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RANDOM FRACTALS

The term *fractal* usually refers to sets which, in some sense, have a self-similar structure. Already in the seventies of the last century Mandelbrot (1982) made a compelling case for the importance of this concept in mathematical modelling. Indeed, some form of self-similarity is common in random sets, in particular those arising from stochastic processes. Therefore studying fractal aspects is an important feature of modern stochastic geometry.

Early progress in fractal geometry often referred to sets with obvious self-similarity, like the fixed points of iterated function systems. These are toy examples, tailor-made to study self-similarity in its tidiest form. An overview of the achievements in this period can be obtained from (Falconer, 2003).

Starting with the work of Taylor in the sixties, researchers were also looking at sets, where self-similarity is more hidden. Such sets often arise in the context of stochastic processes. A beautiful survey of the state of the art in the mid 1980s, written by the protagonist in this area, is (Taylor, 1986). In the last ten years interest in this area has increased considerably, powerful techniques have been developed, and very substantial progress has been made. Typical examples of the fractals studied today are level sets of stochastic processes, the double points of random curves, or the boundary of excursions of random fields. The self-similar nature of these examples is typically less tidy and exploiting it means entering deep into the geometry of the sets.

Very roughly speaking, a set is *self-similar* if it can be decomposed into parts which look like scaled copies of the original set. This definition becomes particularly powerful when 'look like' is interpreted in a statistical sense, i.e. if it can be decomposed into parts which have (up to scaling) the same distribution as the whole set. This idea is naturally linked to trees: Starting from the root we identify the parts in the decomposition as the children of the root. Each part is itself a scaled copy of the whole picture and hence has a decomposition of the same kind as its parent, proceeding like this each point of the fractal has a natural address in the tree.

A crucial tool to bring the self-similarity of a random set to light is therefore its representation in terms of a tree, or sometimes a process on a tree. This technique has been exploited with great success in the last ten years and continues to be a vital tool. My main aim in this chapter is to show that the first step in many deep geometric problems for random sets is to find the self-similarity of the problem and capture it in form of a tree picture. This picture determines the key direction of the argument, although the formalised proof often does not make the tree structure explicit.

Questions in geometry are very often related to the *size* of sets. Other than in classical geometry, random sets can often already be distinguished by the crudest measure of size, which is dimension. The most powerful concept of dimension, but by far not the only one, is *Hausdorff dimension*, introduced almost a century ago by Hausdorff (1918). This concept extends the classical notion of dimension to arbitrary metric spaces, allowing non-integer dimensions for sufficiently irregular sets. The notion is based on a family of measures $\mathcal{H}_s$, $s \geq 0$, the s-Hausdorff measures, which for integer values $s = 1, 2, 3$ coincide with the classical measures of length, area, and volume. The Hausdorff dimension of a set A is the critical value $s > 0$ where the function $s \mapsto \mathcal{H}_s(A)$ jumps from infinity to zero. We do not give a precise definition of Hausdorff measure and dimension here, but refer the reader instead to the excellent book (Falconer, 2003).

The first section of this chapter is devoted to representing self-similarity in terms of trees, and we initially confine ourselves to simple examples. We show how to obtain the Hausdorff dimension of a set from a suitable tree representation and apply this to finding the Hausdorff dimension of the zero set of a linear Brownian motion.

In the second section we move to more sophisticated examples and present two more recent results on the fine structure of planar Brownian motion, which make great use of tree representations. On the one hand we look at the problem of the favourite sites, solved by Dembo, Peres, Rosen, and Zeitouni (2001), and on the other hand we study the multifractal spectrum of the intersection of two paths, a result of Klenke and Mörters (2005). In both cases, rather than giving details of the proof, we emphasise the underlying tree structure. We complete the section with a discussion of an open problem initialised by work of Bass, Burdzy, and Khoshnevisan (1994).

The second example introduces the notion of *probability exponents*, the general use of which we discuss in the third section. This particular aspect of random fractals has gained momentum through the discovery of an explicit formula for the intersection and disconnection exponents by Lawler, Schramm, and Werner (2001) and the subsequent award of the Fields medal to Werner in 2006. Here we discuss work of Lawler (1996a) on the Hausdorff dimension of the Brownian frontier and some closely related results.

1.1 Representing fractals by trees

There is no generally accepted definition of a statistically self-similar set, and we do not attempt to give one. Instead, we define a class of statistically self-similar sets, the *Galton-Watson fractals*, which comprises a number of interesting examples. We prove a formula for the Hausdorff dimension of Galton-Watson fractals, which gives us the opportunity to explore the relationship between branching processes and self-similarity and introduce basic ideas about probability on trees.

The forthcoming book (Lyons and Peres, 2008) gives a comprehensive account of this subject, on which much of this section is based.

1.1.1 *Fractals and trees*

We start with a general approach to capture the self-similar nature of fractals by means of *trees* with weights, so called *capacities*, associated to the edges, and investigate how the Hausdorff dimension of the fractal can be derived from the tree and the capacities.

A *tree* $T = (V, E)$ consists of a finite or countable set V of *vertices* and a set $E \subset V \times V$ of *edges*. For every $v \in V$ the set of *parents* $\{w \in V : (w, v) \in E\}$ consists of exactly one element, denoted by $\bar{v}$, except for exactly one distinguished element, called the *root* $\rho \in V$, which has no parent. For every $v \in V$ there is a unique self-avoiding path from the root to v, called the *ancestral line*, and the number of edges in this path is the *generation* $|v|$ of the vertex $v \in V$. For every vertex $v \in V$ we assume that the set of *children* $\{w \in V : (v, w) \in E\}$ is finite.

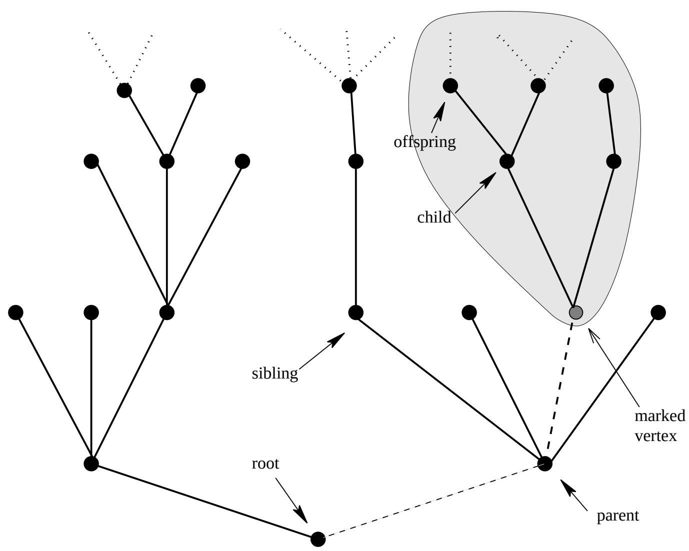

Fig. 1.1. A tree, with a vertex in the second generation marked; its ancestral line is dashed and the tree of its offspring shaded. One of its three siblings, and one of its two children are pointed out, as well as its parent and the root.

The *offspring* of a vertex v is the collection of vertices having v on their ancestral line. These vertices naturally form a subtree $T(v)$ of T. The *siblings* of $v \in V$ are the vertices $u \neq v$ with $\bar{u} = \bar{v}$. A sequence $(v_0, v_1, \ldots)$ of vertices such that $v_0 = \rho$ and $\bar{v_i} = v_{i-1}$ for all $i \geq 1$ is called a *ray* in the tree. The set of rays in T is denoted by ∂T. Finally, a set $\Pi \subset E$ is called a *cutset* if every ray includes an edge from the set Π.

We now describe a way to represent sets by marked trees. Let $T = (V, E)$ be an infinite tree and associate to each vertex $v \in V$ a nonempty, compact set $I_v \subset \mathbb{R}^d$ such that $I_v = \mathrm{cl}\big(\mathrm{int}\, I_v\big)$ and

- if v is a child of u, then $I_v \subset I_u$;
- if u and v are siblings, then $\mathrm{int}\, I_u \cap \mathrm{int}\, I_v = \emptyset$;
- for all rays $\xi = (v_0, v_1, \dots)$ we have $\lim_{n\to\infty} \mathrm{diam}(I_{v_n}) = 0$.

Then the set

$$I(T) = \bigcup_{\xi \in \partial T} \bigcap_{v \in \xi} I_v$$

is represented by the tree T and the marks $\{I_v \colon v \in V\}$. Observe that, except for a possible boundary effect, there is a one-to-one relationship between the points of $I(T)$ and the rays of the tree, which can be interpreted as addresses.

It is easy to see that for every compact subset of $\mathbb{R}^d$ there are many representations, but the idea of the method is to pick one which captures the structure of the set and leads to a simple tree.

We now give a formula for the Hausdorff dimension of sets in terms of the parameters of the tree representation. To this end we have to discuss the notion of flows on trees. Fix a mapping $C \colon E \to [0, \infty]$ representing the *capacities* of the edges. A mapping $\theta \colon E \to [0, c]$ such that

- we have $\sum_{\bar{w}=\rho} \theta\big((\rho, w)\big) = c$,
- for every vertex $v \neq \rho$ we have $\theta\big((\bar{v}, v)\big) = \sum_{\bar{w}=v} \theta\big((v, w)\big)$,
- for every $e \in E$ we have $\theta(e) \le C(e)$,

is called a *flow of strength* $c > 0$ through the tree with capacities C.

