Random Walk Intersections: Large Deviations and Related Topics
by Xia Chen

This book presents an up-to-date account of one of liveliest areas of probability in the past ten years, the large deviation theory of intersections and self-intersections of random walks. The author, one of the protagonists in this area, has collected some of the main techniques and made them accessible to an audience of graduate students and researchers. The book is particularly suitable for researchers who want to make this area their own or who want to apply the methods in the field of stochastic processes in random environments. Being in this situation myself I thoroughly enjoyed reading it.

To explain by an example what this book is about, suppose that $(S_n \colon n = 0, 1, \dots)$ is a symmetric simple random walk on the d-dimensional integer lattice for $d = 2$. The *self-intersection local time* is defined as

$$Q_n := \sum_{1 \le j < k \le n} \mathbb{1}_{\{S_j = S_k\}}.$$

This is a random variable measuring the number of self-intersections of the random walk path up to time n. The asymptotics of the mean of Q_n can be easily calculated as

$$\mathbb{E}Q_n \sim \frac{1}{\sqrt{8}\pi} n \log n$$

There is a law of large numbers which ensures that $Q_n/\mathbb{E}Q_n \to 1$ in probability, and a central limit theorem which states that

$$\frac{1}{n}\Big\{Q_n - \mathbb{E}Q_n\Big\} \overset{\mathrm{d}}{\Longrightarrow} X,$$

where X is a random variable associated with self-intersections of Brownian motion. Among the problems of interest in the theory of random walk intersections is the behaviour of the large deviation probabilities $\mathbb{P}\{Q_n \ge b_n\}$ and $\mathbb{P}\{Q_n - \mathbb{E}Q_n \ge c_n\}$ when $b_n \gg n \log n$ and $c_n \gg n$. For the latter problem one can believe that, at least if c_n is not growing too fast, it is related to the analogous problem for Brownian motion, in our notation the asymptotics of $\mathbb{P}\{X \ge x\}$ as $x \uparrow \infty$, which is of independent interest. One can imagine that a multitude of interesting question emerges from varying the deviation scale or the dimension. Other variations look at the range of a random walk, i.e. the number of sites visited in the first n steps, at intersections of more than one random walk, or at the analogous questions for continuous time random walks or Brownian motion. All these questions have their justification, not only in the richness of the different behaviour that emerges and in the breadth of proof techniques, but also in a variety of applications in probability and statistical mechanics.

After a short chapter surveying some general tools from large deviation theory, the book opens with a discussion of intersection local times for Brownian motion in dimensions $d \le 3$. The construction of this nontrivial object is given in some detail and the renormalized self-intersection local time (essentially the random variable X above) is derived. With this definition at hand Chen starts the discussion of modern material in Chapter 3 with an investigation of the upper tail behaviour of the intersection local time for p independent Brownian motions in subcritical dimensions $p(d-2) < d$. This result is his own [Ch04] but he gives an alternative proof based on the method of high moments first used in this context in [KM02]. It describes the tail of the probability that the intersection local time of p Brownian motions exceeds x asymptotically as $x^{1/p}$ times a constant given in terms of the optimal constant in the Gagliardo-Nirenberg inequality; the appearance of optimal constants for Sobolev-type inequalities is a typical feature of the results in this area. This chapter already shows the great strength of this book: rather than an encyclopaedic approach to the multitude of different results in this area, the author emphasises the various methods of proof, enabling the reader to add new tools to their mathematical toolbox.

Chapter 4 is devoted to the large deviations for self-intersections of a single Brownian path. The author explains his method based on the Feynman-Kac formula and some interesting tricks from the area of probability in Banach spaces. Other techniques introduced here are subadditivity methods and path decompositions, which relate self-intersections of a single Brownian path to intersections of several Brownian paths. In Chapter 5 the author turns to random walks. The focus here is on results relating the random walk and the Brownian motion case by weak convergence, which is a fruitful approach in the subcritical dimensions $p(d-2) < d$. Chapter 6 prepares the ground for the final two chapters by providing a range of mostly combinatorial arguments. In Chapter 7 the focus is on mutual intersections of several walks. Here the techniques developed in Chapters 5 and 6 help understanding the subcritical case, while the exact upper tail behaviour for the total intersection local time of p independent walks in the supercritical case $p(d-2) > d$ is proved using the high moment method. The final chapter is devoted to self-intersections of one random walk. Here we obtain an answer to the example problem picked above: If $n \ll c_n \ll n^2$ then

$$\lim_{n\to\infty} \frac{n}{c_n} \log \mathbb{P}\big\{Q_n - \mathbb{E}Q_n \geq \lambda c_n\big\} = -\lambda\sqrt{2}\kappa^{-4},$$

where κ is the optimal constant in the Gagliardo–Nirenberg inequality

$$\|f\|_4 \leq \kappa\sqrt{\|\nabla f\|_2}\,\sqrt{\|f\|_2} \quad \text{for all } f \in W^{1,2}(\mathbb{R}^2).$$

Smoothing the self-intersection local time plays an important part in the proof, which is taken from [BCR06]. Similar result are available in $d = 3$, but in higher dimensions much is still unknown. This chapter, as all other chapters, ends with extensive comments, references and, particularly valuable, a selection of open problems for the reader to work on.

Let me list some of the things the book does *not* cover. Naturally the author puts special emphasis on the methods used in his own work. Alternative approaches are based on a detailed analysis of the local time field, see e.g. [As08, BHK07], classical concentration of measure inequalities [FMW08], advanced coarse graining techniques [BBH01], or the Dynkin isomorphism theorem, see e.g. [Ca10, MR06]. Although these (and other) techniques are not explicitly covered in the book, a reader who is familiar with the material of the book will have very good background to learn about these methods from the original sources.

The book also does not discuss applications of large deviations for random walk intersections beyond the laws of the iterated logarithm. Volumes could be filled with beautiful material applying results listed in the book to the fractal geometry of Brownian motion, the parabolic Anderson model, random walk in random environment, self-attracting polymers and a variety of other interesting processes. It would of course be asking too much to include all of these, or even a representative selection, in the present book. As it is, the book requires a good level of self-motivation from the reader, but it is definitely the ideal point of entrance for anyone interested in the ongoing research into large deviations for random walk intersections.

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