(\tfrac{Z_{t_k T}}{r_T}, \tfrac{\Phi_{t_k T}(Z_{t_k T})}{a_T}\big) \Big) \\ \Rightarrow \big((Y^{(1)}_{t_1}, Y^{(2)}_{t_1} + q(1 - \tfrac{1}{t_1})|Y^{(1)}_{t_1}|), \ldots, (Y^{(1)}_{t_k}, Y^{(2)}_{t_k} + q(1 - \tfrac{1}{t_k})|Y^{(1)}_{t_k}|) \big) .$$

Proof. First notice, by the continuous mapping theorem, see e.g. [Bil99, Theorem 2.7], we can equivalently show that for $Y_t = (Y^{(1)}_t, Y^{(2)}_t)$

$$\Big(\big(\tfrac{Z_{t_1 T}}{r_T}, \tfrac{\Phi_{t_1 T}(Z_{t_2 T})}{a_T} - q\big(1 - \tfrac{1}{t_1}\big)\tfrac{|Z_{t_1 T}|}{r_T}\big), \ldots, \big(\tfrac{Z_{t_k T}}{r_T}, \tfrac{\Phi_{t_k T}(Z_{t_k T})}{a_T} - q\big(1 - \tfrac{1}{t_k}\big)\tfrac{|Z_{t_k T}|}{r_T}\big) \Big) \\ \Rightarrow \big(Y_{t_1}, \ldots, Y_{t_k}\big) .$$

Define

$$H^* = \big\{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y > -q\big(1 - \tfrac{1}{t_k}\big)|x| \big\} .$$

and recall that, for large T, all components in the vectors above are in H^*. Hence it suffices to show that, for any $A \subset (H^*)^k$ with $\mathrm{Leb}_{k(d+1)}(\partial A) = 0$, we have, as $T \to \infty$,

$$\mathrm{Prob}\Big\{ \big(\tfrac{Z_{t_i T}}{r_T}, \tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q\big(1 - \tfrac{1}{t_i}\big)\tfrac{|Z_{t_i T}|}{r_T}\big)_{i=1}^k \in A \Big\} \to \mathrm{Prob}\big\{ (Y_{t_i})_{i=1}^k \in A \big\} . \tag{48}$$

The remainder of the proof is organised as follows: First, we show that in fact it suffices to show (48) for A intersected with large boxes. Second, we also show that it is enough to consider the maximizer of the variational problem on a large region. These steps let us express the probability in question in terms of the point process $\Pi_T = \{(z/r_t, \Phi_T(z)/a_T) : z \in \mathbb{Z}^d\}$ restricted to a relatively compact set, so that we can invoke the weak convergence of $\Pi_T \Rightarrow \Pi$ and recognize the resulting event in terms of the process $(Y_t \colon t > 0)$.

Step 1. Define a large region

$$B_N = \{ (x, y) \in H^* : |x| \le N, \tfrac{1}{N} - q|x| \le y \le N \} .$$

We claim that we only have to show that

$$\mathrm{Prob}\Big\{ \big(\tfrac{Z_{t_i T}}{r_T}, \tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \tfrac{1}{t_i})\tfrac{|Z_{t_i T}|}{r_T}\big)_{i=1}^k \in B_N^k \cap A \Big\} \to \mathrm{Prob}\big\{ (Y_{t_i})_{i=1}^k \in B_N^k \cap A \big\} , \tag{49}$$

for all N in order to deduce (48). Indeed, using that, by [KLMS09, Lemma 3.2], $\Phi_t(Z_t)$ is an increasing function of t, for all t large enough, we get

$$\begin{aligned} &\mathrm{Prob}\Big\{ \big(\tfrac{Z_{t_i T}}{r_T}, \tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q\big(1 - \tfrac{1}{t_i}\big)\tfrac{|Z_{t_i T}|}{r_T}\big) \notin B_N \text{ for some } i \Big\} \\ &\le \sum_{i=1}^k \mathrm{Prob}\big\{ \tfrac{|Z_{t_i T}|}{r_T} > N \big\} + \mathrm{Prob}\big\{ \tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} < \tfrac{1}{N} \big\} + \mathrm{Prob}\big\{ \tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} + \tfrac{q}{t}\tfrac{|Z_{t_i T}|}{r_T} > N \big\} \\ &\le k \Big[\max_{i=1,\ldots,k} \mathrm{Prob}\big\{ \tfrac{|Z_{t_i T}|}{r_{t_i T}} > N \tfrac{r_T}{r_{t_k T}} \big\} + \mathrm{Prob}\big\{ \tfrac{\Phi_{t_1 T}(Z_{t_1 T})}{a_{t_1 T}} < \tfrac{1}{N}\tfrac{a_T}{a_{t_1 T}} \big\} \\ &\qquad + \mathrm{Prob}\big\{ \tfrac{\Phi_{t_k T}(Z_{t_k T})}{a_{t_k T}} > \tfrac{1}{2} N \tfrac{a_T}{a_{t_k T}} \big\} + \max_{i=1,\ldots,k} \mathrm{Prob}\big\{ \tfrac{|Z_{t_i T}|}{a_{t_i T}} > \tfrac{N t_1}{2q}\tfrac{r_T}{r_{t_k T}} \big\} \Big] \\ &\le C_1\, k \Big[(\tfrac{N}{t_k^{q+1}})^{d-\alpha} + e^{-C_2 (N t_1^q)^{\alpha - d}} + (\tfrac{N}{2 t_k^q})^{d-\alpha} + (\tfrac{N t_1}{2q t_k^{q+1}})^{d-\alpha} \Big] , \end{aligned} \tag{50}$$

where $C_1, C_2 > 0$ are some constants and in the last step we used Lemma 3.3 and the fact that $a_T/a_{t_1 T} \to t_1^{-q}$ and $r_T/r_{t_k T} \to t_k^{-(q+1)}$. Hence, the terms in the last display tend to zero as $N \to \infty$. Similarly, we can bound

