Tossing coins, matching points and shifting Brownian motion

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based on joint work with

Günter Last (Karlsruhe)
Hermann Thorisson (Reykjavik)

Setup of the talk

- Three problems in probability
- Solution of the extra head problem
- Solution of the Poisson matching problem
- Solution of the embedding problem for Brownian motion

Problem 1: The extra head problem

Let $(X_i \colon i \in \mathbb{Z})$ be a two sided sequence of independent fair coin tosses, i.e.

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Liggett (2001) Explicit, nonrandomized solution (I'll show you later).

Problem 2: Matching of Poisson points

Let $\mathcal{R}$ and $\mathcal{B}$ be two independent standard Poisson processes on $\mathbb{R}^d$, constituting a random pattern of red and blue points.

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Holroyd, Pemantle, Peres and Schramm (2008)
Explicit procedure called stable matching (I'll show you later).

- In $d = 1$ a stable matching procedure is optimal.
- In $d = 2$ it is open whether stable matching is optimal.
  An optimal non-randomized solution was given by Timar (2009).
- In $d \geq 3$ stable matching is bad, and the problem is open.

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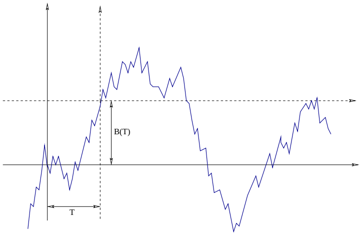

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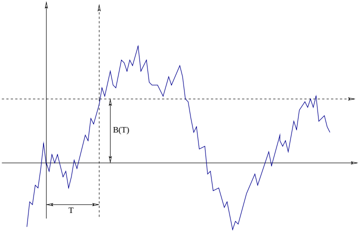

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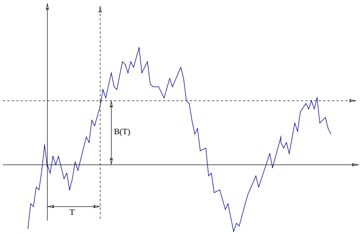

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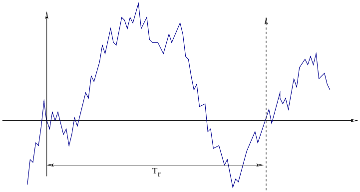
The following are not unbiased shifts:

- first hitting time of a level, or first exit time from an interval;
- fixed times $T \neq 0$.

Problem 3: Shifting Brownian motion

Here is an example of an unbiased shift:
Let $(L^0_t \colon t \geq 0)$ be the local time of B at zero, and $r > 0$. Take

$$T_r := \inf\{t \geq 0 \colon L^0_t = r\}.$$



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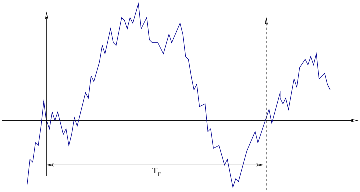

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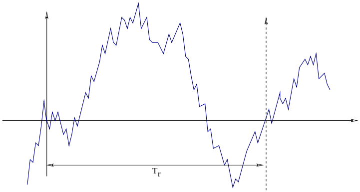

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Last, M ,Thorisson (2012)
Explicit solution T which is also a stopping time with optimal tail behaviour.
Inspired by solutions to Problem 1 (Liggett) and 2 (Holroyd et al.).

Liggett's solution of the extra head problem

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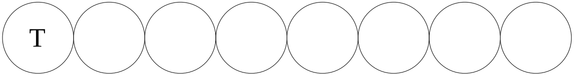

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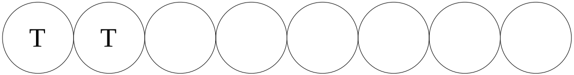

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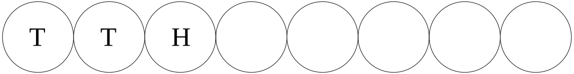

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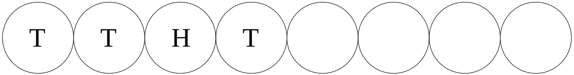

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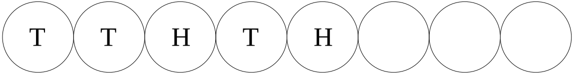