Theorem 1.1 *Suppose that a set $A \subset \mathbb{R}^d$ is represented by a tree T and sets $\{I_v \colon v \in V\}$. Assume additionally that*

$$\inf_{v \neq \rho} \frac{\mathrm{diam}(I_v)}{\mathrm{diam}(I_{\bar{v}})} > 0 \quad \text{and} \quad \inf_{v} \frac{\mathrm{vol}(\mathrm{int}\, I_v)}{\mathrm{diam}(I_v)^d} > 0, \tag{1.1}$$

and, for every $s \ge 0$, define capacities $C_s(e) = \mathrm{diam}(I_v)^s$ if $e = (\bar{v}, v)$. Then

$$\begin{aligned} \dim A &= \inf\Big\{ s \colon \inf_{\Pi \text{ cutset}} \sum_{e \in \Pi} C_s(e) = 0 \Big\} \\ &= \sup\Big\{ s \colon \text{ there is a flow with capacities } C_s \Big\}. \end{aligned}$$

Theorem 1.1 is not hard to prove. The first equality is little more than the definition of Hausdorff dimension, the second is the famous *max-flow min-cut theorem* from graph theory, which, when applied to trees, states that the maximal strength of a flow with capacities C equals the minimal sum of capacities over the edges in a cutset, see (Ford and Fulkerson, 1962).

Example 1.2 The ternary Cantor set can be canonically represented by a binary tree such that I_v is an interval of length $3^{-|v|}$. Assigning capacities $C_s = 3^{-sn}$ to edges with end-vertex in the nth generation, it is easy to see that a necessary and sufficient condition for a flow to exists is $3^s \le 2$. Hence we obtain that the dimension of the Cantor set is $\log 2/\log 3$.

1.1.2 *Galton-Watson fractals*

We now look at random sets given in terms of representations with randomly chosen tree and marks. For this purpose let $X = (N, A_1, \ldots, A_N)$ be a random variable consisting of a nonnegative integer N and weights $0 < A_i \le 1$. We construct a *(weighted) Galton-Watson tree* by sampling, successively for each vertex, an independent copy of X and assigning N children carrying weights $A_1, \ldots, A_N$. We will be concerned with tree representations with the property that the diameter of the set associated with a vertex v is the product of the weights along the ancestral line of v.

We now recall some well-known facts about Galton-Watson trees. The first question is when a Galton-Watson tree can be infinite and hence suitable for representing a set. Excluding the trivial case $\mathbf{P}\{N = 1\} = 1$, we get that

$$p = \mathbf{P}\big\{ \text{tree infinite} \big\} > 0 \quad \text{if and only if} \quad \mathbf{E}N > 1.$$

A slightly less known important fact is the following zero-one-law for Galton-Watson trees. Let A be a set of trees or, equivalently, a property of trees. We say that A is *inherited* if

- every finite tree is in A, and
- if the tree $T \in A$ and $v \in V$ is a vertex of the tree, then $T(v) \in A$.

Then every inherited property A has $\mathbf{P}\{T \in A\} \in \{1 - p, 1\}$ or, equivalently,

$$\mathbf{P}\big\{T \in A \,\big|\, \text{tree infinite}\big\} \in \{0, 1\}.$$

Suppose now that (random) sets $\{I_v : v \in V\}$ are assigned to the vertices of the Galton-Watson tree in the way of a tree representation such that additionally

$$\inf_v \frac{\mathrm{vol}(\mathrm{int}\, I_v)}{\mathrm{diam}(I_v)^d} > 0,$$

and the normalized diameters correspond to the weights in the sense that

$$\frac{\mathrm{diam}(I_v)}{\mathrm{diam}(I_\rho)} = \prod_{i=1}^{n} A(v_i),$$

where $(\rho, v_1, \ldots, v_n)$ are the vertices on the ancestral line of the vertex $v = v_n$ and $A(v_1), \ldots, A(v_n)$ are the associated weights. Then the set $I(T)$ represented by this tree is a *Galton-Watson fractal*.

By Theorem 1.1, to find the Hausdorff dimension of the Galton-Watson fractals, we first need to study the existence of flows on Galton-Watson trees with edge capacities

$$C_s\big((\bar{v}, v)\big) = \prod_{i=1}^{n} A(v_i)^s.$$

The answer to this question is given by the following theorem of Falconer (1986). Note that the excluded case is trivial.

Theorem 1.3. (Falconer's theorem) *Suppose that a weighted Galton-Watson tree is given by the generating variable $X = (N, A_1, \dots, A_N)$, let $s > 0$ and assume that $\sum_{i=1}^{N} A_i^s \neq 1$ with positive probability. Let*

$$\gamma = \mathbf{E}\Big[\sum_{i=1}^{N} A_i^s\Big].$$

(a) If $\gamma \leq 1$ then almost surely no *flow is possible.*
(b) If $\gamma > 1$ then flow is possible almost surely given that the tree is infinite.

Note that in the special case when $A_i = 1$ almost surely, we recover the criterion for trees being finite. We now give a proof of Theorem 1.3, which is due to Falconer (part (a)) and Lyons and Peres (part (b)). The second part of the proof uses the idea of *percolation*, which is another important technique in fractal geometry.

Proof of (a): If $(v_0, \dots, v_n)$ are the vertices on the ancestral line of $w = v_n$ and let $v = v_j$ for some $j \leq n$, we equip the tree $T(v)$ with capacities $C_s^v((\bar{w}, w)) = \prod_{i=j+1}^{n} A(v_i)^s$, and let $\theta(v)$ be the maximal strength of a flow in this subtree. Abbreviating $\theta = \theta(\rho)$ we have

$$\theta = \sum_{\bar{v}=\rho} \big(A(v)^s \wedge (A(v)^s \theta(v))\big) = \sum_{\bar{v}=\rho} A(v)^s \big(1 \wedge \theta(v)\big). \tag{1.2}$$

Now suppose that $\gamma \leq 1$ and suppose $X = (N, A_1, \dots, A_N)$ describes the children of the root and their weights. Using independence, and the fact that θ and $\theta(v)$ have the same distribution for every edge v,

$$\begin{aligned}
\mathbf{E}[\theta] &= \sum_{n=1}^{\infty} \mathbf{E}\big[\theta \mathbf{1}_{\{N=n\}}\big] = \sum_{n=1}^{\infty} \sum_{v=1}^{n} \mathbf{E}\big[A(v)^s (1 \wedge \theta(v)) \mathbf{1}_{\{N=n\}}\big\} \\
&= \sum_{n=1}^{\infty} \sum_{v=1}^{n} \mathbf{E}\big[A(v)^s \mathbf{1}_{\{N=n\}}\big] \mathbf{E}\big[1 \wedge \theta(v)\big] \\
&= \sum_{n=1}^{\infty} \mathbf{E}\big[\sum_{v=1}^{N} A(v)^s \mathbf{1}_{\{N=n\}}\big] \mathbf{E}\big[1 \wedge \theta\big] = \gamma \mathbf{E}[1 \wedge \theta] \leq \mathbf{E}[1 \wedge \theta].
\end{aligned}$$

Hence $\theta \leq 1$ almost surely and $\mathbf{P}\{\theta > 0\} > 0$ only if $\gamma = 1$. This already shows that no flow is possible if $\gamma < 1$. In the case $\gamma = 1$ we get from (1.2) and independence, using that $\theta \leq 1$, that

$$\text{ess sup}\,(\theta) = \text{ess sup}\,\Big(\sum_{v=1}^{N} A(v)^s\Big)\,\text{ess sup}\,(\theta)\,.$$

Hence, if ess sup $(\theta) > 0$ we have ess sup $(\sum_{v=1}^N A(v)^s) = 1$. As $\mathbf{E}[\sum_{v=1}^N A(v)^s] = \gamma = 1$ we must have $\sum_{v=1}^N A(v)^s = 1$, which is the excluded case. Hence $\theta = 0$ almost surely, which means that no flow is possible.

