A CLASS OF WEAKLY SELF-AVOIDING WALKS

PETER MÖRTERS AND NADIA SIDOROVA

Abstract: We define a class of weakly self-avoiding walks on the integers by conditioning a simple random walk of length n to have a p-fold self-intersection local time smaller than n^β, where $1 < \beta < (p+1)/2$. We show that the conditioned paths grow of order n^α, where $\alpha = (p-\beta)/(p-1)$, and also prove a coarse large deviation principle for the order of growth.

1. Introduction and main results

Weakly self-avoiding walks are defined by multiplying the distribution of a simple symmetric random walk path $(S_i \colon 1 \le i \le n)$ on $\mathbb{Z}^d$ with a density which is decreasing in the p-fold intersection local time

$$\Lambda_n(p) = \sum_{0 \le i_1, \dots, i_p \le n} \mathbf{1}\{S_{i_1} = \cdots = S_{i_p}\}$$

of the walk, where $p \in \mathbb{N}$ and $p \ge 2$. In the classical *Domb-Joyce model* this density is given as

$$\frac{1}{Z_n} \exp\big\{ -\tfrac{1}{T} \Lambda_n(2)\big\},$$

where $T > 0$ is a temperature parameter and Z_n is a normalising factor. This model is well-understood in the one-dimensional case, where the resulting polymers grow like $\sim c\,n$ and laws of large numbers, central limit theorems and large deviation results are established, see [2] for a survey.

A natural alternative to the Domb-Joyce model is to choose densities

$$\frac{\mathbf{1}\{\Lambda_n(p) < b_n\}}{\mathbb{P}\big\{\Lambda_n(p) < b_n\big\}}$$

where b_n grows slower than $\mathbb{E}\Lambda_n(p)$. In this paper we study this model in dimension $d = 1$. At a first glance, this may look harder to analyse than the Domb-Joyce model, but it turns out that interesting behaviour kicks in on a coarser scale and we are helped by the fact that on this scale we can understand, to a certain extent, what kind of behaviour of the walk realises the event $\{\Lambda_n(p) < b_n\}$ with maximal probability. As b_n varies between the expectation and the minimum of $\Lambda_n(p)$, we see weakly self-avoiding walks with typical growth of any order between $\sqrt{n}$ and n.

To formulate our results precisely, we define the *local time* of the random walk at $z \in \mathbb{Z}$ by

$$\ell_n(z) = \sum_{i=0}^{n} \mathbf{1}\{S_i = z\},$$

which is exactly the number of visits to z until time n. Note that, for any integer $p > 1$, the p-fold self-intersection local time is

$$\Lambda_n(p) = \sum_{z \in \mathbb{Z}} \ell_n^p(z). \tag{1}$$

Hence, $n \leq \Lambda_n(p) \leq n^p$ and it is not hard to show (see Lemma 3) that $\mathbb{E}\Lambda_n(p) \asymp n^{\frac{p+1}{2}}$ (which means that the ratio of the two sides is bounded away from zero and infinity). We define

$$\bar{S}_n = \max_{0 \leq i \leq n} |S_i|$$

and state a limit theorem for $\bar{S}_n$ under the conditional probability.

Theorem 1 (Law of large numbers)**.** *Let* $\varepsilon_n \downarrow 0$ *such that* $\varepsilon_n^{\frac{2}{p-1}} n \to \infty$*. Then there exist constants* $0 < c < C < \infty$ *such that*

$$\mathbb{P}\Big\{c\,\sqrt{n}\,\varepsilon_n^{-\frac{1}{p-1}} \leq \bar{S}_n \leq C\,\sqrt{n}\,\varepsilon_n^{-\frac{1}{p-1}} \;\Big|\; \Lambda_n(p) \leq \varepsilon_n\, \mathbb{E}\Lambda_n(p)\Big\} \to 1.$$

Remarks:

- The condition $\varepsilon_n^{\frac{2}{p-1}} n \to \infty$ ensures that $\varepsilon_n\, \mathbb{E}\Lambda_n(p)$ grows faster than n, which is a strict lower bound for $\Lambda_n(p)$. Hence the conditioning event has positive probability.
- An important step in the proof of Theorem 1 is to analyse the asymptotic behaviour of the probability of the conditioning event. This result will be formulated as Theorem 3 in the next section.
- It would be interesting to see if there exists a constant $c_* > 0$ such that, in probability, $\bar{S}_n \sim c_*\sqrt{n}\,\varepsilon_n^{-1/(p-1)}$, but the methods of this paper do not allow to show this.
- In Theorem 1, we assume that p is an integer, but it is plausible that the same statement is true for any $p > 1$ if we use (1) as a definition of $\Lambda_n(p)$.

As a particular case of the law of large numbers, if we condition the random walk on the event $\{\Lambda_n(p) \leq n^\beta\}$ for some $1 < \beta < \frac{p+1}{2}$, we obtain a typical growth rate of

$$\frac{\log \bar{S}_n}{\log n} \sim \frac{p - \beta}{p - 1}$$

for the weakly self-avoiding walk. Interestingly, it turns out that the probability of deviations from this behaviour decay with a speed dependent on the size of the deviation. The following coarse large deviation principle describes this behaviour. For its formulation we define $\log^{(2)}(x) = \log|\log x|$ for all $0 < x < 1$, as well as $\log^{(2)}(1) = -\infty$ and $\log^{(2)}(0) = \infty$.

Theorem 2 (Large deviation principle)**.** *Suppose $1 < \beta < \frac{p+1}{2}$. Then, for any $1 > a > \frac{p-\beta}{p-1}$, we have*

$$\lim_{n\uparrow\infty} \frac{1}{\log n} \log^{(2)} \mathbb{P}\Big\{ \frac{\log \bar{S}_n}{\log n} > a \,\Big|\, \Lambda_n(p) \le n^\beta \Big\} = 2a - 1,$$

and, for any $0 < a < \frac{p-\beta}{p-1}$ and $c > 0$ small enough

$$\mathbb{P}\big\{ \bar{S}_n < cn^a \,\big|\, \Lambda_n(p) \le n^\beta \big\} = 0\,.$$

for all sufficiently large n.

Remark: In fact, we prove a stronger (but more technical) result, see Proposition 1 for the precise formulation. In particular, our result shows that the typical growth rate for any walk with $\Lambda_n(p) \le n^\beta$ is $\frac{p-\beta}{p-1}$, while larger growth rates are possible but occur with subexponentially decaying probability, and smaller rates are not possible at all.

