

# GALTON-WATSON TREES WITH VANISHING MARTINGALE LIMIT

NATHANAËL BERESTYCKI, NINA GANTERT, PETER MÖRTERS, NADIA SIDOROVA

**Abstract:** We show that an infinite Galton-Watson tree, conditioned on its martingale limit being smaller than  $\varepsilon$ , agrees up to generation  $K$  with a regular  $\mu$ -ary tree, where  $\mu$  is the essential minimum of the offspring distribution and the random variable  $K$  is strongly concentrated near an explicit deterministic function growing like a multiple of  $\log(1/\varepsilon)$ . More precisely, we show that if  $\mu \geq 2$  then with high probability as  $\varepsilon \downarrow 0$ ,  $K$  takes exactly one or two values. This shows in particular that the conditioned trees converge to the regular  $\mu$ -ary tree, providing an example of entropic repulsion where the limit has vanishing entropy.

**Mathematics Subject Classification (2010):** 60J80 (Primary) 60F10, 60K37.

**Keywords:** Conditioning principle, large deviations, micro-canonical distribution, concentration of measure, sharp thresholds, branching, entropic repulsion.

## 1. INTRODUCTION

The problem of *conditioning principles* can be formulated in the following way: Given that a system comprising a large number of individual components shows highly unlikely *collective* behaviour, describe the conditional law of an *individual* component. This situation arises frequently in statistical mechanics, when an ensemble of particles is subject to some constraint (for example a fixed energy per particle). The distribution of the individual feature given the constraint is then referred to as the *micro-canonical* distribution of the system. The most famous result in this respect is the *Gibbs conditioning principle*, which loosely speaking says that under the condition that the empirical measure

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

of a family of independent random variables  $X_1, \dots, X_n$  with law  $P$  belongs to some convex set  $A$ , the law of  $X_1$  converges to the probability measure  $Q$  that minimizes the relative entropy  $H(Q|P)$  subject to the constraint  $Q \in A$ . There exist several refinements of this result describing rigorously the precise asymptotic strategy by which the random variables realize the large deviation event  $\{L_n \in A\}$ . See the book of Dembo and Zeitouni [6] for more on the classical Gibbs conditioning principle, [5, 7, 14] for refinements, and [8, 11, 12] for further examples of conditioning principles.

The present paper describes such a conditioning principle in the case of Galton–Watson trees with a nondegenerate offspring variable  $N$  satisfying  $P(N = 0) = 0$  and  $EN \log N < \infty$ . Let  $a := EN > 1$  be the mean offspring number. We denote by  $(Z_n : n = 0, 1, \dots)$  the sequence of generation sizes of the Galton–Watson tree and note that by definition  $Z_0 = 1$ . By the Kesten–Stigum theorem the *martingale limit*

$$W := \lim_{n \rightarrow \infty} \frac{Z_n}{a^n}$$

is well-defined and strictly positive almost surely. Note that  $W$  can be seen as a random constant factor in front of a deterministic exponential growth term  $a^n$ , which together determine the leading order asymptotics of the generation size  $Z_n$ . In the framework of the preceding paragraph the quantity  $W$  represents the collective behaviour of the branching individuals and we are interested in the offspring distribution of individual particles given that  $W$  is smaller than a small parameter  $\varepsilon$ .

An important observation is that the offspring distribution of the conditioned tree is not uniform over all generations and the influence of the initial generations far outweighs that of later generations. Indeed, we show that there is a sharp threshold level  $\gamma(\varepsilon)$ , satisfying

$$\gamma(\varepsilon) \sim \frac{\log(1/\varepsilon)}{\log(a/\mu)},$$

such that all individuals up to generation  $\lceil \gamma(\varepsilon) \rceil - 2$  only produce the minimal number  $\mu := \min\{n \in \mathbb{N} : P(N = n) > 0\}$  of offspring. Here  $\sim$  denotes that the ratio of the left and right hand side converges to one. Decomposing the population according to their ancestry in generation  $k$  gives

$$W = \frac{1}{a^k} \sum_{j=1}^{Z_k} W_j,$$

where  $W_1, W_2, \dots$  are independent copies of  $W$ . Using this decomposition for  $k = \lceil \gamma(\varepsilon) \rceil - 1$  and assuming that the tree performs unconditionally from generation  $k$  onwards shows that  $W \sim (\mu/a)^k$  and hence  $\log W \sim \log \varepsilon$ , showing that minimal branching up to generation  $\lceil \gamma(\varepsilon) \rceil - 2$  almost single-handedly delivers the collective requirement.

Our main results confirm and substantially refine this rough picture in the case where the minimal offspring number satisfies  $\mu > 1$ . In this case we can describe  $\gamma(\varepsilon)$  precisely as

$$\gamma(\varepsilon) := \frac{\log(1/\varepsilon)}{\log(a/\mu)} - \frac{\log \log(1/\varepsilon)}{\log \mu} + H(\varepsilon),$$

where  $H$  is a multiplicatively periodic continuous function with period  $a/\mu$ . The first branching producing more than the minimal number of offspring occurs in generation  $\lceil \gamma(\varepsilon) \rceil - 1$  or  $\lceil \gamma(\varepsilon) \rceil$ . We show that for most values of  $\varepsilon$  it occurs in generation  $\lceil \gamma(\varepsilon) \rceil - 1$  and, defining the random variable

$$K := \min \{k \in \mathbb{N} : Z_k > \mu^k\},$$

we find that the size of generation  $K$  is asymptotically still given by  $\mu^K$  with a relatively small additive  $\varepsilon$ -dependent correction. Before describing these results in more detail in Section 2, we now briefly explain the situation in the ‘degenerate’ case  $\mu = 1$ , in which nonexponential growth of the tree is possible. The concentration effect of the random variable  $K$  which holds in the case  $\mu > 1$  is much less pronounced in the case  $\mu = 1$ , but the result can be obtained by soft arguments, whereas the case of general  $\mu > 1$  requires much more subtle reasoning.

## 2. STATEMENT OF THE MAIN RESULTS

We start by describing our results in the case  $\mu = 1$ , for which the analysis is fairly straightforward. In this case we define

$$\gamma(\varepsilon) := \frac{\log(1/\varepsilon)}{\log a}.$$

Our results in this case are summarised by the following proposition.

**Proposition 1.** *There exists  $\lambda > 0$  such that*

$$\limsup_{\varepsilon \downarrow 0} \mathbb{P}(|K - \gamma(\varepsilon)| \geq x | W < \varepsilon) \leq e^{-\lambda x} \quad \text{for all } x \geq 1. \quad (1)$$

In other words the time of the first branching producing more than the minimal number of offspring occurs in generation  $\gamma(\varepsilon)$  with a tight random correction of order one.

Because  $K \uparrow \infty$  this implies that the Galton-Watson tree conditioned on  $W < \varepsilon$  converges (in a sense detailed below) for  $\varepsilon \downarrow 0$  to the regular  $\mu$ -ary tree. This fact, which also holds in the case  $\mu > 1$ , is quite remarkable when seen in a large deviations context. We shall explain this further in the next section, after the first main result is established.

We now come to the main result of this paper, which deals with the case  $\mu \geq 2$ . More precisely, we consider a Galton-Watson tree with offspring probabilities  $p_n = P(N = n)$  and keep the notation established above. We assume that  $\mu = \min\{n \in \mathbb{N} : p_n > 0\} \geq 2$  and also exclude the trivial case  $p_\mu = 1$ . Recall that  $K - 1$  is the first generation where an individual has more than the minimal number of offspring.

**Theorem 2.** *We have*

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(K = \lceil \gamma(\varepsilon) \rceil \text{ or } K = \lceil \gamma(\varepsilon) \rceil + 1 \mid W < \varepsilon) = 1,$$

where

$$\gamma(\varepsilon) := \frac{\log(1/\varepsilon)}{\log(a/\mu)} - \frac{\log \log(1/\varepsilon)}{\log \mu} + H(\varepsilon)$$

and  $H$  is a multiplicatively periodic continuous nonrandom function with period  $a/\mu$ .

Before giving more detailed results on the shape of the conditioned tree, we give an interpretation of Theorem 2 and put it into context. To this end we denote by  $\mathcal{T}$  the space of all rooted trees with the property that every vertex has finite degree. A metric  $d$  on this space is uniquely determined by the requirement that  $d(T_1, T_2) = e^{-n}$ , when  $n$  is maximal with the property that the trees  $T_1$  and  $T_2$  coincide up to the  $n$ th generation. This makes  $(\mathcal{T}, d)$  a complete, separable metric space. The next results also holds when  $\mu = 1$ .

**Corollary 3.** *As  $\varepsilon \downarrow 0$ , conditionally on the event  $\{W < \varepsilon\}$  the tree  $T$  converges in law on  $(\mathcal{T}, d)$  to the regular  $\mu$ -ary tree, i.e., the tree in which every vertex has exactly  $\mu$  offspring.*

*Proof.* The statement is equivalent to  $\lim_{\varepsilon \downarrow 0} \mathbb{P}(Z_k = \mu^k \mid W < \varepsilon) = 1$ , for all  $k \in \mathbb{N}$ . This follows directly from (1) in the case  $\mu = 1$ , and from Theorem 2 in the case  $\mu > 1$ .  $\square$

From the point of view of large deviations theory this result is quite surprising, at least at a first glance. One would expect that the limiting behaviour represents the optimal strategy by which the event  $W = 0$  is realized and that this strategy depends on the details of the law of  $N$ . There seems to be no good reason why in the limit the *growth rate* of the tree should drop dramatically, or in fact why it should drop at all, as we only require the constant

to be small. Above all, the probability of seeing a  $\mu$ -ary tree up to the  $n$ th generation may be arbitrarily small and can certainly be much smaller than those of seeing other trees satisfying  $Z_n \leq \varepsilon a^n$ .

This becomes even more intriguing if the result is put in the context of *entropic repulsion*, an expression used by physicists to convey the idea that entropy maximisation may force certain systems to obey properties that are not obviously imposed on them *a priori*. This phenomenon has been studied mathematically by Bolthausen et al [4] in the context of the two-dimensional harmonic crystal with hard wall repulsion, and by Benjamini and Berestycki [2] and [3], where it is shown that conditioning a one-dimensional Brownian motion on some self-repelling behaviour may force the process to satisfy a strongly amplified version of the constraint. Usually, the reason entropic repulsion may arise is in order to increase the entropy of the system, i.e., make room for fluctuations. Thus the eventual state of the system is a compromise between the energy cost of adopting an unusual behaviour and the entropic benefits. Corollary 3 may be cast in this framework, as it shows that the effect of requiring the constant  $W$  to be small is to reduce the overall exponential growth rate from  $a$  to  $\mu$ . If the limiting state of the system is non-random, as it is the case in our model, what could the entropic benefits possibly be?

