The exact packing measure of Brownian double points

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Abstract: Let $D \subset \mathbb{R}^3$ be the set of double points of a 3-dimensional Brownian motion. We show that, if $\xi = \xi_3(2, 2)$ is the intersection exponent of two packets of two independent Brownian motions, then almost surely, the ϕ-packing measure of D is zero if

$$\int_{0+} r^{-1-\xi} \phi(r)^\xi \, dr < \infty,$$

and infinity otherwise. As an important step in the proof we show up-to-constants estimates for the tail at zero of Brownian intersection local times in dimensions two and three.

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1. Introduction and statement of the results

1.1 A short history of the problem

By an interesting twist of mathematical history, it was *C. Loewner* who communicated the problem of finding the Hausdorff dimension of the range of a Brownian motion to *A.S. Besicovitch*, who in 1951 suggested this problem to his research student *S.J. Taylor*. Taylor not only solved this problem, but went on to investigate dimension problems for a wide range of fractal sets arising from stochastic processes in an impressive body of work spanning half a century. In an influential survey paper [Ta86] Taylor reviews this work and states a large number of open problems in this field, most of which have been solved in the decade following the publication of the paper.

Some problems however withstood his and other people's efforts, for a longer time. Among them is, for example, the *'Mandelbrot problem'* of finding the Hausdorff dimension of the Brownian frontier, the boundary of the unbounded component of the complement of a planar Brownian path. *G.F. Lawler* in [La96b] solved this problem in terms of the (non-)intersection exponents, which describe the asymptotic rate of decay of non-intersection probabilities of independent Brownian paths. Finally, the discovery of the Schramm-Loewner evolution, a class of stochastic processes built using Loewner's equation of 1923, enabled *G.F. Lawler, O. Schramm* and *W. Werner* to calculate these exponents explicitly and fully settle Mandelbrot's conjecture.

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At the time when Taylor's article was written, *J.-F. Le Gall* was studying the exact *Hausdorff*-measure of Brownian double points. Recall that Brownian motion has double points in dimensions $d \leq 3$, but not in dimensions $d \geq 4$. Le Gall showed in [LG86] that $\mathcal{H}^{\phi}(D)$ is σ-finite for the critical gauge function $\phi(r) = r^2[\log(1/r) \log\log\log(1/r)]^2$ in $d = 2$ and $\phi(r) = r[\log\log(1/r)]^2$ in $d = 3$. Soon after that he also identified the exact *packing* measure $\mathcal{P}^{\phi}(D)$ for the double points of a planar Brownian motion, see [LG87, Théorème 5.1]. In this case the measure is always either zero or not σ-finite, and Le Gall showed that it is zero exactly if

$$\int_{0+} \frac{\phi(r)}{[r \log(1/r)]^3} \, dr < \infty \, .$$

But key arguments in his proof are limited to the planar case and in the case of 3-dimensional Brownian motion he could only give estimates for the exact dimension, see [LG87, Théorème 5.3]. The reason for this is, as we shall show below, that this case (and only this case) involves intersection exponents, a concept that was not yet established at the time.

It is our aim in this paper to settle what appears to be the *last open problem* about Brownian motion raised in Taylor's survey paper, see [Ta86, Conjecture C], the packing measure of the set of Brownian double points in $\mathbb{R}^3$. We shall show that, if $\xi = \xi_3(2,2)$ is the intersection exponent of two packets of two independent Brownian motions, then almost surely, the ϕ-packing measure of D is zero if

$$\int_{0+} r^{-1-\xi} \phi(r)^{\xi} \, dr < \infty,$$

and infinity otherwise. In dimension three the intersection exponents have not been calculated explicitly, and there is no evidence of a natural formula for them. Estimates currently available show that $1 < \xi_3(2,2) < 2$, which in particular shows that our result is *not* in line with Taylor's original conjecture.

1.2 Intersection exponents of Brownian motion

To formulate our result we recall here the concept of (non-)intersection exponents.

Suppose $M, N \in \mathbb{N}$ and let $W^{(1)}, \ldots, W^{(M+N)}$ be a family of independent Brownian motions in $\mathbb{R}^d$, $d = 2, 3$, started uniformly on $\partial B(0,1)$. We divide the motions into two packets and, for any vector $\vec{t} = (t^{(1)}, \ldots, t^{(M+N)})$ of nonnegative times, we look at the union of the paths in each packet,

$$\mathfrak{B}^{(1)}(t^{(1)}, \ldots, t^{(M)}) := \bigcup_{i=1}^{M} W^{(i)}([0, t^{(i)}]) \quad \text{and} \quad \mathfrak{B}^{(2)}(t^{(M+1)}, \ldots, t^{(M+N)}) := \bigcup_{i=M+1}^{M+N} W^{(i)}([0, t^{(i)}]).$$

Let $R > 1$ and $t^{(i)} = \tau_R^{(i)}$ be the first hitting time of the sphere $\partial B(0, R)$ by the motion $W^{(i)}$. Then the event that the two packets of Brownian paths fail to intersect has a decreasing probability as $R \uparrow \infty$. Indeed, it is easy, using subadditivity, to show that there exists a constant $\xi_d(M, N)$ such that

$$\mathbb{P}\big\{\mathfrak{B}^{(1)}(\tau_R^{(1)}, \ldots, \tau_R^{(M)}) \cap \mathfrak{B}^{(2)}(\tau_R^{(M+1)}, \ldots, \tau_R^{(M+N)}) = \emptyset\big\} = R^{-\xi_d(M,N)+o(1)}, \text{ as } R \uparrow \infty.$$

The numbers $\xi_d(M, N)$ are called the *intersection exponents*.

Lawler showed in [La96a] that these exponents describe the probability of non-intersection up to constants: There exist constants $0 < c < C < \infty$ such that for all $R \geq 1$,

$$c\, R^{-\xi_d(M,N)} \leq \mathbb{P}\big\{\mathfrak{B}^{(1)}(\tau_R^{(1)}, \ldots, \tau_R^{(M)}) \cap \mathfrak{B}^{(2)}(\tau_R^{(M+1)}, \ldots, \tau_R^{(M+N)}) = \emptyset\big\} \leq C\, R^{-\xi_d(M,N)}. \tag{1.1}$$

In a seminal series of papers [LS01b, LS01c, LS02], Lawler, Schramm and Werner show that

$$\xi_2(M,N) = \frac{\big(\sqrt{24M+1}+\sqrt{24N+1}-2\big)^2-4}{48}. \tag{1.2}$$

As the proof of (1.2) is based on conformal invariance, there is no analogue in $d=3$ and indeed there is no reason to believe that a similarly easy formula holds in this case. The only value of intersection exponents explicitly known in this dimension is

$$\xi_3(1,2) = \xi_3(2,1) = 1\,.$$

Strict concavity of a suitable extension $\lambda \mapsto \xi_3(N,\lambda)$ to nonnegative reals established in [La98], is instrumental in the estimates

$$\tfrac{1}{2} < \xi_3(1,1) < 1, \qquad 1 < \xi_3(2,2) < 2, \tag{1.3}$$

which we shall use in our arguments.

1.3 The packing measure of Brownian double points

We start by introducing the notion of *packing measure*, which is a concept dual to that of Hausdorff measure. It was first introduced by Taylor and Tricot in [TT85]. A *gauge function* is any increasing function $\phi\colon [0,\varepsilon) \to [0,\infty)$ with $\phi(0)=0$.

Suppose $E \subset \mathbb{R}^d$. For every $\delta > 0$, a *δ-packing* of E is a countable collection of *disjoint* open balls

$$B(x_1,r_1), B(x_2,r_2), B(x_3,r_3), \ldots$$

with centres $x_i \in E$ and radii $0 \le r_i \le \delta$. For every gauge function $\phi\colon [0,\varepsilon) \to [0,\infty)$ we introduce the *ϕ-value* of the packing as $\sum_{i=1}^\infty \phi(r_i)$. The *$\phi$-packing number* of E is defined as

$$P^\phi(E) := \lim_{\delta\downarrow 0} P_\delta^\phi(E) \quad \text{ for } P_\delta^\phi(E) := \sup\Big\{ \sum_{i=1}^\infty \phi(r_i) \,:\, (B(x_i,r_i)) \text{ a } \delta\text{-packing of } E\Big\}.$$

Note that the packing number is defined in essentially the same way as the Hausdorff measure with efficient (small) coverings replaced by efficient (large) packings. A difference is that the packing numbers do *not* define a measure on the Borel sets of $\mathbb{R}^d$. However a small modification gives the packing measure, defined by

$$\mathcal{P}^\phi(E) := \inf\Big\{ \sum_{i=1}^\infty P^\phi(E_i) \,:\, E = \bigcup_{i=1}^\infty E_i \Big\}.$$

We are now able to formulate the first main result of this paper.

Theorem 1.1. *Let D be the set of double points of a Brownian motion in $\mathbb{R}^3$ and ϕ any gauge function.*

(i) $\mathcal{P}^\phi(D) = 0$ *almost surely if and only if* $\int_{0+} r^{-1-\xi}\,\phi(r)^\xi\,dr < \infty$,

(ii) $\mathcal{P}^\phi(D) = \infty$ *almost surely if and only if* $\int_{0+} r^{-1-\xi}\,\phi(r)^\xi\,dr = \infty$,

where $\xi = \xi_3(2,2)$ is the intersection exponent.

Remark 1 Our proof also shows that the same integral test applies if D is the intersection of two independent Brownian paths in $\mathbb{R}^3$ with the same starting point.

1.4 Lower tails for intersection local times

Like in many proofs of almost-sure properties of the set of double points of a Brownian motion, the main step is to prove a suitable probability estimate for intersections of *independent* Brownian paths. As our main step is of independent interest, we present it in a more general form than actually needed.

Again, we let $M, N \in \mathbb{N}$ and let $W^{(1)}, \ldots, W^{(M+N)}$ be independent Brownian motions in $\mathbb{R}^d$, $d = 2, 3$ started in the origin. We divide them in two packets of M, resp. N, motions and, for any $\vec{t} = (t^{(1)}, \ldots, t^{(M+N)})$, we denote by $\mathfrak{B}^{(1)}(t^{(1)}, \ldots, t^{(M)})$ and $\mathfrak{B}^{(2)}(t^{(M+1)}, \ldots, t^{(M+N)})$ the union of the paths in each packet, taken up to the times given by the components of $\vec{t}$.