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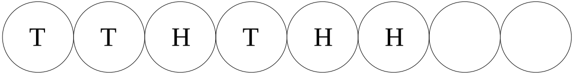

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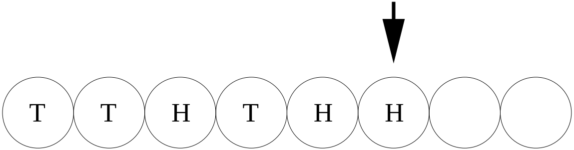

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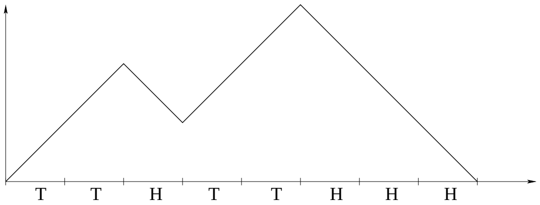
Why is this true?

- Use the sequence of coins to build a random walk.



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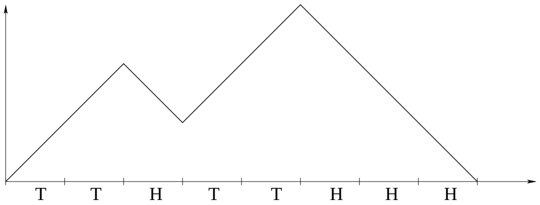

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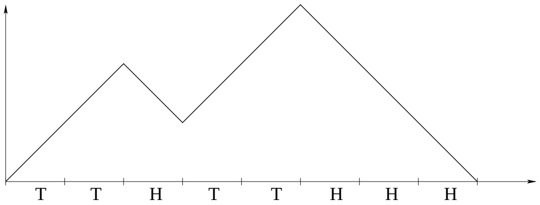

- Shifting the chosen coin to the origin means shifting the random walk by exactly one excursion.
- Reversing a random walk excursion leaves its distribution invariant.

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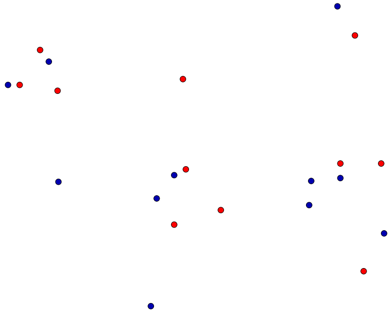

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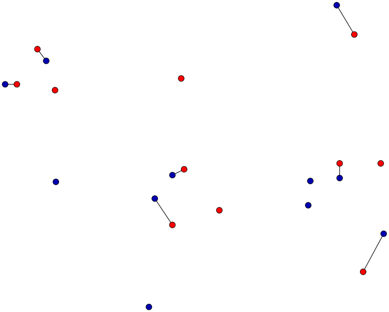

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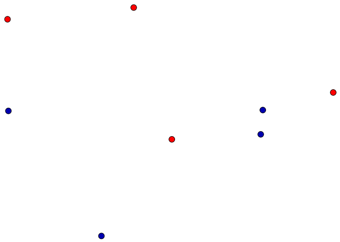

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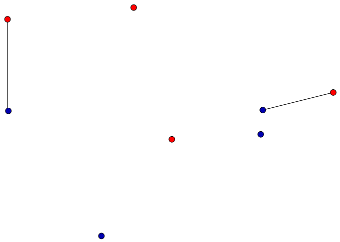

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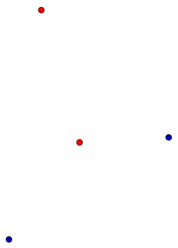

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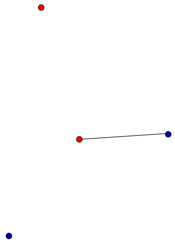

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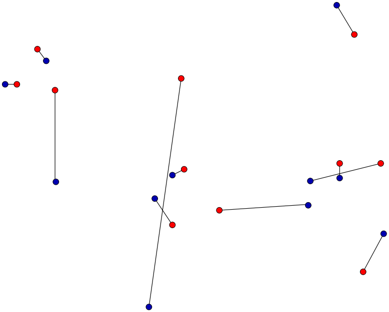

