18.02 Recitation 1

- For a point $A = (a_1, a_2, a_3)$ in three space the vector $\vec{A}$ is given by $\vec{A} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{j}$.
- The length of the vector is $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- Scalar multiplication of vectors: $c\vec{A} = (ca_1, ca_2, ca_3)$
  Addition of vectors: $\vec{A} + \vec{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
- Dot product of two vectors is given by

  $$\vec{A}.\vec{B} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{A}|.|\vec{B}|\cos\theta$$

  where θ is the angle between the two vectors. Hence two vectors are perpendicular $\iff$ $\vec{A}.\vec{B} = 0$.

Problems

1. Is $(1, 1, 1)$ perpendicular to $(1, -1, 1)$? If not find a vector perpendicular to $(1, 1, 1)$.

2. Find a vector perpendicular to both $(1, 1, 1)$ and $(1, -1, 0)$.

3. Consider the triangle with vertices $(0, 2), (3, 2)$ and $(\sqrt{3}, 3)$. Find the angle at the vertex $(0, 2)$.

4. If $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ does this imply that $\vec{B} = \vec{C}$ by cancellation?

See Simmons section 18.2 for more problems.

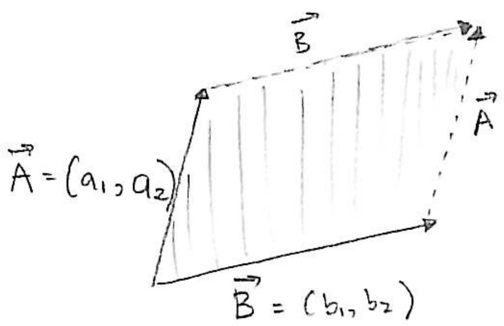


Fig 1.

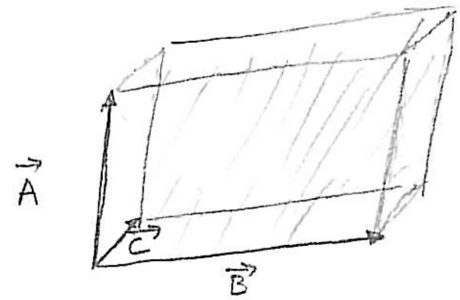


Fig 2.

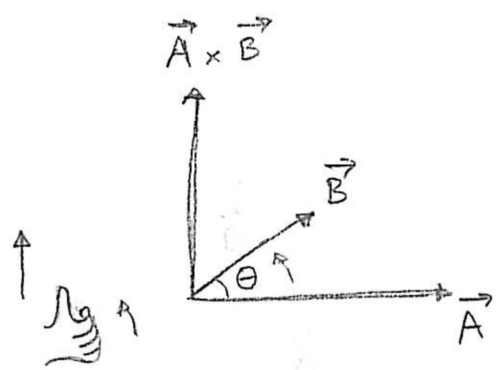


Fig 3.

18.02 Recitation 3

- Matrix multiplication: the ij th entry of AB is the dot product of the ith row vector of A and jth column vector of B. For example

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \cdot \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix} = \begin{bmatrix} a_1p_1 + a_2q_1 + a_3r_1 & a_1p_2 + a_2q_2 + a_3r_2 \\ b_1p_1 + b_2q_1 + b_3r_1 & b_1p_2 + b_2q_2 + b_3r_2 \end{bmatrix}.$$

- Inverse of a 2×2 matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Inverse of a $n \times n$ matrix: If $A = (a_{ij})$ then $A^{-1} = \frac{1}{|A|}(A_{ij})^T$. Here A_{ij} is the signed cofactor of a_{ij} defined as the determinant of the minor M_{ij}. The minor M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the row and column of a_{ij} and the sign is given by the checkerboard rule.
- $2 \times 2/3 \times 3$ matrices correspond to linear transformations of the 2-plane/3-space.
- The equation $ax + by + cz = d$ represents a plane in 3-space with normal vector (a, b, c). It passes through the origin $\iff d = 0$.

Problems

1. (Notes 1F-3) Find all 2×2 matrices such that $A^2 = 0$.

2. (Notes 1G-1) If $A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 3 \\ 0 \end{bmatrix}$ solve $Ax = b$ by finding A^{-1}.

3. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Find the geometric meaning of the linear transformations A, B, AB and BA.

4. (Notes 1E-6) Show that the distance D from the origin to the plane $ax+by+cz = d$ is $D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$.

5. Find the surface area of the regular tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$.

18.02 Recitation 4

- A line in 3-space is represented by 2 linear equations $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ such that the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are not proportional. This geometrically represents the intersection of two planes.
- A parametric equation of a line is of the form $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$. This line passes through the point (x_0, y_0, z_0) and points in the direction (a, b, c). The corresponding non-parametric equation is $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- A parametric equation for a curve is of the form $\vec{r}(t) = (x(t), y(t), z(t))$ where $\vec{r}(t)$ is the position vector. The velocity vector is $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$ and the acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = (\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2})$.
- The arclength from $t = a$ to $t = b$ of the parametric curve is $\int_a^b |\vec{v}(t)|dt$.

Problems

1. (Notes 1E-3) Find the parametric equations for
a) The line through $(1, 0, -1)$ and parallel to $2i - j + 3k$.
b) The line through $(2, -1, -1)$ and perpendicular to the plane $x - y + 2z = 3$.

2. (Notes 1E-5) The line passing through $(1, 1, -1)$ and perpendicular to the plane $x + 2y - z = 3$ intersects the plane $2x - y + z = 1$ at what point?

3. (Notes 1I-3) Describe the motions of the following vector functions as t goes from $-\infty$ to ∞. In each case give a non-parametric (xy-equation) for the curve that the point $P = \vec{r}(t)$ travels along and what part of the curve point P actually traces
a) $\vec{r}(t) = 2\cos^2 ti + \sin^2 tj$
b) $\vec{r}(t) = \cos(2t)i + \cos(t)j$
c) $\vec{r}(t) = (t^2 + 1)i + t^3 j$
d) $\vec{r}(t) = \tan(t)i + \sec(t)j$.

4. (Notes 1J-6) For the helical motion $\vec{r}(t) = a\cos(t)i + a\sin(t)j + (bt)k$ calculate the velocity and acceleration vectors at each point and show that they are perpendicular.

18.02 Recitation 5

- Given a function of several variables $f(x, y, z)$, its partial derivative with respect to x is defined as the limit
$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$
In other words, the partial derivative with respect to x is computed by treating the other variables as constants.
- Partial derivatives satisfy the usual sum and product rules
$$\begin{aligned} \frac{\partial (f+g)}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \\ \frac{\partial (fg)}{\partial x} &= \left(\frac{\partial f}{\partial x}\right) g + f \left(\frac{\partial g}{\partial x}\right). \end{aligned}$$
- Partial derivatives can be takes in any order. That is the mixed partials
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$
are equal.

Problems

1. Find the partial derivatives with respect to x and y for

   (1) xy^2
   (2) $\cos(x + y)$
   (3) $\frac{2y^2}{3x+1}$
   (4) $x \ln(2x + y)$.

2. Check that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for the function $f(x, y) = \sin(x^2 - y)$.

3. Show that the function $f(x, y) = e^x \sin(y)$ satisfies Laplace's equation
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

4. Show that the functions $f(x, t) = \sin(x - t)$ and $f(x, t) = \sin(x + t)$ both satisfy the wave equation
$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0.$$

Tangent plane equation:

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{p_0} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{p_0} (y - y_0)$$