On the intersection of the two packets,

$$S := \mathfrak{B}^{(1)}\big(t^{(1)}, \ldots, t^{(M)}\big) \cap \mathfrak{B}^{(2)}\big(t^{(M+1)}, \ldots, t^{(M+N)}\big),$$

one can define a natural locally finite measure $\ell_{\vec{t}}$, the *(projected) intersection local time*, which can be described symbolically by the formula

$$\ell_{\vec{t}}(A) = \sum_{i=1}^{M} \sum_{j=M+1}^{M+N} \int_A dy \int_0^{t^{(i)}} ds_1 \int_0^{t^{(j)}} ds_2 \; \delta_y(W^{(i)}(s_1))\, \delta_y(W^{(j)}(s_2)), \text{ for } A \subset \mathbb{R}^d \text{ Borel.} \tag{1.4}$$

If $d = 3$ we can permit infinite times, thanks to the transience of Brownian motion in this case. Rigorous constructions of the random measure $\ell_{\vec{t}}$ are reviewed in [KM02, Section 2.1].

Let $U \subset \mathbb{R}^d$ be a bounded and open set containing the origin. Let $R > 0$ be large enough such that U is contained in the centred ball of radius R and let $\vec{t} = (\tau_R^{(1)}, \ldots, \tau_R^{(M+N)})$ be the vector of first hitting times of the sphere $\partial B(0, R)$ by the Brownian motions.

In [KM02, Theorem 1.1] the authors determine the *upper* tails of the random variable $\ell_{\vec{t}}(U)$. It turns out that the upper tails decay stretched exponentially, more precisely,

$$\lim_{a \uparrow \infty} \frac{\log \mathbb{P}\{\ell_{\vec{t}}(U) > a\}}{\sqrt{a}} = -\chi,$$

where $0 < \chi < \infty$ is explicitly given by a variational problem involving R, U, d. For the *lower* tail probabilities however, it is shown in [KM05, Theorem 1.1] that the decay is polynomial, more precisely

$$\lim_{a \downarrow 0} \frac{\log \mathbb{P}\big\{\ell_{\vec{t}}(U) < a\big\}}{-\log a} = -\frac{\xi_d(M, N)}{4 - d}.$$

In the present paper we refine this result by showing that the intersection exponents provide estimates for the lower tails of the intersection local times which are precise up to constants. The following theorem is the second main result of this paper.

Theorem 1.2. *Let $d \in \{2, 3\}$ and $U \subset \mathbb{R}^d$ be a bounded and open set containing the origin. Suppose that $\vec{t}$ is the vector of first exit times of the Brownian motions from a ball $B(0, R)$, which is large enough to contain U, allowing R to be infinite if $d = 3$. Let $\xi = \xi_d(M, N)$. Then there exist constants $0 < \underline{c} < \overline{c} < \infty$, depending on the choice of U, the dimension d and the radius R, such that*

$$\underline{c}\, a^{\frac{\xi}{4-d}} \leq \mathbb{P}\{\ell_{\vec{t}}(U) < a\} \leq \overline{c}\, a^{\frac{\xi}{4-d}} \qquad \text{for all } a \in (0, 1). \tag{1.5}$$

Remark 2 Equation (1.5) holds in the same form, if $\vec{t}$ is a fixed vector of positive times. This can be shown easily using an argument of Lawler, see [La96a, Prop 3.15] or [MO07, Section 3.5].

Remark 3 Theorem 1.1 depends on the case $M = N = 2$ and, crucially, $R = \infty$, which is permitted in the transient case $d = 3$ only. In the case $d = 2$, letting $R \to \infty$ invalidates the lower bound in (1.5) because, for each Brownian motion, the number of returns to $B(0,1)$ between visits of $\partial B(0,2)$ is typically of order $\log R$.

Remark 4 The proof of Theorem 1.2 will be given in Section 2. It uses the idea of the 'branching tree heuristic', which is explained for some easier examples in [MO07]. The heuristic describes the *strategy* by which the Brownian paths achieve the event $\{\ell_{\mathfrak{t}}(U) < a\}$. Loosely speaking, all $M + N$ Brownian paths are running freely until they hit the boundary of the ball $B(0, a^{1/(4-d)})$ for the first time. By this time they have accumulated an intersection local time of the order a. From then on they do not intersect anymore inside U.

2. Proof of the lower tail asymptotics for intersection local times

2.1 Some auxiliary lemmas

We start with an auxiliary lemma about the probability that two Brownian motions in $d = 2, 3$ hit a small ball without their paths intersecting each other beforehand. Throughout the proofs we shall make ample use of the Brownian scaling property, i.e. the fact that for a Brownian motion W with start in $x \in \mathbb{R}^d$ and $r > 0$, the process given by $W'(t) = rW(r^{-2}t)$ is a Brownian motion started in rx. From (1.4) one easily derives that this scaling of the Brownian motions induces the following identity in law

$$\ell_{(\tau_r^{(i)})}(rA) \stackrel{d}{=} r^{4-d}\, \ell_{(\tau_1^{(i)})}(A), \tag{2.1}$$

where $\tau_r^{(i)}$ denotes the first hitting time of the sphere $\partial B(0,r)$ by the Brownian motion $W^{(i)}$. We suppose that under $\mathbb{P}_{(x^{(i)})}$ the processes $W^{(1)}, \ldots, W^{(M+N)}$ are Brownian motions with

$$W^{(1)}(0) = x^{(1)}, \ldots, W^{(M+N)}(0) = x^{(M+N)}.$$

Lemma 2.1. *There exists a constant $C_{2.1} > 0$ such that, for all radii $r, s > 0$ with $r/s > 4$, indices $\mathfrak{k}, \mathfrak{l} \in \{1, \ldots, M+N\}$ and starting points $x^{(1)}, \ldots, x^{(M+N)} \in \partial B(0,r)$,*

$$\mathbb{P}_{(x^{(i)})}\big\{\tau_s^{(\mathfrak{k})}, \tau_s^{(\mathfrak{l})} < \infty,\, W^{(\mathfrak{k})}[0, \tau_s^{(\mathfrak{k})}] \cap W^{(\mathfrak{l})}[0, \tau_s^{(\mathfrak{l})}] = \emptyset\big\} \le C_{2.1}\, (r/s)^{4-2d-\xi_d(1,1)} \big(\log(r/s)\big)^{2(3-d)}.$$

Proof. The proof is similar to that of [KM05, Lemma 2.9]. As $\mathbb{P}_{x^{(i)}}\{\tau_s^{(i)} < \infty\} = (r/s)^{2-d}$, it suffices to prove that

$$\mathbb{P}_{(x^{(i)})}\Big\{W^{(\mathfrak{k})}[0, \tau_s^{(\mathfrak{k})}] \cap W^{(\mathfrak{l})}[0, \tau_s^{(\mathfrak{l})}] = \emptyset \,\Big|\, \tau_s^{(\mathfrak{k})}, \tau_s^{(\mathfrak{l})} < \infty\Big\} \le C_{2.1}\, (r/s)^{-\xi_d(1,1)} \big(\log(r/s)\big)^{2(3-d)}. \tag{2.2}$$

At the expense of replacing the assumption $r/s > 4$ by $r/s > 2$, we may additionally assume that the starting points $(x^{(i)})$ are independent and uniformly distributed on $\partial B(0,r)$. Indeed, from the explicit form of the Poisson kernel we infer that the density of

$$\big(W^{(1)}(\tau_{r/2}^{(1)}), \ldots, W^{(M+N)}(\tau_{r/2}^{(M+N)})\big)$$

with respect to the product of $M + N$ uniform distributions on $\partial B(0, r/2)$ is bounded from infinity and hence the result for arbitrary starting points on $\partial B(0,r)$ follows from the result with starting points independent and uniformly distributed on $\partial B(0, r/2)$.

To prove (2.2) with uniform starting points, we define random times

$$\tau_*^{(i)} := \sup\big\{0 < t < \tau_s^{(i)} \,:\, |W^{(i)}(t)| = r\big\}, \text{ for } i \in \{\mathfrak{k}, \mathfrak{l}\}\,.$$

The paths $e^{(i)}\colon [0, \tau_s^{(i)} - \tau_*^{(i)}] \to \mathbb{R}^d$, $e^{(i)}(t) = W^{(i)}(t + \tau_*^{(i)})$, are Brownian excursions from $\partial B(0,r)$ to $\partial B(0,s)$, and hence the time-reversed paths

$$e_*^{(i)} \,:\, [0, \tau_s^{(i)} - \tau_*^{(i)}] \to \mathbb{R}^d, \quad e_*^{(i)}(t) = e^i(\tau_s^{(i)} - \tau_*^{(i)} - t),$$

are Brownian excursions from $\partial B(0,s)$ to $\partial B(0,r)$, see our appendix for a proof of this and background on excursions. Now fix $\rho = 3s/2$, so that $s < \rho < r$, and define $\sigma^{(i)} = \inf\{t > 0 \,:\, |e_*^{(i)}(t)| = \rho\}$. The processes defined by

$$\overline{W}^{(i)} \,:\, [0, \tau_s^{(i)} - \tau_*^{(i)} - \sigma^{(i)}] \to \mathbb{R}^d, \quad \overline{W}^{(i)}(t) = e_*^{(i)}(\sigma^{(i)} + t),$$

have the law of independent Brownian motions $\tilde{W}^{(i)}$ started in a uniformly chosen point on $\partial B(0,\rho)$ killed upon leaving $B(0,r) \setminus B(0,s)$ and conditioned to hit $\partial B(0,r)$ before $\partial B(0,s)$. Denoting the first hitting times of $\partial B(0,s)$, resp. $\partial B(0,r)$, by the motion $\tilde{W}^{(i)}$ by $\tilde{\tau}_s^{(i)}$, resp. $\tilde{\tau}_r^{(i)}$, we get