The resolution of this apparent paradox comes from understanding the inhomogeneity of the optimal strategy. While the growth rate  $\log a$  is purely asymptotic, i.e. depends only on the offspring numbers *after* any given generation, the growth constant  $W$  depends heavily on the *initial* generations of the tree. Roughly speaking, the collection of trees which form the optimal strategy to achieve  $W < \varepsilon$  have minimal offspring for roughly  $\gamma(\varepsilon)$  generations, which causes high entropic and energetic cost but only for a small number of generations, and then after a while switch to growth with the natural rate  $\log a$ . The initial behaviour ensures that  $W$  is small at a minimal probabilistic cost, because for all but a small number of generations the trees can have their natural growth. The topology on  $\mathcal{T}$  compares trees starting from their root so that in the limit we only see the behaviour in the initial generations. This leads to a limiting object with minimal growth rate at all generations and creates the illusion of a drop in the growth rate for the optimal strategy. A somewhat similar phenomenon is observed by Bansaye and Berestycki [1] in the context of branching processes in random environment, although they consider situations where the growth rate is directly conditioned to be atypical.

In the following two theorems we return to the case  $\mu > 1$  and take a closer look at the *shape of the conditioned tree* and thus on the inhomogeneous strategy underlying the conditioning event  $W < \varepsilon$ . Figure 1 sketches the curve  $\gamma$  and, for each  $\varepsilon$ , the two possible values for  $K$ , namely  $\lceil \gamma(\varepsilon) \rceil$  and  $\lceil \gamma(\varepsilon) \rceil + 1$ , represented by the horizontal lines. Roughly speaking, we will see in Theorem 4 that for most  $\varepsilon$  the random variable  $K$  has a particular non-random value, represented by the thick horizontal lines. For most values of  $\varepsilon$  we have  $K = \lceil \gamma(\varepsilon) \rceil$  and only very occasionally  $K = \lceil \gamma(\varepsilon) \rceil + 1$ . The switch happens when  $\gamma(\varepsilon)$  gets too close to the integer  $\lceil \gamma(\varepsilon) \rceil$ . Then, for a short range of values of  $\varepsilon$ , marked in grey on the zoomed picture,  $K$  is truly random and can take the values  $\lceil \gamma(\varepsilon) \rceil$  and  $\lceil \gamma(\varepsilon) \rceil + 1$ . As  $\varepsilon$  decreases further  $\lceil \gamma(\varepsilon) \rceil$  loses its power, and  $K$  moves to  $\lceil \gamma(\varepsilon) \rceil + 1$ . This, in turn, does not last long because when  $\varepsilon$  decreases just a little more the curve  $\gamma$  crosses an integer level, and then for another long range of  $\varepsilon$  the random variable  $K$  takes the value  $\lceil \gamma(\varepsilon) \rceil$  again.

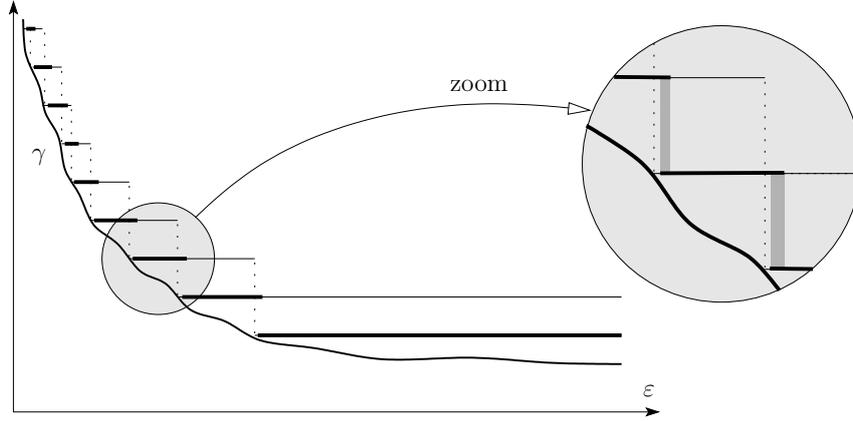


FIGURE 1. The time of the first branching with more than the minimal offspring.

In order to be able to formulate this precisely, we need to identify the different regions of  $\varepsilon$ . Let  $\beta := \log \mu / \log a$  and  $\alpha := \frac{\beta}{1-\beta}$ , and denote  $\{x\} := [x] - x$ , for  $x \in \mathbb{R}$ . Further, denote

$$\omega(\varepsilon) := \frac{\varepsilon^{\alpha(\mu^{-\{\gamma(\varepsilon)\}} - 1)}}{\log(1/\varepsilon)}.$$

It is easy to see that

$$\liminf_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0 \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} \omega(\varepsilon) = \infty,$$

where the liminf is achieved via values of  $\varepsilon$  for which  $\{\gamma(\varepsilon)\}$  is very small, and the limsup is achieved via most other values of  $\varepsilon$ .

In the sequel, if  $(q_j)$  is a sequence of probabilities we write  $0 \prec q_j \prec 1$  to denote that the sequence is asymptotically bounded away from zero and one. For sequences  $(a_j)$ ,  $(b_j)$  of positive numbers we use the symbol  $a_j \asymp b_j$  to denote that  $a_j/b_j$  is asymptotically bounded away from zero and infinity.

**Theorem 4.**

(a) Suppose  $\varepsilon_j \downarrow 0$  such that  $\omega(\varepsilon_j) \rightarrow \infty$ . Then

$$\lim_{j \rightarrow \infty} \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) = 1.$$

(b) Suppose  $\varepsilon_j \downarrow 0$  such that  $\omega(\varepsilon_j) \asymp 1$ , then

$$0 \prec \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j), \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil + 1 \mid W < \varepsilon_j) \prec 1.$$

(c) Suppose  $\varepsilon_j \downarrow 0$  such that  $\omega(\varepsilon_j) \rightarrow 0$ . Then

$$\lim_{j \rightarrow \infty} \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil + 1 \mid W < \varepsilon_j) = 1.$$

**Remark:** It is possible to compute the exact asymptotics in the second regime but we do not want to overload the paper with unpleasant computations.

Next, we address the question of what happens in the generation where the first non-minimal branching occurs. We denote

$$\lambda := \min\{n > \mu : p_n > 0\}. \tag{2}$$

**Theorem 5.** *If  $\varepsilon_j \downarrow 0$  such that  $\omega(\varepsilon_j) \rightarrow \infty$  or  $\omega(\varepsilon_j) \rightarrow 0$ , then*

$$\lim_{j \rightarrow \infty} \frac{Z_K - \mu^K}{\mu^K \varepsilon_j^{\alpha \mu^{\gamma(\varepsilon_j) - K}}} = \left(\frac{\lambda}{\mu} - 1\right) p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}},$$

*in probability under  $\mathbb{P}(\cdot | W < \varepsilon_j)$ .*

**Remarks:**

(a) The influence of the first extra branching on the next generation is very small. Roughly, in generation  $K - 1$  most of the individuals still have the minimal number  $\mu$  of children and only a small proportion of order  $\varepsilon^{\alpha \mu^{\gamma(\varepsilon) - K}}$  have more than  $\mu$  children. It can be seen from the proof that most of these individuals would have exactly  $\lambda$  children.

(b) Not only does  $\omega(\varepsilon)$  govern the transition between the regimes, it also explicitly controls the number of additional children. Indeed, in regime (a) in Theorem 5, when  $K = \lceil \gamma(\varepsilon) \rceil$ , the number of extra individuals in generation  $K$  is of order

$$\mu^K \varepsilon^{\alpha \mu^{\gamma(\varepsilon) - K}} = \mu^{\gamma(\varepsilon) + \{\gamma(\varepsilon)\}} \varepsilon^{\alpha \mu^{-\{\gamma(\varepsilon)\}}}, \quad (3)$$

which is bounded from above and below by constant multiples of  $\omega(\varepsilon)$ . This number can be quite large, but as we approach the end of the regime the number of extra individuals becomes smaller. Eventually, there are no extra individuals which means that there is no more extra branching at time  $\lceil \gamma(\varepsilon) \rceil - 1$ , and the point of transition moves to  $K = \lceil \gamma(\varepsilon) \rceil + 1$ .

(c) We conjecture that the extra branching remains negligible for a few generations (corresponding roughly to the second term in the definition of  $\gamma(\varepsilon)$ ) and after that the tree starts growing at its normal rate.

### 3. PROOF OF PROPOSITION 1

To prove (1), decompose the population according to their ancestry in generation  $K$  and get

$$W = \frac{1}{a^K} \sum_{i=1}^{Z_K} W_i =: \frac{1}{a^K} W', \quad (4)$$

where  $W_i$  are independent copies of  $W$ , independent of  $Z_K$  and  $K$ . Note that, as  $\mu = 1$ , the random variable  $K$  is independent of  $Z_K$  and hence of  $W'$ . Using the abbreviation

$$p_n := P(N = n), \quad \text{for } n \in \mathbb{N},$$

and letting  $\tau := -\log p_1 / \log a$  we get from [9] or an easy argument in [13] that there exist constants  $0 < c_1 < C_1$  such that, for all  $0 < \varepsilon < 1$ ,

$$c_1 \varepsilon^\tau \leq \mathbb{P}(W < \varepsilon) \leq C_1 \varepsilon^\tau.$$

Hence, for  $\ell = \gamma(\varepsilon) - z$ ,  $z > 0$ ,

$$\begin{aligned} \mathbb{P}(K = \ell | W < \varepsilon) &\leq c_1^{-1} \varepsilon^{-\tau} \mathbb{P}(W' < \varepsilon a^\ell) \mathbb{P}(K = \ell) \\ &\leq c_1^{-1} p_1^{-1} p_1^\ell \varepsilon^{-\tau} \mathbb{P}(W < \varepsilon a^\ell)^2 \\ &\leq c_1^{-1} p_1^{-1} C_1^2 \exp(\ell \log p_1 + \tau \log \varepsilon + 2\tau \ell \log a) \\ &= c_1^{-1} p_1^{-1} C_1^2 \exp((\log p_1)z), \end{aligned} \quad (5)$$

where we used (4) in the second inequality. Summing over all  $z \geq x$  gives, for a suitable choice of  $\lambda > 0$ ,

$$\mathbb{P}(K \leq \gamma(\varepsilon) - x \mid W < \varepsilon) \leq e^{-\lambda x}.$$

Conversely note that, making  $\lambda > 0$  smaller if necessary,

$$\mathbb{P}(K > \gamma(\varepsilon) + z \mid W < \varepsilon) \leq c_1^{-1} \varepsilon^{-\tau} p_1^{\gamma(\varepsilon)} p_1^z \leq e^{-\lambda z},$$

completing the proof of (1).