Proof of (b): We first look at a fixed (deterministic) tree T with weights $A(v)$ attached to the vertices. We introduce a family of random variables on this tree T as follows. Independently for every edge $e = (\bar{v}, v) \in E$ we let

$$X(e) = \begin{cases} 1 & \text{with probability } A(v)^s, \\ 0 & \text{with probability } 1 - A(v)^s. \end{cases}$$

The intuition is that an edge e is *open* if $X(e) = 1$ and otherwise closed. We consider the subtree $T^* \subset T$ consisting of all edges which are connected to the root by a path of open edges. Let $Q(T) = P\{T^* \text{ is infinite}\}$. For any cutset Π note that $\sum_{e\in\Pi} C_s(e)$ is the expected number of edges in Π, which are also in T^*. Hence

$$\sum_{e\in\Pi} C_s(e) \geq P\big\{e \in T^* \text{ for some } e \in \Pi\big\} \geq P\big\{T^* \text{is infinite}\big\}.$$

If $\theta(T)$ is the maximal strength of a flow in T, then the last inequality together with the max-flow min-cut theorem shows that

$$Q(T) > 0 \implies \theta(T) > 0. \tag{1.3}$$

Now we use this result for a Galton-Watson tree, by performing a two-step experiment: first sampling the tree T and the reducing it to T^*. As a result of the experiment, T^* is another Galton-Watson tree. Denoting by $v_1, \ldots, v_n$ the children of the root, we get for the mean number of children in T^*,

$$\begin{aligned}
\mathbf{E}\Big[\sum_{i=1}^{N} X\big((\rho, v_i)\big)\Big] &= \mathbf{E}\Big[\sum_{n=1}^{\infty}\sum_{i=1}^{n} X\big((\rho, v_i)\big)\mathbf{1}_{\{N=n\}}\Big] \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbf{E}\big[X\big((\rho, v_i)\big)\mathbf{1}_{\{N=n\}}\big] = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbf{E}\big[A(v_i)^s\mathbf{1}_{\{N=n\}}\big] \\
&= \sum_{n=1}^{\infty} \mathbf{E}\Big[\sum_{i=1}^{N} A(v_i)^s\mathbf{1}_{\{N=n\}}\Big] = \mathbf{E}\Big[\sum_{i=1}^{N} A(v_i)^s\Big] = \gamma.
\end{aligned}$$

If $\gamma > 1$, by the criterion for Galton-Watson trees being infinite, we have

$$0 < \mathbf{P}\big\{T^* \text{ is infinite } \big\} = \mathbf{E}\big[Q(T)\big].$$

Hence $Q(T) > 0$ with positive probability, and by (1.3) we infer that $\theta(T) > 0$ with positive probability. In other words, $\mathbf{P}\{\theta(T) = 0\} < 1$. As the event $\{\theta(T) = 0\}$ is inherited, we infer from the Galton-Watson zero-one-law that we have $\theta(T) > 0$, almost surely on the tree being infinite. □

Up to some technicalities, the dimension formula for Galton-Watson fractals, found independently by Falconer (1986) and Mauldin and Williams (1986), now follows by combining Falconer's theorem and the dimension formula for tree representations, Theorem 1.1.

Theorem 1.4. (Hausdorff dimension of Galton-Watson fractals) *Suppose that $I(T)$ is a Galton-Watson fractal associated with a weighted Galton-Watson tree with generating variable $X = (N, A_1, \dots, A_N)$. Then, almost surely on the event $\{I(T) \neq \emptyset\}$,*

$$\dim I(T) = \min\Big\{ s \,:\, \mathbf{E}\Big[\sum_{i=1}^{N} A_i^s\Big] \le 1 \Big\}.$$

An interesting corollary comes from the fact that in the *critical case* $\gamma = 1$ flow is impossible unless we are in the excluded case $\sum_{i=1}^N A_i^s = 1$, in which flow is obviously possible.

Corollary 1.5 *If $\dim I(T) = s$ and $\sum_{i=1}^N A_i^s \neq 1$ with positive probability, then $\mathcal{H}_s(I(T)) = 0$ almost surely.*

We now exploit our main result by giving formulas for the Hausdorff dimension of a variety of sets. The main example, presented in some detail, is the zero set of a linear Brownian motion, which we study avoiding the use of local times.

Example 1.6 We define *percolation fractals*, or *percolation limit sets*. Fix the ambient dimension d, a parameter $p \in (0,1)$ and an integer $n \ge 2$. Divide $[0,1]^d$ into n^d nonoverlapping compact subcubes of equal sidelength. Keep each independently with probability p, and remove the rest. Apply the same procedure to the remaining cubes ad infinitum. The remaining set is a Galton-Watson fractal which has a generating random variable $(N, A_1, \dots, A_N)$, where N is binomial with parameters n^d and p, and A_i deterministic with $A_i = 1/n$. The probability that it is nonempty is positive if and only if $p > 1/n^d$. Moreover,

$$\mathbf{E}\Big[\sum_{i=1}^{N} A_i^s\Big] = \Big(\frac{1}{n}\Big)^s \mathbf{E}N = \frac{n^d p}{n^s}.$$

This is ≤ 1 if and only if $s \ge d + \frac{\log p}{\log n}$. Hence, almost surely on $\{I(T) \neq \emptyset\}$,

$$\dim I(T) = d + \frac{\log p}{\log n}.$$

Example 1.7 We compare the following two random fractals: On the one hand a percolation fractal based on dividing the unit interval $[0,1]$ into three non-overlapping intervals of length $1/3$ and keeping each with probability $p = 2/3$, on the other hand the random fractal obtained by dividing $[0,1]$ into three non-overlapping intervals of length $1/3$ and keeping two randomly chosen intervals out of the three, proceeding like this ad infinitum.

In both cases we obtain fractals of Hausdorff dimension $s = \log 2/\log 3$. To see this in the second case just observe that the 3-adic coding tree of the fractal is the dyadic tree, exactly as in the case of the ordinary ternary Cantor set. Corollary 1.5 indicates a significant difference between the two examples. Whereas for the first case, by the corollary, the s-Hausdorff measure is zero, one can show that in the second case the s-Hausdorff measure is strictly positive. This can be seen from the fact that there exists a flow on the coding tree with capacities $C_s(\bar{v}, v) = |I_v|^s$ in the second example, whilst there is none in the first.

1.1.3 *The dimension of the zero-set of Brownian motion*

We now use the theory developed so far to calculate the dimension of the *zero-set* of a Brownian motion $W\colon [0,1] \to \mathbb{R}$. The idea of this proof is based on Galton-Watson fractals and is due to Graf et al. (1988).

A first step is to make the problem more symmetric by looking at a *Brownian bridge* instead of a Brownian motion. There are several ways of defining a Brownian bridge B from a Brownian motion W:

- The process $B(t) = W(t) - tW(1)$, for $t \in [0,1]$, is a Brownian bridge.
- Let $T = \sup\{t < 1\colon W(t) = 0\}$, then the process $C(t) = \sqrt{1/T}\, W(tT)$, for $t \in [0,1]$, is also a Brownian bridge.

Note that for a given sample path W of Brownian motion the two bridges B and C have quite different sample paths. From the second definition it is easy to see that the dimension of the zero set of a Brownian bridge and of a Brownian motion have the same law. An important property of the Brownian bridge is *symmetry*: If $\{B(t)\colon 0 \le t \le 1\}$ is a Brownian bridge, then so is the process $\{\tilde{B}(t)\colon 0 \le t \le 1\}$ defined by $\tilde{B}(t) = B(1-t)$.

To study the dimension of the zero set of a Brownian bridge, define

$$T_1 = \sup\{t \le 1/2 \,:\, B(t) = 0\} \text{ and } T_2 = \inf\{t \ge 1/2 \,:\, B(t) = 0\}.$$

By symmetry the random variables T_1 and $1 - T_2$ have the same distribution (but they are not independent). The interval (T_1, T_2) does not contain any zeros, and we remove it from $[0,1]$, which leaves us with two random intervals $[0, T_1]$ at the left and $[T_2, 1]$ on the right. Moreover, it is not hard to show that the process

$$\left\{ \sqrt{1/T_1}\, B(tT_1) \,:\, 0 \le t \le 1 \right\}$$

is a Brownian bridge, which is independent of $\{B(t) \,:\, t \ge T_1\}$.

Now we can represent the zero set of the Brownian bridge as a Galton-Watson fractal: we start with the interval $[0,1]$ and remove the interval (T_1, T_2). To the left of the removed interval, we have an independent Brownian bridge

$$\big\{\sqrt{1/T_1}\, B(tT_1) \,:\, 0 \le t \le 1\big\}.$$

By symmetry, we also have an independent Brownian bridge

$$\big\{\sqrt{1/(1-T_2)}\, B(1-t(1-T_2)) \,:\, 0 \le t \le 1\big\}.$$

to the right of the removed interval. If we apply the same procedure on each of the remaining bridges, we iteratively construct the zero set of the Brownian bridge by removing all gaps. The essence of all this is the following:

Lemma 1.8 *The zero set of a Brownian bridge B is a Galton Watson fractal with generating random variable $X = (2, T_1, 1-T_2)$. Hence* $\dim\{t \in [0,1] : B(t) = 0\} = \alpha$, *where α is the unique solution of*

$$\mathbf{E}\big[T_1^\alpha + (1-T_2)^\alpha\big] = 1.$$

We can now calculate the dimension by evaluating this expectation for the right value of α.

Lemma 1.9

$$\mathbf{E}[\sqrt{T_1} + \sqrt{1-T_2}] = 1.$$

Proof By symmetry of the Brownian bridge, T_1 and $1-T_2$ have the same distribution, hence it suffices, to show that $\mathbf{E}[\sqrt{1-T_2}] = 1/2$. We have, using the definition of the Brownian bridge and the time inversion property of Brownian motion,

$$\begin{aligned} T_2 &= \inf\big\{1/2 \le t \le 1 \colon B(t) = 0\big\} \\ &= \inf\big\{1/2 \le t \le 1 \,:\, W(t) - tW(1) = 0\big\} \\ &\overset{d}{=} \inf\big\{1/2 \le t \le 1 \,:\, tW(1/t) - tW(1) = 0\big\} \\ &= \inf\big\{1/2 \le t \le 1 \colon W(1/t) - W(1) = 0\big\} \\ &= 1/\sup\big\{1 \le s \le 2 \colon W(s) - W(1) = 0\big\}. \end{aligned}$$

As $\{W(s) - W(1) \,:\, s \ge 1\}$ has the same law as $\{W(s-1) \,:\, s \ge 1\}$, we have

$$T_2 \overset{d}{=} \frac{1}{1 + \sup\big\{0 \le t \le 1 \,:\, W(t) = 0\big\}}$$

and, in particular,

$$\mathbf{E}\sqrt{1-T_2} = \int_0^1 \sqrt{\frac{x}{1+x}}\, f(x)\, dx\,,$$

where f is the density of the random variable $L = \sup\big\{0 \le t \le 1 \,:\, W(t) = 0\big\}$. This random variable has the arcsine-distribution, which can be verified using the *reflection principle* of Brownian motion, see e.g. (Mörters and Peres, 2008). We get that

$$\begin{aligned}\mathbf{E}\sqrt{1-T_2} &= \frac{1}{\pi}\int_0^1 \sqrt{\frac{x}{1+x}}\,\frac{dx}{\sqrt{x(1-x)}}\\ &= \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\arcsin(1) = \frac{1}{2},\end{aligned}$$

which completes the proof of Lemma 1.9. □

We have thus proved the following result.