$$\begin{aligned}
\mathbb{P}\big\{W^{(\mathfrak{k})}[0,\tau_s^{(\mathfrak{k})}] \cap W^{(\mathfrak{l})}[0,\tau_s^{(\mathfrak{l})}] = \emptyset \,\big|\, \tau_s^{(\mathfrak{k})} < \infty, \tau_s^{(\mathfrak{l})} < \infty\big\} \\
\le \mathbb{P}\big\{W^{(\mathfrak{k})}[\tau_*^{(\mathfrak{k})}, \tau_s^{(\mathfrak{k})}] \cap W^{(\mathfrak{l})}[\tau_*^{(\mathfrak{l})}, \tau_s^{(\mathfrak{l})}] = \emptyset \,\big|\, \tau_s^{(\mathfrak{k})} < \infty, \tau_s^{(\mathfrak{l})} < \infty\big\} \\
\le \mathbb{P}\Big\{\tilde{W}^{(\mathfrak{k})}[0,\tilde{\tau}_r^{(\mathfrak{k})}] \cap \tilde{W}^{(\mathfrak{l})}[0,\tilde{\tau}_r^{(\mathfrak{l})}] = \emptyset \,\Big|\, \tilde{\tau}_r^{(\mathfrak{k})} < \tilde{\tau}_s^{(\mathfrak{k})}, \tilde{\tau}_r^{(\mathfrak{l})} < \tilde{\tau}_s^{(\mathfrak{l})}\Big\} \\
\le \mathbb{P}\Big\{\tilde{W}^{(\mathfrak{k})}[0,\tilde{\tau}_r^{(\mathfrak{k})}] \cap \tilde{W}^{(\mathfrak{l})}[0,\tilde{\tau}_r^{(\mathfrak{l})}] = \emptyset\Big\}\, \mathbb{P}\{\tilde{\tau}_r^{(1)} < \tilde{\tau}_s^{(1)}\}^{-2},
\end{aligned} \tag{2.3}$$

where we have used the trivial fact that $\mathbb{P}(A \,|\, B) \le \mathbb{P}(A)/\mathbb{P}(B)$ and independence in the last step. By (1.1) and Brownian scaling,

$$\mathbb{P}\big\{\tilde{W}^{(\mathfrak{k})}[0,\tilde{\tau}_r^{(\mathfrak{k})}] \cap \tilde{W}^{(\mathfrak{l})}[0,\tilde{\tau}_r^{(\mathfrak{l})}] = \emptyset\big\} \le C\,(r/s)^{-\xi_d(1,1)}. \tag{2.4}$$

Moreover, recalling the exit probabilities from concentric spheres,

$$\mathbb{P}\big\{\tilde{\tau}_r^{(i)} < \tilde{\tau}_s^{(i)}\big\} = \begin{cases} \frac{\log(\rho/s)}{\log(r/s)} & \text{if } d = 2, \\ \frac{(r/s)-(r/\rho)}{(r/s)-1} & \text{if } d = 3, \end{cases}$$

and that $\rho = 3s/2$, we get, for a suitable constant $c_{(2.5)} > 0$, the lower bound

$$\mathbb{P}\big\{\tilde{\tau}_r^{(i)} < \tilde{\tau}_s^{(i)}\big\} \ge c_{(2.5)}\, \log(r/s)^{d-3}\,. \tag{2.5}$$

The result (2.2) follows by plugging (2.4) and (2.5) into (2.3). □

The next lemma is an upper bound for the lower tail probability of the intersection local time, which serves as an a-priori estimate in our proof.

Lemma 2.2. *For any $\varepsilon > 0$ there exists a constant $C_{2.2}(\varepsilon) > 0$ such that, for any $r > s > 0$, $\mathfrak{k}, \mathfrak{l} \in \{1, \dots, M+N\}$ and $x^{(1)}, \dots, x^{(M+N)} \in \partial B(0,s)$,*

$$\mathbb{P}_{(x^{(i)})}\{\ell_r^{(\mathfrak{k},\mathfrak{l})} \le s^{4-d}\} \le C_{2.2}(\varepsilon)\ (r/s)^{-\xi_d(1,1)+\varepsilon}\,.$$

where $\ell_r^{(\mathfrak{k},\mathfrak{l})}$ denotes the total intersection local time of the paths $W^{(\mathfrak{k})}[0,\tau_r^{(\mathfrak{k})}]$ and $W^{(\mathfrak{l})}[0,\tau_r^{(\mathfrak{l})}]$.

Proof. This is shown in [KM05, Section 2.2]. The relevant estimate is derived in the large display on the top of page 1271. In our notation it states that, for fixed $\varepsilon > 0$, there are (ε-dependent) constants $0 < \delta < 1$ (called r there) and $\eta > 1$ (called $4r^{1-m}$ there) such that for any $n \in \mathbb{N}$ with $\delta^n < 1/\eta$ and starting points $x^{(i)} \in \partial B(0, \eta\delta^n)$,

$$\mathbb{P}_{(x^{(i)})}\big\{\ell_1^{(\mathfrak{k},\mathfrak{l})}(B(0,1)) < \delta^{(4-d)n}\big\} \le C_{(2.6)}\ \delta^{n(\xi_d(1,1)-\varepsilon)}\,. \tag{2.6}$$

where the constant $C_{(2.6)}$ depends on $\varepsilon > 0$, but not on n or the starting points.

For $0 < s/r < 1$ we find n such that $\delta^{n+1} < s/r \le \delta^n$. We may assume that s/r is small enough so that n satisfies $\delta^n < 1/\eta$. Hence, whenever $x^{(i)} \in \partial B(0, s/r)$,

$$\begin{aligned}\mathbb{P}_{(x^{(i)})}\big\{\ell_1^{(\mathfrak{k},\mathfrak{l})}(B(0,1)) < (s/r)^{4-d}\big\} &\le \mathbb{E}_{(x^{(i)})}\Big[\mathbb{P}_{\big(W^{(i)}(\tau^{(i)}_{\eta\delta^n})\big)}\big\{\ell_1^{(\mathfrak{k},\mathfrak{l})}(B(0,1)) < \delta^{(4-d)n}\big\}\Big]\\ &\le C_{(2.6)}\ \delta^{n(\xi_d(1,1)-\varepsilon)} \le C_{2.2}(\varepsilon)\ (r/s)^{-\xi_d(1,1)+\varepsilon}\,,\end{aligned}$$

and the claim follows by scaling the Brownian motions by a spatial factor of r as in (2.1). □

For the formulation of the next lemma define, for $0 < s < r$, the packets

$$\mathfrak{B}^{(1)} := \bigcup_{i=1}^{M} W^{(i)}[0, \tau_r^{(i)}], \quad \mathfrak{B}^{(2)} := \bigcup_{i=M+1}^{M+N} W^{(i)}[0, \tau_r^{(i)}],$$

and, for small $\varepsilon > 0$, define neighbourhoods of the starting points of the packets,

$$\mathfrak{E}^{(1)} := \bigcup_{i=1}^{M} B(x^{(i)}, \varepsilon s), \quad \mathfrak{E}^{(2)} := \bigcup_{i=M+1}^{M+N} B(x^{(i)}, \varepsilon s)\,.$$

The lemma, which is due to Lawler, estimates the probability that packets started on $\partial B(0,s)$ do not intersect before hitting $\partial B(0,r)$ and also do not enter $B(0,s)$ except in a small neighbourhood of their respective starting points.

Lemma 2.3. *For all $\varepsilon > 0$ there is a constant $C_{2.3}(\varepsilon) > 0$ such that, for all $0 < s < r$ and $x^{(1)}, \dots, x^{(M+N)} \in \partial B(0,s)$ with $|x^{(i)} - x^{(j)}| > s\varepsilon$ for $i \ne j$,*

$$\mathbb{P}_{(x^{(i)})}\Big\{\mathfrak{B}^{(1)} \cap \mathfrak{B}^{(2)} = \emptyset,\ \mathfrak{B}^{(1)} \cap B(0,s) \subset \mathfrak{E}^{(1)},\ \mathfrak{B}^{(2)} \cap B(0,s) \subset \mathfrak{E}^{(2)}\Big\} \ge C_{2.3}(\varepsilon)\,\big(r/s\big)^{-\xi}\,.$$

Proof. This is shown in [La96a, Lemma 3.7]. □

2.2 The upper bound of the lower tail asymptotics

In this section we prove the upper bound in Theorem 1.2. By scaling we may assume that U contains the unit ball $B(0,1) \subset \mathbb{R}^d$, $d = 2, 3$, and by monotonicity we may even assume that $U = B(0,1)$ and $R = 1$. We abbreviate $\xi = \xi_d(M,N)$ and write

$$\mathbb{P}\big\{\ell_{\overline{t}}(B(0,1)) \le 2^{-(4-d)n}\big\} = a(n)\, 2^{-n\,\xi}\,.$$

This defines $a(n)$, $n = 0, 1, 2, \dots$ and our aim is to show that $a(n)$ is bounded, which would immediately imply the upper bound in Theorem 1.2 in the considered cases for the choice of $\overline{c} = 2^{\xi} \max a(n)$.