#### 4. NOTATION AND BACKGROUND

In this section we prepare the proof of our main theorems. We start by introducing some additional notation and background from the paper [10] by Fleischmann and Wachtel, on which our proofs are based. In the sequel, we often omit the argument  $\varepsilon$  from  $\gamma(\varepsilon), \omega(\varepsilon)$  and similar expressions to shorten the formulas. We always assume that  $\varepsilon$  is small enough. Let

$$\kappa(\varepsilon) := \left\lfloor \frac{\log(1/\varepsilon)}{\log(a/\mu)} \right\rfloor, \quad (6)$$

and denote  $y(\varepsilon) := \varepsilon(a/\mu)^{\kappa(\varepsilon)} \in (\mu/a, 1]$ . Let

$$\varphi(z) := \mathbb{E}e^{-zW}, \quad \text{for } z \in \mathbb{C}, \operatorname{Re}(z) \geq 0,$$

be the Laplace transform of  $W$  and let

$$f(s) := \sum_{j=0}^{\infty} p_j s^j, \quad \text{for } s \in [0, 1],$$

be the offspring generating function of the Galton-Watson tree. Denote  $f_0(z) := z$  and  $f_m(z) := f(f_{m-1}(z))$ , for  $m \in \mathbb{N}$ . The *logarithmic Böttcher function* is defined by

$$b(s) := \lim_{m \rightarrow \infty} \mu^{-m} \log f_m(s), \quad \text{for } s \in (0, 1].$$

That the limit exists in the Böttcher case follows for instance from Lemma 10 in [10]. Note that  $b \circ \varphi < 0$  on  $(0, \infty)$  and recall from Lemma 17 in [10] that the function  $(b \circ \varphi)'$  increases from  $-\infty$  to 0 on  $(0, \infty)$ . Therefore, for any  $q \in [1, 2]$ , there exists a unique  $u_q(\varepsilon) > 0$  such that

$$(b \circ \varphi)'(u_q) = -y/q, \quad (7)$$

where  $y = y(\varepsilon)$  is defined under (6). Observe that since the ranges of  $y$  and  $q$  are bounded we have  $u_q \in [u_*, u^*]$  for some  $0 < u_* < u^*$  for all  $\varepsilon$  and  $q$ . Define

$$\sigma_q^2(\varepsilon) := \frac{d^2}{du^2}(b \circ \varphi)(u_q) > 0, \quad (8)$$

where the positivity follows from Lemma 17 in [10].

Let  $d \in \{-1, 0, 1\}$  and

$$n(\varepsilon) := \kappa(\varepsilon) - \lceil \gamma(\varepsilon) \rceil - d. \quad (9)$$

Observe that  $n(\varepsilon) \rightarrow \infty$ ,  $n(\varepsilon)/\kappa(\varepsilon) \rightarrow 0$ , and that  $\kappa - n \in \{\lceil \gamma \rceil, \lceil \gamma \rceil + 1\}$  if and only if  $d \in \{0, 1\}$ . Note that  $n$  depends on  $d$ . This dependence is omitted in the notation but we always make it clear if a particular value of  $d$  is used. If no explicit assumption is made about  $d$ , then it is arbitrary (but independent of  $\varepsilon$ ). Recall (2) and denote

$$H(\varepsilon) := \frac{1}{\log \mu} \log \left( - \frac{b(\varphi(u_1)) y^\alpha (\lambda - \mu)}{\alpha} \right). \quad (10)$$

Since  $y$  is continuous and multiplicatively periodic with period  $a/\mu$  so is  $u_1 = u_1(\varepsilon)$  and thus so is  $H$ .

Observe that we may extend the domain of all functions  $f_n$  to complex variables  $z$  with  $|z| \leq 1$ . Denote  $\mathcal{D}(\delta, \theta) := \{z \in \mathbb{C} : 0 < |z| \leq 1 - \delta, |\arg z| \leq \theta\}$ , for  $\delta \in (0, 1), \theta \in (0, \pi)$ . By Lemma 10 in [10] for every  $\delta \in (0, 1)$  there is  $\theta \in (0, \pi)$  such that  $f_m(z) \neq 0$  for all  $m$  and  $z \in \mathcal{D}(\delta, \theta)$  and  $b$  can be extended to an analytic function on  $\mathcal{D}(\delta, \theta)$  by the uniformly converging series

$$b(z) = \log z + \sum_{j=0}^{\infty} \mu^{-j-1} \log \frac{f_{j+1}(z)}{f_j(z)^\mu}.$$

Observe that on  $\mathcal{D}(\delta, \theta)$  we have

$$\mu^{-m} \log f_m(z) = \log z + \sum_{j=0}^{m-1} \mu^{-j-1} \log \frac{f_{j+1}(z)}{f_j(z)^\mu}$$

and denote

$$\psi_m(z) := b(z) - \mu^{-m} \log f_m(z) - \frac{\mu^{-m}}{\mu - 1} \log p_\mu, \quad \text{for } z \in \mathcal{D}(\delta, \theta).$$

It is easy to see that

$$\psi_m(z) = \sum_{j=m}^{\infty} \mu^{-j-1} \log \frac{f_{j+1}(z)}{p_\mu f_j(z)^\mu}. \quad (11)$$

This implies, in particular, that  $\psi_m(s) > 0$  for all  $s \in (0, 1)$  and all  $m \in \mathbb{N}$ . Our next aim is to describe the asymptotic behaviour of  $\psi_m$  and  $\psi'_m$  as  $m \rightarrow \infty$ . In the sequel we use the Landau symbols  $o(f)$  and  $O(f)$  to denote nonnegative functions, whose actual definition can change at every occurrence, with the property that when divided by  $f$  they converge to zero, respectively stay bounded from above. By Lemma 10 in [10],

$$f_m(z) = p_\mu^{-\frac{1}{\mu-1}} \exp \{ \mu^m b(z) + O(e^{-\delta \mu^m}) \}, \quad (12)$$

that is,  $\psi_m(z) = O(\mu^{-m} e^{-\delta \mu^m})$  uniformly on  $\mathcal{D}(\delta, \theta)$  as  $m \rightarrow \infty$ . In the next lemma we compute a much more precise asymptotics for  $\psi_m$ .

**Lemma 6.** *As  $m \rightarrow \infty$ ,*

$$\psi_m(z) = p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}} \mu^{-m-1} \exp \{ (\lambda - \mu) \mu^m b(z) \} (1 + o(1))$$

and

$$\psi'_m(z) = \mu^m \psi_m(z) O(1)$$

uniformly on compact subsets of  $\mathcal{D}(\delta, \theta)$ .

*Proof.* Using  $f_{j+1}(z) = f(f_j(z))$  we obtain

$$\frac{f_{j+1}(z)}{p_\mu f_j(z)^\mu} = 1 + \sum_{l=1}^{\infty} \frac{p_{\mu+l}}{p_\mu} f_j^l(z). \quad (13)$$

As  $j \rightarrow \infty$ , we have  $f_j(z) \rightarrow 0$  uniformly on  $\mathcal{D}(\delta, \theta)$ , and hence also

$$\frac{f_{j+1}(z)}{p_\mu f_j(z)^\mu} = 1 + \frac{p_\lambda}{p_\mu} f_j^{\lambda-\mu}(z) (1 + o(1)) \quad (14)$$

and

$$\log \frac{f_{j+1}(z)}{p_\mu f_j(z)^\mu} = \frac{p_\lambda}{p_\mu} f_j^{\lambda-\mu}(z) (1 + o(1)).$$

Substituting this into (11) and taking (12) into account we get

$$\begin{aligned} \psi_m(z) &= \frac{p_\lambda}{p_\mu} \sum_{j=m}^{\infty} \mu^{-j-1} f_j^{\lambda-\mu}(z) (1 + o(1)) \\ &= p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}} \sum_{j=m}^{\infty} \mu^{-j-1} \exp\{(\lambda-\mu)\mu^j b(z)\} (1 + o(1)) \\ &= p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}} \mu^{-m-1} \exp\{(\lambda-\mu)\mu^m b(z)\} (1 + o(1)). \end{aligned}$$

Substituting (13) into (11) and differentiating the uniformly converging series of analytic functions, we get

$$\psi'_m(z) = \sum_{j=m}^{\infty} \mu^{-j-1} \left(1 + \sum_{l=1}^{\infty} \frac{p_{\mu+l}}{p_\mu} f_j^l(z)\right)^{-1} \sum_{l=1}^{\infty} l \frac{p_{\mu+l}}{p_\mu} f_j^{l-1}(z) f'_j(z).$$

Using the leading term of the asymptotics (14) and

$$\sum_{l=1}^{\infty} l \frac{p_{\mu+l}}{p_\mu} f_j^{l-1}(z) = f_j^{\lambda-\mu-1}(z) O(1),$$

as  $j \rightarrow \infty$  uniformly on  $\mathcal{D}(\delta, \theta)$ , we obtain

$$\begin{aligned} \psi'_m(z) &= \sum_{j=m}^{\infty} \mu^{-j-1} f_j^{\lambda-\mu-1}(z) f'_j(z) O(1) = \sum_{j=m}^{\infty} (\mu^{-j} \log f_j)'(z) f_j^{\lambda-\mu}(z) O(1) \\ &= \sum_{j=m}^{\infty} \exp\{(\lambda-\mu)\mu^j b(z)\} O(1) = \exp\{(\lambda-\mu)\mu^m b(z)\} O(1) = \mu^m \psi_m(z) O(1), \end{aligned}$$

where we have used the Weierstrass theorem to justify the convergence of the derivatives of uniformly converging analytic functions, and also absorbed a factor  $b'(z)$  into  $O(1)$ .  $\square$

## 5. THE LOWER TAIL OF SUMS OF INDEPENDENT COPIES OF $W$

The main result in [10] yields the following fine lower tail behaviour of  $W$ .

**Lemma 7.** *As  $\varepsilon \rightarrow 0$ , recalling (7) and (8),*

$$\mathbb{P}(W < \varepsilon) = p_\mu^{-\frac{1}{\mu-1}} \frac{1}{\sigma_1 u_1 \sqrt{2\pi}} \mu^{-\frac{\kappa}{2}} \exp\{\mu^\kappa (b(\varphi(u_1)) + y u_1) + o(1)\}.$$

*Proof.* By Theorem 1 in [10] we have, as  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(W < \varepsilon) = L(\varepsilon) \varepsilon^{\frac{\alpha}{2}} \exp\{-M(\varepsilon) \varepsilon^{-\alpha} + o(1)\}, \quad (15)$$

where  $M$  and  $L$  are positive multiplicatively periodic functions with period  $a/\mu$  given by

$$\begin{aligned} M(\varepsilon) &:= -\varepsilon^\alpha \min_{v>0} \{b(\varphi(v)) + v\varepsilon\}, \\ L(\varepsilon) &:= p_\mu^{-\frac{1}{\mu-1}} \frac{y^{-\frac{\alpha}{2}}}{\sigma_1 u_1 \sqrt{2\pi}}, \end{aligned}$$

see formula (142) and (155) in [10]. Using  $\varepsilon(a/\mu)^\kappa = y$ ,  $a = \mu^{\frac{1}{\beta}}$ , the definition of  $u_1$ , the periodicity of  $M$  and the convexity of  $b \circ \varphi$  (see Lemma 17 in [10]), we have

$$-M(\varepsilon)\varepsilon^{-\alpha} = -M(y)\varepsilon^{-\alpha} = y^\alpha \varepsilon^{-\alpha} (b(\varphi(u_1)) + yu_1) = \mu^\kappa (b(\varphi(u_1)) + yu_1)$$

and  $y^{-\frac{\alpha}{2}} \varepsilon^{\frac{\alpha}{2}} = \mu^{-\frac{\kappa}{2}}$ , which completes the proof.  $\square$

Recall that in our calculation (5) for the case  $\mu = 1$  we used a crude estimate to bound the lower tail probability of  $W'$ , the sum of finitely many independent copies  $W_1, W_2, \dots$  of the limiting variable  $W$ . While this estimate holds in general, it is insufficient in the case  $\mu > 1$ . The main goal of this section is to establish a fine result describing the lower tails of the sum of independent copies of  $W$  in this case. The proof uses three technical lemmas which, for the reader's convenience, are stated and proved after the presentation of the main argument.