Theorem 1.10 *Almost surely,*

$$\dim\{t \in [0,1] \colon W(t) = 0\} = \dim\{t \in [0,1] \colon B(t) = 0\} = \tfrac{1}{2}.$$

1.2 Fine properties of stochastic processes

In this section we discuss two deeper results, which were solved using the tree approach. We also state an interesting open problem, which may be suitable for a treatment based on these ideas.

1.2.1 *Favourite points of planar Brownian motion*

Suppose $(W(t) \colon 0 \le t \le 1)$ is a planar Brownian motion, and denote by

$$T(A) = \int_0^1 \mathbf{1}_{\{W(s)\in A\}}\,ds$$

the *occupation time* of the path in $A \subset \mathbb{R}^2$. A famous problem of Erdős and Taylor (stated in 1960 for the analogous random walk case) is to find the asymptotics of the occupation time around the favourite points,

$$T^*(\epsilon) = \max_{x\in\mathbb{R}^2} T\big(\mathcal{B}(x,\epsilon)\big) \qquad \text{as } \epsilon \downarrow 0\,.$$

This problem was solved by Dembo, Peres, Rosen, and Zeitouni (2001) exploiting the deep self-similar structure of the Brownian path using tree ideas.

Theorem 1.11 *Almost surely,* $T^*(\epsilon) \sim 2\epsilon^2 \log^2 \epsilon$ *as* $\epsilon \downarrow 0$.

A detailed account of the proof of this and some closely related results can be found in (Dembo, 2005). Other than the original paper (Dembo et al., 2001), this highly recommended source also discusses the tree analogy in depth. In our account we focus entirely on a rough sketch of this analogy. This captures the main idea of the proof, but neglects a lot of (often interesting) technical details.

Recall that a planar Brownian motion is neighbourhood recurrent, i.e. any ball is visited infinitely often as time goes to infinity. The main difficulty in the proof of Theorem 1.11 lies in the fact that the occupation time in a ball $\mathcal{B}(x, \epsilon)$ is accumulated during a large number of excursions from its boundary, whose lengths vary across a large range of scales. This leads to a complicated dependence between $T(\mathcal{B}(x, \epsilon))$ and $T(\mathcal{B}(y, \epsilon))$, even if x and y are relatively far away. The main merit of the tree picture is to organise this dependence structure in a natural fashion.

If a ball is visited often, by the law of large numbers, the time spent in the ball can be well approximated by the number of excursions from its boundary. To be more precise, let $x \in \mathbb{R}^2$ and consider a sequence of decreasing radii such that $\epsilon_k/\epsilon_{k-1} = k^{-3}$. Fix $a > 0$ and let N_k^x be the number of excursions from $\partial\mathcal{B}(x, \epsilon_{k-1})$ to $\partial\mathcal{B}(x, \epsilon_k)$ before time one. We call $x \in \mathbb{R}^s$ an *n-perfect* point if

$$N_k^x \approx 3ak^2 \log k \quad \text{for all } k \in \{2, \ldots, n\}\,.$$

During an excursion from $\partial\mathcal{B}(x, \epsilon_k)$ to $\partial\mathcal{B}(x, \epsilon_{k-1})$ the path spends on average about $3\epsilon_k^2 \log k$ time units in the ball $\mathcal{B}(x, \epsilon_k)$, and these times are all independent, so that a law of large numbers applies. As $\log(1/\epsilon_k) \approx 3k \log k$, we get that if x is n-perfect then it is *n-favourite* in the sense that

$$\frac{T\big(\mathcal{B}(x, \epsilon)\big)}{\epsilon^2 \log^2 \epsilon} \approx a \quad \text{for all } \epsilon_2 \geq \epsilon \geq \epsilon_n\,. \tag{1.4}$$

Strictly speaking the n-perfect points are only a subset of the n-favourite points, but the difference is small enough for us to neglect this distinction from now on. Note that, by definition, if x is n-perfect, it is also m-perfect for all $m \leq n$.

We now focus on the favourite points inside a square S of sidelength ϵ_1. We partition S into $(\epsilon_n/\epsilon_1)^{-2} = (n!)^6$ non-overlapping squares $S(n, i)$ of sidelength ϵ_n with centres $x_{n,i}$. This decomposition yields a natural tree representation of the cube S, with squares $S(n, i)$ associated to the vertices in the nth generation, such that any vertex is offspring of another one, if its associated square is contained in that of the other. Observe that in this tree, denoted T, any vertex of the kth generation has exactly $(k + 1)^6$ children.

In a rough approximation, which needs to be refined in the actual proof, we represent the set

$$\Big\{ x \in S\colon \lim_{\epsilon \downarrow 0} \frac{T\big(\mathcal{B}(x, \epsilon)\big)}{\epsilon^2 \log^2 \epsilon} \approx a \Big\}$$

by the tree T_a consisting of all vertices in the nth generation corresponding to squares with n-perfect centre. Here we *neglect* the fact that, because of the different centres, a square of sidelength ϵ_n with n-perfect centre may be contained in a square of sidelength ϵ_k, $k \leq n$, whose centre fails to be k-perfect. As most squares are sufficiently far away from the boundary of their parental square, this approximation turns out to be safe. We therefore have to show that, almost surely, the tree T_a is infinite if $a < 2$ and finite if $a > 2$.

To get hold of the squares with perfect centre, we fix a square $S(n,i)$ and map the planar Brownian curve onto a homogeneous Markov chain $(Z_k\colon k \in \mathbb{N})$ with values on the set $\{1,\dots,n\}$. This Markov chain is started in $Z_0 = n$, and the transition probabilities of the Markov chain are given, for $j \geq 1$, as

$$P\big\{Z_j = \ell \,\big|\, Z_{j-1} = k\big\} = \begin{cases} 1 & \text{if } k = n,\ \ell = n-1, \\ p_k & \text{if } 1 < k < n,\ \ell = k-1, \\ 1 - p_k & \text{if } 1 < k < n \text{ and } \ell = k+1, \\ 1 & \text{if } k = \ell = 1, \end{cases}$$

for

$$p_k = \frac{\log \epsilon_{k+1} - \log \epsilon_{k+2}}{\log \epsilon_k - \log \epsilon_{k+2}}.$$

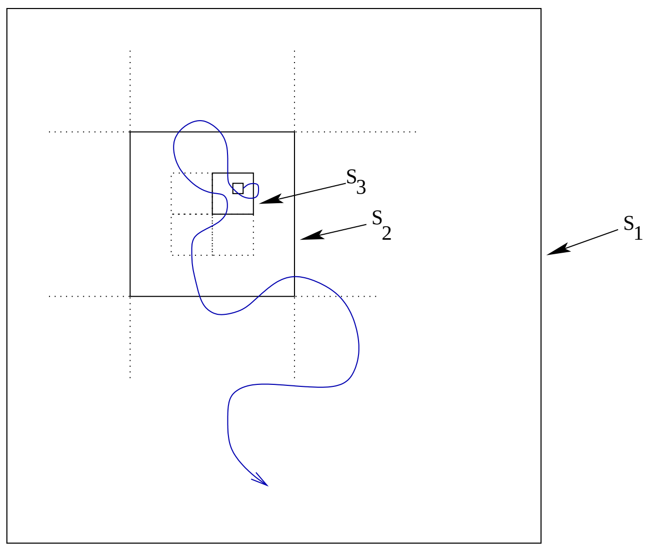

FIG. 1.2. Brownian motion moving between squares. In this picture $n = 4$ and the shown path yields the chain $4,3,4,3,2,3,2$.

The rationale behind this choice is that, if $S = S_1 \supset S_2 \supset \cdots \supset S_n = S(n,i)$ is the sequence of construction squares containing $S(n,i)$, we follow the Brownian curve from the first time it hits the boundary of S_n and, as indicated in Figure 1.2, whenever the motion moves from the boundary of S_k to the boundary of $S_{k\pm1}$, the chain moves from state k to $k \pm 1$. If squares are approximated by concentric balls of the same diameter, the probability that a Brownian motion started on the sphere of radius ϵ_{k+1} hits the sphere of radius ϵ_k before the sphere of radius ϵ_{k+2} is given by p_k, see e.g. (Mörters and Peres, 2008). The motion is stopped once it leaves S, which makes one an absorbing state.