Coming back to the beginning of this introduction, we see that, if $p = 2$, the Domb-Joyce model can be interpreted as a limiting case of our models when $\beta \downarrow 1$, in which case both the self-intersection local time $\Lambda_n(2)$, and the growth of the polymer are linear in n. Hence the self-avoidance in the Domb-Joyce model is significantly stronger than in our models.

2. Lower deviations for self-intersection local times

As an important step in the proof, we compute the asymptotics for the probability of lower deviations of $\Lambda_n(p)$ from its expectation. This is also of independent interest.

Theorem 3. *Let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$ and $\varepsilon_n^{\frac{2}{p-1}} n \to \infty$. Then,*

$$-\log \mathbb{P}\left(\Lambda_n(p) \le \varepsilon_n \mathbb{E}\Lambda_n(p)\right) \asymp \varepsilon_n^{-\frac{2}{p-1}}.$$

Remark: An analogous lower deviation regime also exists for self-intersections of planar random walks, see [1, Theorem 1.2]. Results for the lower deviations of self-intersection local times of one-dimensional Brownian motion are discussed in [4].

In the following, we fix $p > 1$ and abbreviate $\Lambda_n = \Lambda_n(p)$. We assume that $\varepsilon_n > 0$ is such that

$$\varepsilon_n \to 0 \qquad \text{and} \qquad \varepsilon_n^{\frac{2}{p-1}} n \to \infty.$$

We prepare the proof of Theorem 3 with two easy lemmas dealing with the curtailed Green function

$$G_n(z) = \mathbb{E}\ell_n(z) = \sum_{i=0}^{n} \mathbb{P}\big(S_i = z\big) \qquad \text{for } z \in \mathbb{Z}.$$

The first lemma is elementary and is included for the sake of self-containment.

Lemma 1. *The curtailed Green function has the following properties:*

(1) $\sum_{z\in\mathbb{Z}} G_n(z) = n$;

(2) $\sum_{z\in\mathbb{Z}} G_n^2(z) \asymp n^{\frac{3}{2}}$;

(3) $G_n(0) \asymp \sqrt{n}$ *and* $G_n(1) \asymp \sqrt{n}$.

Proof. Formula (1) is trivial, and (2) is proved in [3, (3.4)]. To show (3), we use the Stirling formula

$$n! = \exp\big\{n \log n - n + \log\sqrt{2\pi n} + \delta_n\big\},$$

where $\delta_n \to 0$. We have

$$\begin{aligned} G_n(0) &= \sum_{i=0}^{n} \mathbb{P}(S_i = 0) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{2i}{i} 2^{-2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(2i)!}{(i!)^2} 2^{-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \exp\big\{2i\log(2i) - 2i + \log\sqrt{4\pi i} \\ &\qquad\qquad + \delta_{2i} - 2i\log i + 2i - 2\log\sqrt{2\pi i} - 2\delta_i - 2i\log 2\big\} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \exp\big\{ \log\sqrt{4\pi i} + \delta_{2i} - 2\log\sqrt{2\pi i} - 2\delta_i\big\} \asymp \sum_{i=0}^{\lfloor n/2 \rfloor} i^{-\frac{1}{2}} \asymp \sqrt{n} \end{aligned}$$

and

$$G_n(1) = \sum_{i=0}^{n} \mathbb{P}(S_i = 1) = \sum_{i=1}^{\lceil n/2 \rceil} \binom{2i-1}{i} 2^{-2i+1} = \sum_{i=0}^{\lceil n/2 \rceil} \binom{2i}{i} 2^{-2i} - 1.$$

This implies $G_n(0) - 1 \le G_n(1) \le G_{n+1}(0) - 1$, which gives $G_n(1) \asymp \sqrt{n}$. □

Lemma 2. *For every $q \in \mathbb{N}$, we have*

$$G_{\lfloor n/q \rfloor}(z)\, G_{\lfloor n/q \rfloor}^{q-1}(0) \le \mathbb{E}\ell_n^q(z) \le q!\, G_n(z)\, G_n^{q-1}(0).$$

Proof. We first prove the lower bound. We have

$$\begin{aligned}
\mathbb{E}\ell_n^q(z) &= \mathbb{E}\sum_{i_1=0}^{n}\cdots\sum_{i_q=0}^{n}\prod_{j=1}^{q}\mathbf{1}\{S_{i_j}=z\} \\
&\geq \sum_{0\leq i_1\leq \frac{n}{q}}\ \sum_{i_1\leq i_2\leq i_1+\frac{n}{q}}\cdots\sum_{i_{q-1}\leq i_q\leq i_{q-1}+\frac{n}{q}}\mathbb{P}\big(S_{i_1}=z\big)\prod_{j=2}^{q}\mathbb{P}\big(S_{i_j-i_{j-1}}=0\big) \\
&= \sum_{0\leq i_1\leq \frac{n}{q}}\cdots\sum_{0\leq i_q\leq \frac{n}{q}}\mathbb{P}\big(S_{i_1}=z\big)\prod_{j=2}^{q}\mathbb{P}\big(S_{i_j}=0\big) \\
&= \Big(\sum_{i=0}^{\lfloor n/q\rfloor}\mathbb{P}\big(S_i=z\big)\Big)\Big(\sum_{i=0}^{\lfloor n/q\rfloor}\mathbb{P}\big(S_i=0\big)\Big)^{q-1} = G_{\lfloor n/q\rfloor}(z)\,G_{\lfloor n/q\rfloor}^{q-1}(0).
\end{aligned}$$

For the upper bound, we get

$$\begin{aligned}
\mathbb{E}\ell_n^q(z) &\leq q!\sum_{0\leq i_1\leq\cdots\leq i_q\leq n}\mathbb{P}\big(S_{i_1}=\cdots=S_{i_q}=z\big) \\
&= q!\sum_{0\leq i_1\leq\cdots\leq i_q\leq n}\mathbb{P}\big(S_{i_1}=z\big)\prod_{j=2}^{q}\mathbb{P}\big(S_{i_j-i_{j-1}}=0\big) \\
&\leq q!\sum_{0\leq i_1,\ldots,i_q\leq n}\mathbb{P}\big(S_{i_1}=z\big)\prod_{j=2}^{q}\mathbb{P}\big(S_{i_j}=0\big) = q!\,G_n(z)\,G_n^{q-1}(0),
\end{aligned}$$

which completes the proof. □

In the next two lemmas, we study the asymptotic behaviour of the first two moments of Λ_n.