For all integers $0 \le j < k$ we look at the packets

$$\mathfrak{B}^{(1)}(j,k) := \bigcup_{i=1}^{M} W^{(i)}([\tau^{(i)}_{2^{-k}}, \tau^{(i)}_{2^{-j}}]) \quad \text{and} \quad \mathfrak{B}^{(2)}(j,k) := \bigcup_{i=M+1}^{M+N} W^{(i)}([\tau^{(i)}_{2^{-k}}, \tau^{(i)}_{2^{-j}}]),$$

and for $k = \infty$,

$$\mathfrak{B}^{(1)}(j,\infty) := \bigcup_{i=1}^{M} W^{(i)}([0, \tau^{(i)}_{2^{-j}}]) \quad \text{and} \quad \mathfrak{B}^{(2)}(j,\infty) := \bigcup_{i=M+1}^{M+N} W^{(i)}([0, \tau^{(i)}_{2^{-j}}])\,.$$

We denote by $\ell_{j,k}(\,\cdot\,)$ the intersection local time of the two packets $\mathfrak{B}^{(1)}(j,k)$ and $\mathfrak{B}^{(2)}(j,k)$. We shall also use $\ell_{j,k}$ without argument to denote the total intersection local time of the packets. We let

$$\sigma := \min\big\{ j \in \mathbb{N} \,:\, \mathfrak{B}^{(1)}(0,j) \cap \mathfrak{B}^{(2)}(0,j) \neq \emptyset \big\}\,.$$

Then, using (1.1) in the second step,

$$\begin{aligned} \mathbb{P}\big\{ \ell_{\vec{t}}(B(0,1)) \le 2^{-(4-d)n} \big\} &\le \mathbb{P}\big\{ \sigma > n \big\} + \sum_{j=1}^{n} \mathbb{P}\big\{ \ell_{\vec{t}}(B(0,1)) \le 2^{-(4-d)n},\, \sigma = j \big\} \\ &\le C\, 2^{-n\xi} + \sum_{j=1}^{n} \mathbb{P}\big\{ \ell_{\vec{t}}(B(0,1)) \le 2^{-(4-d)n},\, \sigma = j \big\}\,. \end{aligned} \tag{2.7}$$

To study the right hand side of (2.7) we now formulate a lemma, which will be crucial in the proof.

Lemma 2.4. *There exists a sequence $\beta(k), k \in \mathbb{N}$, of positive numbers such that $\sum \beta(k) < \infty$ and a constant $C_{2.4} > 0$ such that, for any $j \le n$ and $x^{(1)}, \ldots, x^{(M+N)} \in \partial B(0, 2^{-j})$,*

$$\mathbb{P}_{(x^{(i)})}\big\{ \ell_{0,j} \le 2^{-(4-d)n},\, \sigma = j \big\} \le C_{2.4}\, 2^{-j\xi}\, \beta(n-j)\,.$$

Before proving the lemma, let us see how to complete the proof of the upper bound with its help. We first use the strong Markov property at the stopping times $\tau^{(i)}_{2^{-j}}$, then Lemma 2.4 to estimate the inner probability, and finally Brownian scaling to estimate the remaining term,

$$\begin{aligned} \mathbb{P}\big\{ \ell_{\vec{t}}(B(0,1)) \le 2^{-(4-d)n},\, \sigma = j \big\} &\le \mathbb{E}\Big\{ 1\{\ell_{j,\infty} \le 2^{-(4-d)n}\}\, \mathbb{P}_{(W^{(i)}(\tau^{(i)}_{2^{-j}}))}\big\{ \ell_{0,j} \le 2^{-(4-d)n},\, \sigma = j \big\} \Big\} \\ &\le C_{2.4}\, 2^{-j\xi}\, \beta(n-j)\, \mathbb{P}\big\{ \ell_{j,\infty} \le 2^{-(4-d)n} \big\} \\ &\le C_{2.4}\, 2^{-n\xi}\, \beta(n-j)\, a(n-j)\,. \end{aligned}$$

Plugging this into (2.7) we obtain a recursion formula for $a(n)$, precisely for some positive finite constant $\varkappa > 0$,

$$a(n) \le \varkappa \sum_{j=1}^{n+1} \beta(n-j)\, a(n-j) \qquad \text{for } n \ge 0,\;\; a(-1) = \beta(-1) = 1\,.$$

To see that such an $a(n)$ must be bounded we study $\tilde{a}(n)$ defined by $\tilde{a}(-1) := 1$ and

$$\tilde{a}(n) := \varkappa \sum_{j=1}^{n+1} \beta(n-j)\, \tilde{a}(n-j) \qquad \text{for } n \ge 0\,.$$

Note that $a(n) \le \tilde{a}(n)$, so that it suffices to show that $\tilde{a}(n)$ is bounded. By induction we see that $\tilde{a}(n) = \tilde{a}(n-1)\big[1 + \varkappa\,\beta(n-1)\big]$, and therefore, for $n \ge 0$,

$$\tilde{a}(n) = \tfrac{\varkappa}{\varkappa+1} \prod_{j=0}^{n} [1 + \varkappa\,\beta(j-1)]\,,$$

which is bounded in n as the series $\sum \beta(j)$ converges. This completes the proof of the upper bound subject to the proof of Lemma 2.4, which we give now.

Proof of Lemma 2.4. Denote by $\mathfrak{D}$ the collection of dyadic cubes of sidelength 2^{-n} which intersect $B(0, 2^{-j+1})$. Denote by $m(D)$ the minimal distance of any of the starting points of the motions to the centre of the cube D, and let $\mathfrak{D}' = \{D \in \mathfrak{D} \,:\, m(D) > 2^{-n+2}\}$ be those cubes well away from the starting points of the motions. For a given $D \in \mathfrak{D}$ we denote by $\tau_r^{(i)}(D)$ the first hitting time of $\partial B(x, r)$ by the motion $W^{(i)}$, where x is the centre of the cube D.

On the event $\{\sigma = j\}$ there exist $\mathfrak{k} \in \{1, \ldots, M\}$ and $\mathfrak{l} \in \{M+1, \ldots, M+N\}$ such that the $\mathfrak{k}^{\text{th}}$ and $\mathfrak{l}^{\text{th}}$ motion intersect in $B(0, 2^{-j+1})$, i.e.,

$$W^{(\mathfrak{k})}[0, \tau_1^{(\mathfrak{k})}] \cap W^{(\mathfrak{l})}[0, \tau_1^{(\mathfrak{l})}] \cap B(0, 2^{-j+1}) \ne \emptyset\,.$$

To identify a specific location of the intersection we temporarily write

$$\tau := \inf\big\{t > 0 \,:\, W^{(\mathfrak{k})}(t) \in W^{(\mathfrak{l})}[0, \tau_1^{(\mathfrak{l})}]\big\}\,.$$

Then $W^{(\mathfrak{k})}(\tau) \in B(0, 2^{-j+1})$ is an intersection point and hence there exists $D \in \mathfrak{D}$ such that $W^{\mathfrak{k}}(\tau) \in D$. If $D \in \mathfrak{D}'$, neither of the motions start in D and hence we have

$$W^{(\mathfrak{k})}[0, \tau_{2^{-n}}^{(\mathfrak{k})}(D)] \cap W^{(\mathfrak{l})}[0, \tau_{2^{-n}}^{(\mathfrak{l})}(D)] = \emptyset\,.$$

We define $\sigma_r^{(i)} = \tau_r^{(i)}$ if $i \ne \mathfrak{k}, \mathfrak{l}$ and otherwise

$$\sigma_r^{(i)} = \inf\big\{t > \tau_{2^{-n}}^{(i)}(D) \,:\, |W^{(i)}(t)| = r\big\}\,.$$

Let ℓ^D be the total intersection local time of the paths $W^{(\mathfrak{k})}[\tau_{2^{-n}}^{(\mathfrak{k})}(D), \sigma_{2^{-j+2}}^{(\mathfrak{k})}]$ and $W^{(\mathfrak{l})}[\tau_{2^{-n}}^{(\mathfrak{l})}(D), \sigma_{2^{-j+2}}^{(\mathfrak{l})}]$. The pathwise argument outlined so far implies that

$$\begin{aligned}
&\mathbb{P}_{(x^{(i)})}\big\{\ell_{0,j} \le 2^{-(4-d)n},\, \sigma = j\big\} \\
&\quad \le \sum_{\mathfrak{k}=1}^{N} \sum_{\mathfrak{l}=N+1}^{M+N} \sum_{D \in \mathfrak{D}} \mathbb{E}_{(x^{(i)})}\Big[1\big\{\tau_{2^{-n}}^{(\mathfrak{k})}(D) < \tau_1^{(\mathfrak{k})}, \tau_{2^{-n}}^{(\mathfrak{l})}(D) < \tau_1^{(\mathfrak{l})}\big\} \\
&\qquad\qquad \times 1\big\{D \notin \mathfrak{D}' \text{ or } W^{(\mathfrak{k})}[0, \tau_{2^{-n}}^{(\mathfrak{k})}(D)] \cap W^{(\mathfrak{l})}[0, \tau_{2^{-n}}^{(\mathfrak{l})}(D)] = \emptyset\big\} \\
&\qquad\qquad \times 1\{\ell^D \le 2^{-(4-d)n}\}\, \mathbb{P}_{(W^{(i)}(\sigma_{2^{-j+2}}^{(i)}))}\big\{\mathfrak{B}^{(1)}(0, j-2) \cap \mathfrak{B}^{(2)}(0, j-2) = \emptyset\big\}\Big]\,.
\end{aligned}$$

The innermost probability can be bounded (uniformly in the starting points) by $C2^{-(j-2)\xi}$, so that the remaining expectation depends only on the Brownian motions $W^{(\mathfrak{k})}$ and $W^{(\mathfrak{l})}$. Using the strong Markov property at times $\tau_{2^{-n}}^{(\mathfrak{k})}(D)$, $\tau_{2^{-n}}^{(\mathfrak{l})}(D)$ we obtain the upper bound of

$$\begin{aligned}
C2^{-(j-2)\xi} \sum_{\mathfrak{k}=1}^{N} \sum_{\mathfrak{l}=N+1}^{M+N} \sum_{D \in \mathfrak{D}} \mathbb{E}_{(x_i)}\Big[1\big\{\tau_{2^{-n}}^{\mathfrak{k}}(D) < \tau_1^{\mathfrak{k}},\, \tau_{2^{-n}}^{\mathfrak{l}}(D) < \tau_1^{\mathfrak{l}}\big\} \\
\times 1\big\{D \notin \mathfrak{D}' \text{ or } W^{(\mathfrak{k})}[0, \tau_{2^{-n}}^{(\mathfrak{k})}(D)] \cap W^{(\mathfrak{l})}[0, \tau_{2^{-n}}^{(\mathfrak{l})}(D)] = \emptyset\big\} \\
\times \mathbb{P}_{W^{(\mathfrak{k})}(\tau_{2^{-n}}^{(\mathfrak{k})}(D)), W^{(\mathfrak{l})}(\tau_{2^{-n}}^{(\mathfrak{l})}(D))}\big\{\ell^D \le 2^{-(4-d)n}\big\}\Big].
\end{aligned}$$