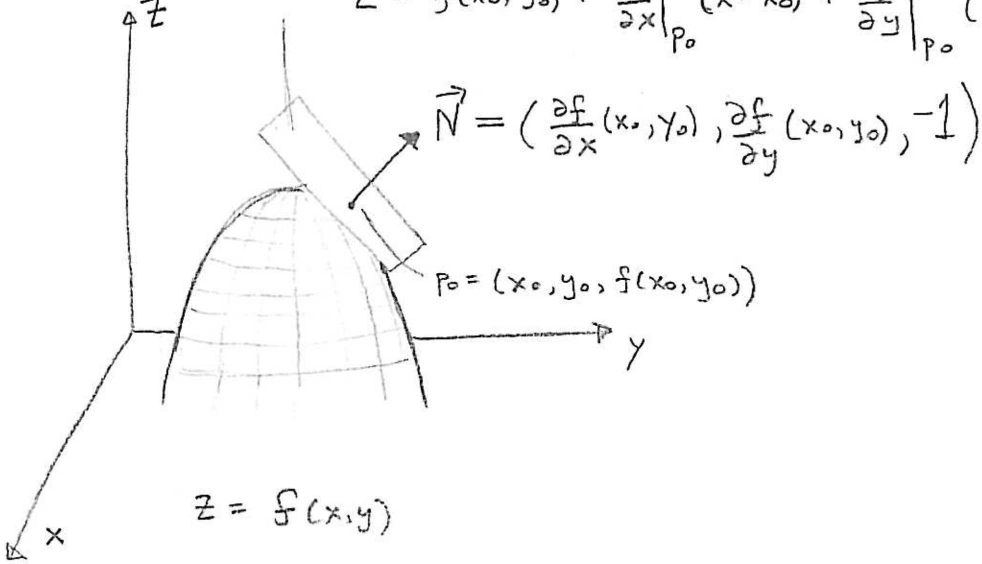
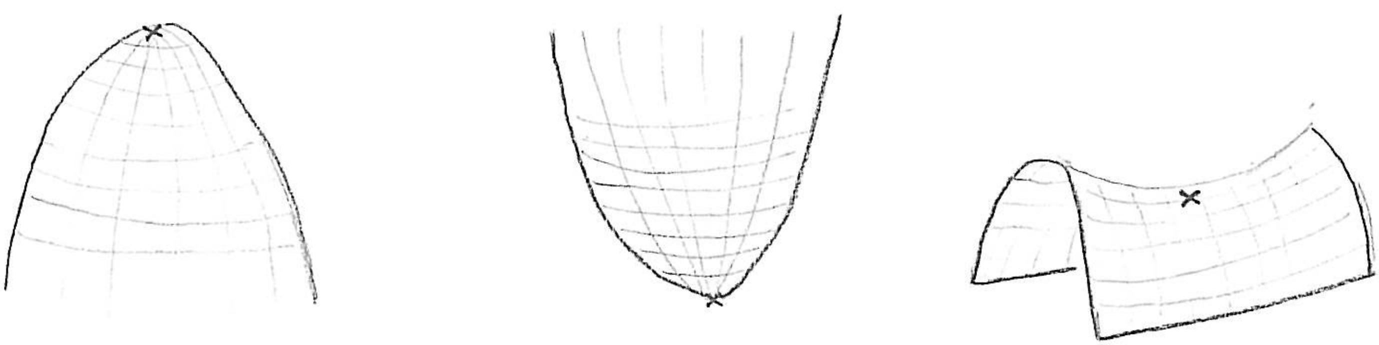


Figure 1.

Critical points



Local Maxima　　Local minima　　Saddle point.

Figure 2.

18.02 Recitation 7

- Given data points (x_i, y_i) for $i = 1, 2, \ldots, n$, the *least squares line* is the line $y = mx + b$ that best fits the data in the following sense:
  (i) consider the deviations $d_i = y_i - (mx_i + b)$ of the predicted value $mx_i + b$ from the true value x_i for each of these data points
  (ii) the least squares line minimizes the sum of the squares of these deviations

$$D(m, b) = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2.$$

- $D(m, b)$ can be minimized as a function of m and b to get the coefficients m and b. These are generally the unique solutions of the pair of linear equations

$$\left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right) m = \left(\sum_{i=1}^{n} x_i y_i\right) \tag{1}$$

$$nb + \left(\sum_{i=1}^{n} x_i\right) m = \left(\sum_{i=1}^{n} y_i\right). \tag{2}$$

Problems

1. (Notes 2G-1) Find the least squares line which best fits the data points

   (a) $(0, 0), (0, 2), (1, 3)$
   (b) $(0, 0), (1, 2), (2, 1)$

2. (Notes 2G-2) Show that the equations (1) and (2) for the least squares line have a unique solution unless all x_i are equal. Explain geometrically why this exception occurs. (Hint: $n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i \neq j} (x_i - x_j)^2$.)

3. (Notes 2G-4) What linear equations in a, b, c does the method of least squares lead to, when you use it to fit a linear function $z = a + bx + cy$ to a set of data points $(x_i, y_i, z_i), i = 1, \ldots, n$.