Lemma 3. $\mathbb{E}\Lambda_n \asymp n^{\frac{p+1}{2}}$.

Proof. Applying Lemmas 1 and 2 with $q=p$, we obtain

$$\mathbb{E}\Lambda_n = \sum_{z\in\mathbb{Z}}\mathbb{E}\ell_n^p(z) \geq G_{\lfloor n/p\rfloor}^{p-1}(0)\sum_{z\in\mathbb{Z}}G_{\lfloor n/p\rfloor}(z) \asymp n^{\frac{p+1}{2}},$$

and

$$\mathbb{E}\Lambda_n \leq p!\,G_n^{p-1}(0)\sum_{z\in\mathbb{Z}}G_n(z) \asymp n^{\frac{p+1}{2}},$$

which completes the proof. □

Lemma 4. *There exists $c>0$ such that $\mathbb{E}\Lambda_n^2 \leq cn^{p+1}$ for all n.*

Proof. We have

$$
\begin{aligned}
\mathbb{E}\Lambda_n^2 &= \mathbb{E}\Big(\sum_{0\le i_1,\dots,i_p\le n} \mathbf{1}\{S_{i_1}=\cdots=S_{i_p}\}\Big)^2\\
&= \sum_{z,w\in\mathbb{Z}} \sum_{\substack{0\le i_1,\dots,i_p\le n\\ 0\le j_1,\dots,j_p\le n}} \mathbb{P}(S_{i_1}=\cdots=S_{i_p}=z, S_{j_1}=\cdots=S_{j_p}=w)\\
&\le (2p)! \sum_{z,w\in\mathbb{Z}} \sum_{0\le l_1\le\cdots\le l_{2p}\le n} \sum_{\substack{A\subset\{1,\dots,2p\}\\ |A|=p}} \prod_{i=1}^{2p} \mathbb{P}\big(S_{l_i-l_{i-1}} = a_i^{z,w}(A)\big), \qquad (2)
\end{aligned}
$$

where $l_0=0$ and, for all $2\le i\le 2p$,

$$
a_1^{z,w}(A) = \begin{cases} z, & \text{if } 1\in A,\\ w, & \text{if } 1\notin A, \end{cases} \qquad a_i^{z,w}(A) = \begin{cases} z-w, & \text{if } i\in A, i-1\notin A,\\ w-z, & \text{if } i\notin A, i-1\in A,\\ 0, & \begin{array}{l}\text{if } i,i-1\in A \text{ or}\\ \text{if } i,i-1\notin A,\end{array} \end{cases}
$$

Observe that if $1\in A$ then $a_1^{z,w}=z$ and there is an index $i(A)\in\{2,\dots,2p\}$ such that $a_{i(A)}^{z,w}=w-z$. On the other hand, if $1\notin A$ then $a_1^{z,w}=w$ and there is an index $i(A)\in\{2,\dots,2p\}$ such that $a_{i(A)}^{z,w}=z-w$.
Further, denote $k_i=l_i-l_{i-1}$ and notice that

$$
\mathbb{P}\big(S_{k_i}=a_i^{z,w}(A)\big) \le \mathbb{P}\big(S_{k_i}\in\{0,1\}\big).
$$

We will use this estimate to bound all the factors in the product in (2) except those numbered 1 and $i(A)$. This gives

$$
\begin{aligned}
\mathbb{E}\Lambda_n^2 &\le (2p)! \sum_{z,w\in\mathbb{Z}} \sum_{0\le k_1,\dots,k_{2p}\le n} \sum_{\substack{A\subset\{1,\dots,2p\}\\ |A|=p}} \prod_{i=1}^{2p} \mathbb{P}\big(S_{k_i}=a_i^{z,w}(A)\big)\\
&\le (2p)! \sum_{z,w\in\mathbb{Z}} \sum_{0\le k_1,\dots,k_{2p}\le n} \sum_{\substack{A\subset\{1,\dots,2p\}\\ |A|=p}} \prod_{i\notin\{1,i(A)\}} \mathbb{P}\big(S_{k_i}=a_i^{z,w}(A)\big)\\
&\qquad \times \Big[\mathbf{1}\{1\in A\}\mathbb{P}\big(S_{k_1}=z\big)\mathbb{P}\big(S_{k_{i(A)}}=w-z\big)\\
&\qquad\qquad + \mathbf{1}\{1\notin A\}\,\mathbb{P}\big(S_{k_1}=w\big)\mathbb{P}\big(S_{k_{i(A)}}=z-w\big)\Big].
\end{aligned}
$$

Rearranging the terms, we obtain

$$\begin{aligned}
\mathbb{E}\Lambda_n^2 &\le (2p)! \sum_{z,w\in\mathbb{Z}} \sum_{0\le k_1,\dots,k_{2p}\le n} \frac{1}{2}\binom{2p}{p} \prod_{i=3}^{2p} \mathbb{P}\big(S_{k_i}\in\{0,1\}\big) \\
&\qquad \Big[\mathbb{P}\big(S_{k_1}=z\big)\mathbb{P}\big(S_{k_2}=w-z\big)+\mathbb{P}\big(S_{k_1}=w\big)\mathbb{P}\big(S_{k_2}=z-w\big)\Big] \\
&\le (2p)!^2\Big[\sum_{0\le k_3,\dots,k_{2p}\le n}\prod_{i=3}^{2p}\mathbb{P}\big(S_{k_i}\in\{0,1\}\big)\Big] \\
&\qquad \times \Big[\sum_{0\le k_1,k_2\le n}\sum_{z,w\in\mathbb{Z}}\mathbb{P}\big(S_{k_1}=z\big)\,\mathbb{P}\big(S_{k_2}=w\big)\Big] \\
&= (2p)!^2 n^2\Big[\sum_{k=0}^{n}\mathbb{P}\big(S_k\in\{0,1\}\big)\Big]^{2p-2} \\
&= (2p)!^2 n^2\big[G_n(0)+G_n(1)\big]^{2p-2} \asymp n^{p+1},
\end{aligned}$$

where the last line follows from Lemma 1. $\square$

We fix some small number $0<\eta<1$. Define

$$B_n^\eta=\left\{\sup_{0\le i\le n}\Big|\frac{S_i}{\sqrt{n}}-\eta\Big|<1, 1<\frac{S_n}{\sqrt{n}}<1+\eta\right\}.$$