$$\begin{aligned}
&\mathrm{Prob}\big\{(Y_{t_i})_{i=1}^k \in A \setminus B_N^k\big\} \le \sum_{i=1}^k \mathrm{Prob}\big\{Y_{t_i} \notin B_N\big\} \\
&\le \sum_{i=1}^k \big[\mathrm{Prob}\big\{|Y_{t_i}^{(1)}| > N\big\} + \mathrm{Prob}\big\{ -q(1-\tfrac{1}{t_k})|Y_{t_i}^{(1)}| \le Y_{t_i}^{(2)} \le \tfrac{1}{N} - q|Y_{t_i}^{(1)}|\big\} + \mathrm{Prob}\big\{Y_{t_i}^{(2)} > N\big\}\big] \\
&\le k\big[\mathrm{Prob}\big\{|Y_{t_k}^{(1)}| > N\big\} + \mathrm{Prob}\big\{|Y_{t_1}^{(1)}| \le \tfrac{t_k}{qN}\big\} + \mathrm{Prob}\big\{Y_1^{(2)} > N\big\}\big]\,,
\end{aligned}$$

where we used that $|Y_t^{(1)}|$ is an increasing function in t and $Y_1^{(2)} \ge Y_t^{(2)}$ for all t by construction. By definition of $(Y_t \colon t > 0)$ all the probabilities tend to zero, as $N \to \infty$, and hence if we can show (49) we can also deduce (48).

Step 2. Denote, for $K > N$ by $Z_{tT}^{K,T}$ the point satisfying

$$\Phi_{tT}(Z_{tT}^{K,T}) = \max\big\{\Phi_{tT}(z) \,:\, t\xi(z) \ge z \text{ and } \big(\tfrac{z}{r_T}, \tfrac{\Phi_T(z)}{a_T}\big) \in B_K\big\}\,,$$

where in case of a tie we take the one with the larger ℓ^1 norm. We claim that if K is large, $Z_{tT}^{K,T}$ agrees with high probability with the global maximizer Z_{tT}. Indeed, we find that

$$\begin{aligned}
\mathrm{Prob}\big\{&\text{there exists } i \text{ with } Z_{t_i T}^{K,T} \ne Z_{t_i T}\big\} \le \sum_{i=1}^k \mathrm{Prob}\big\{\big(\tfrac{Z_{t_i T}}{r_T}, \tfrac{\Phi_T(Z_{t_i T})}{a_T}\big) \notin B_K\big\} \qquad (51)\\
&\le k \max_{i=1,\dots,k} \Big[\mathrm{Prob}\big\{\tfrac{|Z_{t_i T}|}{r_T} \ge K\big\} + \mathrm{Prob}\big\{\tfrac{\Phi_T(Z_T)}{r_T} > K\big\} + \mathrm{Prob}\big\{\tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} < \tfrac{1}{K}\big\}\Big]\,,
\end{aligned}$$

where for the last term, we use that by Lemma 2.3, we can express

$$\tfrac{\Phi_{t_i T}(Z_{t_i T})}{a_T} = \tfrac{\Phi_T(Z_{t_i T})}{a_T} + q(1-\tfrac{1}{t_i})\tfrac{|Z_{t_i T}|}{r_T} + \mathrm{error}(T)\,,$$

where the error term tends to 0. Hence, as in (50), we can use Lemma 3.3 to show that the expression (51) tends to zero if we first let T and then $K \to \infty$.

Step 3. Using the point process we want to express the probability

$$\begin{aligned}
&\mathrm{Prob}\Big\{\big(\tfrac{Z_{t_i T}^{K,T}}{r_T}, \tfrac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q(1-\tfrac{1}{t_i})\tfrac{|Z_{t_i T}^{K,T}|}{r_T}\big)_{i=1}^k \in B_N^k \cap A\Big\} \\
&\quad = \int_{A \cap B_N^k} \mathrm{Prob}\Big\{\tfrac{Z_{t_i T}^{K,T}}{r_T} \in \mathrm{d}x_i, \tfrac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q(1-\tfrac{1}{t_i})\tfrac{|Z_{t_i T}^{K,T}|}{r_T} \in \mathrm{d}y_i \text{ for all } i\Big\}\,,
\end{aligned} \qquad (52)$$

in the limit as $T \to \infty$. First note that by Lemma 2.3 we have that, for any $t \in [t_1, t_k]$,

$$\tfrac{\Phi_{tT}(z)}{a_T} = \tfrac{\Phi_T(z)}{a_T} + q(1-\tfrac{1}{t})\tfrac{|z|}{r_T} + \delta_{1-t}\big(T, \tfrac{z}{r_T}, \tfrac{\Phi_T(z)}{a_T}\big)\,,$$

where the error δ_{1-t} goes to 0 uniformly for all z such that $(\tfrac{z}{r_T}, \tfrac{\Phi_T(z)}{a_T}) \in B_K$ and also uniformly for all $t \in [t_1, t_k]$. Recall also that Π_T converges weakly to Π on H^*. Now, as the restriction to large boxes ensures that we are only dealing with the point process on relatively compact sets, we can in the limit as $T \to \infty$ express the condition

$$\tfrac{Z_{t_i T}^{K,T}}{r_T} = x_i, \tfrac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q(1-\tfrac{1}{t_i})\tfrac{|Z_{t_i T}^{K,T}|}{r_T} = y_i$$

by requiring that Π has an atom in (x_i, y_i) and all other points (x, y) of Π restricted to B_K satisfy $y + q(1-\tfrac{1}{t_i})|x| \le y_i + q(1-\tfrac{1}{t_i})|x_i|$. Therefore, if we denote by $\mathcal{C}_{t_i}(x_i, y_i)$ the open cone of all

points $(x,y) \in H^*$ satisfying $y + q(1-\frac{1}{t_i})|x| > y_i + q(1-\frac{1}{t_i})|x_i|$, we can express the probability in (52) in the limit as

$$\lim_{T\to\infty} \mathrm{Prob}\Big\{ \big(\tfrac{Z^{K,T}_{t_iT}}{r_T}, \tfrac{\Phi_{t_iT}(Z^{K,T}_{t_iT})}{a_T} - q(1-\tfrac{1}{t_i})\tfrac{|Z^{K,T}_{t_iT}|}{r_T}\big)_{i=1}^k \in B_N^k \cap A \Big\}$$
$$= \int_{A\cap B_N^k} \mathrm{Prob}\Big\{ \Pi|_{B_K}(\mathrm{d}x_i\,\mathrm{d}y_i) = 1 \text{ for } i=1,\dots,k,\, \Pi|_{B_K}\Big(\bigcup_{i=1}^k \mathcal{C}_{t_i}(x_i,y_i)\Big) = 0\Big\}.$$