Summarising, the square $S(n,i)$ is kept in the construction if and only if the associated Markov chain satisfies

$$\sum_{\ell=1}^{\infty} 1\{Z_\ell = k-1, Z_{\ell+1} = k\} \approx 3ak^2 \log k \qquad \text{for all } k = 2, \dots, n.$$

The picture given so far suffices to show that $\partial T_a = \emptyset$ if $a > 2$. Indeed, using a Markov chain calculation, one can see that, for any vertex $v \in V$ with $|v| = n$ we have $\mathbf{P}\{v \in T_a\} \approx (n!)^{-3a}$, and hence, looking at the expected number of retained vertices,

$$\mathbf{E}\#\big\{v \in T_a \colon |v| = n\big\} \approx (n!)^{6-3a} \longrightarrow 0 \qquad \text{if } a > 2\,.$$

For the lower bound first moment arguments as above are insufficient and we need to look at the more complex picture arising when two squares are considered simultaneously. For this purpose we fix $a < 2$ and two vertices v, w from the nth generation of T, whose oldest common ancestor is in generation $0 < m < n$. To get hold of $\mathbf{P}\{v, w \in T_a\}$ we look at the Markov chain $(Z_n \colon n \in \mathbb{N})$ on the branching set $\{1, \dots, n\} \dot\cup \{m+1, \dots, n\}$ shown in Figure 1.3. The chain can only change branch when it moves up from state m, and in this case each branch is chosen with the same probability. Otherwise the transition probabilities are the same as before, where we allow ourselves an abuse of notation by using the same symbol for the *distinct* states on the two branches of the state space that emerge from state m.

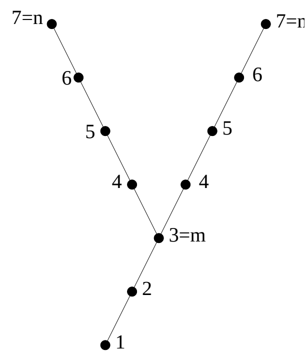

FIG. 1.3. The statespace of the Markov chain as a branching structure.

The rationale behind this chain is that the state j on the left branch represents the construction square of sidelength ϵ_j containing the square representing v, and the state $j > k$ on the right branch represents the construction square of sidelength ϵ_j containing the square representing w. The transition probabilities mimic the consecutive visits of the boundaries of these squares by the Brownian curve, though this mapping is imprecise about excursions between squares of radius ϵ_{m+1} and ϵ_m. This effect turns out to be negligible.

A Markov chain calculation shows that

$$\mathbf{P}\big\{v, w \in T_a\big\} \approx (n!)^{-6a}\,(k!)^{3a}\,,$$

and from this we obtain a constant $C > 0$ and a bound on the variance

$$\mathbf{E}\big[\big(\#\big\{v \in T_a \colon |v| = n\big\}\big)^2\big] \le C\,\mathbf{E}\big[\#\big\{v \in T_a \colon |v| = n\big\}\big]^2\,.$$

We can therefore use the Paley-Zygmund inequality to derive, for any $0 < \lambda < 1$,

$$\mathbf{P}\Big\{\#\big\{v \in T_a : |v| = n\big\} \ge (1-\lambda)\mathbf{E}\big[\#\big\{v \in T_a \colon |v| = n\big\}\big]\Big\}$$
$$\ge \lambda^2\,\frac{\mathbf{E}[\#\{v \in T_a \colon |v| = n\}]^2}{\mathbf{E}[(\#\{v \in T_a \colon |v| = n\})^2]} \ge C^{-1}\,\lambda^2 > 0\,.$$

Recall that, if $a < 2$, we have

$$\mathbf{E}\big[\#\big\{v \in T_a \colon |v| = n\big\}\big] \longrightarrow \infty,$$

and hence this argument shows that $\partial T_a \neq \emptyset$ with positive probability. A self-similarity argument (not unlike the Galton-Watson zero-one law) shows that this must therefore hold with probability one.

Let me emphasise the importance of the correct choice of the scales (ϵ_k) for the success of the tree approximation. If the ratio $\epsilon_{k-1}/\epsilon_k$ is chosen significantly smaller, the excursion counts typically do not reflect the occupation times at all radii and centres; observe that we need the equivalence analogous to (1.4) *simultaneously* for all squares of sidelength ϵ_k, $k \in \{2, \dots, n\}$, so that a rigorous proof requires a much more quantitative approach to this part of the argument than our informal discussion suggests. Conversely, if the ratio $\epsilon_{k-1}/\epsilon_k$ is chosen significantly larger, we lose the necessary control over the occupation times for intermediate radii.

Finally, a note of caution: Turning this picture into a full proof of Theorem 1.11 still requires skill and a lot of work, as we oversimplified at many places. Nevertheless, the tree representation gives a neat organisation of the complicated dependencies, which greatly helps understanding and solving this hard problem.

1.2.2 *The multifractal spectrum of intersection local time*

The *multifractal spectrum* is an important means of describing the fine structure of a fractal measure, see (Mörters, 2008) for a subjective discussion of its importance in the context of stochastic processes. For a precise definition, fix a locally finite measure μ, which may be random or non-random. The value $f(a)$ of the multifractal spectrum is the Hausdorff dimension of the set of points x with local dimension

$$\lim_{r \downarrow 0} \frac{\log \mu(\mathcal{B}(x, r))}{\log r} = a, \tag{1.5}$$

where $\mathcal{B}(x, r)$ denotes the open ball of radius r centred in x. In some cases of interest, the limit in (1.5) has to be replaced by liminf or limsup to obtain an interesting nontrivial spectrum.

Examples of multifractal spectra for measures arising in probability are the occupation measures of stable subordinators (Hu and Taylor, 1997), the states of super-Brownian motion (Perkins and Taylor, 1998), and the harmonic measure on a Brownian path (Lawler, 1997). The example we study here has some likeness with the first two examples, for which a similar tree analogy could be built, though details in the proof invariably differ considerably.

We look at two independent planar Brownian motions $(W_t^{(1)} \colon 0 \leq t \leq 1)$ and $(W_t^{(2)} \colon 0 \leq t \leq 1)$ and study the intersection set

$$\mathcal{S} = \left\{ x \in \mathbb{R}^2 \colon \text{ there exist } 0 \leq s, t \leq 1 \text{ with } W_s^{(1)} = W_t^{(2)} = x \right\}.$$

The natural measure on $\mathcal{S}$ is the *intersection local time* μ defined symbolically by

$$\mu(A) = \int_A dx \prod_{i=1}^{2} \int_0^1 ds \; \delta_0(W_s^{(i)} - x)\,,$$

Rigorous definitions of μ can be given by approximation of the 'delta-function' δ_0, but also as a suitable Hausdorff measure on $\mathcal{S}$. Technical details of the construction are not of interest to us here.

Theorem 1.12 *For every* $2 \leq a \leq \frac{70}{11}$ *we have, almost surely,*

$$\dim \Big\{ x \in \mathcal{S} \colon \limsup_{r \downarrow 0} \frac{\log \mu(\mathcal{B}(x,r))}{\log r} = a \Big\} = \frac{1}{12}\Big(\frac{70}{a} - 11 \Big)\,.$$

Moreover, there are no points with local dimension $a < 2$ or $a > \frac{70}{11}$ in any sense (liminf, limsup, or lim). At least heuristically, all the results concerning values $a \geq 2$ can be read off a tree picture, which we describe below. The full proof of the result, which is inspired by this tree picture but does not make explicit use of it, can be found in Klenke and Mörters (2005).

As $\mathcal{S}$ is the intersection of two independent sets of full dimensions (the Brownian paths) it is not surprising that $\dim \mathcal{S} = 2$ and therefore $\mu(\mathcal{B}(x,r)) \approx r^2$ for typical points $x \in \mathcal{S}$. Fix $a > 2$ for the remainder of this section. For the points $x \in \mathcal{S}$ with

$$\mu(\mathcal{B}(x,r)) \approx r^a \ll r^2$$

we expect that

- the ball $\mathcal{B}(x,r)$ is visited only once by each Brownian motion,
- the intersection local time spent in $\mathcal{B}(x,r)$ during this visit is small.

Due to the first item, the recurrence effects that were so crucial in the proof of Theorem 1.11 do not play a rôle here. Indeed, here we can assume that for disjoint balls $\mathcal{B}(x,r)$ and $\mathcal{B}(y,r)$ the events $\{\mu(\mathcal{B}(x,r)) \approx r^a\}$ and $\{\mu(\mathcal{B}(y,r)) \approx r^a\}$ are essentially independent. This simplifies the informal discussion immensely, but making this argument rigorous is one of the main difficulties in the proof of Theorem 1.12, which we do not discuss here.

The remainder of our discussion of this example is based on this independence (or locality) assumption. Fix a square $S \subset \mathbb{R}^2$ of unit sidelength and pick a large integer m. Divide the square into m^2 squares of sidelength $1/m$, and keep a square if it contains a point of $\mathcal{S}$, then repeat this procedure with any square kept, and so on at infinitum. Identifying the squares kept in the procedure with vertices in a tree $T = (V, E)$, we obtain a tree representation of $\mathcal{S} \cap S$.