Lemma 5. $\mathbb{P}(B_n^\eta)\asymp 1$;

Proof. By Donsker's invariance principle there is a standard one-dimensional Brownian motion $(B_t)_{0\le t\le 1}$ defined on the same probability space as $(S_i)_{i\in\mathbb{N}_0}$ such that

$$\mathbb{P}\left(\sup_{0\le i\le n}\Big|\frac{S_i}{\sqrt{n}}-B_{i/n}\Big|>\delta\right)\to 0, \tag{3}$$

for any $\delta>0$. Further,

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{0\le i\le n}\big|\tfrac{S_i}{\sqrt{n}}-\eta\big|<1, 1<\tfrac{S_n}{\sqrt{n}}<1+\eta\,\Big|\,\sup_{0\le i\le n}\big|\tfrac{S_i}{\sqrt{n}}-B_{i/n}\big|\le\delta\Big] \\
&\ge \mathbb{P}\Big[\sup_{0\le t\le 1}\big|B_t-\eta\big|<1-\delta, 1+\delta<B_1<1+\eta-\delta\,\Big|\,\sup_{0\le i\le n}\big|\tfrac{S_i}{\sqrt{n}}-B_{i/n}\big|\le\delta\Big] \\
&\to \mathbb{P}\Big(\sup_{0\le t\le 1}\big|B_t-\eta\big|<1-\delta, 1+\delta<B_1<1+\eta+\delta\Big)>0,
\end{aligned}$$

for $\delta<\min\{1-\eta,\eta/2\}$, which, together with (3), implies the statement. $\square$

Abbreviate $a_n = \mathbb{E}\Lambda_n$ and let (m_n) be the sequence of natural numbers such that $2^{m_n} \le n < 2^{m_n+1}$.

Lemma 6 (Upper bound)**.** *There is a constant $C > 0$ such that*

$$\log \mathbb{P}\big(\Lambda_n \le \varepsilon_n a_n\big) \le -C\, \varepsilon_n^{-\frac{2}{p-1}}.$$

Proof. Let (k_n) be the sequence of natural numbers such that

$$c\varepsilon_n^{\frac{2}{p-1}} n \le 2^{k_n} < 2c\varepsilon_n^{\frac{2}{p-1}} n,$$

where the constant c will be specified later. Note that, by choice of ε_n, we have $2^{k_n} \to \infty$. To obtain an upper bound for $\mathbb{P}(\Lambda_n \le \varepsilon_n a_n)$, we only count self-intersections occurring within $2^{m_n-k_n}$ disjoint intervals of length 2^{k_n}. We fix n and, for each $1 \le j \le 2^{m_n-k_n}$, denote

$$S_i^{(n,j)} = S_{(j-1)2^{k_n}+i} - S_{(j-1)2^{k_n}}, \quad 0 \le i \le 2^{k_n}.$$

They are simple symmetric random walks starting at zero, which are independent. Denote, for $z \in \mathbb{Z}$,

$$\ell_{2^{k_n}}^{(n,j)}(z) = \sum_{i=0}^{2^{k_n}} \mathbf{1}\{S_i^{(n,j)} = z\} = \sum_{i=(j-1)2^{k_n}}^{j2^{k_n}} \mathbf{1}\{S_i = z + S_{(j-1)2^{k_n}}\}.$$

For each $1 \le j \le 2^{m_n-k_n}$, denote by

$$Y_{nj} = \sum_{i_1=(j-1)2^{k_n}}^{j2^{k_n}} \cdots \sum_{i_p=(j-1)2^{k_n}}^{j2^{k_n}} \mathbf{1}\{S_{i_1} = \cdots = S_{i_p}\}$$

the number of p-fold self-intersections in the j-th interval. We have

$$\begin{aligned} Y_{nj} &= \sum_{z\in\mathbb{Z}} \sum_{i_1=(j-1)2^{k_n}}^{j2^{k_n}} \cdots \sum_{i_p=(j-1)2^{k_n}}^{j2^{k_n}} \mathbf{1}\{S_{i_1} = \cdots = S_{i_p} = z + S_{(j-1)2^{k_n}}\} \\ &= \sum_{z\in\mathbb{Z}} \Big[\ell_{2^{k_n}}^{(n,j)}(z)\Big]^p, \end{aligned}$$

and by Lemma 3 we get

$$\mathbb{E}Y_{nj} = \mathbb{E}\Lambda_{2^{k_n}} \asymp 2^{\frac{k_n(p+1)}{2}}.$$

Notice that

$$\Lambda_n \ge \sum_{j=1}^{2^{m_n-k_n}} \big(Y_{nj} - 1\big)$$

and so

$$\mathbb{P}\left(\Lambda_n \le \varepsilon_n a_n\right) \le \mathbb{P}\Big(\sum_{j=1}^{2^{m_n-k_n}} Y_{nj} \le \varepsilon_n a_n + n\Big) \sim \mathbb{P}\Big(\sum_{j=1}^{2^{m_n-k_n}} Y_{nj} \le \varepsilon_n a_n\Big) \tag{4}$$

(it is easy to see that the equivalence follows from $n = o(\varepsilon_n a_n)$). Using Markov's inequality and the independence of Y_{nj} for fixed n and different j, we obtain, for each $s > 0$,

$$\mathbb{P}\Big(\sum_{j=1}^{2^{m_n-k_n}} Y_{nj} \leq \varepsilon_n a_n\Big) \leq \exp\{s\varepsilon_n a_n\}\,\mathbb{E}\exp\Big\{-s\sum_{j=1}^{2^{m_n-k_n}} Y_{nj}\Big\}$$

$$= \exp\Big\{s\varepsilon_n a_n + 2^{m_n-k_n}\log\mathbb{E}e^{-s\Lambda_{2^{k_n}}}\Big\}. \tag{5}$$

It is easy to check that $e^x \leq 1 + x + x^2$ for all $x < 0$ and so

$$\mathbb{E}e^{-s\Lambda_{2^{k_n}}} \leq 1 - s\mathbb{E}\Lambda_{2^{k_n}} + s^2\mathbb{E}\Lambda_{2^{k_n}}^2.$$