Now, we can remove the restriction of the point process to B_K, by letting $K\to\infty$ and noting that the probability that for some $(x_i,y_i) \in A\cap B_N^k$ and some $i=1,\dots,k$ the point process Π has a point in the set $\mathcal{C}_{t_i}(x_i,y_i)\cap B_K^c$ can be bounded from above by the probability that Π has a point in the set

$$\big\{(x,y)\in\mathbb{R}^{d+1} : y > \tfrac{1}{N} - q(1-\tfrac{1}{t_k})|x| \text{ and } (y>K \text{ or } |x|>K)\big\}.$$

But the intensity measure ν of Π gives finite mass to this region, so that we can conclude that the probability of the latter event tends to zero as $K\to\infty$. Hence, we can combine this observation with the estimate in (51) and letting first $T\to\infty$ and then $K\to\infty$, to deduce that

$$\begin{aligned}\lim_{T\to\infty} \mathrm{Prob}\Big\{ \big(\tfrac{Z_{t_iT}}{r_T}, \tfrac{\Phi_{t_iT}(Z_{t_iT})}{a_T} - q(1-\tfrac{1}{t_i})\tfrac{|Z_{t_iT}|}{r_T}\big)_{i=1}^k \in B_N^k \cap A \Big\}&\\ = \int_{A\cap B_N^k} \mathrm{Prob}\Big\{\Pi(\mathrm{d}x_i\,\mathrm{d}y_i) = 1 \text{ for } i=1,\dots,k, \Pi\Big(\bigcup_{i=1}^k \mathcal{C}_{t_i}(x_i,y_i)\Big) = 0\Big\}&\\ = \mathrm{Prob}\{(Y_{t_i})_{i=1}^k \in B_N^k \cap A\},&\end{aligned}$$

where in the last step we used the definition of Y. For an illustration of the event under the integral, see also Figure 4. Thus we have completed the proof of the lemma. □

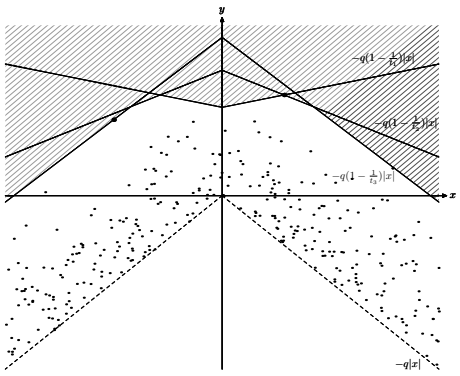


Figure 4: Calculation of finite-dimensional distributions at times $t_1 < 1 < t_2 < t_3$. The event that $Y_{t_i} = (x_i,y_i)$ translates to the condition that the point process Π has an atom in each of the points (x_i,y_i), but does not contain any points in the union of open cones with 'slope' $-q(1-\frac{1}{t_i})$ whose boundaries touch the points (x_i,y_i) (as indicated by the shaded region).

4.2 Tightness

To prepare the tightness argument we prove two auxiliary lemmas. For fixed $0<a<b$ the first lemma gives us control on the probability that the maximizer makes small jumps during the time interval $[aT,bT]$. The second shows that, with arbitrarily high probability, during this time the maximizer stays within a box with sidelength a multiple of r_T.

Lemma 4.3. *Let τ_i denote the jump times of the process $(Z_t : t \geq aT)$ in increasing order. Then*

$$\liminf_{T\to\infty} \mathrm{Prob}\big\{\tau_{i+1} - \tau_i \geq \delta T \text{ for all jump times } \tau_i \in [aT, bT]\big\} \geq p(\delta)\,,$$

where $p(\delta) \to 1$ as $\delta \downarrow 0$.

Proof. Cover the interval $[aT, bT]$ by small subintervals of length δT by setting $x_i = aT + i\delta T$ for $i = 0, \ldots, N+1$, for $N = \lceil (b-a)/\delta \rceil$. We estimate

$$\begin{aligned}
&\mathrm{Prob}\big\{\tau_{i+1} - \tau_i < \delta T \text{ for some jump times } \tau_i \in [aT, bT]\big\} \\
&\quad \leq \sum_{j=0}^{N-1} \mathrm{Prob}\big\{\tau_{i+1} - \tau_i < \delta T \text{ for some jump times } \tau_i \in [x_j, x_{j+1}]\big\} \\
&\quad \leq \sum_{j=0}^{N-1} \mathrm{Prob}\big\{Z_t \text{ jumps more than once in the interval } [x_j, x_{j+2}]\big\}\,.
\end{aligned}$$

Hence, taking the limit $T \to \infty$, we have that

$$\limsup_{T\to\infty} \mathrm{Prob}\big\{\tau_{i+1} - \tau_i < \delta T \text{ for some jump time } \tau_i \in [aT, bT]\big\} \leq N\, \tilde{p}(2\delta)\,,$$

where

$$\tilde{p}(\delta) := \limsup_{T\to\infty} \mathrm{Prob}\big\{Z_t \text{ jumps more than once in the interval } [T, (1+\delta)T]\big\}.$$

