A class of random measures and their fractal geometry

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We show that a wide class of random measures have very similar fractal geometry and argue that this can be traced back to their similar local hitting, scaling and conditioning behaviour. This is a contribution to, and significant extension of, an exciting research programme initiated by Kallenberg in [4].

Typical random measures Ξ belonging to our class are

- occupation measures of stable subordinators with stability index $0 < \alpha < 1$,
- states of a Dawson-Watanabe superprocesses in $\mathbb{R}^d$, $d \geq 2$,
- intersection local times of two Brownian paths in $\mathbb{R}^d$, $d = 2, 3$.

Very roughly, with some modification in the critical cases $d = 2$, the following basic common properties of these examples can be identified:

- The *local hitting* properties are related to the *local intensity*, i.e. for some *scaling index* $\alpha > 0$ we have,
$$\epsilon^\alpha\, \mathbb{P}\{\Xi B_\epsilon(x) > 0\} \sim \mathbb{E}\Xi(B_\epsilon(x)).$$
- Given that Ξ charges a small ball B, its neighbourhood looks like a translation of the *Palm distribution* $\mathbb{P}^0$ associated with a stationary version of the process.
- Given that Ξ charges two balls with distance of larger order than their size, the behaviour of Ξ inside these balls is (up to constant factors) *conditionally independent*.
- *Local self-similarity* holds with scaling index α,
$$\Xi(r\,\cdot) \approx r^\alpha \Xi(\,\cdot\,) \qquad \text{under } \mathbb{P}^0.$$
- There is a finite *annular lacunarity index* ξ such that
$$\mathbb{P}^0\{\Xi(B_1 \setminus B_r) = 0\} \approx r^\xi \qquad \text{as } r \downarrow 0.$$

The indices associated with our examples are the stability index α and $\xi = 2\alpha$ in the case of stable subordinators; $\alpha = 2$, $\xi = 4$ for the superprocess example; and in the intersection example $\alpha = 2$, $\xi = \frac{35}{12}$ if $d = 2$, $\alpha = 1$, $1 < \xi < 2$ unknown if $d = 3$. The lacunarity index in the planar case of the intersection example goes back to the seminal work of Lawler, Schramm and Werner.

Coming to the fractal geometry, in all our examples, the measure Ξ can be approximated by the Lebesgue measure on ϵ-neighbourhoods of the support. More precisely, let
$$S(\epsilon) = \big\{x \in \mathbb{R}^d \colon \Xi(B_\epsilon(x)) > 0\big\}.$$
Then, at least in probability, as $\epsilon \downarrow 0$,
$$\phi(\epsilon)\,\mathrm{Leb}\big(\,\cdot\, \cap S(\epsilon)\big) \longrightarrow \Xi$$
for a suitable function of the form $\phi(\epsilon) = \epsilon^{\alpha-d}\, L(\epsilon)$, where L is a slowly varying correction required in the critical cases. See [6] for the subordinators, [8] for intersections, and [5] for the superprocess case. In the subordinator case the result

was probably known to the pioneers of local time, like Paul Lévy, as early as the 1940s.

All our examples have an interesting *multifractal spectrum* that does not conform to the classical multifractal spectrum of statistical physics. While

$$\liminf_{r\downarrow 0} \frac{\log \Xi(B_r(x))}{\log r} = \alpha \quad \text{for all } x \in S,$$

we have variations of the limsup behaviour. For every $\alpha \le a \le \frac{\xi\alpha}{\xi-\alpha}$,

$$\dim\Big\{x \in S\colon \limsup_{r\downarrow 0} \frac{\log \Xi(B_r(x))}{\log r} = a\Big\} = \alpha - \xi + \frac{\xi\alpha}{a}\,.$$

This is shown in [3] for subordinators, [13] for superprocesses and [7] for intersections. Note that the latter paper includes intersections of Brownian paths in the critical dimension $d = 2$, but the critical case for superprocesses is still open.

An *average density*, as introduced by Bedford and Fisher [1], can be defined in the non-critical cases as

$$\lim_{\epsilon\downarrow 0} \frac{1}{\log(1/\epsilon)} \int_\epsilon^1 \frac{\Xi(B_r(x))}{r^\alpha}\,\frac{dr}{r} = D_2 \qquad \text{for } \Xi\text{-almost every } x.$$

In the critical cases this order-two average diverges, but an order-three average

$$\lim_{\epsilon\downarrow 0} \frac{1}{\log\log(1/\epsilon)} \int_\epsilon^{1/e} \frac{\Xi(B_r(x))}{r^\alpha L(r)}\,\frac{dr}{r\log(1/r)} = D_3 \qquad \text{exists for } \Xi\text{-almost every } x.$$

See [2] for subordinators, [10] for intersections and [12] for superprocesses.

Finally, and only in the non-critical cases, we have an integral test for the *packing measures* of the support S,

$$\mathcal{P}^\psi(S) = \begin{cases} 0 \\ \infty \end{cases} \qquad \text{iff} \qquad \int_{0+} r^{-1-\xi}\psi(r)^{\frac{\xi}{\alpha}}\,dr \begin{cases} < \infty, \\ = \infty. \end{cases}$$

See [14] for subordinators, [9] for superprocesses, and [11] for intersections.

At this moment, proofs rely on specific features of the examples, in particular on the Markov property. It is an interesting challenge for the future to provide proofs that follow directly from the hitting, scaling and conditioning properties of the random measures, and to add further examples of different flavour.

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