Using $\log(1+x) \leq x$ for all $x > -1$, we obtain

$$\log\mathbb{E}e^{-s\Lambda_{2^{k_n}}} \leq -s\mathbb{E}\Lambda_{2^{k_n}} + s^2\mathbb{E}\Lambda_{2^{k_n}}^2. \tag{6}$$

Combining (4), (5), and (6), we get

$$\begin{aligned} &\mathbb{P}\big(\Lambda_n \leq \varepsilon_n a_n\big) \\ &\leq \min_{s>0}\ \exp\Big\{-s\Big[2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n\Big] + s^2 2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}}^2\Big\}. \end{aligned} \tag{7}$$

The optimal value of s is given by

$$s = \Big[2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n\Big]\,2^{k_n-m_n-1}\,\big[\mathbb{E}\Lambda_{2^{k_n}}^2\big]^{-1}.$$

By Lemma 3 there are constants $c_1, c_2 > 0$ such that

$$\mathbb{E}\Lambda_{2^{k_n}} \geq c_1 2^{\frac{k_n(p+1)}{2}} \qquad \text{and} \qquad a_n = \mathbb{E}\Lambda_n \leq c_2 n^{\frac{p+1}{2}}.$$

By choice of the sequences m_n and k_n, we obtain

$$\begin{aligned} 2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n &\geq c_1 2^{m_n} 2^{\frac{k_n(p-1)}{2}} - c_2\varepsilon_n n^{\frac{p+1}{2}} \\ &\geq \varepsilon_n n^{\frac{p+1}{2}}\big(c^{\frac{p-1}{2}}c_1 2^{-1} - c_2\big) > 0, \end{aligned} \tag{8}$$

where the last inequality holds if we choose c large enough. Computing the corresponding value in (7), we obtain

$$\mathbb{P}\left(\Lambda_n \leq \varepsilon_n a_n\right) \leq \exp\Big\{-\frac{\big[2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n\big]^2}{2^{m_n-k_n+2}\mathbb{E}\Lambda_{2^{k_n}}^2}\Big\}. \tag{9}$$

By Lemma 3 we have $\varepsilon_n a_n \asymp \varepsilon_n n^{\frac{p+1}{2}}$ and, using the choice of k_n and m_n and again Lemma 3 we also have

$$2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} \asymp n 2^{\frac{k_n(p-1)}{2}} \asymp \varepsilon_n n^{\frac{p+1}{2}},$$

which, together with (8), implies

$$2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n \asymp \varepsilon_n n^{\frac{p+1}{2}}. \tag{10}$$

Further, by Lemma 4 there is a constant c_3 such that

$$2^{m_n-k_n+2}\mathbb{E}\Lambda_{2^{k_n}}^2 \leq c_3 n 2^{k_n p} \asymp \varepsilon_n^{\frac{2p}{p-1}} n^{p+1}. \tag{11}$$

Finally, combining (9), (10), and (11), we obtain

$$\log \mathbb{P}\left(\Lambda_n \leq \varepsilon_n a_n\right) \leq -\frac{\left[2^{m_n-k_n}\mathbb{E}\Lambda_{2^{k_n}} - \varepsilon_n a_n\right]^2}{2^{m_n-k_n+2}\mathbb{E}\Lambda_{2^{k_n}}^2} \leq -C\varepsilon_n^{-\frac{2}{p-1}},$$

for some $C > 0$. □

Lemma 7 (Lower bound)**.** *There exist constants $c, C > 0$ such that, for each g_n satisfying $g_n \geq c$, and*

$$g_n n^{-\frac{1}{2}}\varepsilon_n^{-\frac{1}{p-1}} \longrightarrow 0,$$

one has

$$\log \mathbb{P}\big(\Lambda_n \leq \varepsilon_n a_n,\, \bar{S}_n > g_n n^{\frac{1}{2}}\varepsilon_n^{-\frac{1}{p-1}}\big) \geq -C\varepsilon_n^{-\frac{2}{p-1}} g_n^2.$$

Proof. Let (k_n) be a sequence of even natural numbers such that $2^{k_n} < n$ and $k_n \to \infty$, which will be specified later. To prove the lower bound, we describe a strategy of a random walk, the probability of which is large enough to provide the required bound, which implies that the p-fold self intersection local time is small and the maximal displacement is large.

For this purpose, we divide the time interval into $2^{m_n-k_n}$ time sub-intervals of length 2^{k_n} and observe the path on the coarse time scale (that is, at times $i2^{k_n}$, $0 \leq i \leq 2^{m_n-k_n}$). As a strategy, we consider the event that on the coarse scale the path moves up in each step, whereas on the fine scale the path behaves typically. Hence, in one coarse time step, the path moves up by a distance of order $2^{k_n/2}$. Then we optimise over (k_n). This strategy guarantees that the path will have almost no self-intersections at times belonging to different sub-intervals. $\bar{S}_n$ will be large, because the path is forced to go up (instead of fluctuating) on the coarse scale.

Let $0 < \eta < 1/2$ be fixed. Denote by

$$A_{nj}^{\eta} = \Big\{ \sup_{(j-1)2^{k_n} \leq i \leq j2^{k_n}} \big|\tfrac{S_i - S_{(j-1)2^{k_n}}}{2^{k_n/2}} - \eta\big| < 1,\, 1 < \tfrac{S_{j2^{k_n}} - S_{(j-1)2^{k_n}}}{2^{k_n/2}} < 1+\eta \Big\}$$

the event that the random walk, considered on the i-th sub-interval, stays at distance of order $2^{k_n/2}$ from its starting point and moves up by a distance of order $2^{k_n/2}$ during the whole time. Further, denote by

$$A_n^{\eta} = \bigcap_{j=1}^{2^{m_n-k_n+1}} A_{nj}^{\eta},$$

the event that this happens on each sub-interval.