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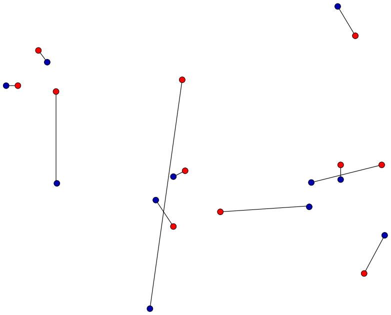

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We let $F(r)$ be the expected number of red points in the box $[0,1]^d$ matched to a blue point at distance no more than r, and define a random variable X denoting the typical matching length as

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Holroyd, Pemantle, Peres and Schramm (2008)

If $d = 1, 2$ every shift-invariant matching of red and blue points satisfies

$$\mathbb{E}^* X^{d/2} = \infty.$$

The stable matching in $d = 1$ has the property that

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How is this related to the extra head problem?

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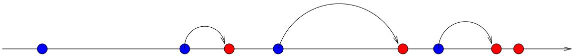

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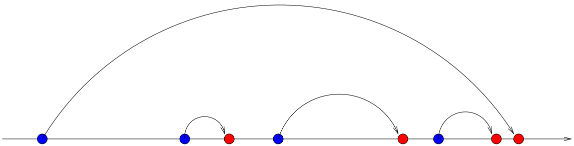

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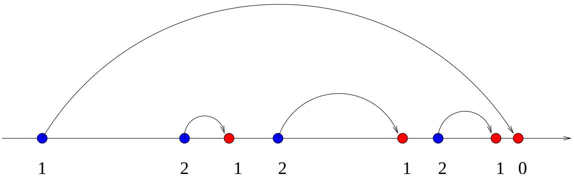

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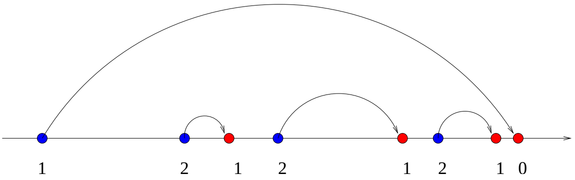

- Starting from a blue point count the number of blue and red points as you move along the line, the matching red point is the first instance when we have counted the same number of red and blue points.
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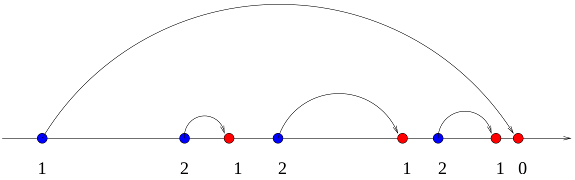

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- The length X corresponds to the first return time to the origin of the associated random walk (with exponential holding times).
- The return time satisfies $\mathbb{P}^*\{X > r\} \leq Cr^{-\frac{1}{2}}$.

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Tool: There exist stochastic processes $(L_t^x \colon t \geq 0)$ on the line called local times such that the value L_t^x quantifies how much time B has spent at level x up to time t. In formulas

$$\int_0^t f(B_s)\, ds = \int_{\mathbb{R}} f(x)\, L_t^x\, dx.$$

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Tool: There exist stochastic processes $(L_t^x \colon t \geq 0)$ on the line called local times such that the value L_t^x quantifies how much time B has spent at level x up to time t. In formulas

$$\int_0^t f(B_s)\, ds = \int_{\mathbb{R}} f(x)\, L_t^x\, dx.$$

Given a probability measure ν we can generalise the family of local times to a process $(L_t^\nu \colon t \geq 0)$ given by

$$L_t^\nu = \int L_t^x\, d\nu(x),$$

called the additive functional with Revuz-measure ν.

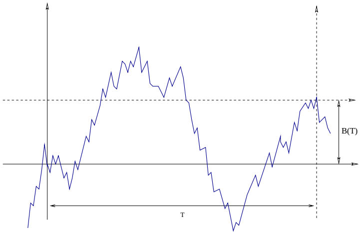
Shifting Brownian motion

Theorem 1: Last, M and Thorisson (2012)

Suppose that ν is a probability measure with $\nu\{0\} = 0$. Then

$$T = \inf\{t > 0 \colon L_t^0 = L_t^\nu\}$$

is an unbiased shift and $B(T)$ has law ν.