Again we find a uniform upper bound for the inner probability: Note that the distance from the centre of the cube D to the sphere $\partial B(0,2^{-j+2})$ is at least 2^{-j}. Hence, using Lemma 2.2, for fixed $\varepsilon>0$,

$$\mathbb{P}_{W^{(\mathfrak{k})}(\tau^{(\mathfrak{k})}_{2^{-n}}(D)),W^{(\mathfrak{l})}(\tau^{(\mathfrak{l})}_{2^{-n}}(D))}\{\ell^D \le 2^{-(4-d)n}\} \le C_{2.2}(\varepsilon)\,[2^{n-j}]^{-\xi_d(1,1)+\varepsilon}\,.$$

Once this estimate is carried out, we note that, if $D\in\mathfrak{D}'$ and hence $2^n m(D)>4$, by Lemma 2.1,

$$\begin{aligned}\mathbb{P}_{(x^{(i)})}\Big\{\tau^{(\mathfrak{k})}_{2^{-n}}(D)<\tau^{(\mathfrak{k})}_1,\ \tau^{(\mathfrak{l})}_{2^{-n}}(D)<\tau^{(\mathfrak{l})}_1,\ W^{(\mathfrak{k})}[0,\tau^{(\mathfrak{k})}_{2^{-n}}(D)]\cap W^{(\mathfrak{l})}[0,\tau^{(\mathfrak{l})}_{2^{-n}}(D)]=\emptyset\Big\}&\\ \le C_{2.1}\,[2^n m(D)]^{4-2d-\xi_d(1,1)}[\log(2^n m(D))]^{2(3-d)}&\,.\end{aligned}$$

Summarising, we have shown that, for some constant $C_{(2.8)}(\varepsilon)>0$,

$$\begin{aligned}\mathbb{P}_{(x^{(i)})}\{\ell_{0,j}\le 2^{-(4-d)n},\ \sigma=j\} &\le C_{(2.8)}(\varepsilon)\,2^{-j\xi}\,[2^{n-j}]^{-\xi_d(1,1)+\varepsilon}\\ &\times\Big(\sum_{\mathfrak{k}=1}^{N}\sum_{\mathfrak{l}=N+1}^{M+N}\sum_{D\in\mathfrak{D}'}[2^n m(D)]^{4-2d-\xi_d(1,1)}\,[\log(2^n m(D))]^{2(3-d)}+\sum_{\mathfrak{k}=1}^{N}\sum_{\mathfrak{l}=N+1}^{M+N}\#\big(\mathfrak{D}\setminus\mathfrak{D}'\big)\Big)\,.\end{aligned}\tag{2.8}$$

Observe that the double sum in (2.8) is bounded from above, while for $k\ge 4$ the number of cubes D with $(k-\frac12)2^{-n}\le m(D)<(k+\frac12)2^{-n}$ is bounded by a constant multiple of k^{d-1}. Therefore the triple sum in (2.8) can be bounded by a constant multiple of

$$\sum_{k=4}^{2^{n-j+4}} k^{d-1}\,k^{4-2d-\xi_d(1,1)}\,[\log k]^{2(3-d)} \le 2^{(n-j)[4-d-\xi_d(1,1)]}\int_0^{16} x^{3-d-\xi_d(1,1)}\,(1\vee\log x)^{2(3-d)}dx\,.$$

If $d=2$ we note that $\xi_2(1,1)=5/4$ so that $3-d-\xi_d(1,1)=-1/4>-1$ and the integral converges. We thus obtain the required upper bound with $\beta(n-j):=2^{(n-j)[-\frac12+\varepsilon]}$ so that $\varepsilon<\frac12$ gives $\sum\beta(k)<\infty$.

If $d=3$ recall that $\frac12<\xi_3(1,1)<1$ by (1.3). Hence the integral is bounded and we obtain the upper bound with $\beta(n-j):=2^{(n-j)[1-2\xi_3(1,1)+\varepsilon]}$. For sufficiently small $\varepsilon>0$, we get $\sum\beta(k)<\infty$. □

2.3 The lower bound of the lower tail asymptotics

To give the proof of the lower bound in Theorem 1.2, the argument in [KM05] could be strengthened to give up-to-constants estimates. We prefer to follow an alternative route here, which draws heavily on the result of Lawler, which we stated as Lemma 2.3.

By scaling we may assume that U is contained in the unit ball $B(0,\frac12)$, and by monotonicity we may even assume that $U=B(0,\frac12)$ and $R>1$. It thus suffices to prove that, for some constant $\underline{c}>0$,

$$\mathbb{P}\{\ell_{\bar{t}}(B(0,\tfrac12))<2^{-n(4-d)}\}\ge \underline{c}\,2^{-n\xi}\,.$$

Recall that $\tau^{(i)}_{2^{-n}}$ is the first time $W^{(i)}$ hits the sphere $\partial B(0,2^{-n})$, the notation $\mathfrak{B}^{(i)}(j,k)$ for the packets of Brownian paths and $\ell_{j,k}$ for their intersection local times, which was introduced in the beginning of the previous section. We also write

$$\mathfrak{E}^{(1)}(n,\varepsilon):=\bigcup_{i=1}^{M}B\big(W^{(i)}(\tau^{(i)}_{2^{-n}}),2\varepsilon2^{-n}\big),\qquad\text{and}\qquad \mathfrak{E}^{(2)}(n,\varepsilon):=\bigcup_{i=M+1}^{M+N}B\big(W^{(i)}(\tau^{(i)}_{2^{-n}}),2\varepsilon2^{-n}\big)\,.$$

Then, by scaling, for a fixed small $\varepsilon>0$,

$$\begin{aligned}&\mathbb{P}\{\ell_{n,\infty}<2^{-n(4-d)},\ \mathfrak{B}^{(1)}(n,\infty)\cap\mathfrak{E}^{(2)}(n,\varepsilon)=\emptyset,\mathfrak{B}^{(2)}(n,\infty)\cap\mathfrak{E}^{(1)}(n,\varepsilon)=\emptyset\}\\ &\quad=\mathbb{P}\{\ell_{0,\infty}<1,\ \mathfrak{B}^{(1)}(0,\infty)\cap\mathfrak{E}^{(2)}(0,\varepsilon)=\emptyset,\ \mathfrak{B}^{(2)}(0,\infty)\cap\mathfrak{E}^{(1)}(0,\varepsilon)=\emptyset\}=:\delta>0\,.\end{aligned}\tag{2.9}$$

By Lemma 2.3 there is a constant $c(\varepsilon) > 0$ such that, whenever $x^{(1)}, \ldots, x^{(M+N)} \in \partial B(0, 2^{-n})$ satisfy $|x^{(i)} - x^{(j)}| > 2\varepsilon 2^{-n}$, then

$$\begin{aligned} \mathbb{P}_{(x^{(i)})}\big\{ \mathfrak{B}^{(1)}(0,n) \cap \mathfrak{B}^{(2)}(0,n) = \emptyset,\ \mathfrak{B}^{(1)}(0,n) \cap B(0,2^{-n}) \subset \mathfrak{E}^{(1)}(n,\varepsilon), \\ \mathfrak{B}^{(2)}(0,n) \cap B(0,2^{-n}) \subset \mathfrak{E}^{(2)}(n,\varepsilon)\big\} \geq c(\varepsilon)\, 2^{-n\xi}\,. \end{aligned} \tag{2.10}$$

Note that whenever Brownian paths $W^{(1)}, \ldots, W^{(M+N)}$ started at the origin satisfy the events considered on the left hand side of (2.9) and (2.10), the packet $\mathfrak{B}^{(1)}(0,n)$ cannot intersect $\mathfrak{B}^{(2)}(n,\infty)$, and the packet $\mathfrak{B}^{(2)}(0,n)$ cannot intersect $\mathfrak{B}^{(1)}(n,\infty)$. Finally, there exists $\eta = \eta(R) > 0$ such that, for all $x^{(1)}, \ldots, x^{(M+N)} \in \partial B(0,1)$,

$$\mathbb{P}_{(x^{(i)})}\big\{ \tau_{1/2}^{(i)} > \tau_R^{(i)} \text{ for all } i \in \{1, \ldots, M+N\} \big\} \geq \eta\,. \tag{2.11}$$

Finally, if Brownian paths $W^{(1)}, \ldots, W^{(M+N)}$ started at the origin satisfy the events considered on the left hand side of (2.9) and (2.10), and they do not return to $B(0,1/2)$ after the times $\tau_1^{(1)}, \ldots, \tau_1^{(M+N)}$, they also satisfy $\ell_{\vec{t}}(B(0,1/2)) < 2^{-n(4-d)}$. This gives a lower bound of $\delta\eta c(\varepsilon)\, 2^{-n\xi}$ for the probability of this event, as required.

3. Proof of the exact packing measure of double points

3.1 The lower bound for the packing measure

In this section we prove Theorem 1.1 (ii). The essential ingredient in the proof is an analysis of the lower envelope of the intersection local time of two independent Brownian motions in $\mathbb{R}^3$. We shall show the following.

Proposition 3.1. *Let ℓ be the intersection local time of two independent Brownian motions in $\mathbb{R}^3$ started in the origin and running for positive and negative times until they hit the ball of radius one. Then, for any gauge function ϕ satisfying*

$$\int_{0+} r^{-1-\xi} \phi(r)^\xi \, dr = \infty, \tag{3.1}$$

we have

$$\mathbb{P}\Big\{ \liminf_{r \downarrow 0} \frac{\ell(B(0,r))}{\phi(r)} = 0 \Big\} = 1.$$

We also use the following lemma, which is a special case of [MS99, Theorem 3.2]. For its formulation we denote by $\mathcal{M}(\mathbb{R}^3)$ the space of all finite measures on $\mathbb{R}^3$ endowed with the weak topology, and for a measure $\mu \in \mathcal{M}(\mathbb{R}^3)$ and $u \in \mathbb{R}^3$ define $T^u\mu$ by $T^u\mu(A) = \mu(u + A)$. Let ℓ be as in Proposition 3.1 and denote by ℓ^* the intersection local time of two Brownian motions running for one unit of time.