Using the independence of the events A^{η}_{nj}, for $j = 1 \ldots, 2^{m_n-k_n+1}$, we obtain by Lemma 5

$$\begin{aligned} \log \mathbb{P}(A^{\eta}_n) &= \log \prod_{j=1}^{2^{m_n-k_n+1}} \mathbb{P}(A^{\eta}_{nj}) = 2^{m_n-k_n+1} \log \mathbb{P}(B^{\eta}_{2^{k_n}}) \\ &\asymp -n 2^{-k_n}. \end{aligned} \tag{12}$$

Consider the event A^{η}_n. Notice that on this event we have

$$\bar{S}_n > 2^{m_n-k_n}(1-\eta)2^{\frac{k_n}{2}} > 2^{m_n-\frac{k_n}{2}-1} > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}},$$

where the last inequality is satisfied if we choose k_n in such a way that

$$2^{k_n} < n\varepsilon_n^{\frac{2}{p-1}}/(16 g_n^2) \longrightarrow \infty. \tag{13}$$

Thus, the strategy leads to the desired growth of the walks. We now check that it also gives the right self-intersection local times. Note that for any $1 \le j_1, j_2 \le 2^{m_n-k_n+1}$ such that $|j_2 - j_1| \ge 2$ the j_1-th and j_2-th pieces of length 2^{k_n} of (S_i) do not intersect. Indeed, let $(j_1-1)2^{k_n} \le i_1 \le j_1 2^{k_n}$ and $(j_2-1)2^{k_n} \le i_1 \le j_1 2^{k_n}$. Then

$$\begin{aligned} S_{i_2} &> S_{(j_2-1)2^{k_n}} - (1-\eta)2^{\frac{k_n}{2}} > S_{(j_2-2)2^{k_n}} + \eta 2^{\frac{k_n}{2}} \\ &\ge S_{j_1 2^{k_n}} + \eta 2^{\frac{k_n}{2}} > S_{(j_1-1)2^{k_n}} + (1+\eta)2^{\frac{k_n}{2}} > S_{i_1}. \end{aligned}$$

For each $1 \le j \le 2^{m_n-k_n+1}$, define independent simple random walks starting at zero by

$$S^{(n,j)}_i = S_{(j-1)2^{k_n}+i} - S_{(j-1)2^{k_n}}, \qquad \text{for } 0 \le i < 2^{k_n}.$$

Denote by $\ell^{(n,j)}_{2^{k_n}-1}(z)$ the local time of $S^{(n,j)}$ at z, and define independent random variables

$$Y_{nj} = \sum_{z\in\mathbb{Z}} \big[\ell^{(n,j)}_{2^{k_n}-1}(z)\big]^p, \qquad \text{for } j \in \{1, \ldots, 2^{m_n-k_n+1}\}.$$

As $n < 2^{m_n+1}$ we have, on the event A_n, that

$$\begin{aligned} \Lambda_n &\le \sum_{j=1}^{2^{m_n-k_n+1}-1} \sum_{i_1=(j-1)2^{k_n}}^{(j+1)2^{k_n}-1} \cdots \sum_{i_p=(j-1)2^{k_n}}^{(j+1)2^{k_n}-1} \mathbf{1}\{S_{i_1} = \cdots = S_{i_p}\} \\ &= \sum_{j=1}^{2^{m_n-k_n+1}-1} \sum_{z\in\mathbb{Z}} \Big(\sum_{i=(j-1)2^{k_n}}^{(j+1)2^{k_n}-1} \mathbf{1}\{S_i = z\} \Big)^p. \end{aligned}$$

Using the inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$, which holds for all $a,b\ge 0$ and $p\in\mathbb{N}$, we obtain

$$\begin{aligned}\Lambda_n &\le 2^{p-1}\sum_{j=1}^{2^{m_n-k_n+1}-1}\sum_{z\in\mathbb{Z}}\Big[\Big(\sum_{i=(j-1)2^{k_n}}^{j2^{k_n}-1}\mathbf{1}\{S_i=z\}\Big)^p+\Big(\sum_{i=j2^{k_n}}^{(j+1)2^{k_n}-1}\mathbf{1}\{S_i=z\}\Big)^p\Big]\\ &=2^{p-1}\sum_{j=1}^{2^{m_n-k_n+1}-1}\big[Y_{nj}+Y_{n(j+1)}\big]\le 2^p\sum_{j=1}^{2^{m_n-k_n+1}}Y_{nj}.\end{aligned}$$

Let $Z^{(\eta)}_{nj}$, $1\le j\le 2^{m_n-k_n+1}$ be a family of independent random variables such that $Z^{(\eta)}_{nj}$ has the same distribution as Y_{nj} conditioned on A^η_{nj}. Since A^η_{nj} are independent for all j, and Y_{nj_1} is independent of $A^\eta_{nj_2}$ for all $j_1\ne j_2$, we obtain

$$\begin{aligned}\mathbb{P}\big(\Lambda_n\le\varepsilon_n a_n,\bar S_n>g_n n^{\frac12}\varepsilon_n^{-\frac{1}{p-1}}\big)&\ge\mathbb{P}\big[\Lambda_n\le\varepsilon_n a_n\,\big|\,A^\eta_n\big]\mathbb{P}(A^\eta_n)\\ &\ge\mathbb{P}\Big[2^p\sum_{j=1}^{2^{m_n-k_n+1}}Y_{nj}\le\varepsilon_n a_n\,\Big|\bigcap_{j=1}^{2^{m_n-k_n+1}}A^\eta_{nj}\Big]\mathbb{P}(A^\eta_n)\\ &=\mathbb{P}\Big(2^p\sum_{j=1}^{2^{m_n-k_n+1}}Z^{(\eta)}_{nj}\le\varepsilon_n a_n\Big)\mathbb{P}(A^\eta_n).\end{aligned}\tag{14}$$