**Proposition 8.** *As  $\varepsilon \rightarrow 0$ , with  $\kappa$  and  $n$  defined in (6) and (9),*

$$\begin{aligned} \mathbb{P}\left(\sum_{j=1}^{q\mu^{\kappa-n}} W_j < \varepsilon a^{\kappa-n}\right) \\ = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \frac{1}{\sigma_q u_q \sqrt{2\pi q}} \mu^{-\frac{\kappa}{2}} \exp\{\mu^\kappa (qb(\varphi(u_q)) - q\psi_n(\varphi(u_q)) + yu_q)\} I_q(\varepsilon), \end{aligned} \quad (16)$$

uniformly in  $q \in [1, 2]$  such that  $q\mu^{\kappa-n} \in \mathbb{N}$ , where  $I_q(\varepsilon)$  has the following properties:

- it is uniformly bounded in  $q$  and  $\varepsilon$ ;
- if  $\varepsilon_j \rightarrow 0$  such that  $\mu^{\kappa(\varepsilon_j)} \psi_n(\varepsilon_j)(\varphi(u_1(\varepsilon_j))) = O(1)$  then  $I_1(\varepsilon_j) \rightarrow 1$ .

*Proof.* We have

$$\mathbb{E}e^{-t(W_1 + \dots + W_{q\mu^{\kappa-n}})} = \varphi^{q\mu^{\kappa-n}}(t)$$

and so by the inversion formula for distribution functions

$$\mathbb{P}\left(\sum_{j=1}^{q\mu^{\kappa-n}} W_j < \varepsilon a^{\kappa-n}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\tau \varepsilon a^{\kappa-n}}}{i\tau} \varphi^{q\mu^{\kappa-n}}(-i\tau) d\tau.$$

Changing the integration contour we get for arbitrary  $p > 0$

$$\mathbb{P}\left(\sum_{j=1}^{q\mu^{\kappa-n}} W_j < \varepsilon a^{\kappa-n}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(p-i\tau)\varepsilon a^{\kappa-n}} - 1}{p - i\tau} \varphi^{q\mu^{\kappa-n}}(p - i\tau) d\tau.$$

Substituting  $p = u_q a^n$  and  $\tau = ta^n$  and using the Poincaré functional equation  $\varphi(az) = f(\varphi(z))$ , we obtain

$$\begin{aligned} \mathbb{P}\left(\sum_{j=1}^{q\mu^{\kappa-n}} W_j < \varepsilon a^{\kappa-n}\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\varepsilon a^\kappa (u_q - it)} - 1}{u_q - it} \varphi^{q\mu^{\kappa-n}}(a^n(u_q - it)) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{y\mu^\kappa (u_q - it)} - 1}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt. \end{aligned} \quad (17)$$

Recall that  $u_* > 0$  has been fixed in such a way that  $u_q \geq u_*$  for all  $\varepsilon$  and  $q$ . By Lemma 15 from [10] there is a constant  $c > 0$  such that for all  $\theta \in (0, c]$ ,

$$\varphi(v - it) \in \mathcal{D}(1 - \varphi(u_*), \theta/c), \quad \text{for all } v \geq u_*, |t| \leq \theta.$$

By Lemma 10 from [10] there is  $\theta_1 > 0$  such that for all  $0 < \theta < \theta_1$  the function  $b$  and so all functions  $\psi_n$  are analytic on  $\mathcal{D}(1 - \varphi(u_*), \theta/c)$ . This implies, in particular, that  $\frac{\partial^3}{\partial t^3} b(\varphi(v - it))$

is bounded on the set  $\{v \geq u_*, |t| \leq \theta\}$  and that the family  $\frac{\partial^3}{\partial t^3} \psi_n(\varphi(v - it))$  is uniformly bounded on the set  $\{v \geq u_*, |t| \leq \theta\}$ , where uniformity follows from the fact that the  $\psi_n$  are analytic and converge uniformly to zero.

Expanding in a Taylor series in  $t$  and using the definition of  $u_q$  and  $\sigma_q^2$  we get

$$\begin{aligned} b(\varphi(u_q - it)) &= b(\varphi(u_q)) - it(b \circ \varphi)'(u_q) - \frac{t^2}{2} \frac{d^2}{du^2}(b \circ \varphi)(u_q) + O(t^3) \\ &= b(\varphi(u_q)) + \frac{ity}{q} - \frac{\sigma_q^2 t^2}{2} + O(t^3), \end{aligned} \quad (18)$$

and

$$\begin{aligned} \psi_n(\varphi(u_q - it)) &= \psi_n(\varphi(u_q)) - it(\psi_n \circ \varphi)'(u_q) - \frac{t^2}{2} \frac{d^2}{du^2}(\psi_n \circ \varphi)(u_q) + O(t^3) \\ &= \psi_n(\varphi(u_q)) - ita_q - \frac{s_q t^2}{2} + O(t^3), \end{aligned} \quad (19)$$

with

$$a_q(\varepsilon) := (\psi_n \circ \varphi)'(u_q) \quad \text{and} \quad s_q(\varepsilon) := \frac{d^2}{du^2}(\psi_n \circ \varphi)(u_q),$$

as  $t \rightarrow 0$ , uniformly in  $\varepsilon$  and  $q$ . Observe that  $s_q \rightarrow 0$  as  $\varepsilon \downarrow 0$  uniformly in  $q$  and so  $\sigma_q^2 - s_q > 0$  for all  $\varepsilon > 0$  small enough and all  $q$ . We fix  $\theta < \theta_1$  so that for all  $t \leq \theta$  the functions  $O$  in (18) and (19) satisfy  $|O(t^3)| < \sigma_q^2 t^2 / 8$ . Let

$$\rho(\varepsilon) := \kappa \mu^{-\frac{\kappa}{2}}.$$

For  $\varepsilon$  small enough we split the integral in (17) as

$$\begin{aligned} &2\pi \mathbb{P}\left(\sum_{j=1}^{q\mu^{\kappa-n}} W_j < \varepsilon a^{\kappa-n}\right) \\ &= \int_{-\rho}^{\rho} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt + \int_{|t| \in [\rho, \theta]} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \\ &\quad + \int_{|t| \geq \theta} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt - \int_{-\infty}^{\infty} \frac{1}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt. \end{aligned} \quad (20)$$

The third and fourth integrals on the right hand side of (20) are negligible by Lemmas 10 and 9 below, respectively. This is due to the fact that in the desired formula (16)  $q\psi_n(\varphi(u_q)) \rightarrow 0$  uniformly in  $q$ , and  $y u_q$  is positive and uniformly bounded away from zero. We now show that the second integral is also negligible, and that the first one has the required asymptotics.

By definition of  $\psi_n$ , we have

$$f_n(z) = p_\mu^{-\frac{1}{\mu-1}} \exp\{\mu^n (b(z) - \psi_n(z))\}$$

and so

$$f_n^{q\mu^{\kappa-n}}(z) = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp\{q\mu^\kappa (b(z) - \psi_n(z))\}. \quad (21)$$

To estimate the second integral on the right hand side of (20) we use (18) and (19) with the uniform error bounds and get

$$\begin{aligned} & \mathcal{R}e[qb(\varphi(u_q - it)) - q\psi_n(\varphi(u_q - it)) + y(u_q - it)] \\ & \leq qb(\varphi(u_q)) - q\psi_n(\varphi(u_q)) + yu_q - \frac{q\sigma_q^2 t^2}{4} \leq qb(\varphi(u_q)) + yu_q - \frac{\sigma_q^2 t^2}{4}, \end{aligned}$$

for all  $|t| \leq \theta$ , using that  $\psi_n$  is positive. Hence

$$\begin{aligned} & \left| \int_{|t| \in [\rho, \theta]} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \right| \\ & = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \left| \int_{|t| \in [\rho, \theta]} \frac{1}{u_q - it} \exp \left\{ \mu^\kappa (qb(\varphi(u_q - it)) - q\psi_n(\varphi(u_q - it)) + y(u_q - it)) \right\} dt \right| \\ & \leq \frac{2\theta}{u_q} p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp \left\{ \mu^\kappa \left( qb(\varphi(u_q)) + yu_q - \frac{\sigma_q^2 \rho^2}{4} \right) \right\} \\ & = o(1) p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \mu^{-\frac{\kappa}{2}} \exp \left\{ \mu^\kappa (qb(\varphi(u_q)) + yu_q) \right\} \end{aligned}$$

uniformly in  $q$  since

$$\exp \left\{ -\frac{\mu^\kappa \sigma_q^2 \rho^2}{4} \right\} = \exp \left\{ -\frac{\kappa^2 \sigma_q^2}{4} \right\} = o(1) \mu^{-\frac{\kappa}{2}}.$$

Now consider the first integral on the r.h.s. of (20), which is the only one contributing to the asymptotics. Using (18), (19), (21), and dropping the  $O$  terms since  $\mu^\kappa \rho^3 \rightarrow 0$  we get

$$\begin{aligned} & \int_{-\rho}^{\rho} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \\ & = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \int_{-\rho}^{\rho} \frac{1}{u_q - it} \exp \left\{ \mu^\kappa (qb(\varphi(u_q - it)) - q\psi_n(\varphi(u_q - it)) + y(u_q - it)) \right\} dt \\ & = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \frac{1 + o(1)}{u_q} \exp \left\{ \mu^\kappa (qb(\varphi(u_q)) - q\psi_n(\varphi(u_q)) + yu_q) \right\} \\ & \quad \times \int_{-\rho}^{\rho} \exp \left\{ itq\mu^\kappa a_q - \frac{q\mu^\kappa (\sigma_q^2 - s_q)t^2}{2} \right\} dt. \end{aligned}$$

Using the substitution  $\tau = t\mu^{\frac{\kappa}{2}} \sqrt{(\sigma_q^2 - s_q)q}$ , we obtain

$$\begin{aligned} & \int_{-\rho}^{\rho} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \\ & = p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \frac{\sqrt{2\pi}}{\sigma_q u_q \sqrt{q}} \mu^{-\frac{\kappa}{2}} \exp \left\{ \mu^\kappa (qb(\varphi(u_q)) - q\psi_n(\varphi(u_q)) + yu_q) \right\} I_q, \end{aligned}$$

where

$$I_q(\varepsilon) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{-\kappa\sqrt{(\sigma_q^2 - s_q)q}}^{\kappa\sqrt{(\sigma_q^2 - s_q)q}} \exp \left\{ i\tau\mu^{\frac{\kappa}{2}} a_q \sqrt{\frac{q}{\sigma_q^2 - s_q}} - \frac{\tau^2}{2} \right\} d\tau.$$

It is easy to see that the absolute value of the integral on the right hand side is bounded by  $\sqrt{2\pi}$ . Since it is clearly nonnegative, we get the uniform bound  $I_q \leq 1 + o(1)$ . If

$\mu^\kappa \psi_n(\varphi(u_1)) = O(1)$  then  $I_1 \rightarrow 1$  by Lemma 11 with

$$\begin{aligned}\rho_1(\varepsilon) &:= \kappa \sqrt{\sigma_1^2 - s_1} \rightarrow \infty \\ \rho_2(\varepsilon) &:= \mu^{\frac{\kappa}{2}} a_1 \sqrt{\frac{1}{\sigma_1^2 - s_1}} = O(1) \mu^{\frac{\kappa}{2} + n} \psi_n(\varphi(u_1)) \rightarrow 0,\end{aligned}$$

where the last line follows from Lemma 6.  $\square$

**Lemma 9.** *There is  $c > 0$  such that*

$$\left| \int_{-\infty}^{\infty} \frac{1}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \right| \leq c p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp\{\mu^\kappa qb(\varphi(u_q))\},$$

for any  $q \in [1, 2]$  such that  $q\mu^{\kappa-n} \in \mathbb{N}$  and any  $\varepsilon > 0$ .