Lemma 3.2. *Consider a Borel set $M \subset \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ with the following properties:*

- *If $(\mu, x) \in M$ and $\nu \in \mathcal{M}(\mathbb{R}^d)$ are such that $\nu = \mu$ on $B(x, \varepsilon)$ for some $\varepsilon > 0$, then $(\nu, x) \in M$,*
- *if $(\mu, x) \in M$, then $(T^u\mu, x - u) \in M$ for all $u \in \mathbb{R}^d$.*

Then

$$\mathbb{P}_{(0,0)}\Big\{ (\ell, 0) \in M \Big\} > 0$$

implies that, for every choice of initial points x_1, x_2,

$$\mathbb{P}_{(x_1,x_2)}\Big\{ (\ell^*, x) \in M \text{ for } \ell^*\text{-almost every point } x \Big\} = 1\,.$$

Proof of the lower bound in Theorem 1.1. Let us first see how Proposition 3.1 implies the lower bound in Theorem 1.1. We suppose that ϕ satisfies (3.1) and apply Lemma 3.2 to the set

$$M = \Big\{ (\mu, x) \,:\, \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{\phi(r)} = 0 \Big\},$$

which implies that, almost surely, for ℓ^*-almost every x,

$$\liminf_{r \downarrow 0} \frac{\ell^*(B(x,r))}{\phi(r)} = 0 \,.$$

By [LT86, Lemma 1] this implies that, for the set S of intersections of the two paths, $\mathcal{P}^\phi(S) = \infty$ provided that $\ell^*(S) > 0$, which by [KM05, Proposition 2.3], is equivalent to $S \neq \emptyset$.

Now look at a single Brownian path $W\colon [0,\infty) \to \mathbb{R}^3$. Then

$$W^{(1)}\colon [0,1] \to \mathbb{R}^3, \quad W^{(1)}(t) = W(1+t) - W(1)$$
$$W^{(2)}\colon [0,1] \to \mathbb{R}^3, \quad W^{(2)}(t) = W(1-t) - W(1)$$

are independent Brownian motions with the same starting point and therefore we have just shown that $\mathcal{P}^\phi(S) = \infty$ almost surely. The set S is however a subset of D and this completes the proof of the lower bound in Theorem 1.1 subject to the proof of Proposition 3.1. □

We prepare the proof of Proposition 3.1 with a lemma, which is a combination of facts due to Lawler, see [La96a, (2)] or [La98, Theorem 2.1] and the proofs given there. Supposing now that the Brownian motions $W^{(i)}$ are two-sided, we define, for $s > 0$ the hitting times

$$T_+^{(i)}(s) := \inf\big\{ t > 0 \colon |W^{(i)}(t)| = s \big\} \quad \text{ and } \quad T_-^{(i)}(s) := \sup\big\{ t < 0 \colon |W^{(i)}(t)| = s \big\}.$$

Lemma 3.3. *For $0 < s < r < \infty$ define packets of Brownian motions,*

$$\mathfrak{B}_{s,r}^{(i)} := W^{(i)}\big([T_+^{(i)}(s), T_+^{(i)}(r)] \cup [T_-^{(i)}(r), T_-^{(i)}(s)] \big), \qquad \textit{for } i = 1, 2;$$

fix a small $\varepsilon > 0$ and let

$$\mathfrak{E}_s^{(i)} := B(W^{(i)}(T_+^{(i)}(s)), \varepsilon s) \cup B(W^{(i)}(T_-^{(i)}(s)), \varepsilon s) \qquad \textit{for } i = 1, 2.$$

Let ℓ be the intersection local time running (for positive and negative time) until the first hitting of $\partial B(0,r)$. Then there exist $0 < c < C < \infty$, independent of r and s, such that

$$\mathbb{P}\big\{ \ell(B(0,r)) < s,\, \mathfrak{B}_{s,r}^{(1)} \cap \mathfrak{B}_{s,r}^{(2)} = \emptyset \big\} \leq C \Big(\frac{s}{r} \Big)^\xi$$

and

$$\mathbb{P}\big\{ \ell(B(0,r)) < s,\, \mathfrak{B}_{s,r}^{(1)} \cap \mathfrak{B}_{s,r}^{(2)} = \emptyset \textit{ and }$$
$$\mathfrak{B}_{0,s}^{(i)} \cap \mathfrak{E}_s^{(3-i)} = \emptyset,\, \mathfrak{B}_{s,r}^{(i)} \cap \big(B(0,s) \setminus \mathfrak{E}_s^{(i)} \big) = \emptyset \textit{ for } i = 1, 2 \big\} \geq c \Big(\frac{s}{r} \Big)^\xi.$$

Proof. The upper bound follows from (1.1) by simply forgetting the condition on the intersection local time. For the lower bound note that the probability that up to the first hitting of $\partial B(0,s)$ the intersection local time is less than s, and simultaneously $\mathfrak{B}_{0,s}^{(1)} \cap \mathfrak{E}_s^{(2)} = \emptyset$ and $\mathfrak{B}_{0,s}^{(2)} \cap \mathfrak{E}_s^{(1)} = \emptyset$ is bounded from below by a positive constant (depending only on ε, which was fixed). Moreover, Lemma 2.3 shows that, starting from starting points at least εs apart, the probability that $\mathfrak{B}_{s,r}^{(1)} \cap \mathfrak{B}_{s,r}^{(2)} = \emptyset$ *and*

$$\mathfrak{B}_{s,r}^{(i)} \cap \big(B(0,s) \setminus \mathfrak{E}_s^{(i)} \big) = \emptyset, \text{ for both } i = 1, 2,$$

is bounded below by a constant multiple of $\big(\frac{s}{r}\big)^\xi$. Clearly, this combination of events does imply $\ell(B(0,r)) < s$, and so the proof is complete. □

Proof of Proposition 3.1. For notational convenience we may assume that ϕ is defined on $[0,1]$. We let $h(r) := \phi(r)/r$ for $0<r<1$. By [Be03, Lemma 1] the condition (3.1) implies that

$$\sum_{n=1}^{\infty} h(2^{-n})^{\xi} = \infty. \tag{3.2}$$

As this is equivalent to $\sum h(2^{-n})^{\xi} \wedge 1 = \infty$ we may assume, without losing generality, that $\phi(r) \le r$ for all $0<r<1$.

We denote by $\ell_{n,\infty}$ the total intersection local time of the Brownian motions $W^{(1)}$, $W^{(2)}$ until they first hit the sphere of radius 2^{-n}, i.e. up to time $T_+^{(i)}(2^{-n})$ for positive times, resp. $T_-^{(i)}(2^{-n})$ for negative times. For $i=1,2$ we define the packets

$$\mathfrak{B}^{(i)}(n) := W^{(i)}\big([T_+^{(i)}(\phi(2^{-n})), T_+^{(i)}(2^{-n+1})] \cup [T_-^{(i)}(2^{-n+1}), T_-^{(i)}(\phi(2^{-n}))]\big)$$

the initial pieces

$$\mathfrak{I}^{(i)}(n) := W^{(i)}\big([0, T_+^{(i)}(\phi(2^{-n}))] \cup [T_-^{(i)}(\phi(2^{-n})), 0]\big),$$

and the union of balls

$$\mathfrak{E}^{(i)}(n) := B\big(W^{(i)}(T_+^{(i)}(\phi(2^{-n}))), \varepsilon\phi(2^{-n})\big) \cup B\big(W^{(i)}(T_-^{(i)}(\phi(2^{-n}))), \varepsilon\phi(2^{-n})\big).$$

We also define the stopping times

$$S_+^{(i)}(n) := \inf\big\{t > T_+^{(i)}(2^{-n+1}) \colon |W^{(i)}(t)| = 2^{-n}\big\},$$

for positive times, and

$$S_-^{(i)}(n) := \sup\big\{t < T_+^{(i)}(2^{-n+1}) \colon |W^{(i)}(t)| = 2^{-n}\big\},$$

for negative times. The hitting probabilities of spheres by a Brownian motion in $d=3$ give that there exists $c_{(3.3)} > 0$ such that

$$\mathbb{P}\big\{S_+^{(i)}(n) > T_+^{(i)}(1),\, S_-^{(i)}(n) < T_-^{(i)}(1)\big\} \ge c_{(3.3)} \qquad \text{for all } n \ge 1. \tag{3.3}$$

Note that (3.3) makes use of the fact that $d=3$. We now look at the events

$$\begin{aligned} E(n) := \big\{ & \ell_{n,\infty} < \phi(2^{-n}),\, \mathfrak{B}^{(1)}(n) \cap \mathfrak{B}^{(2)}(n) = \emptyset \text{ and } \mathfrak{I}^{(i)}(n) \cap \mathfrak{E}^{(3-i)}(n) = \emptyset, \\ & \mathfrak{B}^{(i)}(n) \cap \big(B(0,r) \setminus \mathfrak{E}^{(i)}(n)\big) = \emptyset,\, S_+^{(i)}(n) > T_+^{(i)}(1),\, S_-^{(i)}(n) < T_-^{(i)}(1) \text{ for } i=1,2\big\}. \end{aligned}$$

By Lemma 3.3, Inequality (3.3) and independence there exist (new) constants $0<c<C<\infty$, such that

$$c\, h(2^{-n})^{\xi} \le \mathbb{P}\big(E(n)\big) \le C\, h(2^{-n})^{\xi}\,. \tag{3.4}$$

We now show that there exists a constant $C_{(3.5)}$, such that for $n \ge m$,

$$\mathbb{P}\big(E(n) \cap E(m)\big) \le C_{(3.5)}\, h(2^{-n})^{\xi}\, \big[h(2^{-m})^{\xi} + 2^{-(n-m)\xi}\big]\,. \tag{3.5}$$

Indeed, we look at two cases: *First* assume that $\phi(2^{-m}) \ge 2^{-n+1}$. Then

$$\mathbb{P}\big\{\mathfrak{B}^{(1)}(n) \cap \mathfrak{B}^{(2)}(n) = \emptyset\big\} \le C\, h(2^{-n})^{\xi}\,.$$

The analogous bound applies to the probability that $\mathfrak{B}^{(1)}(m) \cap \mathfrak{B}^{(2)}(m) = \emptyset$. As the latter bound is independent of the starting points of the packets, by independence, we get an upper bound

$$\mathbb{P}\big(E(n) \cap E(m)\big) \le C^2\, h(2^{-n})^{\xi} h(2^{-m})^{\xi}\,.$$

Second assume that $\phi(2^{-m}) \le 2^{-n+1}$. Introducing the packets,

$$\mathfrak{B}^{(i)}(n,m) := W^{(i)}\big([T_+^{(i)}(\phi(2^{-n})), T_+^{(i)}(2^{-m+1})] \cup [T_-^{(i)}(2^{-m+1}), T_-^{(i)}(\phi(2^{-n}))]\big),$$

for $i = 1, 2$, we get that

$$E(n) \cap E(m) \subset \big\{\ell_{m,\infty} < \phi(2^{-n}),\, \mathfrak{B}^{(1)}(n,m) \cap \mathfrak{B}^{(2)}(n,m) = \emptyset\big\}.$$

The upper bound in Lemma 3.3 ensures that

$$\mathbb{P}\big(E(n) \cap E(m)\big) \le C 2^{-(n-m)\xi} h(2^{-n})^\xi,$$

and this completes the proof of (3.5).