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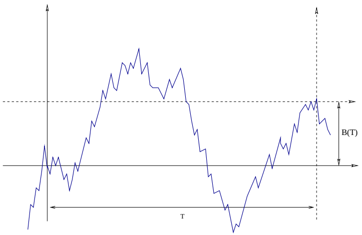

- T is also a stopping time.

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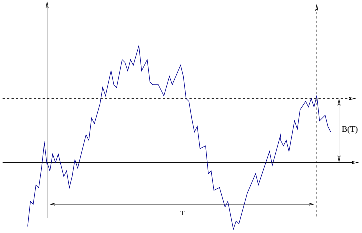

- The embedding property has been observed in a different context by Bertoin, Le Gall (1992).

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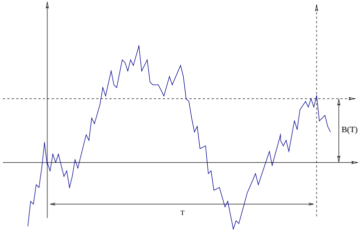

- If $\nu\{0\} > 0$, say $\nu = \epsilon\delta_0 + (1-\epsilon)\mu$ with $\mu\{0\} = 0$, then T is an unbiased shift embedding μ.

Shifting Brownian motion

Theorem 2: Last, M and Thorisson (2012)

Any stopping time T which is an unbiased shift embedding some ν with $\nu\{0\} = 0$ satisfies

$$\mathbb{E}T^{\frac{1}{4}} = \infty.$$

If $\int |x| \, d\nu(x) < \infty$ then the T constructed in Theorem 1 satisfies

$$\mathbb{E}T^{\alpha} < \infty \qquad \text{for all } \alpha < \frac{1}{4}.$$

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- Open: Is there a better T which is not a stopping time?
- If $\nu = \delta_0$ there exists unbiased shifts $T \neq 0$ with exponential moments.

How is this related to the matching problems?

General theory of Last and Thorisson (2009) shows that

A random time T is an unbiased shift embedding ν.

if and only if

The mapping $\tau\colon \mathbb{R} \to \mathbb{R}$ given by $\tau(t) = T \circ \theta_t + t$ satisfies
$\ell^0 \circ \tau^{-1} = \ell^\nu$ $\mathbb{P}$-almost surely.

Here ℓ^ν is the random measure on the line with distribution function given by $(L_t^\nu\colon t \in \mathbb{R})$, and $\mathbb{P}$ is the 'law' of stationary Brownian motion.

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New problem: Find a matching τ between the two random measures ℓ^0 and ℓ^ν. This is a continuous version of the Poisson matching problem on the line!

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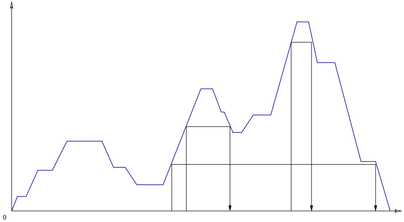

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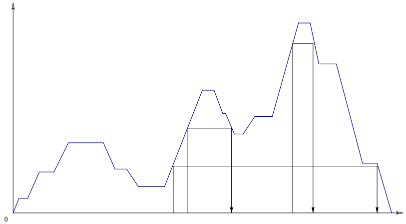

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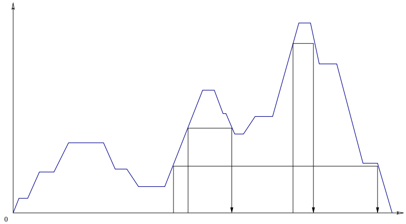

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Warning! This function f, not the Brownian motion, is the analogue of the random walk appearing in the extra head and matching problems!

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$$U_r := \inf\{t > 0\colon L_t^0 + L_t^\nu = r\}$$

with respect to a clock which does not tick during the flat pieces of f, and defining

$$\tilde{f}(r) := f(U_r), \qquad \text{for all } r > 0,$$

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As $U_r \sim r^2$ by Brownian scaling, the return times for the original f have tails of order $t^{-\frac{1}{4}}$.

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We look at the ℓ^0 mass that is not matched within an interval

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If $\mathbb{E}_0(L^0_T)^{\frac{1}{2}} < \infty$ the RHS would be of strictly smaller order than $t^{\frac{1}{2}}$,
contradicting the central limit theorem.