*Proof.* Observe that  $f_n^{q\mu^{\kappa-n}}(z)/z$  is a series with non-negative coefficients and so an increasing function on  $(0, 1)$ . Since  $|\varphi(u_q - it)| \leq \varphi(u_q) \leq \varphi(u_*)$  we have

$$\begin{aligned}\left| \int_{-\infty}^{\infty} \frac{1}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \right| &\leq \frac{1}{u_q} \int_{-\infty}^{\infty} f_n^{q\mu^{\kappa-n}}(|\varphi(u_q - it)|) dt \\ &= \frac{1}{u_q} \int_{-\infty}^{\infty} \frac{f_n^{q\mu^{\kappa-n}}(|\varphi(u_q - it)|)}{|\varphi(u_q - it)|} |\varphi(u_q - it)| dt \\ &\leq \frac{f_n^{q\mu^{\kappa-n}}(\varphi(u_q))}{u_* \varphi(u_*)} \int_{-\infty}^{\infty} |\varphi(u_q - it)| dt.\end{aligned}$$

The integral is uniformly bounded by Lemma 16 in [10]. Lemma 13 from the same paper implies the estimate

$$f_n^{q\mu^{\kappa-n}}(\varphi(u_q)) < p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp\{\mu^\kappa qb(\varphi(u_q))\},$$

which completes the proof.  $\square$

**Lemma 10.** *For any  $\theta > 0$  there are  $\delta > 0$  and  $c > 0$  such that*

$$\left| \int_{|t| \geq \theta} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \right| \leq c p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp\{\mu^\kappa (qb(\varphi(u_q)) + yu_q - \delta)\}$$

for any  $q \in [1, 2]$  such that  $q\mu^{\kappa-n} \in \mathbb{N}$  and any  $\varepsilon$ .

*Proof.* Following the proof of Lemma 16 in [10], we use the fact that, for each  $w \in [u_*, u^*]$ ,  $t \mapsto \varphi(w - it)/\varphi(w)$  is the characteristic function of some absolutely continuous law (Cramér transform), the continuity of the mapping  $(w, t) \mapsto \varphi(w - it)/\varphi(w)$ , and the compactness of  $[u_*, u^*]$  to conclude that there is a constant  $\eta$  such that

$$|\varphi(u_q - it)| < (1 - \eta)\varphi(u_q) \quad \text{for all } |t| > \theta.$$

Using the monotonicity of  $z \mapsto f_n^{q\mu^{\kappa-n}}(z)/z$  on  $(0, 1)$ , we get

$$\begin{aligned}|f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it))| &\leq f_n^{q\mu^{\kappa-n}}(|\varphi(u_q - it)|) \\ &= \frac{f_n^{q\mu^{\kappa-n}}(|\varphi(u_q - it)|)}{|\varphi(u_q - it)|} |\varphi(u_q - it)| \leq \frac{f_n^{q\mu^{\kappa-n}}((1 - \eta)\varphi(u_q))}{(1 - \eta)\varphi(u_q)} |\varphi(u_q - it)|.\end{aligned}$$

Using Lemmas 13 and 16 from [10] we obtain, for some  $c > 0$ ,

$$\begin{aligned} \left| \int_{|t| \geq \theta} \frac{e^{y\mu^\kappa(u_q - it)}}{u_q - it} f_n^{q\mu^{\kappa-n}}(\varphi(u_q - it)) dt \right| &\leq e^{u_q y \mu^\kappa} \frac{f_n^{q\mu^{\kappa-n}}((1-\eta)\varphi(u_q))}{(1-\eta)u_*\varphi(u_*)} \int_{-\infty}^{\infty} |\varphi(u_q - it)| dt \\ &\leq c p_\mu^{-\frac{q\mu^{\kappa-n}}{\mu-1}} \exp\{\mu^\kappa(qb((1-\eta)\varphi(u_q)) + u_q y)\}. \end{aligned}$$

By Lemma 14 in [10] we have  $b'(s) \geq 1/s > 1$  on  $(0, 1)$ . Hence  $b(\varphi(u_q)) - b((1-\eta)\varphi(u_q)) \geq \eta\varphi(u_q) \geq \eta\varphi(u_*)$ . Picking  $\delta = \eta\varphi(u_2)$  we obtain the desired estimate since  $\varphi(u_q) \geq \varphi(u_2)$ .  $\square$

**Lemma 11.** *If  $\rho_1 \rightarrow \infty$  and  $\rho_2 \rightarrow 0$  then*

$$\int_{-\rho_1}^{\rho_1} \exp\left\{i\tau\rho_2 - \frac{\tau^2}{2}\right\} d\tau = \sqrt{2\pi} + o(1).$$

*Proof.* We have

$$\int_{-\rho_1}^{\rho_1} \exp\left\{i\tau\rho_2 - \frac{\tau^2}{2}\right\} d\tau = e^{-\rho_2^2/2} \int_{-\rho_1}^{\rho_1} \exp\left\{-\frac{(\tau - i\rho_2)^2}{2}\right\} d\tau.$$

Denote by  $\Gamma^1(\varepsilon)$  the straight path in  $\mathbb{C}$  going from  $-\rho_1 - i\rho_2$  to  $-\rho_1$  and by  $\Gamma^2(\varepsilon)$  the straight path in  $\mathbb{C}$  going from  $\rho_1$  to  $\rho_1 - i\rho_2$ . Since  $z \mapsto \exp\{-z^2/2\}$  is an entire function we have

$$\int_{-\rho_1}^{\rho_1} \exp\left\{-\frac{(\tau - i\rho_2)^2}{2}\right\} d\tau = \int_{-\rho_1}^{\rho_1} e^{-\tau^2/2} d\tau + \int_{\Gamma^1 \cup \Gamma^2} e^{-z^2/2} dz.$$

Obviously, the first integral converges to  $\sqrt{2\pi}$ . The second integral tends to zero since the length of  $\Gamma^1 \cup \Gamma^2$  goes to zero and  $|e^{-z^2/2}| \leq e^{-(\rho_1^2 - \rho_2^2)/2} \rightarrow 0$  on  $\Gamma^1 \cup \Gamma^2$ .  $\square$

## 6. TIME OF THE FIRST NON-MINIMAL BRANCHING

In this section we prove Theorems 2 and 4. The key idea, just as in the case  $\mu = 1$ , is to combine a decomposition of the population according to their ancestry in a suitably chosen generation with the tail estimate for sums of independent copies of  $W$ .

**Lemma 12.** *Fix  $d \in \{-1, 0, 1\}$ .*

(a) *If  $\varepsilon_j \downarrow 0$  such that  $\mu^{\kappa(\varepsilon_j)}\psi_{n(\varepsilon_j)}(\varphi(u_1(\varepsilon_j))) \rightarrow 0$ , then*

$$\mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 1.$$

(b) *If  $\varepsilon_j \downarrow 0$  such that  $\mu^{\kappa(\varepsilon_j)}\psi_{n(\varepsilon_j)}(\varphi(u_1(\varepsilon_j))) \asymp 1$ , then*

$$0 \prec \mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \prec 1.$$

(c) *If  $\varepsilon_j \downarrow 0$  such that  $\mu^{\kappa(\varepsilon_j)}\psi_{n(\varepsilon_j)}(\varphi(u_1(\varepsilon_j))) \rightarrow \infty$ , then*

$$\mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0.$$

(Recall that  $0 \prec q_j \prec 1$  means that the sequence  $q_j$  is uniformly bounded away from 0 and 1).

*Proof.* Decomposing the tree according to ancestry in generation  $\kappa - n$ , we get

$$\begin{aligned} & \mathbb{P}(K > \kappa(\varepsilon) - n(\varepsilon), W < \varepsilon) \\ &= \mathbb{P}(Z_{\kappa-n} = \mu^{\kappa-n}, W < \varepsilon) = \mathbb{P}\left(Z_{\kappa-n} = \mu^{\kappa-n}, \sum_{i=1}^{\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) \\ &= \mathbb{P}(Z_{\kappa-n} = \mu^{\kappa-n}) \mathbb{P}\left(\sum_{i=1}^{\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) = p \mu^{\frac{\mu^{\kappa-n}-1}{\mu-1}} \mathbb{P}\left(\sum_{i=1}^{\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right). \end{aligned}$$

Hence, combining Proposition 8 with  $q = 1$  and Lemma 7, we obtain

$$\begin{aligned} \mathbb{P}(K > \kappa(\varepsilon) - n(\varepsilon) \mid W < \varepsilon) &= p \mu^{\frac{\mu^{\kappa-n}-1}{\mu-1}} \mathbb{P}\left(\sum_{i=1}^{\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) \mathbb{P}(W < \varepsilon)^{-1} \\ &= \exp\left\{-\mu^{\kappa(\varepsilon)} \psi_n(\varepsilon)(\varphi(u_1(\varepsilon))) + o(1)\right\} I_1(\varepsilon). \end{aligned}$$

In case (a) and (b) we have  $I_1(\varepsilon_j) \rightarrow 1$  and the result follows. In case (c) we use that  $I_1(\varepsilon_j)$  is bounded and therefore  $\mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0$ .  $\square$

It remains to analyse the expression  $\mu^\kappa \psi_n(\varphi(u_1))$  for different sequences  $\varepsilon_n \downarrow 0$ . We prepare this by collecting three auxiliary facts.

**Lemma 13.** *As  $\varepsilon \downarrow 0$  we have*

$$\begin{aligned} \text{(a)} \quad & \mu^{\kappa-n} \asymp \frac{\varepsilon^{-\alpha}}{\log(1/\varepsilon)}, \\ \text{(b)} \quad & \exp\left\{(\lambda - \mu) \mu^n b(\varphi(u_1))\right\} = \varepsilon^{\alpha \mu^{-\{\gamma\}-d}}, \\ \text{(c)} \quad & \mu^\kappa \psi_n(\varphi(u_1)) \asymp \frac{\varepsilon^{\alpha(\mu^{-\{\gamma\}-d-1})}}{\log(1/\varepsilon)}. \end{aligned}$$

(Recall that  $\{\gamma\} = \lceil \gamma \rceil - \gamma$ .)