Thus it remains to show that $\tilde{p}(\delta)/\delta \to 0$ as $\delta \to 0$. We use notation and ideas from Section 2, which tell us in particular that, as $T \to \infty$, if we fix $(Z_T/r_T, \Phi_T(Z_T)/a_T) = (x, y)$ then the probability that $(Z_t : t \geq T)$ jumps more than once in the interval $[T, (1+\delta)T]$ is bounded from above by the probability that the point process Π has no points in the set $D_0(|x|, y)$ and at least two points in the set $D_\delta(|x|, y) \setminus D_0(|x|, y)$. To make this bound rigorous, one has restrict the process $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes, let $T \to \infty$ and then the size of the boxes go to infinity and finally justify interchanging the limit. Details are very similar to Lemma 2.2 and Lemma 4.2 and are therefore omitted. Using this observation, we obtain the bound

$$\begin{aligned}
&\limsup_{T\to\infty} \mathrm{Prob}\big\{Z_t \text{ jumps more than once in the interval } [T, (1+\delta)T]\big\} \\
&\quad \leq \int_{y\geq 0}\int_{x\in\mathbb{R}^d} \mathrm{Prob}\big\{\Pi(\mathrm{d}x\,\mathrm{d}y) = 1,\, \Pi(D_0(|x|, y)) = 0,\, \Pi(D_\delta(|x|, y) \setminus D_0(|x|, y)) \geq 2\big\} \\
&\quad = \int_{y\geq 0}\int_{x\in\mathbb{R}^d} e^{-\nu(D_0(|x|,y))}\big(1 - e^{-f_\delta(|x|,y)} - f_\delta(|x|, y)e^{-f_\delta(|x|,y)}\big)\,\nu(\mathrm{d}x\,\mathrm{d}y) \\
&\quad = \tfrac{2^d}{(d-1)!}\int_0^\infty\int_0^\infty e^{-\nu(D_0(r,y))}\big(1 - e^{-f_\delta(r,y)} - f_\delta(r, y)e^{-f_\delta(r,y)}\big)\,\frac{\alpha r^{d-1}}{(y+qr)^{\alpha+1}}\,\mathrm{d}r\mathrm{d}y\,, \quad (53)
\end{aligned}$$

where $f_\delta(r, y) = \nu(D_\delta(r, y)) - \nu(D_0(r, y))$. It remains to be shown that the right hand side divided by δ converges to zero. As we would like to invoke the dominated convergence theorem, we show that this term is bounded by an integrable function. We have

$$\begin{aligned}
&e^{-\nu(D_0(r,y))}\tfrac{1}{\delta}\big(1 - e^{-f_\delta(r,y)} - f_\delta(r, y)e^{-f_\delta(r,y)}\big)\,\tfrac{\alpha r^{d-1}}{(y+qr)^{\alpha+1}} \\
&\qquad \leq e^{-\nu(D_0(r,y))}\tfrac{1}{\delta}\big(\nu(D_\delta(r, y)) - \nu(D_0(r, y))\big)\tfrac{\alpha r^{d-1}}{(y+qr)^{\alpha+1}}\,.
\end{aligned}$$

Recall from (16) that $\nu(D_\delta(r, y)) = \vartheta y^{d-\alpha}\varphi_\delta(v)^{-1}$ with $y + qr = \frac{y}{v}$ and φ_δ given by (14), and $\vartheta = 2^d B(\alpha - d, d)/q^d(d-1)!$. Next, we estimate the part of the integrand that depends on δ, so for $\tilde{B}(x) := \tilde{B}(x, \alpha - d, d)$ we consider

$$\begin{aligned}
\tfrac{1}{\delta}(\varphi_\delta(v)^{-1} - \varphi_0(v)^{-1}) &= \tfrac{1}{\delta}\big(\tfrac{(1+\delta)^\alpha}{(\delta+v)^{\alpha-d}}v^{\alpha-d}\tilde{B}\big(\tfrac{v+\delta}{1+\delta}\big) - \tilde{B}(v)\big) \\
&\leq \tfrac{1}{\delta}\big((1+\delta)^\alpha - 1\big) + \tfrac{1}{\delta}\big(\tilde{B}\big(\tfrac{v+\delta}{1+\delta}\big) - \tilde{B}(v)\big)\,.
\end{aligned}$$

As the first term is $\le 2\alpha$ for all $\delta \le \delta_0$ for some small δ_0 (independent of r, y), we can concentrate on the second term. Now, we can use the definition of $\tilde{B}$ to write

$$\begin{aligned}\tfrac{1}{\delta}\big(\tilde{B}\big(\tfrac{v+\delta}{1+\delta}\big) - \tilde{B}(v)\big) &= \tfrac{1}{\delta}\int_v^{\frac{v+\delta}{1+\delta}} u^{\alpha-d-1}(1-u)^{d-1}\mathrm{d}u \le \tfrac{1}{\delta}\int_v^{\frac{v+\delta}{1+\delta}} u^{\alpha-d-1}\mathrm{d}u \\ &\le \tfrac{1}{\delta}\big(\tfrac{v+\delta}{1+\delta} - v\big)\max\{v^{\alpha-d-1},1\} \le \max\{v^{\alpha-d-1},1\}\,.\end{aligned}$$

Combining the last three displays we obtain a majorant for the integrand in (53) divided by δ, which does not depend on $0 < \delta < \delta_0$. To show that this majorant is integrable we calculate