To connect the intersection local time μ to this tree representation, we recall a result of Le Gall (1986), which states that μ can be recovered from the volume of the Wiener sausages around the two Brownian paths, more precisely

$$\lim_{\epsilon \downarrow 0} \frac{(\log \epsilon)^2}{\pi^2} \operatorname{vol}\big(S_\epsilon^{(1)} \cap S_\epsilon^{(2)} \cap A\big) = \mu(A)\,,$$

where $S_\epsilon^{(i)} = \{x \in \mathbb{R}^2 \colon |W_t^{(i)} - x| \le \epsilon \text{ for some } 0 \le t \le 1\}$. This suggests that, given a square $v \in V$, we have that

$$\mu(v) \approx \lim_{n\to\infty} \frac{Z_n(v)}{m^{2n} n^{-2}}\,,$$

where $Z_n(v)$ is the number of offspring of v in the nth generation. Note that the mean number of children of a vertex in the nth generation is of order $\approx m^2(\frac{n-1}{n})^2$ and hence is generation dependent.

Instead of looking for a strong analogy and discussing generation dependent offspring distributions, for this exposition we sacrifice precision in favour of simplicity and claim that in this analogous case the most interesting features of the original problem are still present. More precisely, we look at a Galton-Watson tree such that every vertex has a mean number m^2 of children, and discuss the multifractal spectrum of the *branching measure* $\tilde\mu$ on its boundary, defined by

$$\tilde\mu\big(\mathcal{B}(v)\big) = \lim_{n\to\infty} \frac{Z_n(v)}{m^{2n}}\,,$$

where $\mathcal{B}(v)$ is the set of rays passing through the vertex v. Fixing some $b > 1$, we endow ∂T with the metric such that the distance of two rays is b^{-n}, where n is the generation of their last common ancestor. In this metric, the set $\mathcal{B}(v)$ is the ball centred in v of radius $b^{-|v|}$, so that for the choice of $b = m$ this corresponds to the sidelength and therefore, up to a constant, to the diameter of the represented square.

We state a general result for the multifractal spectrum of Galton-Watson trees with generating variable N and finite mean, which is taken from Mörters and Shieh (2004).

Theorem 1.13 *Suppose* $P\{N = 0\} = 0$ *and* $0 < P\{N = 1\} < 1$. *Define*

$$a = \log EN > 0 \qquad \textit{and} \qquad \tau = -\log P\{N = 1\}/\log EN > 0.$$

Then, for all $a \le \theta \le a\big(1 + \frac{1}{\tau}\big)$, *almost surely,*

$$\dim \Big\{(v_0, v_1, \dots) \in \partial T \,:\, \limsup_{n\to\infty} \frac{\log \tilde\mu(\mathcal{B}(v_n))}{-n} = \theta\Big\} = \frac{a}{\log b}\Big(\frac{a}{\theta}\,(1+\tau) - \tau\Big).$$

Before looking at the structure of this result in more detail, let us adapt the parameters of our tree representation in good faith. We have already noted that $EN = m^2$ and by construction we have $N \geq 1$ so that the conditions of Theorem 1.13 are satisfied. For the metric we would like to choose $b = m$, and the remaining parameter is $P\{N = 1\}$, which we write as $m^{-\eta}$ for some $\eta > 0$, which we discuss later. We obtain $a = 2\log m$, $\tau = \eta/2$ and hence a predicted spectrum of

$$\dim\Big\{x \in \mathcal{S}\colon \limsup_{r\downarrow 0} \frac{\log \mu(\mathcal{B}(x,r))}{\log r} = \theta\Big\} = 2\left(\frac{2}{\theta}\,(1+\frac{\eta}{2}) - \frac{\eta}{2}\right).$$

Note that neither side of this equation has any dependence on m, which gives us a handle on η, which we only have to determine asymptotically for $m \uparrow \infty$.

To do this we require knowledge of a *probability exponent*, roughly defined as the rate of decay (as $r \uparrow \infty$) of the probability of an increasingly unlikely event involving Brownian paths running until they exit the ball $\mathcal{B}(0,r)$. Various kinds of exponents can be defined and used in fractal geometry, see Lawler (1999).

In the present case we need an intersection exponent. To define these, suppose $k, m \geq 1$ are integers, and $(W_s^{(i)}\colon s \geq 0)$ for $i \in \{1,\dots,k+m\}$ are independent Brownian motions started on the unit sphere $\partial\mathcal{B}(0,1)$, and stopped upon leaving $\mathcal{B}(0,e^n)$, i.e. at times $T_n^{(i)} = \inf\{s > 0\colon |W_s^{(i)}| = e^n\}$. We denote by

$$\mathfrak{B}_n^{(1)} = \bigcup_{i=1}^{k} \big\{W_s^{(i)}\colon 0 \leq s \leq T_n^{(i)}\big\},$$
$$\mathfrak{B}_n^{(2)} = \bigcup_{i=k+1}^{k+m} \big\{W_s^{(i)}\colon 0 \leq s \leq T_n^{(i)}\big\},$$

two packets of paths, and assume that the starting points in different packets are different. Denote by V_n the event that the two packets $\mathfrak{B}_n^{(1)}$ and $\mathfrak{B}_n^{(2)}$ do not intersect each other. The *intersection exponents* are defined by the requirement that there exist constants $0 < c < C$ such that

$$c\,\exp\{-n\,\xi(k,m)\} \leq \mathbf{P}(V_n) \leq C\,\exp\{-n\,\xi(k,m)\}\,.$$

Lawler (1996b) showed that the intersection exponents $\xi(k,m)$ are well-defined by this requirement, and some years later the (highly nontrivial) techniques of stochastic Loewner evolution (SLE) enabled Lawler, Schramm and Werner to give the explicit values

$$\xi(k,m) = \frac{(\sqrt{24k+1} + \sqrt{24m+1} - 2)^2 - 4}{48}.$$

For a short survey of the key steps in this development and some other early applications of the SLE technique, see (Lawler et al., 2001).

Let us explain how intersection exponents help identifying $P\{N=1\}$ in our tree model. Suppose S is any square containing an intersection point (hence corresponding to a vertex in the tree). The event $\{N=1\}$ means that the Brownian paths intersect in one of the m^2 congruent nonoverlapping subsquares which cover S, but nowhere else in S.

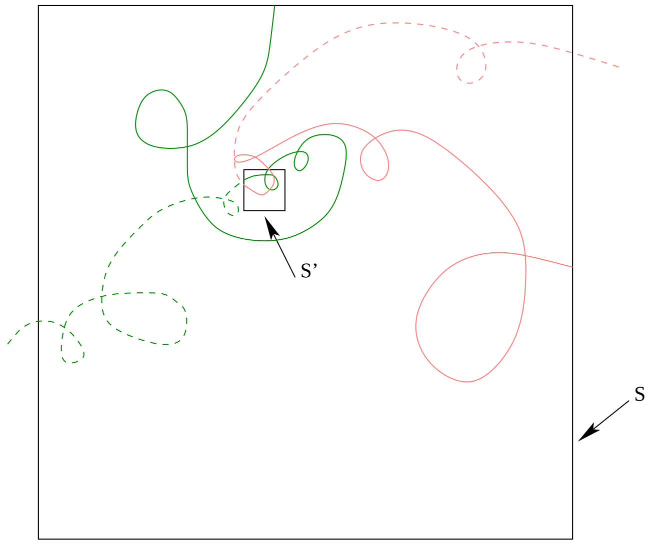

FIG. 1.4. Paths realising the event $\{N=1\}$, the initial parts of both paths are dashed.

We now fix one f these subsquares, say S', and assume (without significant loss of generality, as in the previous section) that it is located sufficiently far away from the boundary of S. We split both motions at the first time they hit $\partial S'$ and apply time-reversal to the initial part of each motion. Though the reversed part are strictly speaking not Brownian motions, they are sufficiently similar to treat them as such. Then we are faced with *four* Brownian motions started at $\partial S'$, which we divide in two packets of two, with each packet consisting of the (time-reversed) first part and the (non-reversed) second part of the same original motion. We consider all motions up to the first time when they hit ∂S, which is at distance of order m times the typical distance of the starting points. Hence, applying Brownian scaling, we get

$$\mathbf{P}\Big\{ \text{ paths do not intersect outside } S' \Big\} \approx m^{-\xi(2,2)}.$$

As these events are disjoint for the m^2 different squares $S' \subset S$ we can sum the probabilities and obtain

$$P\Big\{N=1\Big\} \approx m^{2-\xi(2,2)} \qquad \text{as } m \uparrow \infty.$$

Hence $\eta = \xi(2,2) - 2$ and plugging this into the prediction yields

$$\dim\Big\{x \in \mathcal{S}\colon \limsup_{r\downarrow 0} \frac{\log \mu(\mathcal{B}(x,r))}{\log r} = \theta\Big\} = 2\,\frac{\xi(2,2)}{\theta} + 2 - \xi(2,2)\,.$$

Using the known value $\xi(2,2) = \frac{35}{12}$ of the intersection exponents gives the precise formula claimed in Theorem 1.12.

As in the previous example, a note of caution is necessary: The tree analogy is very suitable to develop an intuition for the problem and guess the right multifractal spectrum. However, in setting up the tree analogy, we have gone too far to prove Theorem 1.12 by justification of the steps undertaken in this simplification and it is preferable to start this proof from scratch. The original problem needs serious treatment before some form of the claimed locality assumption can be exploited, and it seems to be impossible to carry out the proof without using the full power of the strong Markov property of the two Brownian motions.