With (3.5) at hand we use the Kochen-Stone version of the Borel-Cantelli lemma, see [KS64]. Using (3.4) and (3.2), we have

$$\sum_{n=1}^\infty \mathbb{P}\big(E(n)\big) \ge c \sum_{n=1}^\infty h(2^{-n})^\xi = \infty. \tag{3.6}$$

We use (3.5) to estimate

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{\sum_{m=1}^k \sum_{n=1}^k \mathbb{P}(E(n) \cap E(m))}{\Big(\sum_{n=1}^k \mathbb{P}(E(n))\Big)^2} \\
&\le \liminf_{k\to\infty} \sum_{m=1}^k 2 \sum_{n=m}^k \frac{C_{(3.5)}\, h(2^{-n})^\xi \big(h(2^{-m})^\xi + 2^{-(n-m)\xi}\big)}{c^2 \Big(\sum_{n=1}^k h(2^{-n})^\xi\Big)^2} \qquad (3.7)\\
&\le \frac{2C_{(3.5)}}{c^2}\Big(1 + \frac{1}{1-2^{-\xi}} \liminf_{k\to\infty} \Big(\sum_{n=1}^k h(2^{-n})^\xi\Big)^{-1}\Big) = \frac{2C_{(3.5)}}{c^2}.
\end{aligned}$$

By the Kochen-Stone lemma, (3.6) and (3.7) imply that $\mathbb{P}\big\{E(n) \text{ infinitely often }\big\} > 0$, and, together with Blumenthal's zero-one law, this implies that

$$\mathbb{P}\Big\{\liminf_{r\downarrow 0} \frac{\ell(B(0,r))}{\phi(r)} \le 1\Big\} = 1.$$

By applying this to a gauge function $\tilde\phi$ such that $\lim_{r\downarrow 0} \tilde\phi(r)/\phi(r) = 0$ and such that (3.1) still holds, the proof of Proposition 3.1 is completed. ■

3.2 The upper bound for the packing measure

In this section we give a proof of Theorem 1.1 (i). We now assume that the gauge function satisfies

$$\int_{0+} r^{-1-\xi} \phi(r)^\xi \, dr < \infty\,. \tag{3.8}$$

and, for convenience, that ϕ is defined on the entire interval $[0,1]$. By [Be03, Lemma 1], this implies that there exists an integer $N > 1$ and some $0 < \rho < 1$ such that

$$\sum_{n=1}^\infty N^{n\xi}\, \phi(\rho\, N^{-n})^\xi < \infty\,. \tag{3.9}$$

Hence $\phi(\rho\, N^{-n}) < \rho N^{-n-2}/\sqrt{3}$ for all but finitely many n, and we may therefore assume, without losing generality, that

$$h(r) := \frac{\phi(r)}{r} \le \frac{1}{N^2\sqrt{3}}, \qquad \text{for all } 0 < r < 1. \tag{3.10}$$

We start by looking at the intersection S of the paths of two independent Brownian motions $W^{(1)}$ and $W^{(2)}$, each running for infinite time $t^{(1)} = t^{(2)} = \infty$, and first show that, almost surely, $\mathcal{P}^\phi(S) = 0$. By ℓ_∞ we denote the intersection local time of the two motions. The key ingredient here is the following variant of the upper bound in Theorem 1.2.

Lemma 3.4. *There exists a constant $C_{3.4} > 0$ such that, for any $r > 0$, starting points $x^{(1)}, x^{(2)}$ with $|x^{(i)}| \geq r$, and $0 < a < 1$,*

$$\mathbb{P}_{x^{(1)},x^{(2)}}\big\{\ell_\infty(B(0,r)) < ra \,\big|\, \tau^{(1)}_{ra} < \infty,\, \tau^{(2)}_{ra} < \infty\big\} \leq C_{3.4}\, a^\xi\,,$$

where, as before, $\xi = \xi_3(2,2)$.

Proof. To avoid repetition of arguments we only give a sketch of the proof. By considering the intersection local times only from the first hitting times of $\partial B(0,r)$, we may assume that $|x^{(i)}| = r$ and by scaling as in (2.1) we may further assume that $r = 1$. Finally, it suffices to prove the result for the case $0 < a < 1/4$, as otherwise the right hand side is bounded away from zero.

There are *two* options for the main part of the proof: One can *either* repeat the arguments used to prove the upper bound in Theorem 1.2, which is given in Section 2.2, dividing the Brownian motion according to the hitting times of spheres with decreasing, rather than increasing radii. Then the first term on the right hand side of (2.7) can be estimated using Lemma 2.1 instead of (1.1), and all further arguments can be carried out with only obvious changes.

Alternatively one can use the time-reversal technique of Lemma 2.1 to derive the result from the upper bound in Theorem 1.2. More precisely, a brief inspection of the setup of the proof of the upper bound in Theorem 1.2, given in Section 2.2, shows that, for a suitable $C_{(3.11)} > 0$, the bound

$$\mathbb{P}_{x^{(1)},x^{(2)}}\{\ell_\infty(B(0,1)) < b\} \leq C_{(3.11)}\, b^\xi \qquad \text{for } 0 < b < 1, \tag{3.11}$$

still holds if the starting points are located on $\partial B(0,b)$ instead of the origin. We may now continue by using time-reversal, exactly as in the proof of Lemma 2.1: We show that we may assume that the starting points are uniformly distributed on $\partial B(0,1)$, then pass to the intersection local times of the Brownian excursions embedded in our paths. Time-reversing the excursions and dropping a small inital part, we arrive at the intersection local times of Brownian paths, started in $\partial B(0,\eta a)$ for some $\eta > 1$, and stopped upon hitting $\partial B(0,1)$. The result then follows by applying (3.11) with $b = \eta a$. $\square$

Fix a cube $\mathtt{C}$ at positive distance from the starting points $W^{(1)}(0)$ and $W^{(2)}(0)$, it clearly suffices to show that $\mathcal{P}^\phi(S \cap \mathtt{C}) = 0$. For ease of notation we assume that $\mathtt{C} = [0,1)^3$. Let $\mathfrak{D}(n)$ be the class of N-adic cubes of sidelength N^{-n}, consisting of the cubes

$$D = \prod_{i=1}^{3} \big[k_i N^{-n}, (k_i+1)N^{-n}\big), \qquad \text{for some } k_i \in \{0,\dots,N^n - 1\}.$$

For any $n \in \mathbb{N}$ let $k = k(n) \in \mathbb{N}$ be such that

$$N^{-n-k-1} < \phi\big(\rho\, N^{-n}\big) \leq N^{-n-k}. \tag{3.12}$$

For any $D \in \mathfrak{D}(n+k)$ there is a unique compact cube D^* with the same centre and sidelength $(\rho/\sqrt{3})\, N^{-n-1}$, and (3.10) ensures that $D^* \supset D$.

We now consider an arbitrary packing $(B(x_i, r_i))$ of S consisting of balls contained in $\mathtt{C}$. Associate to each ball $B(x_i, r_i)$ in the packing the integer $n_i \in \mathbb{N}$ such that,

$$\rho\, N^{-n_i-1} < r_i \le \rho\, N^{-n_i}, \tag{3.13}$$

and the unique cube $D_i \in \mathfrak{D}(n_i + k(n_i))$ with $x_i \in D_i$. Then $D_i^* \subset B(x_i, r_i)$, because the diameter $\rho\, N^{-n_i-1}$ of D_i^* is smaller than r_i and $x_i \in D_i^*$. Moreover, D_i is hit by both motions. Abbreviating $k_i = k(n_i)$ and using (3.12) we infer that

$$\sum_{i=1}^{\infty} \phi(r_i) \le \sum_{i=1}^{\infty} N^{-n_i-k_i}\, 1\{\ell_\infty(D_i^*) \ge N^{-n_i-k_i}\} + \sum_{i=1}^{\infty} N^{-n_i-k_i}\, 1\{\ell_\infty(D_i^*) \le N^{-n_i-k_i}\}. \tag{3.14}$$

As the balls $B(x_i, r_i)$ and hence the cubes D_i^* are disjoint, we obtain for the *first term* in (3.14),

$$\sum_{i=1}^{\infty} N^{-n_i-k_i}\, 1\{\ell_\infty(D_i^*) \ge N^{-n_i-k_i}\} \le \sum_{i=1}^{\infty} \ell_\infty(D_i^*) \le \ell_\infty(\mathtt{C}). \tag{3.15}$$

To treat the *second term* in (3.14) we define the collection of cubes,

$$\mathfrak{S}(n+k) = \big\{ D \in \mathfrak{D}(n+k) \,:\, D \text{ is hit by both motions, and } \ell_\infty(D^*) \le N^{-n-k} \big\}.$$

Observe that

$$\sum_{i=1}^{\infty} N^{-n_i-k_i}\, 1\{\ell_\infty(D_i^*) \le N^{-n_i-k_i}\} \le \sum_{n=1}^{\infty} N^{-n-k(n)}\, \#\mathfrak{S}(n+k(n)) =: X\,. \tag{3.16}$$