We show that the first probability on the right hand side converges to one. By Lemma 5 we have

$$\begin{aligned}\mathbb{E}Z^{(\eta)}_{nj}&=\frac{\mathbb{E}[Y_{nj}\mathbf{1}\{A^\eta_{nj}\}]}{\mathbb{P}(A^\eta_{nj})}\le\frac{\mathbb{E}\Lambda_{2^{k_n}-1}}{\mathbb{P}(B^\eta_{2^{k_n}-1})}\asymp 2^{\frac{k_n(p+1)}{2}},\\ \mathbb{E}\big[(Z^{(\eta)}_{nj})^2\big]&=\frac{\mathbb{E}[Y^2_{nj}\mathbf{1}\{A^\eta_{nj}\}]}{\mathbb{P}(A^\eta_{nj})}\le\frac{\mathbb{E}\Lambda^2_{2^{k_n}-1}}{\mathbb{P}(B^\eta_{2^{k_n}-1})}\le c_1 2^{k_n(p+1)},\end{aligned}\tag{15}$$

for some $c_1>0$. Observe that

$$\begin{aligned}&\mathbb{P}\Big(2^p\sum_{j=1}^{2^{m_n-k_n+1}}Z^{(\eta)}_{nj}>\varepsilon_n a_n\Big)\\ &=\mathbb{P}\Big(2^{-m_n+k_n-1}\sum_{j=1}^{2^{m_n-k_n+1}}Z^{(\eta)}_{nj}-\mathbb{E}Z^{(\eta)}_{n1}>2^{-p-m_n+k_n-1}\varepsilon_n a_n-\mathbb{E}Z^{(\eta)}_{n1}\Big)\\ &\le\mathbb{P}\Big(|2^{-m_n+k_n-1}\sum_{j=1}^{2^{m_n-k_n+1}}Z^{(\eta)}_{nj}-\mathbb{E}Z^{(\eta)}_{n1}|>2^{-p-m_n+k_n-1}\varepsilon_n a_n-\mathbb{E}Z^{(\eta)}_{n1}\Big).\end{aligned}$$

By Lemma 3 and (15), there are constants $c_2,c_3>0$ such that

$$a_n\ge c_2 n^{\frac{p+1}{2}}\qquad\text{and}\qquad\mathbb{E}Z^{(\eta)}_{n1}\le c_3 2^{\frac{k_n(p+1)}{2}},$$

which implies

$$2^{-p-m_n+k_n-1}\varepsilon_n a_n - \mathbb{E}Z_{n1}^{(\eta)} \geq 2^{-p-1} n^{\frac{p-1}{2}} 2^{k_n}\varepsilon_n c_2 - c_3 2^{\frac{k_n(p+1)}{2}} > 0,$$

where the last inequality holds if we choose

$$2^{k_n} < n\varepsilon_n^{\frac{2}{p-1}}(c_2 2^{-p-1}/c_3). \tag{16}$$

Let $c = (2^{p-3}c_3/c_2)^{\frac{1}{2}}$, and note that, for all $g_n \geq c$, the inequality (13) implies (16). We now choose k_n to be the even number satisfying

$$n\varepsilon_n^{\frac{2}{p-1}}/(64g_n^2) \leq 2^{k_n} < n\varepsilon_n^{\frac{2}{p-1}}/(16g_n^2),$$

so that (13) and (16) are satisfied, $2^{k_n} < n$ and $2^{k_n} \to \infty$, where the latter follows from the growth condition imposed on g_n.

Using the Chebyshev inequality, we obtain that

$$\mathbb{P}\Big(2^p \sum_{j=1}^{2^{m_n-k_n+1}} Z_{nj}^{(\eta)} > \varepsilon_n a_n\Big) \leq \frac{\operatorname{Var} Z_{n1}^{(\eta)}}{2^{m_n-k_n+1}[2^{-p-m_n+k_n-1}\varepsilon_n a_n - \mathbb{E}Z_{n1}^{(\eta)}]^2}.$$

It follows from the choice of (k_n) that

$$[2^{-p-m_n+k_n-1}\varepsilon_n a_n - \mathbb{E}Z_{n1}^{(\eta)}]^2 \geq c_4 \varepsilon_n^{\frac{2(p+1)}{p-1}} n^{p+1} g_n^{-4},$$

for some $c_4 > 0$ independent of g_n. From (15) we obtain, for some $c_5, c_6 > 0$ independent of g_n,

$$\operatorname{Var} Z_{n1}^{(\eta)} \leq c_5 2^{k_n(p+1)} \leq c_6 \varepsilon_n^{\frac{2(p+1)}{p-1}} n^{p+1} g_n^{-2(p+1)}.$$

Using these two formulas and the asymptotics for k_n, we obtain

$$\mathbb{P}\Big(2^p \sum_{j=1}^{2^{m_n-k_n+1}} Z_{nj}^{(\eta)} > \varepsilon_n a_n\Big) \leq c_7 g_n^{-2p} \varepsilon_n^{\frac{2}{p-1}} \leq c_8 \varepsilon_n^{\frac{2}{p-1}} \longrightarrow 0,$$

where $c_7, c_8 > 0$ are independent of g_n. Hence

$$\mathbb{P}\Big(2^p \sum_{j=1}^{2^{m_n-k_n+1}} Z_{nj}^{(\eta)} \leq \varepsilon_n a_n\Big) \geq 1 - c_8 \varepsilon_n^{\frac{2}{p-1}} \to 1.$$

It follows now from (14) and (12) that

$$\begin{aligned}
&\log \mathbb{P}\left(\Lambda_n \leq \varepsilon_n a_n, \bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}\right) \\
&\quad \geq \log \mathbb{P}\Big(2^p \sum_{j=1}^{2^{m_n-k_n+1}} Z_{nj}^{(\eta)} \leq \varepsilon_n a_n\Big) + \log \mathbb{P}(A_n^\eta) \\
&\quad \geq \log(1 - c_8 \varepsilon_n^{\frac{2}{p-1}}) - c_9 n 2^{-k_n} \geq -C g_n^2 \varepsilon_n^{-\frac{2}{p-1}},
\end{aligned}$$

with some $c_9, C > 0$ independent of g_n. $\square$

Proof of Theorem 3. Let $g_n = c$ be a constant sequence, where c is taken from Lemma 7. Then the assumptions on the growth of g_n hold by choice of ε_n. Hence we can use Lemma 7 to obtain a lower bound,

$$\log \mathbb{P}\big(\Lambda_n \le \varepsilon_n a_n\big) \ge \log \mathbb{P}\Big(\Lambda_n \le \varepsilon_n a_n, \bar{S}_n > c n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}\Big) \ge -C c^2 \varepsilon_n^{-\frac{2}{p-1}}.$$

The upper bound from Lemma 6 completes the proof. $\square$

3. Growth of the weakly self-avoiding walk

We now state a more general version of Theorem 2, which also includes Theorem 1. The result of Theorem 2 follows immediately by specialising to the case $\varepsilon_n = n^{\beta-(p+1)/2}$.