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{-\nu(D_0(r,y))}y^{d-\alpha}\left(2\alpha + \max\left\{(\tfrac{y}{y+qr})^{\alpha-d-1},1\right\}\right)\tfrac{\alpha\vartheta r^{d-1}}{(y+qr)^{\alpha+1}}\,\mathrm{d}r\mathrm{d}y \\ &\quad\le \tfrac{\alpha\vartheta}{q^d}\int_0^\infty\int_0^1 v^{\alpha-d}(1-v)^{d-1}y^{2(d-\alpha)-1}3\alpha\max\{v^{\alpha-d-1},1\}e^{-\vartheta y^{d-\alpha}}\,\mathrm{d}v\,\mathrm{d}y \\ &\quad= \tfrac{3\,\alpha^2\vartheta}{q^d}\int_0^1 v^{\alpha-d}(1-v)^{d-1}\max\{v^{\alpha-d-1},1\}\mathrm{d}v\int_0^\infty y^{2(d-\alpha)-1}e^{-\vartheta y^{d-\alpha}}\mathrm{d}y \\ &\quad\le \tfrac{3\,\alpha^2}{q^d(\alpha-d)}\max\{B(\alpha-d+1,d), B(2(\alpha-d),d)\},\end{aligned}$$

so that the proof is completed by applying the dominated convergence theorem. □

Lemma 4.4. *For fixed $0 < a < b$, we have that*

$$\lim_{\kappa\to\infty}\limsup_{T\to\infty}\mathrm{Prob}\Big\{\sup_{t\in[aT,bT]}\frac{|Z_t|}{r_T} \ge \kappa\Big\} = 0\,.$$

Proof. Fix a jump time τ of Z_t. By Lemma 2.8 we have $\xi(Z_\tau^{(1)}) > \xi(Z_\tau^{(2)})$. In particular, we have, using that $\chi(z) = x - \rho\log x$ is increasing on $x > \rho$,

$$\Phi_\tau(Z_\tau^{(1)}) \ge \xi(Z_\tau^{(1)}) - \tfrac{1}{\tau}|Z_\tau^{(1)}|\log\xi(Z_\tau^{(1)}) > \xi(Z_\tau^{(2)}) - \tfrac{1}{\tau}|Z_\tau^{(1)}|\log\xi(Z_\tau^{(2)})\,.$$

Since $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$, we thus obtain that

$$\xi(Z_\tau^{(2)}) - \tfrac{1}{\tau}|Z_\tau^{(2)}|\log\xi(Z_\tau^{(2)}) + \tfrac{1}{\tau}\eta(Z_\tau^{(2)}) > \xi(Z_\tau^{(2)}) - \tfrac{1}{\tau}|Z_\tau^{(1)}|\log\xi(Z_\tau^{(2)})\,.$$

Hence using that $\eta(z) \le |z|\log d$, we find that

$$|Z_\tau^{(2)}| < |Z_\tau^{(1)}|\big(1 - \tfrac{\log d}{\log\xi(Z_\tau^{(2)})}\big)^{-1} < |Z_\tau^{(1)}|\big(1 - \tfrac{\log d}{q\log\tau(1+o(1))}\big)^{-1}\,,$$

where we invoked [KLMS09, Lemma 3.2] to deduce that eventually $\xi(Z_t^{(2)}) > a_t(\log t)^{-1}$. Hence, denoting by N_T the number of jumps of Z_t in the interval $[aT, bT]$, we have that for T large enough

$$\sup_{t\in[aT,bT]}|Z_t| \le (1 - \tfrac{2\log d}{q\log aT})^{-N_T}|Z_{bT}|\,. \tag{54}$$

Fix $\varepsilon > 0$. By Lemma 4.3, we can choose $\delta > 0$ such that,

$$\liminf_{T\to\infty}\mathrm{Prob}\big\{\tau_{i+1} - \tau_i \ge \delta T \text{ for all jump times } \tau_i \in [aT, bT]\big\} \ge 1 - \tfrac{\varepsilon}{4}\,.$$

If all jump times τ_i in $[aT, bT]$ satisfy $\tau_{i+1} - \tau_i \ge \delta T$, then $N_T \le \frac{b-a}{\delta} + 1$ and hence

$$\sup_{t\in[aT,bT]}|Z_t| \le (1 - \tfrac{2\log d}{q\log aT})^{-\frac{b-a}{\delta}-1}|Z_{bT}|\,.$$

Therefore, for any $\kappa > 1$, we can estimate that

$$\begin{aligned}
&\mathrm{Prob}\Big\{ \sup_{t\in[aT,bT]} \tfrac{|Z_t|}{r_T} \geq \kappa \Big\} \\
&\quad \leq \mathrm{Prob}\Big\{ (1 - \tfrac{2\log d}{q\log aT})^{-\frac{b-a}{\delta}-1} \tfrac{|Z_{bT}|}{r_T} \geq \kappa \Big\} + \mathrm{Prob}\big\{ \tau_{i+1} - \tau_i < \delta T \text{ for some } 1 \leq i \leq N_T \big\} \\
&\quad \leq \mathrm{Prob}\Big\{ \tfrac{|Z_{bT}|}{r_{bT}} \geq \kappa b^{-(q+1)}(1+o(1)) \Big\} + \tfrac{\varepsilon}{2} \\
&\quad \leq \big(1+\tfrac{\varepsilon}{2}\big)\, \mathrm{Prob}\big\{ |Y_1^{(1)}| \geq \kappa b^{-(q+1)} \big\} + \tfrac{\varepsilon}{2}\,,
\end{aligned}$$

for all all t sufficiently large, where we use that $Z_t/r_t \Rightarrow Y_1^{(1)}$. Hence, by choosing κ large enough, the latter expression can be made smaller than ε, which completes the proof. □

To prove tightness we use the following characterization (see,e.g., [Bil99, Thm. 13.2]). A family $(P_T : T \geq 1)$ of probability measures on $D([a,b])$ is tight if and only if the following two conditions are satisfied:

$$\begin{aligned}
&\text{(i)} \quad \lim_{\kappa\to\infty} \limsup_{T\to\infty} P_T\{x \,:\, \|x\| \geq \kappa\} = 0, \\
&\text{(ii)} \quad \text{for any } \epsilon > 0 \text{ we have } \lim_{\delta\to 0} \limsup_{T\to\infty} P_T\{x \,:\, w'_x(\delta) \geq \epsilon\} = 0\,.
\end{aligned} \tag{55}$$