*Proof.* Observe that it follows from  $a = \mu^{1/\beta}$  and the definition of  $y$  that  $\mu^\kappa = (y/\varepsilon)^\alpha$ . By definition of  $\gamma$  and  $n$  we have  $\mu^{n-\kappa} \asymp \mu^{-\gamma} = \varepsilon^\alpha \log(1/\varepsilon) \mu^{-H}$ , which implies (a). By the definition of  $H$ , see (10), we have  $\mu^H = -b(\varphi(u_1)) y^\alpha \alpha^{-1} (\lambda - \mu)$ . Combining these facts we obtain

$$\begin{aligned} (\lambda - \mu) \mu^n b(\varphi(u_1)) &= (\lambda - \mu) \mu^{\kappa-\gamma-\{\gamma\}-d} b(\varphi(u_1)) \\ &= (\lambda - \mu) \log(1/\varepsilon) y^\alpha \mu^{-H-\{\gamma\}-d} b(\varphi(u_1)) = \alpha \mu^{-\{\gamma\}-d} \log \varepsilon, \end{aligned} \quad (22)$$

which proves (b). By Lemma 6, part (a) and (22) we have

$$\mu^\kappa \psi_n(\varphi(u_1)) \asymp \mu^{\kappa-n-1} \exp\left\{(\lambda - \mu) \mu^n b(\varphi(u_1))\right\} \asymp \frac{\varepsilon^{\alpha(\mu^{-\{\gamma\}-d-1})}}{\log(1/\varepsilon)},$$

proving (c).  $\square$

*Proof of Theorem 2.* Let  $d = -1$ , so that  $\kappa - n = \lceil \gamma \rceil - 1$ . By Lemma 13 (c),  $\mu^\kappa \psi_n(\varphi(u_1)) \rightarrow 0$  since  $\mu^{-\{\gamma\}+1} - 1 \geq 0$ . Hence Lemma 12 implies

$$\mathbb{P}(K > \lceil \gamma \rceil - 1 \mid W < \varepsilon) = \mathbb{P}(K > \kappa - n \mid W < \varepsilon) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0. \quad (23)$$

Now let  $d = 1$ , so that  $\kappa - n = \lceil \gamma \rceil + 1$ . Again, by Lemma 13 (c),  $\mu^\kappa \psi_n(\varphi(u_1)) \rightarrow \infty$  as now  $\mu^{-\lceil \gamma \rceil - 1} - 1 < 0$ . Hence Lemma 12 implies

$$\mathbb{P}(K > \lceil \gamma \rceil + 1 \mid W < \varepsilon) = \mathbb{P}(K > \kappa - n \mid W < \varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (24)$$

The statement now follows from (23) and (24).  $\square$

*Proof of Theorem 4.* Let  $d = 0$ . Then  $\kappa - n = \lceil \gamma \rceil$ . By Lemma 13 (c) we have

$$\mu^\kappa \psi_n(\varphi(u_1)) \asymp \omega.$$

In case (a) of Theorem 4, we have  $\omega(\varepsilon_j) \rightarrow \infty$  and so  $\mu^\kappa \psi_n(\varphi(u_1)) \rightarrow \infty$  by Lemma 13. Hence Lemma 12 implies  $\mathbb{P}(K > \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) = \mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0$ . Together with Theorem 2 we get  $\mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) \rightarrow 1$ .

In case (b) we have  $\omega(\varepsilon_j) \asymp 1$  and so  $\mu^\kappa \psi_n(\varphi(u_1)) \asymp 1$  by Lemma 13. Hence Lemma 12 implies that  $\mathbb{P}(K > \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) = \mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j)$  is asymptotically equivalent to  $\exp\{-\mu^\kappa \psi_n(\varphi(u_1))\}$ . Together with Theorem 2 we infer that  $0 \prec \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) \prec 1$  and  $0 \prec \mathbb{P}(K = \lceil \gamma(\varepsilon_j) \rceil + 1 \mid W < \varepsilon_j) \prec 1$ , as required.

In case (c)  $\omega(\varepsilon_j) \rightarrow 0$  and so  $\mu^\kappa \psi_n(\varphi(u_1)) \rightarrow 0$  by Lemma 13. Hence Lemma 12 implies  $\mathbb{P}(K > \lceil \gamma(\varepsilon_j) \rceil \mid W < \varepsilon_j) = \mathbb{P}(K > \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 1$ . Together with Theorem 2 this implies the statement.  $\square$

## 7. EXTRA OFFSPRING IN THE CRITICAL GENERATION

In this section we prove Theorem 5. Denote  $\mathcal{J} := \{j \geq \lambda : p_j \neq 0\}$  and

$$\mathcal{M} := \{(m_j)_{j \in \mathcal{J}} : m_j \in \mathbb{N} \cup \{0\} \text{ for all } j \in \mathcal{J}\}.$$

For each  $m \in \mathcal{M}$ , denote

$$|m| := \sum_{j \in \mathcal{J}} m_j \quad \text{and} \quad \langle m \rangle := \sum_{j \in \mathcal{J}} (j - \mu) m_j \in \mathbb{N} \cup \{0, \infty\}.$$

For each  $j \in \mathcal{J}$ , denote by  $M_j$  the number of individuals in generation  $K-1$  having precisely  $j$  children and let  $M := (M_j)_{j \in \mathcal{J}}$ . The strategy of the proof is as follows. We first show that  $Z_K = \mu^K + \langle M \rangle$ , see (28). We then prove that, conditioned on the event  $W < \varepsilon$ , the random variable  $\langle M \rangle$  is, with high probability, in a certain interval, see (29). Not surprisingly, in order to show (29), we have to give the asymptotic behaviour of  $\mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon)$ , see (36), resulting in (45), which has to be optimized over  $m$ .

For each  $t > 0$ , denote

$$\mathcal{M}_t := \{m \in \mathcal{M} : \langle m \rangle < t\}.$$

**Lemma 14.** *The cardinality of  $\mathcal{M}_t$  satisfies  $|\mathcal{M}_t| = e^{o(t)}$  as  $t \rightarrow \infty$ .*

*Proof.* For each  $n \in \mathbb{N}$  and  $t > 0$ , denote

$$S_{n,t} = \{m \in (\mathbb{N} \cup \{0\})^n : \sum_{i=1}^n m_i < t\}.$$

Let  $Q_n = [0, 1]^n$  be the unit  $n$ -dimensional cube based in the origin. Then

$$|S_{n,t}| = \text{vol} \left\{ \bigcup_{m \in S_{n,t}} (m + Q) \right\} \leq \text{vol} \left\{ x \in [0, \infty)^n : \sum_{i=1}^n x_i < t + n \right\} = \frac{(t+n)^n}{n!}. \quad (25)$$

On the other hand,

$$|S_{n,t}| = \sum_{0 \leq j < t} \left| \left\{ m \in (\mathbb{N} \cup \{0\})^n : \sum_{i=1}^n m_i = j \right\} \right| \leq \sum_{0 \leq j < t} n^j \leq \int_0^t n^x dx \leq n^{t+1}. \quad (26)$$

The former estimate is useful for large  $t$ , the latter for large  $n$ .

Let  $r: (0, \infty) \rightarrow \mathbb{N}$  be such that  $r_t = o(t/\log t)$  and  $\log t = o(r_t)$  as  $t \rightarrow \infty$ . With the convention  $m_i = 0$  if  $i \notin \mathcal{J}$  we get, for large  $t$ ,

$$\mathcal{M}_t \subset \left\{ m \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : (\lambda - \mu) \sum_{i=1}^{r_t} m_i < t, (r_t - \mu) \sum_{i=r_t+1}^{\lfloor t+\mu \rfloor} m_i < t, m_i = 0 \text{ for all } i > t + \mu \right\}.$$

Using (25) and (26) we get

$$|\mathcal{M}_t| \leq |S_{r_t, \frac{t}{\lambda-\mu}}| |S_{\lfloor t+\mu \rfloor - r_t, \frac{t}{r_t-\mu}}| \leq |S_{r_t, t}| |S_{\lfloor t+\mu \rfloor - r_t, \frac{t}{r_t-\mu}}| \leq \frac{(t+r_t)^{r_t}}{r_t!} (t+\mu-r_t)^{\frac{t}{r_t-\mu}+1}.$$

This leads to

$$\begin{aligned} |\mathcal{M}_t| &= \exp \left\{ r_t \log(t+r_t) - r_t \log r_t + r_t + \frac{t+\mu}{r_t-\mu} \log(t-r_t) + o(t) \right\} \\ &= \exp \left\{ r_t \log t - r_t \log r_t + r_t + \frac{t+\mu}{r_t-\mu} \log t + o(t) \right\} = e^{o(t)}. \end{aligned}$$

□

**Lemma 15.** For  $q \in [1, 2]$  and  $\varepsilon > 0$  let  $h(q) := q(b \circ \varphi)(u_q) + yu_q$ . Then

$$h(q) \leq h(1) + (b \circ \varphi)(u_1)(q-1).$$

*Proof.* Since  $b \circ \varphi$  is analytic we get, using (7),

$$\frac{\partial h}{\partial q}(q) = (b \circ \varphi)(u_q) \quad \text{implying} \quad \frac{\partial h}{\partial q}(1) = (b \circ \varphi)(u_1)$$

and

$$\frac{\partial^2 h}{\partial q^2}(q) = (b \circ \varphi)'(u_q) \frac{\partial u_q}{\partial q}.$$

Since  $(b \circ \varphi)'$  is analytic and increasing from  $-\infty$  to 0 on  $(0, \infty)$ , equation (7) implies that  $u_q$  is increasing in  $q$  and so the derivative  $\partial u_q / \partial q$  is nonnegative. Since  $b \circ \varphi$  is negative we have  $\partial^2 h / \partial q^2(q) \leq 0$  for all  $q$  and  $\varepsilon$ . Now the statement of the lemma follows from the Taylor expansion of  $h$  at the point  $q = 1$ . □

Denote

$$N(\varepsilon) := \mu^{\kappa(\varepsilon) - n(\varepsilon) - 1} \quad (27)$$

and let

$$\Phi_j(\varepsilon) := p_j p_\mu^{-\frac{j-1}{\mu-1}} N \exp \left\{ (j-\mu) \mu^n b(\varphi(u_1)) \right\}, \quad \text{for } j \in \mathcal{J}.$$

Note that all  $\Phi_j$ ,  $j \neq \lambda$ , are negligible with respect to  $\Phi_\lambda$  to the extent that, for any  $c \in \mathbb{R}$ ,

$$\sum_{j \in \mathcal{J} \setminus \{\lambda\}} \Phi_j e^{cj} = o(\Phi_\lambda).$$

*Proof of Theorem 5.* Recall that for each  $j \in \mathcal{J}$ ,  $M_j$  is the number of individuals in generation  $K - 1$  having precisely  $j$  children. Write  $M := (M_j)_{j \in \mathcal{J}}$ . Then

$$Z_K = \mu(\mu^{K-1} - |M|) + \sum_{j \in \mathcal{J}} jM_j = \mu^K + \langle M \rangle. \quad (28)$$

Observe that, by Lemma 13 (b), we have  $(\lambda - \mu)\Phi_\lambda = C \mu^{\kappa-n} \varepsilon^{\alpha\mu^{\gamma-\kappa+n}}$ , where

$$C = \left(\frac{\lambda}{\mu} - 1\right) p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}}.$$

Let  $\delta > 0$  be small enough. By Theorems 2 and 4 it suffices to show that

$$\mathbb{P}(\langle M \rangle \notin ((\lambda - \mu - \delta)\Phi_\lambda(\varepsilon_j), (\lambda - \mu + \delta)\Phi_\lambda(\varepsilon_j)), K = \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0, \quad (29)$$

for  $d = 0$  in the case  $\omega(\varepsilon_j) \rightarrow \infty$  and for  $d = 1$  in the case  $\omega(\varepsilon_j) \rightarrow 0$ . By Lemma 13,

$$\Phi_\lambda(\varepsilon_j) \asymp \frac{\varepsilon_j^{\alpha(\mu^{-\{\gamma\}}-1)}}{\log(1/\varepsilon_j)} = \omega(\varepsilon_j) \rightarrow \infty \quad \text{for } d = 0 \text{ in the case } \omega(\varepsilon_j) \rightarrow \infty, \quad (30)$$

$$\Phi_\lambda(\varepsilon_j) \asymp \frac{\varepsilon_j^{\alpha(\mu^{-\{\gamma\}-1}-1)}}{\log(1/\varepsilon_j)} \rightarrow \infty \quad \text{for } d = 1 \text{ in the case } \omega(\varepsilon_j) \rightarrow 0. \quad (31)$$

Hence in both cases  $\Phi_\lambda(\varepsilon_j) \rightarrow \infty$ .