Proposition 1. *Let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$ and $\varepsilon_n^{\frac{2}{p-1}} n \to \infty$. There exists $c_1 > 0$ such that eventually,*

$$\mathbb{P}\big(\bar{S}_n \le c_1 n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}} \;\big|\; \Lambda_n(p) \le \varepsilon_n \mathbb{E}\Lambda_n(p)\big) = 0,$$

and there exists $c_2 > 0$ such that

$$-\log \mathbb{P}\big(\bar{S}_n \ge c_2 n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}} \;\big|\; \Lambda_n(p) \le \varepsilon_n \mathbb{E}\Lambda_n(p)\big) \asymp \varepsilon_n^{-\frac{2}{p-1}}.$$

In particular, $\bar{S}_n \asymp n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}$ in probability.
Moreover, for any $g_n \ge c_2$ such that $g_n n^{-\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}} \to 0$, one has

$$-\log \mathbb{P}\big(\bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}} \;\big|\; \Lambda_n(p) \le \varepsilon_n \mathbb{E}\Lambda_n(p)\big) \asymp g_n^2 \varepsilon_n^{-\frac{2}{p-1}}.$$

Proof of Proposition 1. Denote $f_n = c_1 n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}$, where $c_1 > 0$ will be specified later. On the event $\{\bar{S}_n < f_n\}$, we have $\ell_n(z) = 0$ for $|z| \ge f_n$, and

$$\sum_{|z|<f_n} \ell_n(z) = n. \tag{17}$$

Consider the function φ defined on the simplex $S = \{x \in \mathbb{R}^m \colon x_i \ge 0 \,\forall i, x_1 + \cdots + x_m = a\}$ by $\varphi(x) = x_1^p + \cdots + x_m^p$. As φ has the global minimum at the point $(a/m, \ldots, a/m)$ we have $x_1^p + \cdots + x_m^p \ge a^p m^{1-p}$. Applying this together with the condition (17), we obtain eventually

$$\Lambda_n = \sum_{|z|<f_n} \ell_n^p(z) \ge n^p (2\lceil f_n \rceil - 1)^{1-p} \ge n^p 3^{1-p} f_n^{1-p} = (3c_1)^{1-p} \varepsilon_n n^{\frac{p+1}{2}}.$$

Since $a_n = \mathbb{E}\Lambda_n \asymp n^{\frac{p+1}{2}}$ by Lemma 3, one can pick c_1 small enough in order to ensure that

$$\big\{\Lambda_n \le \varepsilon_n a_n\big\} \subset \big\{\bar{S}_n > f_n\big\},$$

which finishes the proof of the first statement.

Now fix g_n such that $g_n \geq c_2$, and $g_n n^{-1/2} \varepsilon_n^{-1/(p-1)} \to 0$ (where $c_2 > 0$ will be specified later). Define

$$f_n = g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}.$$

By the reflection principle we have

$$\mathbb{P}(\bar{S}_n > f_n) \geq \mathbb{P}(\max_{0 \leq i \leq n} S_i > f_n) = 2\mathbb{P}(S_n > f_n),$$

and

$$\begin{aligned} \mathbb{P}\big(\bar{S}_n > f_n\big) &\leq \mathbb{P}\big(\max_{0 \leq i \leq n} S_i > f_n\big) + \mathbb{P}\big(\min_{0 \leq i \leq n} S_i < -f_n\big) \\ &= 2\,\mathbb{P}\big(\max_{0 \leq i \leq n} S_i > f_n\big) = 4\,\mathbb{P}\big(S_n > f_n\big). \end{aligned}$$

Hence $\mathbb{P}(\bar{S}_n > f_n) \asymp \mathbb{P}(S_n > f_n)$. Further, the Azuma-Hoeffding inequality gives

$$\log \mathbb{P}\big(S_n > f_n\big) \leq -\tfrac{f_n^2}{2n}.$$

In particular, this implies

$$\begin{aligned} &\log \mathbb{P}\big(\bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}, \Lambda_n \leq \varepsilon_n a_n\big) \\ &\quad \leq \log \mathbb{P}\big(\bar{S}_n > f_n\big) \leq -\frac{f_n^2}{2n} + \log 4 = -\frac{g_n^2}{2} \varepsilon_n^{-\frac{2}{p-1}} + \log 4. \end{aligned} \tag{18}$$

The corresponding lower bound is given by Lemma 7. For $c_2 > c$, we have

$$\log \mathbb{P}\big(\bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}, \Lambda_n \leq \varepsilon_n a_n\big) \geq -C \varepsilon_n^{-\frac{2}{p-1}} g_n^2, \tag{19}$$

where C is independent of g_n. Recall that, by Theorem 3,

$$\log \mathbb{P}\big(\Lambda_n \leq \varepsilon_n a_n\big) \asymp -\varepsilon_n^{-\frac{2}{p-1}},$$

which, together with (18) and (19), implies

$$\begin{aligned} &\log \mathbb{P}\big(\bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}} \;\big|\; \Lambda_n \leq \varepsilon_n a_n\big) \\ &\quad = \log\big(\bar{S}_n > g_n n^{\frac{1}{2}} \varepsilon_n^{-\frac{1}{p-1}}, \Lambda_n \leq \varepsilon_n a_n\big) - \log \mathbb{P}\big(\Lambda_n \leq \varepsilon_n a_n\big) \\ &\quad \asymp -g_n^2 \varepsilon_n^{-\frac{2}{p-1}}, \end{aligned}$$

if c_2 is chosen large enough so that the first probability dominates even in the case when g_n is constant. □

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References

[1] Bass, R. F., X. Chen, and J. Rosen, Moderate deviations and law of the iterated logarithm for the renormalized self-intersection local times of planar random walks. *El. Journal Probab.* 11:993-1030 (2006).

[2] van der Hofstad, R., and W. König, A survey of one-dimensional polymers. *J. Stat. Phys.* 103:915-944 (2001).

[3] Lawler, G. F., *Intersections of random walks*, Birkhäuser, Boston (1991).

[4] Mörters, P., and M. Ortgiese, Small value probabilities via the branching tree heuristic. To appear in *Bernoulli* (2008).

Peter Mörters
University of Bath
Department of Mathematical Sciences
Claverton Down
Bath BA2 7AY
United Kingdom
maspm@bath.ac.uk

Nadia Sidorova
University College London
Department of Mathematics
Gower Street
London WC1E 6BT
United Kingdom
n.sidorova@ucl.ac.uk