Here, $\|x\|$ is the uniform norm, i.e. $\|x\| = \sup_{t\in[a,b]} |x(t)|$, and the modulus $w'_x(\delta)$ is defined as

$$w'_x(\delta) = \inf_{\{t_i\}} \max_{1\leq i\leq v} w_x[t_{i-1}, t_i)\,,$$

where the infimum runs over all partitions $a = t_0 < t_1 < \cdots < t_v = b$ of $[a,b]$ satisfying $\min_{1\leq i\leq v}(t_i - t_{i-1}) > \delta$ and w_x is the modulus of continuity defined for an interval $I \subset [a,b]$ as

$$w_x(I) = \sup_{s,t\in I} |x(s) - x(t)|\,.$$

Lemma 4.5. *For any $0 < a < b$, the family $\{\mathrm{Prob}_T : T \geq 1\}$ is a tight family of probability measures on $D([a,b])$, where Prob_T is the law of*

$$V_T = \Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T} \big) \,:\, t \in [a,b] \Big)\,,$$

under Prob.

Proof. We have to check the two conditions in (55).

(i) First recall from [KLMS09, Lemma 3.2] that eventually for all t, the function $t \mapsto \Phi_t(Z_t)$ is increasing, so that we can assume throughout the proof that this property holds for all $t \geq aT$. Note that

$$\|V_T\| = \sup_{t\in[a,b]} \big\{ \big|\tfrac{Z_{tT}}{r_T}\big| + \big|\tfrac{\Phi_{tT}(Z_{tT})}{a_T}\big| \big\} = \sup_{t\in[a,b]} \big\{ \tfrac{|Z_{tT}|}{r_T} \big\} + \tfrac{\Phi_{bT}(Z_{bT})}{a_T}\,.$$

Therefore, we find that for any $\kappa > 0$

$$\mathrm{Prob}\big\{ \|V_T\| \geq \kappa \big\} \leq \mathrm{Prob}\Big\{ \sup_{t\in[a,b]} \tfrac{|Z_{tT}|}{r_T} \geq \tfrac{\kappa}{2} \Big\} + \mathrm{Prob}\big\{ \tfrac{\Phi_{bT}(Z_{bT})}{a_T} \geq \tfrac{\kappa}{2} \big\}\,. \tag{56}$$

Now, by Lemma 4.4 and the weak convergence of $\Phi_t(Z_t)/a_t \Rightarrow Y_1^{(2)}$, we can deduce that the above expressions tend to zero, if we first let $T \to \infty$ and then $\kappa \to \infty$.

(ii) Fix $\delta > 0$ and a partition $(t_i)_{i=0}^v$ of $[a,b]$ such that $\delta < t_{i+1} - t_i < 2\delta$ and such that all the jump times of $(Z_{tT} : t \in [a,b])$ are some of the t_i. This is possible if all the jump times τ_i of Z_t in $[aT, bT]$ satisfy $\tau_{i+1} - \tau_i \geq \delta T$, an event which by Lemma 4.3 has probability tending to 1 if we first let $T \to \infty$ and then $\delta \to 0$. Thus, we can work on this event from now on.

First, using that Z_{tT} does not jump in $[t_{i-1}, t_i)$ and the fact that $\Phi_t(Z_t)$ is increasing and $t \mapsto \xi(Z_t)$ nondecreasing by Lemma 2.8, we can estimate

$$\begin{aligned} w_{V_T}[t_{i-1}, t_i) &= \sup_{s,t\in[t_{i-1},t_i)} \big|\tfrac{Z_{tT}}{r_T} - \tfrac{Z_{sT}}{r_T}\big| + \sup_{s,t\in[t_{i-1},t_i)} \big|\tfrac{\Phi_{tT}(Z_{tT})}{a_T} - \tfrac{\Phi_{sT}(Z_{sT})}{a_T}\big| \\ &= \tfrac{1}{a_T}\big(\Phi_{t_iT}(Z_{t_{i-1}T}) - \Phi_{t_{i-1}T}(Z_{t_{i-1}T})\big) \\ &= \tfrac{1}{a_T}\big(\tfrac{1}{t_{i-1}T} - \tfrac{1}{t_iT}\big)\,\big(|Z_{t_{i-1}}|\log\xi(Z_{t_{i-1}}) - \eta(Z_{t_{i-1}})\big) \\ &\le \tfrac{2\delta}{a^2} \sup_{s\in[a,b]} \{\tfrac{|Z_{sT}|}{r_T}\} \tfrac{\log\xi(Z_{bT})}{\log T}\,. \end{aligned}$$

Now, recall that, by (38), we can bound $\xi(Z_t) \le a_t \log t$ eventually for all t so that together with $\log a_T = (q + o(1))\log T$ we obtain

$$w'_{V_T}(\delta) \le \tfrac{2\delta}{a^2} \sup_{s\in[a,b]} \{\tfrac{|Z_{sT}|}{r_T}\} \tfrac{\log\xi(Z_{bT})}{\log T} \le \tfrac{2q\delta}{a^2} \sup_{s\in[a,b]} \{\tfrac{|Z_{sT}|}{r_T}\}(1 + o(1))\,.$$

Finally, we can use Lemma 4.4 to deduce that

$$\lim_{\delta\downarrow 0} \limsup_{T\to\infty} \mathrm{Prob}\big\{w'_{V_T}(\delta) \ge \epsilon\big\} \le \lim_{\delta\downarrow 0} \limsup_{T\to\infty} \mathrm{Prob}\Big\{\tfrac{2q\delta}{a^2} \sup_{s\in[a,b]} \{\tfrac{|Z_{sT}|}{r_T}\}(1 + o(1)) \ge \epsilon\Big\} = 0\,,$$

so that also the second part of the criterion (55) is satisfied. □

4.3 Functional limit theorem for the maximizer of the solution profile

In this section, we prove Theorem 1.6 by translating the functional limit theorem from the maximizer of the variational problem to the maximizer of the solution profile. We prove both parts (a) and (b) simultaneously. The main argument is contained in the following lemma.