We prove (29) by showing that

$$\mathbb{P}(\langle M \rangle \leq (\lambda - \mu - \delta)\Phi_\lambda(\varepsilon_j), K = \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0, \quad (32)$$

$$\mathbb{P}((\lambda - \mu + \delta)\Phi_\lambda(\varepsilon_j) \leq \langle M \rangle < 3e^\lambda \Phi_\lambda(\varepsilon_j), K = \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0, \quad (33)$$

$$\mathbb{P}(3e^\lambda \Phi_\lambda(\varepsilon_j) \leq \langle M \rangle < N(\varepsilon_j)/2, K = \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0, \quad (34)$$

$$\mathbb{P}(\langle M \rangle \geq N(\varepsilon_j)/2, K = \kappa(\varepsilon_j) - n(\varepsilon_j) \mid W < \varepsilon_j) \rightarrow 0. \quad (35)$$

The rest of the proof is split into five steps. In Step 1, we find an asymptotic formula for the conditional probabilities  $\mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon)$  for  $m \in \mathcal{M}$ . Then we prove (32), (33), (34), and (35) in the next four steps.

*Step 1. An asymptotic formula.*

Let  $m \in \mathcal{M}$  be such that  $m_j \neq 0$  for some  $j \in \mathcal{J}$  and  $|m| \leq N$ . In particular, this means that only finitely many of the  $m_j$  are non-zero. Denote

$$q(m) := 1 + \langle m \rangle \mu^{n-\kappa} \quad \text{and} \quad \bar{q}(m) := 2 \wedge q(m),$$

where  $\wedge$  stands for the minimum.

For each  $j \in \mathcal{J}$ , denote by  $\widetilde{M}_j(\varepsilon)$  the number of individuals in generation  $\kappa - n - 1$  having precisely  $j$  children. Let  $\widetilde{M}(\varepsilon) := (\widetilde{M}_j(\varepsilon))_{j \in \mathcal{J}}$ . Again we drop the dependence on  $\varepsilon$  from this notation whenever convenient. Observe that  $K = \kappa - n$  and  $M = m$  imply  $Z_{\kappa-n} = q(m)\mu^{\kappa-n}$  and so we have

$$\begin{aligned} \mathbb{P}(M = m, K = \kappa - n, W < \varepsilon) &= \mathbb{P}\left(Z_{\kappa-n-1} = N, \widetilde{M} = m, \sum_{i=1}^{q(m)\mu^{\kappa-n}} W_i < \varepsilon d^{\kappa-n}\right) \\ &= \mathbb{P}(Z_{\kappa-n-1} = N) \mathbb{P}(\widetilde{M} = m \mid Z_{\kappa-n-1} = N) \mathbb{P}\left(\sum_{i=1}^{\bar{q}(m)\mu^{\kappa-n}} W_i < \varepsilon d^{\kappa-n}\right). \end{aligned}$$

This yields

$$\begin{aligned} & \mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon) \\ &= \mathbb{P}(Z_{\kappa-n-1} = N) \mathbb{P}(\widetilde{M} = m \mid Z_{\kappa-n-1} = N) \mathbb{P}\left(\sum_{i=1}^{\bar{q}(m)\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) \mathbb{P}(W < \varepsilon)^{-1}. \end{aligned} \quad (36)$$

For the *first* term in (36), we have

$$\mathbb{P}(Z_{\kappa-n-1} = N) = p_\mu^{1+\mu+\dots+\mu^{\kappa-n-2}} = p_\mu^{\frac{N-1}{\mu-1}}. \quad (37)$$

We can compute the *second* term in (36) as

$$P(\widetilde{M} = m \mid Z_{\kappa-n-1} = N) = p_\mu^N \frac{N!}{(N - |m|)!} \prod_{j \in \mathcal{J}} \frac{1}{m_j!} \left(\frac{p_j}{p_\mu}\right)^{m_j}.$$

Observe that

$$\frac{\sqrt{N}}{\sqrt{N - |m|}} \prod_{\substack{j \in \mathcal{J} \\ m_j \neq 0}} \frac{1}{\sqrt{m_j}} \leq \begin{cases} \sqrt{2} & \text{if } |m| < N/2, \\ \sqrt{N} & \text{if } |m| < N. \end{cases}$$

Combining this with Stirling's formula we obtain, uniformly for  $|m| < N/2$ ,

$$\begin{aligned} & P(\widetilde{M} = m \mid Z_{\kappa-n-1} = N) \\ &= O(1) p_\mu^N \exp \left\{ N \log N - (N - |m|) \log(N - |m|) - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \frac{p_j}{p_\mu} \right\} \\ &= O(1) p_\mu^N \exp \left\{ |m| \log N - N \left(1 - \frac{|m|}{N}\right) \log \left(1 - \frac{|m|}{N}\right) - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \frac{p_j}{p_\mu} \right\} \\ &= O(1) p_\mu^N \exp \left\{ |m| \log N + |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \frac{p_j}{p_\mu} \right\}, \end{aligned} \quad (38)$$

since  $(1-x) \log(1-x) \geq -x$  for all  $0 \leq x \leq 1$  (we use the convention  $0 \log 0 = 0$ ). Similarly,

$$\begin{aligned} & P(\widetilde{M} = m \mid Z_{\kappa-n-1} = N) \\ &= O(1) \sqrt{N} p_\mu^N \exp \left\{ |m| \log N + |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \frac{p_j}{p_\mu} \right\} \end{aligned} \quad (39)$$

uniformly for all  $|m| \leq N$ .

To compute the *third* term in (36) we use Proposition 8 and get

$$\mathbb{P}\left(\sum_{i=1}^{\bar{q}(m)\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) = O(1) p_\mu^{-\frac{\bar{q}(m)\mu N}{\mu-1}} \mu^{-\frac{\kappa}{2}} \exp \left\{ \mu^\kappa h(\bar{q}(m)) - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\}.$$

Applying Lemma 15 yields

$$\begin{aligned} & \mathbb{P}\left(\sum_{i=1}^{\bar{q}(m)\mu^{\kappa-n}} W_i < \varepsilon a^{\kappa-n}\right) \\ &= O(1) p_\mu^{-\frac{\bar{q}(m)\mu N}{\mu-1}} \mu^{-\frac{\kappa}{2}} \exp \left\{ \mu^\kappa h(1) + \langle m \rangle \mu^n (b \circ \varphi)(u_1) - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\}. \end{aligned} \quad (40)$$

The *fourth* term in (36) is given by Lemma 7,

$$\mathbb{P}(W < \varepsilon)^{-1} = O(1) p_\mu^{\frac{1}{\mu-1}} \mu^{\frac{\kappa}{2}} \exp \left\{ -\mu^\kappa h(1) \right\}. \quad (41)$$

Combining (36), (37), (38), (40), and (41) we obtain

$$\begin{aligned}
& \mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon) \\
&= O(1) p_\mu^{-\frac{\langle m \rangle}{\mu-1}} \exp \left\{ |m| \log N + |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \frac{p_j}{p_\mu} \right. \\
&\quad \left. + \langle m \rangle \mu^n (b \circ \varphi)(u_1) - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\} \\
&= O(1) \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \left( p_j p_\mu^{-\frac{j-1}{\mu-1}} N \exp \{ (j - \mu) \mu^n b(\varphi(u_1)) \} \right) \right. \\
&\quad \left. - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\} \\
&= O(1) \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\}, \tag{42}
\end{aligned}$$

uniformly in  $m$  such that  $|m| < N/2$ .

If the condition  $|m| < N/2$  is not satisfied we need to replace (38) by the rougher estimate (39) in the previous computation. This gives

$$\begin{aligned}
& \mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon) \\
&= O(1) \sqrt{N} \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j - \mu^\kappa \bar{q}(m) \psi_n(\varphi(u_{\bar{q}(m)})) \right\} \\
&= O(1) \sqrt{N} \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j \right\} \tag{43}
\end{aligned}$$

uniformly for all  $|m| \leq N$ , since the last term in the second line of (43) is positive.

*Step 2. Proof of (32).*

Consider all  $m$  such that  $\langle m \rangle \leq (\lambda - \mu - \delta) \Phi_\lambda$ . Observe that in this case, for  $\varepsilon > 0$  small enough,  $q(m) \leq 2$  and  $|m| \leq \langle m \rangle < N/2$ . By Lemma 6 we have

$$\begin{aligned}
& \mu^\kappa q(m) \psi_n(\varphi(u_{q(m)})) \\
&= p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}} N \exp \{ (\lambda - \mu) \mu^n b(\varphi(u_{q(m)})) \} (1 + o(1)) \\
&= p_\lambda p_\mu^{-\frac{\lambda-1}{\mu-1}} N \exp \{ (\lambda - \mu) \mu^n b(\varphi(u_1)) + \langle m \rangle \mu^{2n-\kappa} O(1) \} (1 + o(1)) \\
&= \Phi_\lambda + o(\Phi_\lambda). \tag{44}
\end{aligned}$$

since  $\mu^{\kappa-2n} \asymp \varepsilon^{-\alpha} (\log(1/\varepsilon))^{-2}$  and so  $\Phi_\lambda \mu^{2n-\kappa} = o(1)$  according to (30) and (31).