However, an inspection of the proof of Theorem 1.13 gives structural insight, which is directly applicable to the proof of Theorem 1.12. Indeed, given a vertex v with $|v| = n$, the event

$$\big\{\tilde\mu(\mathcal{B}(v)) \approx e^{-n\theta}\big\}$$

is typically coming up when $Z_k(v) = 1$ for $k = n\theta/\log EN$, i.e. when the vertex v has just one offspring for k generations. This fact can be translated directly into the Brownian world. A point $x \in \mathcal{S}$ typically satisfies

$$\limsup_{r\downarrow 0} \frac{\log \mu(\mathcal{B}(x,r))}{\log r} \approx a$$

if there exists a sequence $r_n \downarrow 0$ of radii such that

$$\big(\mathcal{B}(x,r_n) \setminus \mathcal{B}(x,r_n^{a/2})\big) \cap \mathcal{S} = \emptyset\,.$$

The occurence of large empty annuli at selected radii is also key to the understanding of the multifractal spectrum of super-Brownian motion (Perkins and Taylor, 1998). Hence, despite greatly oversimplifying the situation, the tree approach gives valuable insight into the original problem, which can be exploited directly in the proof.

1.2.3 *Points of infinite multiplicity*

In this section we turn our attention to an attractive *unsolved* problem. It is known for a long time that planar Brownian motion has points of multiplicity p, for any positive integer p. Moreover, Dvoretzky et al. (1958) have shown that, almost surely, there exist points of uncountably infinite multiplicity, see (Le Gall, 1992) or (Mörters and Peres, 2008) for modern proofs. These arguments can also be used to show that the Hausdorff dimension of the set of points of uncountably infinite multiplicity is still two.

How far can we go, before we see a reduction in the dimension? A natural way is to count the number of excursions from a point. To be explicit, let $(W_s : s \geq 0)$ be a planar Brownian motion and fix $x \in \mathbb{R}^2$ and $\epsilon > 0$. Let $S_{-1} = 0$ and, for any integer $j \geq 0$, let $T_j = \inf\{s > S_{j-1} : W_s = x\}$ and $S_j = \inf\{s > T_j : |W_s - x| \geq \epsilon\}$. Then define

$$N_\epsilon^x = \max\big\{j \geq 0 : T_j < \infty\big\},$$

which is the number of excursions from x hitting $\partial\mathcal{B}(x, \epsilon)$. Observe that $N_\epsilon^x = 0$ for almost every point on the curve (with respect to the occupation time T introduced in Section 1.2.1) and that

$$\lim_{\epsilon \downarrow 0} N_\epsilon^x = \infty \qquad \Longleftrightarrow \qquad x \text{ has infinite multiplicity.}$$

It is therefore a natural question to ask how rapidly N_ϵ^x can go to infinity when $\epsilon \downarrow 0$. A partial answer is given in the following theorem of Bass et al. (1994).

Theorem 1.14

(a) Let $0 < a < \frac{1}{2}$. Then, almost surely,

$$\dim\Big\{x \in \mathbb{R}^2 : \lim_{\epsilon \downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} = a\Big\} \geq 2 - a\,.$$

(b) Let $0 < a < 2e$. Then, almost surely,

$$\dim\Big\{x \in \mathbb{R}^2 : \lim_{\epsilon \downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} = a\Big\} \leq 2 - \frac{a}{e}\,.$$

(c) Almost surely, for every $x \in \mathbb{R}^2$, we have

$$\limsup_{\epsilon \downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} \leq 2e.$$

Note, for comparison, that for a linear Brownian motion, almost surely, for every $x \in \mathbb{R}$, we have

$$\lim_{\epsilon \downarrow 0} (4\epsilon)\, M_\epsilon^x = L_t^x,$$

where M_ϵ^x is the number of excursions from x hitting $\{x - \epsilon, x + \epsilon\}$ before time t, and L_t^x is the local time at x, see e.g. (Mörters and Peres, 2008).

The proof of parts (b) and (c) of Theorem 1.14 is fairly straightforward, though the statements are certainly not optimal. The delicate part is the lower bound, given in (a). This argument is based on the construction of a local time, a nondegenerate measure on the set

$$\Big\{x \in \mathbb{R}^2 : \lim_{\epsilon \downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} = a\Big\}.$$

The restriction to values $a < 1/2$ is due to the use of L^2-estimates and appears to be of a technical nature. It is believed that the following conjecture is true.

Conjecture 1.1 *Almost surely,*

$$\max_{x\in\mathbb{R}^2} \lim_{\epsilon\downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} = 2.$$

Moreover, for any $0 < a < 2$, *almost surely,*

$$\dim\Big\{x \in \mathbb{R}^2 \colon \lim_{\epsilon\downarrow 0} \frac{N_\epsilon^x}{\log(1/\epsilon)} = a\Big\} = 2 - a\,.$$

This is still an open problem. Hope for its solvability comes from the fact that one can represent the dependence structure of the random variables N_ϵ^x in a tree picture similar to the one indicated in Section 1.2.1. However, because the Brownian path is required to return *exactly* to a given point, this problem has much less inherent continuity than the two previous ones, and therefore appears to be much harder.

1.3 More on the planar Brownian path

We have seen in the second example of the previous section that in some cases, once the tree technique has been exploited, there remains a serious challenge to identify the rate of decay of certain probabilities associated with the underlying process. This challenge can be formalised in the notion of *probability exponents*, and in this section we give further evidence of their use in fractal geometry, following ideas surveyed in Lawler (1999).

1.3.1 *The Mandelbrot conjecture*

We look at a famous example, the *Mandelbrot conjecture*: Let $(W_s \colon 0 \le s \le 1)$ be a planar Brownian motion running for one time unit and consider the complement of its path,

$$\big\{x \in \mathbb{R}^2 \colon x \neq W_s \text{ for any } 0 \le s \le 1\big\}.$$

This set is open and can be decomposed into connected components, exactly one of which is unbounded. We denote this component by U and define its boundary ∂U as the *frontier* of the Brownian path. The frontier can be seen as the set of points on the Brownian path which are accessible from infinity and is therefore also called the *outer boundary* of Brownian motion.

According to a frequently told legend, Mandelbrot, when presented with a simulation of the Brownian frontier, cast a brief glance at the picture and immediately identified its dimension as 4/3, see (Mandelbrot, 1982). However, a more rigorous confirmation of this conjecture took a long time. In the late nineties Bishop et al. (1997) showed that the frontier has Hausdorff dimension strictly larger than one, and about the same time Lawler (1996a) identified the Hausdorff dimension in terms of a disconnection exponent.

The disconnection exponents $\xi(k)$, $k \in \mathbb{N}$, can be defined as follows: Suppose $(W_s^{(i)} \colon s \geq 0)$ for $i \in \{1, \dots, k\}$ are independent Brownian motions started on the unit sphere $\partial \mathcal{B}(0,1)$, and stopped upon leaving $\mathcal{B}(0, e^n)$, i.e. at times $T_n^{(i)} = \inf\{s > 0 \colon |W_s^{(i)}| = e^n\}$. We denote by

$$\mathfrak{B}_n = \bigcup_{i=1}^{k} \big\{ W_s^{(i)} \colon 0 \leq s \leq T_n^{(i)} \big\}$$

the union of the paths, and by V_n the event that $\mathfrak{B}_n$ does not disconnect the origin from infinity, i.e. the origin is in the unbounded connected component of the complement of $\mathfrak{B}_n$. The *disconnection exponents* are defined by the requirement that there exist constants $0 < c < C$ such that

$$c \, \exp\{-n \, \xi(k)\} \leq \mathbf{P}(V_n) \leq C \, \exp\{-n \, \xi(k)\} \, .$$

Lawler (1996a) showed that the disconnection exponents $\xi(k)$ are well-defined by this requirement, and —just as in the case of intersection exponents— Lawler, Schramm and Werner found the explicit values

$$\xi(k) = \frac{(\sqrt{24k+1} - 1)^2 - 4}{48} .$$

Note that this is in line with the intersection exponents as (formally, because of our requirement that m be an integer)

$$\lim_{m \downarrow 0} \xi(k, m) = \xi(k),$$

and this corresponds to the observation that if $\mathfrak{B}_n$ disconnects the origin from infinity, no further independent packet (no matter how slim, i.e. how small m) started at the origin can reach $\partial \mathcal{B}(0, e^n)$ without intersecting $\mathfrak{B}_n$. This can be made rigorous by extending the definition of intersection exponents to noninteger arguments.

In Lawler (1996a) the dimension of the frontier was identified to be $2 - \xi(2)$, so that Mandelbrot's conjecture follows.

Theorem 1.15 *Almost surely, the Hausdorff dimension of the frontier is*

$$\dim \partial U = \tfrac{4}{3} \, .$$

It is not hard to paint a tree picture that makes the connection of the disconnection exponents and the frontier clear. This time we prefer to work in the time domain and use the following striking result of Kaufman, for a proof see e.g. (Mörters and Peres, 2008).

Lemma 1.16. (Kaufman's lemma) *Suppose $d \geq 3$ and $(W_s \colon s \in [0,1])$ is a d-dimensional Brownian motion. Then, almost surely, for every $A \subset [0,1]$,*

$$\dim \big\{ W_s \colon s \in A \big\} = 2 \, \dim A \, .$$

Note that the ‘dimension doubling’ rule holds simultaneously for all sets $A \subset [0,1]$ with a single exceptional set of probability zero. It can therefore be applied to any random set A, which makes Kaufman’s lemma a powerful tool.