Lemma 4.6. *As $T \to \infty$, the difference process*

$$\Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_t(Z_{tT})}{a_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T} + \tfrac{q}{t}\tfrac{|Z_{tT}|}{r_T}\big) - \big(\tfrac{X_{tT}}{r_T}, \tfrac{1}{a_T}\tfrac{\log U(tT)}{tT}, \tfrac{\xi(X_{tT})}{a_T}\big) : t > 0\Big)$$

tends to zero in probability.

Proof. Denoting the difference process above by $(D_T(t) : t > 0)$ it suffices to show that, for any $0 < a < b$, there exist time-changes $\lambda_T : [a, b] \to [a, b]$ such that as $T \uparrow \infty$, in probability,

$$\sup_{t\in[a,b]} |\lambda_T(t) - t| \Rightarrow 0 \quad \text{and} \quad \sup_{t\in[a,b]} \|D_T(\lambda_T(t))\| \Rightarrow 0.$$

Fix $0 < a < b$. Note that, by Proposition 2.1,

$$\lim_{\gamma\downarrow 0} \lim_{T\to\infty} \mathrm{Prob}\big\{Z_{aT} = Z_{aT(1+\gamma)} \text{ and } Z_{bT(1-\gamma)} = Z_{bT}\big\} = 1. \tag{57}$$

so that we can henceforth assume that $0 < \gamma < \frac{b-a}{4}$ is given such that the event above holds. Let $(\sigma_i, i = 0, 1, \ldots)$ be the jump times of $(X_t : t \ge aT)$ and $(\tau_i, i = 0, 1, \ldots)$ be the jump times of $(Z_t : t \ge aT)$, both in increasing order. Recall from the discussion in Section 3.3 that if T is large enough then the jump times always occur in pairs which are close together, i.e. for $\beta > 1 + \frac{1}{\alpha-d}$ each connected component of the set $\mathcal{E}(\beta)$, defined in (35), contains exactly one jump time of each of the two processes. In particular, by Lemma 3.10, there exists $\delta > 0$ such that

$$\frac{|\sigma_i - \tau_i|}{\tau_i} \le (\log\tau_i)^{-\delta} \le (\log aT)^{-\delta} < \gamma\,, \tag{58}$$

so that under the event in (57) there exists

$$N = \max\big\{i\colon \sigma_i \in [aT, bT]\big\} = \max\big\{i\colon \tau_i \in [aT, bT]\big\}.$$

Denote $s_i = \sigma_i/T$ and $t_i = \tau_i/T$ and define $\lambda = \lambda_T\colon [a,b] \to \mathbb{R}$ such that $\lambda(a) = a, \lambda(b) = b$ and $\lambda(s_i) = t_i$ for all $i = 0, \dots, N$, and linear between these points. Then

$$\begin{aligned}\sup_{t\in[a,b]} |\lambda(t) - t| &= \sup_{i=0,\dots,N} |\lambda(s_i) - s_i| = \sup_{i=0,\dots,N} \tfrac{1}{T}|\tau_i - \sigma_i| \le b \sup_{i=0,\dots,N} \tfrac{|\tau_i - \sigma_i|}{\tau_i} \\ &\le b \sup_{i=0,\dots,N} (\log \tau_i)^{-\delta} \le b(\log aT)^{-\delta}\,,\end{aligned} \tag{59}$$

which converges to 0 when $T \to \infty$, as required.

We now look at the individual components of the process D_T. For the *first component*, we simply observe that the time-change is set up in such way that $X_{tT} = Z_{\lambda(t)T}$ for all $t \in [a,b]$. For the *second component*, we split

$$\begin{aligned}&\tfrac{1}{a_T}\big|\tfrac{\log U(tT)}{tT} - \Phi_{\lambda(t)T}(Z_{\lambda(t)T})\big| \\ &\quad\le \tfrac{1}{a_T}\big|\tfrac{\log U(tT)}{tT} - \Phi_{tT}(Z_{tT})\big| + \tfrac{1}{a_T}\big|\Phi_{tT}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{tT})\big| \\ &\qquad + \tfrac{1}{a_T}\big|\Phi_{\lambda(t)T}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{\lambda(t)T})\big|\,,\end{aligned} \tag{60}$$

and look at the three terms separately. For the first term, we use Propositions 4.2 and 4.4 from [KLMS09] to conclude that there exists $\delta' > 0$ and $C > 0$ such that almost surely, for all sufficiently large t,

$$\Phi_t(Z_t) - 2d + o(1) \le \frac{1}{t}\log U(t) \le \Phi_t(Z_t) + Ct^{q-\delta'}\,,$$

so that the first term in (60) tends to 0 uniformly for all $t \in [a,b]$. For the second term, we use the bound $\eta(z) \le |z| \log d$, the bound (59) for the time-change, and that, by (38) combined with [KLMS09, Lemma 3.2], there exists a $\delta' > 0$ such that

$$a_t(\log t)^{-\delta'} \le \xi(Z_t) \le a_t(\log t)^{\delta'}. \tag{61}$$

This gives, for T large enough and all $t \in [a,b]$,

$$\begin{aligned}\tfrac{1}{a_T}\big|\Phi_{tT}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{tT})\big| &= \tfrac{1}{a_T}\big|\tfrac{1}{tT} - \tfrac{1}{\lambda(t)T}\big|\,\big||Z_{tT}|\log\xi(Z_{tT}) - \eta(Z_{tT})\big| \\ &\le \tfrac{1}{a^2}|\lambda(t) - t| \sup_{t\in[a,b]} \tfrac{|Z_{tT}|}{r_T \log T} \max\{|\log\xi(Z_{bT})|, 2d\} \\ &\le (1 + o(1))\tfrac{qb}{a^2}(\log aT)^{-\delta} \sup_{t\in[a,b]} \tfrac{|Z_{tT}|}{r_T}\,,\end{aligned}$$