Combining (42) and (44) we get, uniformly in  $m$ ,

$$\begin{aligned}
& \mathbb{P}(M = m, K = \kappa - n \mid W < \varepsilon) \\
&= O(1) \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j - \Phi_\lambda + o(\Phi_\lambda) \right\}. \tag{45}
\end{aligned}$$

It is easy to see that the function in the brackets achieves its maximum at  $m$  given by  $m_j = \Phi_j$ . However, this  $m$  does not satisfy the condition  $\langle m \rangle \leq (\lambda - \mu - \delta) \Phi_\lambda$ , and so the maximum over the admissible domain is achieved on the boundary  $\langle m \rangle = (\lambda - \mu - \delta) \Phi_\lambda$ . Using Lagrange multipliers, we obtain that the maximum is attained for  $m_j = \Phi_j e^{c(j-\mu)}$  for

some  $c < 0$  (depending on  $\varepsilon$ ) such that

$$\sum_{j \in \mathcal{J}} (j - \mu) \Phi_j e^{c(j-\mu)} = (\lambda - \mu - \delta) \Phi_\lambda.$$

Since all  $\Phi_j$  with  $j \neq \lambda$  are negligible with respect to  $\Phi_\lambda$  we have that  $(\lambda - \mu) e^{c(\lambda - \mu)} \sim \lambda - \mu - \delta$  and so  $c$  is separated from zero. Substituting the maximiser into (45) and neglecting all  $\Phi_j$  with  $j \neq \lambda$  we get

$$\begin{aligned} \mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) \\ = O(1) \exp \left\{ -\Phi_\lambda (1 - e^{c(\lambda - \mu)} + c(\lambda - \mu) e^{c(\lambda - \mu)}) + o(\Phi_\lambda) \right\}. \end{aligned} \quad (46)$$

Observe that the function  $\rho(x) = 1 - e^x + x e^x$  is decreasing from 1 to 0 on  $(-\infty, 0]$ . Since  $c$  is negative and separated from zero we obtain

$$\mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) = O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \}.$$

with some  $\theta > 0$ , uniformly in  $m$ . Finally, by Lemma 14,

$$\begin{aligned} \mathbb{P}(\langle M \rangle \leq (\lambda - \mu - \delta) \Phi_\lambda, K = \kappa - n | W < \varepsilon) &= O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \} |\mathcal{M}_{(\lambda - \mu - \delta) \Phi_\lambda}| \\ &= O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \} = o(1). \end{aligned}$$

*Step 3. Proof of (33).*

Now consider all  $m$  such that  $(\lambda - \mu + \delta) \Phi_\lambda \leq \langle m \rangle < 3e^\lambda \Phi_\lambda$ . The estimates  $q(m) \leq 2$  and  $|m| < N/2$  as well as the asymptotics (44) and (45) remain true in this case and, similarly to the previous step, the maximum of the function in the brackets in (45) over the region  $\langle m \rangle \geq (\lambda - \mu + \delta) \Phi_\lambda$  is attained on the boundary  $\langle m \rangle = (\lambda - \mu + \delta) \Phi_\lambda$  at  $m$  given by  $m_j = \Phi_j e^{c(j-\mu)}$  for some  $c > 0$ , depending on  $\varepsilon$  but bounded away from zero.

We use (46), which is true in this case as well, and the fact that  $\rho$  is increasing from 0 to  $\infty$  on  $[0, \infty)$  to obtain  $\mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) = O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \}$  with some  $\theta > 0$ , uniformly in  $m$ . Finally, by Lemma 14,

$$\begin{aligned} \mathbb{P}((\lambda - \mu + \delta) \Phi_\lambda \leq \langle M \rangle < 3e^\lambda \Phi_\lambda, K = \kappa - n | W < \varepsilon) \\ = O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \} |\mathcal{M}_{3e^\lambda \Phi_\lambda}| \\ = O(1) \exp \{ -\theta \Phi_\lambda + o(\Phi_\lambda) \} = o(1). \end{aligned}$$

*Step 4. Proof of (34).*

Here we consider all  $m$  satisfying  $3e^\lambda \Phi_\lambda \leq \langle m \rangle < N/2$ . Then again  $|m| < N/2$  and  $q(m) \leq 2$ . Since the last term in (42) is positive we have

$$\begin{aligned} \mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) &= O(1) \exp \left\{ |m| - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j \right\} \\ &= O(1) \prod_{j \in \mathcal{J}} e^{-j m_j} \exp \left\{ \sum_{j \in \mathcal{J}} (j+1) m_j - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j \right\}. \end{aligned} \quad (47)$$

The maximum of the function in the brackets is achieved for  $m_j = \Phi_j e^j$ , which does not satisfy the condition  $\langle m \rangle \geq 3e^\lambda \Phi_\lambda$ . It is easy to see that the maximum over the region  $\langle m \rangle \geq 3e^\lambda \Phi_\lambda$  is achieved on the boundary  $\langle m \rangle = 3e^\lambda \Phi_\lambda$  for  $m$  given by  $m_j = \Phi_j e^{j+c(j-\mu)}$ , and  $c > 0$  is such that

$$\sum_{j \in \mathcal{J}} \Phi_j e^{j+c(j-\mu)} = 3e^\lambda \Phi_\lambda. \quad (48)$$

Substituting the maximiser into (47) we obtain

$$\mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) = O(1) \prod_{j \in \mathcal{J}} e^{-jm_j} \exp \left\{ - \sum_{j \in \mathcal{J}} (c(j - \mu) - 1) \Phi_j e^{j+c(j-\mu)} \right\}.$$

Since  $e^{\lambda+c(\lambda-\mu)} \sim 3e^\lambda$  according to (48) we have  $c(\lambda - \mu) - 1 \sim \log 3 - 1 > 0$  and so

$$\mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) = O(1) \exp \left\{ - \theta \Phi_\lambda + o(\Phi_\lambda) \right\} \prod_{j \in \mathcal{J}} e^{-jm_j}$$

for some  $\theta > 0$ , uniformly in  $m$ . Hence

$$\begin{aligned} & \mathbb{P}(3e^\lambda \Phi_\lambda \leq \langle M \rangle < N/2, K = \kappa - n | W < \varepsilon) \\ &= O(1) \exp \left\{ - \theta \Phi_\lambda + o(\Phi_\lambda) \right\} \sum_{m \in \mathcal{M}_{N/2}} \prod_{j \in \mathcal{J}} e^{-jm_j}, \end{aligned}$$

and the right hand side is  $o(1)$  as

$$\sum_{m \in \mathcal{M}_{N/2}} \prod_{j \in \mathcal{J}} e^{-jm_j} \leq \prod_{j \in \mathcal{J}} \sum_{m_j=0}^{\infty} e^{-jm_j} = \prod_{j \in \mathcal{J}} \frac{1}{1 - e^{-j}} < \infty. \quad (49)$$

*Step 5. Proof of (35).*

Finally consider all  $m$  such that  $\langle m \rangle \geq N/2$ . Using (43) we obtain

$$\begin{aligned} & \mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) \\ &= O(1) \sqrt{N} \prod_{j \in \mathcal{J}} e^{-jm_j} \exp \left\{ \sum_{j \in \mathcal{J}} (j+1)m_j - \sum_{j \in \mathcal{J}} m_j \log m_j + \sum_{j \in \mathcal{J}} m_j \log \Phi_j \right\}. \end{aligned} \quad (50)$$

Similarly to the previous step, the maximum of the function in the brackets over the region  $\langle m \rangle \geq N/2$  is achieved on the boundary  $\langle m \rangle = N/2$  at  $m$  given by  $m_j = \Phi_j e^{j+c(j-\mu)}$ , where  $c > 0$  is such that

$$\sum_{j \in \mathcal{J}} \Phi_j e^{j+c(j-\mu)} = N/2. \quad (51)$$

Substituting the maximiser into (50) we obtain

$$\begin{aligned} & \mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) \\ &= O(1) \sqrt{N} \prod_{j \in \mathcal{J}} e^{-jm_j} \exp \left\{ - \sum_{j \in \mathcal{J}} (c(j - \mu) - 1) \Phi_j e^{j+c(j-\mu)} \right\}. \end{aligned}$$

Now (51) implies that  $c \rightarrow \infty$  and so  $c(j - \mu) - 1 \geq c(\lambda - \mu) - 1 \geq 1$  for all  $j$  eventually. Hence

$$\begin{aligned} \mathbb{P}(M = m, K = \kappa - n | W < \varepsilon) &= O(1) \sqrt{N} \exp \left\{ - \sum_{j \in \mathcal{J}} \Phi_j e^{j+c(j-\mu)} \right\} \prod_{j \in \mathcal{J}} e^{-jm_j} \\ &= O(1) \sqrt{N} e^{-\frac{N}{2}} \prod_{j \in \mathcal{J}} e^{-jm_j} = o(1) \prod_{j \in \mathcal{J}} e^{-jm_j}. \end{aligned}$$

From this we can conclude that

$$\mathbb{P}(\langle M \rangle \geq N/2, K = \kappa - n | W < \varepsilon) = o(1) \sum_{m \in \mathcal{M}} \prod_{j \in \mathcal{J}} e^{-jm_j} = o(1),$$

using again that the sum is finite, similarly to (49).  $\square$

**Acknowledgments:** The first author is supported by EPSRC grants EP/G055068/1 and EP/IO3372X/1.

## REFERENCES

- [1] Bansaye, V., Berestycki, J.; Large deviations for branching processes in random environment. *Markov Proc. Rel. Fields.* 15, 493–524 (2009).
- [2] Benjamini, I., Berestycki, N.; Random paths with bounded local time. *J. Europ. Math. Soc.* 12, 819–854 (2010).
- [3] Benjamini, I., Berestycki, N.; An integral test for the transience of a Brownian path with limited local time. *Ann. Inst. H. Poincaré (B): Probab. Stat.*, 47, 539–558 (2011).
- [4] Bolthausen, E., Deuschel, J.-D., Giacomin, G.; Entropic repulsion and the maximum of the two dimensional harmonic crystal. *Ann. Probab.*, 29, 1670–1692 (2001).
- [5] Csizsár, I.; Sanov property, generalized  $I$ -projection and a conditional limit theorem. *Ann. Probab.* 12, 768–793 (1984).
- [6] Dembo, A., Zeitouni, O.; *Large deviations techniques and applications*. Second edition. Applications of Mathematics 38. Springer-Verlag, New York, 1998.
- [7] Dembo, A., Zeitouni, O.; Refinements of the Gibbs conditioning principle. *Probab. Theory Related Fields* 104, 1–14 (1996).
- [8] Deuschel, J.-D., Stroock, D. W., Zessin, H.; Microcanonical distributions for lattice gases. *Comm. Math. Phys.* 139, 83–101 (1991).
- [9] Dubuc, S. La densité de la loi limite d’un processus en cascade expansif. *Z. Wahrscheinlichkeitsth.* 19, 281–290 (1971).
- [10] Fleischmann, K., Wachtel, V.; On the left tail asymptotics for the limit law of supercritical Galton–Watson processes in the Böttcher case, *Ann. Inst. Henri Poincaré Probab. Stat.* 45, 201–225 (2009).
- [11] McKean, H. P., Vaninsky, K. L.; Brownian motion with restoring drift: the petit and micro-canonical ensembles. *Comm. Math. Phys.* 160, 615–630 (1994).
- [12] Meda, A., Ney, P.; The Gibbs conditioning principle for Markov chains. In: *Perplexing problems in probability*, 385–398, Progr. Probab., 44, Birkhäuser, Boston, 1999.
- [13] Mörters, P., Ortgiese, M.; Small value probabilities via the branching tree heuristic. *Bernoulli* 14, 277–299 (2008).
- [14] Stroock, D. W., Zeitouni, O.; Microcanonical distributions, Gibbs states, and the equivalence of ensembles. In: *Random walks, Brownian motion, and interacting particle systems*, 399–424, Progr. Probab., 28, Birkhäuser, Boston, 1991.

Nathanaël Berestycki: Statistical Laboratory, DPMMS, University of Cambridge. Wilberforce Road, Cambridge CB3 0WB. United Kingdom.

Nina Gantert: Technische Universität München, Fakultät für Mathematik, Boltzmannstraße 3, 85748 Garching bei München. Germany.

Peter Mörters: Department of Mathematical Sciences, University of Bath. Claverton Down, Bath BA2 7AY. United Kingdom.

Nadia Sidorova: Department of Mathematics, University College London. Gower Street, London WC1E 6BT. United Kingdom.