We now look at the decomposition of the unit interval $[0,1]$ into 2^n nonoverlapping intervals of equal length. Any such interval $[j2^{-n},(j+1)2^{-n}]$ is associated to a vertex in a representing tree T if the set

$$\mathfrak{B}_n^{(j)} = \big\{W_s \colon 0 \le s \le j2^{-n} \text{ or } (j+1)2^{-n} \le s \le 1\big\}$$

does not disconnect $\big\{W_s \colon j2^{-n} \le s \le (j+1)2^{-n}\big\}$ from infinity. With the rule that a vertex v is an offspring of w if the interval associated to v is contained in that associated to w, this constitutes a tree representation of the set

$$I(T) = \{s \in [0,1] \colon W_s \in \partial U\}.$$

This representation does not make $I(T)$ a Galton-Watson fractal, but the following lemma taken from (Lawler, 1999, Lemma 1 and 2) indicates how, in a special situation, the independence conditions of Theorem 1.4 can be weakened.

Lemma 1.17 *Given a family of (not necessarily independent) zero-one valued random variables*

$$\big\{Y(j_1,\dots,j_n) \colon j_i \in \{0,1\},\, n \in \mathbb{N}\big\},$$

we build a random fractal A iteratively. Let $\mathcal{S}_0 = \{[0,1]\}$ and, given a collection $\mathcal{S}_n$ of compact intervals of length 2^{-n}, construct a collection $\mathcal{S}_{n+1}$ by

- *splitting each interval in $\mathcal{S}_{n-1}$ into two nonoverlapping intervals of half the length,*
- *adding any interval thus constructed to the collection $\mathcal{S}_n$ if*

$$Y(j_1,\dots,j_n) = 1,$$

where $\sum_{i=1}^n j_i 2^{-i}$ is the left endpoint of the interval.

Define the random fractal as

$$A = \bigcap_{n=1}^{\infty} \bigcup_{I \in \mathcal{S}_n} I\,.$$

Then

(i) If $\displaystyle\sum_{j_1,\dots,j_n=0}^{1} \mathbf{E} \prod_{k=1}^{n} Y(j_1,\dots,j_k) \le C\, 2^{(1-\alpha)n}$ *for some $C>0$, then*

$$\dim A \le 1-\alpha \qquad \textit{almost surely.}$$

(ii) If, for some $0 < c < C < \infty$ *and* $\epsilon > 0$,

$$c\,2^{-\alpha n} \le \mathbf{E}\prod_{k=1}^{n} Y(j_1,\dots,j_k) \le C\,2^{-\alpha n} \qquad \text{for all } \epsilon \le \sum_{i=1}^{n} j_i 2^{-i} \le 1-\epsilon,$$

and

$$\mathbf{E}\prod_{k=1}^{n} Y(j_1,\dots,j_k)Y(i_1,\dots,i_k) \le C\,2^{-2\alpha n}\Big(\sum_{i=1}^{n}(i_i-j_i)2^{-i}\Big)^{-\alpha},$$

for all $\epsilon \le \sum_1^n j_i 2^{-i} < \sum_1^n i_i 2^{-i} \le 1-\epsilon$, *then*

$$\dim A \ge 1-\alpha \qquad \text{with positive probability.}$$

This lemma exploits a tree representation of A with the tree T given as a subtree of a binary tree with vertices in the nth generation canonically denoted by $(j_1,\dots,j_n)$. Such a vertex is contained in T if and only if

$$\prod_{k=1}^{n} Y(j_1,\dots,j_k) = 1\,.$$

The set attached to the vertex $(j_1,\dots,j_n)$ is the closed interval of length 2^{-n} with left endpoint $\sum_1^n j_i 2^{-i}$, and the number of children of this vertex is

$$Y(j_1,\dots,j_n,0) + Y(j_1,\dots,j_n,1)\,.$$

Supposing that all the random variables $Y(j_1,\dots,j_k)$ are independent, we get

$$\begin{aligned}\min\Big\{s\colon \mathbf{E}\big\{2^{-s}Y(j_1,\dots,j_n,0) + 2^{-s}Y(j_1,\dots,j_n,1)\big\} \le 1\Big\}\\ = \min\big\{s\colon 2^{-s-\alpha+1} \le 1\big\} = 1-\alpha,\end{aligned}$$

confirming that this generalises a special case of Theorem 1.4.

In our case, we let $Y(j_1,\dots,j_n) = 1$ if and only if the set

$$\big\{W_s\colon 0 \le s \le \textstyle\sum_{i=1}^n j_i 2^{-i} \text{ or } \sum_{i=1}^n j_i 2^{-i} + 2^{-n} \le s \le 1\big\}$$

does not disconnect

$$\{W_s\colon \sum_{i=1}^{n} j_i 2^{-i} \le s \le \sum_{i=1}^{n} j_i 2^{-i} + 2^{-n}\}$$

from infinity. Using Brownian scaling and the definition of the disconnection exponents, one can show that the probability of this event is of order $2^{-\frac{n}{2}\xi(2)}$, and that the second condition in (ii) also holds. This argument therefore gives that

$$\dim\big\{s \in [0,1]\colon W_s \in \partial U\big\} = 1 - \tfrac{1}{2}\,\xi(2) = \tfrac{2}{3},$$

with positive probability, and an application of Kaufman's lemma gives $\dim \partial U = \frac{4}{3}$ with positive probability. Some nontrivial extra work is required to show that this actually holds with probability one.

1.3.2 *More on the geometry of the Brownian frontier*

There are a variety of subsets of Brownian paths whose Hausdorff dimensions can be expressed in terms of different probability exponents. Examples with known exponents, like cutpoints, pioneer points and cone points of planar Brownian motion are given in (Lawler, 1999) and (Lawler et al., 2001). Here we sketch two results, which reveal further details about the geometry of the frontier.

To begin with, it is easy to observe that the Brownian frontier contains double points of the Brownian motion. The argument, which is probably due to Paul Lévy, goes roughly like this: If it did not, then by construction the frontier would just be a stretch of the original Brownian path. This would however imply that it had double points, which is a contradiction.

Knowing that there are double points on the frontier, it is natural to ask, whether the frontier contains *triple points*. This problem was solved by Burdzy and Werner (1996).

Theorem 1.18 *Almost surely, there are* no *triple points on the frontier of a planar Brownian motion.*

A second natural question that comes up is *how many* double points one can find on the Brownian frontier. Surprisingly, it turns out that while the set

$$D = \big\{x \in \mathbb{R}^2 \colon x = W_s = W_t \text{ for distinct } s, t \in [0,1]\big\}$$

of double points has full dimension on the entire path, it does not have full dimension on the frontier. The following curious result is due to Kiefer and Mörters (2008).

Theorem 1.19 *Almost surely, the set of double points on the Brownian frontier satisfies*

$$\dim\big(D \cap \partial U\big) = \frac{\sqrt{97}+1}{24}\,.$$

In the proof of this result, a spatial approach is preferable. In this context it is natural to consider Brownian motion up to the first exit time τ from a big ball, rather than up to time one (it is not hard to see that this is equivalent). We fix a compact square S_0 of unit sidelength inside this ball, and a small $\epsilon > 0$. Let $\mathcal{S}_n$ be the collection containing those of the 2^{2n} nonoverlapping subsquares $S \subset S_0$ of sidelength 2^{-n}, which satisfy

- the Brownian motion $(W_s \colon s \geq 0)$ hits S, then moves to distance ϵ from S, and then hits S again before time τ;
- the union of the paths outside the square S does not disconnect its boundary ∂S from infinity.

Then we have

$$S_0 \cap D \cap \partial U = \bigcup_{\epsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{S \in \mathcal{S}_n} S,$$

and the Hausdorff dimension can be determined (with positive probability) by verifying a first and second moment criterion analogous to (i), (ii) in Lemma 1.17.

For the first moment criterion, we have to show that the probability that a cube of sidelength 2^{-n} is in $\mathcal{S}_n$ is bounded from above and below by constant multiples of $2^{-n\xi(4)}$. Indeed, we may use three stopping times to split the path into four pieces: They are the first hitting time of S, the first time afterwards where the path has moved to distance ϵ from the square, the first hitting time of S after that. If we reverse the first and third part in time, the four pieces are sufficiently close to Brownian paths started on the boundary of the cube and running for one time unit, to infer that the probability of disconnection is, up to a factor which is polynomial in n, of order $2^{-n\xi(4)}$. At this place the proof is a bit more delicate than the arguments in Section 1.2.2, because a careful control of the polynomial factors is required.

This first moment argument, a slightly more sophisticated one for the second moment, and a tree framework similar to the one above, show that, with positive probability,

$$\dim\big(D \cap \partial U\big) = 2 - \xi(4) = \frac{\sqrt{97}+1}{24}\,.$$

Again, some more work is required to show that this holds almost surely.

For Theorem 1.18 the argument is easier, as no lower bound is needed. Using no more than the Borel-Cantelli lemma one can infer that for

$$T = \big\{x \in \mathbb{R}^2 \colon x = W_r = W_s = W_t \text{ for distinct } r, s, t \in [0,1]\big\}$$

we have $T \cap \partial U = \emptyset$ almost surely, if $\xi(6) > 2$. The merit of the paper (Burdzy and Werner, 1996) is mostly in providing this estimate long before the SLE-technology allowed the precise calculation of this value.

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