and the right hand side tends to zero in probability by Lemma 4.4. In order to deal with the last term in (60), note that if $t \in (s_i \vee t_i, s_{i+1} \wedge t_{i+1})$ for some $i = 0, \dots, N-1$, then $Z_{tT} = Z_{\lambda(t)T}$ so that the term vanishes. Otherwise, if $t \in [s_i \wedge t_i, s_i \vee t_i]$, then tT is in the set of transition times $\mathcal{E}$ as discussed in Section 3.3 and we find that $\{Z_{tT}, Z_{\lambda(t)T}\} \subset \{Z^{(1)}_{\lambda(t)T}, Z^{(2)}_{\lambda(t)T}\}$ and also that there exists $\beta > 1 + \frac{1}{\alpha - d}$ such that

$$\begin{aligned}\tfrac{1}{a_T}\big|\Phi_{\lambda(t)T}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{\lambda(t)T})\big| &\le \tfrac{1}{a_T}\big(\Phi_{\lambda(t)T}(Z^{(1)}_{\lambda(t)T}) - \Phi_{\lambda(t)T}(Z^{(2)}_{\lambda(t)T})\big) \\ &\le \tfrac{a_{\lambda(t)T}}{a_T}(\log\lambda(t)T)^{-\beta} \\ &\le b^q(1 + o(1))(\log aT)^{-\beta}\,,\end{aligned}$$

which tends to zero uniformly in $t \in [a,b]$ completing the discussion of the second component.

Finally, we consider the *third component*. Using (61) and that $Z_{tT} = X_{\lambda^{-1}(t)T}$, we estimate

$$\begin{aligned}\big|\tfrac{\Phi_{tT}(Z_{tT})}{a_T} + \tfrac{q}{t}\tfrac{|Z_{tT}|}{r_T} - \tfrac{\xi(X_{\lambda^{-1}(t)T})}{a_T}\big| &= \big|\tfrac{q}{t}\tfrac{|Z_{tT}|}{r_T} - \tfrac{|Z_{tT}|}{tr_T}\tfrac{\log\xi(Z_{tT})}{\log T} + \tfrac{\eta(Z_{tT})}{tr_T\log T}\big| \\ &\le C'\tfrac{1}{a}\sup_{t\in[a,b]}\big\{\tfrac{|Z_{tT}|}{r_T}\big\}\tfrac{\log\log bT}{\log T}\,,\end{aligned}$$

where C' is some constant depending on a, b. By Lemma 4.4, the right hand side converges in probability to zero, which completes the proof of the lemma. □

Proof of Theorem 1.6. By a classic result on weak convergence, see e.g. [Bil99, Thm. 3.1], the previous lemma ensures that the processes

$$\Big(\big(\tfrac{X_{tT}}{r_T}, \tfrac{1}{a_T}\tfrac{\log U(tT)}{tT}, \tfrac{\xi(X_{tT})}{a_T}\big) \,:\, t>0\Big) \text{ and } \Big(\big(\tfrac{Z_{tT}}{r_T}, \tfrac{\Phi_t(Z_{tT})}{a_T}, \tfrac{\Phi_{tT}(Z_{tT})}{a_T} + \tfrac{q}{t}\tfrac{|Z_{tT}|}{r_T}\big) \,:\, t>0\Big),$$

have the same limit, which was identified in Proposition 4.1, as

$$\Big(\big(Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{t}\big)|Y_t^{(1)}|, Y_t^{(2)} + \tfrac{d}{\alpha-d}|Y_t^{(1)}|\big) \,:\, t>0\Big).$$

Hence, projecting onto the first and third component proves (a), and projecting onto the second component and noting that all involved processes are continuous proves (b). □

Proof of Proposition 1.4. We focus on the one-dimensional distributions, as the higher dimensional case works analogously. Fix $t>0$ and let f be a continuous, bounded nonnegative function on $\mathbb{R}^d$. Denote

$$\xi_{tT}(f) := \big(\tfrac{T}{\log T}\big)^{\frac{\alpha d}{\alpha-d}} \int v\big(tT, \big(\tfrac{T}{\log T}\big)^{\frac{\alpha}{d-\alpha}}x\big)\, f(x)\,\mathrm{d}x = \sum_{y\in\mathbb{Z}^d} v(tT,y)\, f\big(\tfrac{y}{r_T}\big).$$

It suffices to show that the Laplace functionals converge, i.e.

$$\lim_{T\uparrow\infty} \mathbb{E}\big[e^{-\xi_{tT}(f)}\big] = \mathbb{E}\big[e^{-f(Y_t)}\big]\,.$$

Let $\kappa>0$ be an upper bound for f. For small $\varepsilon>0$ and $\delta = \frac{\log(1+\frac{\varepsilon}{2})}{\kappa}$, consider the event $A_\delta = \{v(tT, Z_{tT}) > 1-\delta\}$. Since $v(tT, Z_{tT}) \Rightarrow 1$, we can choose T_0 large enough such that $\mathrm{Prob}(A_\delta) > 1-\frac{\varepsilon}{2}$ for all $T\ge T_0$. We estimate

$$\mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}\big[e^{-\xi_{tT}(f)}\big] \le \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}\big[e^{-f(\frac{Z_{tT}}{r_T})}\mathbb{1}_{A_\delta}\big]e^{-\delta\kappa} \le \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}\big[e^{-f(\frac{Z_{tT}}{r_T})}\big] + \varepsilon\,,$$

and

$$\mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-\xi_{tT}(f)}] \ge \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}\big[e^{-(1-\delta)f(\frac{Z_{tT}}{r_T})}\big] - \tfrac{\varepsilon}{2} \ge \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}\big[e^{-f(\frac{Z_{tT}}{r_T})}\big] - \varepsilon.$$

As $\frac{Z_{tT}}{r_T} \Rightarrow Y_t$ by Proposition 4.1, the statement follows. □

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