Note that the random variable X on the right hand side of (3.16) is independent of the choice of the packing. We now show that it is almost surely finite. For this purpose recall from Lemma 3.4, that there exists a constant $C_{(3.17)}$ such that, for all $n \in \mathbb{N}$ and $D \in \mathfrak{D}(n+k)$,

$$\mathbb{P}\big\{ \ell_\infty(D^*) \le N^{-n-k} \,\big|\, D \text{ is hit by both motions} \big\} \le C_{(3.17)}\, N^{-k\xi}\,. \tag{3.17}$$

and recall that, for a constant $C_{(3.18)}$ only depending on the distance of $\mathtt{C}$ to the starting points,

$$\mathbb{P}\{ D \text{ is hit by both motions} \big\} \le C_{(3.18)}\, N^{-2(n+k)}. \tag{3.18}$$

Using (3.17), (3.18) and the fact that $\#\mathfrak{D}(n+k) = N^{3(n+k)}$, we get

$$\begin{aligned} \mathbb{E}X = \sum_{n=1}^{\infty} N^{-n-k(n)} & \sum_{D \in \mathfrak{D}(n+k(n))} \mathbb{P}\{ D \text{ is hit by both motions} \big\} \\ & \qquad\qquad \times \mathbb{P}\big\{ \ell_\infty(D^*) \le N^{-n-k(n)} \,\big|\, D \text{ is hit by both motions} \big\} \\ & \le C_{(3.18)} \sum_{n=1}^{\infty} N^{-3(n+k(n))} \sum_{D \in \mathfrak{D}(n+k(n))} \mathbb{P}\big\{ \ell_\infty(D^*) \le N^{-n-k(n)} \,\big|\, D \text{ is hit by both motions} \big\} \\ & \le C_{(3.17)} C_{(3.18)} \sum_{n=1}^{\infty} N^{-k(n)\xi} \le C_{(3.19)} \sum_{n=1}^{\infty} N^{n\xi} \phi(\rho N^{-n})^{\xi}, \end{aligned} \tag{3.19}$$

where $C_{(3.19)} > 0$ is a suitable constant. As the series on the right of (3.19) converges by (3.9), we infer that $\mathbb{E}X < \infty$, and hence $X < \infty$ almost surely.

We hence see from (3.14) that

$$\sum_{i=1}^{\infty} \phi(r_i) \le \ell_\infty(\mathtt{C}) + X < \infty \qquad \text{almost surely,}$$

and, as the δ-packing was chosen arbitrarily, we infer that $\mathcal{P}^\phi(S\cap \mathtt{C}) \le P^\phi(S\cap \mathtt{C}) < \infty$. Applying this result to $\tilde\phi$ with $\lim_{r\downarrow 0} \tilde\phi(r)/\phi(r) = \infty$ such that (3.8) still holds, gives $\mathcal{P}^\phi(S\cap \mathtt{C}) = 0$ almost surely.

Finally, we look at a single Brownian path $W\colon [0,1) \to \mathbb{R}^3$. Let $\mathrm{Dy}(n) := \{k2^{-n}\colon k = 0,\dots,2^n-1\}$, for $n\in\mathbb{N}$, and $\mathrm{Dy}(0) := \{0,1\}$. For each $x \in \mathrm{Dy}(n)\setminus \mathrm{Dy}(n-1)$ there exists a minimal $y = y(x) \in \mathrm{Dy}(n-1)$ with $x < y$, and a maximal $z = z(x) \in \mathrm{Dy}(n-1)$ with $x > z$. Then $[0,1)\times[0,1)$ can be decomposed (up to the diagonal) into countably many subsets $I\times J$, $J\times I$ where

$$I = [x, y(x)) \text{ and } J = [z(x), x), \qquad \text{for } n\ge 1, x\in \mathrm{Dy}(n)\setminus \mathrm{Dy}(n-1).$$

For each such set, the previous result can be applied to the two independent Brownian motions

$$\begin{aligned} &W^{(1)}\colon [0,2^{-n}) \to \mathbb{R}^3 \quad W^{(1)}(t) = W(x+t) - W(x)\\ &W^{(2)}\colon (0,2^{-n}] \to \mathbb{R}^3 \quad W^{(2)}(t) = W(x-t) - W(x), \end{aligned}$$

for $x\in \mathrm{Dy}(n)\setminus \mathrm{Dy}(n-1)$. Note that D is the union (over all decomposition sets) of the intersections of the paths of these independent Brownian motions. Hence, almost surely, under (3.8), we have $\mathcal{P}^\phi(D) = 0$, and this completes the proof.

4. Appendix: Brownian excursions between spheres

In this section we recall facts on Brownian excursions between concentric spheres, which were used in our proofs. Fix a dimension $d\ge 2$, radii $s\neq r$, and take a Brownian motion $(W(t)\colon t\ge 0)$ started uniformly on the sphere $\partial B(0,s)\subset\mathbb{R}^d$ and, if $d\ge 3$ and $s > r$, condition the path on eventually hitting $\partial B(0,r)$. Define random times

$$\tau := \inf\big\{t > 0 \,:\, |W(t)| = r\big\} \quad \text{ and } \quad \tau^* := \sup\big\{t < \tau \,:\, |W(t)| = s\big\}.$$

Then the random path $e\colon [0,\tau-\tau^*]\to\mathbb{R}^d$ defined by $e(t) = W(t+\tau^*)$ is called a *Brownian excursion* from $\partial B(0,s)$ to $\partial B(0,r)$ and $\zeta := \tau - \tau^*$ is called its *lifetime.*

Proposition 4.1. *Let e be a Brownian excursion from $\partial B(0,s)$ to $\partial B(0,r)$ and let ζ be its lifetime. Then its time-reversal $e^*\colon [0,\zeta]\to\mathbb{R}^d$, which is defined by $e^*(t) = e(\zeta - t)$, is a Brownian excursion from $\partial B(0,r)$ to $\partial B(0,s)$.*

Proof. Without loss of generality we may assume that $0 < r < s$. We invoke the skew-product representation, which states that the underlying Brownian motion $(W(t)\colon t\ge 0)$ can be written as $W(t) = \beta(H_t)\,R(t)$, where

- the *radial* part $(R(t)\colon t\ge 0)$ is a Bessel process with index $\nu = \frac{d}{2} - 1$ started at $R(0) = s$;
- the *spherical* part $(\beta(u)\colon u\ge 0)$ is a Brownian motion on the unit sphere $\partial B(0,1)\subset\mathbb{R}^d$, started uniformly at random, and independent of the radial part;
- the *winding clock* is $H_t = \int_0^t R(u)^{-2}\,du$.

Hence, to prove our result, it suffices to consider the radial parts of the processes. Conditioning $(R(t)\colon 0 \leq t \leq \tau)$ on $\tau < \infty$ (if $d \geq 3$) turns it into a Bessel process of index $-\nu$ stopped at its first hitting time of level r. By the Williams reversal theorem, see [W74], the time reversal $(R(\tau - t)\colon 0 \leq t \leq \tau - \tau^*)$ is a Bessel process with index ν considered between its last passage time at level r and its first passage time at level s. This readily implies the result. □

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Bibliography

[Be03] A. Berlinkov. Exact packing dimension in random recursive constructions. *Probab. Theory Relat. Fields* 126, 477–496 (2003).

[KM05] A. Klenke and P. Mörters. The multifractal spectrum of Brownian intersection local time. *Ann. Probab.* 33, 1255-1301 (2005).

[KS64] S. B. Kochen and S. J. Stone. A note on the Borel-Cantelli problem. *Illinois J. Math.* 8, 248–251 (1964).

[KM02] W. König and P. Mörters. Brownian intersection local times: upper tails and thick points. *Ann. Probab.* 30, 1605–1656 (2002).

[La96a] G. F. Lawler. Hausdorff dimension of cut points for Brownian motion. *Electron. Journal Probab.* 1, Paper 2, pp 1-20 (1996).

[La96b] G. F. Lawler. The dimension of the frontier of planar Brownian motion. *Electron. Comm. Probab.* 1, Paper 5, pp 29–47 (1996).

[La98] G. F. Lawler. Strict concavity of the intersection exponent for Brownian motion in two and three dimensions. *Math. Phys. Electron. Journal* 4, Paper 5, pp 1-67 (1998).

[LS01b] G. F. Lawler, O. Schramm and W. Werner. Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.* 187, 237-273 (2001).

[LS01c] G. F. Lawler, O. Schramm and W. Werner. Values of Brownian intersection exponents, II: Plane exponents. *Acta Math.* 187 , 275–308 (2001).

[LS02] G. F. Lawler, O. Schramm and W. Werner. Values of Brownian intersection exponents. III: Two-sided exponents. *Ann. Inst. Henri Poincaré* 38, 109-123 (2002).

[LG86] J.-F. Le Gall. The exact Hausdorff measure of Brownian multiple points. In: *Seminar on Stochastic Processes* (1986), Birkhäuser, Basel, pp 107–137.

[LG87] J.-F. Le Gall. Temps locaux d'intersection et points multiples des processus de Lévy *Seminaire de Probabilités XXI*, Lecture Notes in Math. 1247 (1987).

[LT86] J.-F. Le Gall and S.J. Taylor. The packing measure of planar Brownian motion. In: *Seminar on Stochastic Processes* (1986), Birkhäuser, Basel, pp 139–147.

[MO07] P. Mörters and M. Ortgiese. Small value probabilities via the branching tree heuristic. *Bernoulli* (to appear).

[MS99] P. Mörters and N.-R. Shieh. Small scale limit theorems for Brownian intersection local times. *Electron. Journal Probab.* 4, Paper 9, pp 1–23 (1999).

[Ta86] S.J. Taylor. The measure theory of random fractals. *Math. Proc. Cambridge Phil. Soc.* 100, 383–406 (1986).

[TT85] S.J. Taylor and C. Tricot. Packing measure and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.* 288, 679–699 (1985).

[W74] D. Williams Path decomposition and continuity of local time for one-dimensional diffusions I. *Proc. London Math. Soc.(3)* 28, 738–768 (1974).