18.02 Recitation 8

- Given a function of two variables $f(x, y)$, its Hessian in the matrix of second derivatives
$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$
- Given a critical point $p = (x_0, y_0)$ of the function $f(x, y)$, it is characterized by the second-derivative test as follows (see figure 1)
  (i) If $\det H_f = f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ then p is a local minimum,
  (ii) If $\det H_f = f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ then p is a local maximum,
  (iii) If $\det H_f = f_{xx} f_{yy} - f_{xy}^2 < 0$ then p is a saddle point.
- A maximum/minimum of a function $f(x, y)$ over a given region occurs at a local maximum/minimum in the interior of the region or a point on the boundary.
- The differential of a function $f(x, y, x)$ is the formal expression
$$df = f_x dx + f_y dy + f_z dz.$$
- Chain rule: If $f(x, y, z)$ is a function of x, y and z while x, y and z are functions of t then
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

Problems

1. (Notes 2H-1) Find all the critical points of the following functions and classify them

(b) $3x^2 + xy + y^2 - x - 2y + 4$
(c) $2x^4 + y^2 - xy + 1$
(d) $x^3 - 3xy + y^3$.

2. Find the maximum and minimum values attained by the function $f(x, y) = xy - x - y + 3$ at the points of the triangular region R in the xy-plane with vertices at $(0, 0), (2, 0)$ and $(0, 4)$.

3. (Notes 2H-7) Find the maximum and minimum points of the function $2x^2 - 2xy + y^2 - 2x$ on the rectangle $R = \{(x, y) | 0 \leq x \leq 2, -1 \leq y \leq 2\}$.

4. Use the chain rule to find $\frac{\partial f}{\partial t}$ for the composite function $f(x(t), y(t))$. Also check your answer by explicitly writing f as a function of t.

(a) $f = \ln(x^2 + y^2), x = \sin(t), y = \cos(t)$
(b) $f = \frac{3xy}{x^2 - y^2}, x = t^2, y = 3t$
(c) $f = e^{-x^2 - y^2}, x = t, y = \sqrt{t}$
(d) $\sin(xy), x = t, y = t^4$.

Hints/Answers

1.

(b) $(0,1)$ local minimum.
(c) $(0,0)$ saddle, $(\frac{1}{4},\frac{1}{8})$ local minimum, $(-\frac{1}{4},-\frac{1}{8})$ local minimum.
(d) $(0,0)$ saddle, $(1,1))$ local minimum.

2. Maximum: $f(0,0)=3$, Minimum: $f(0,4)=-1$.

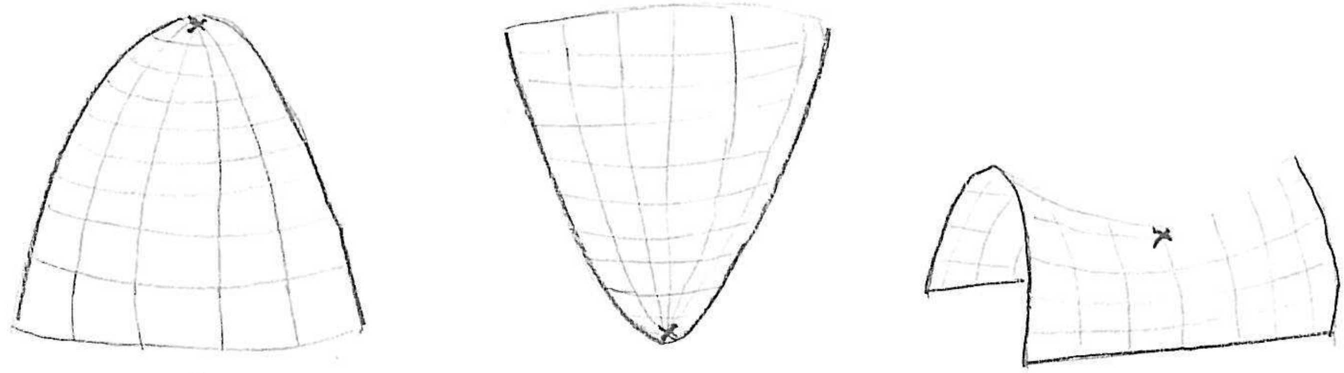
3. Maximum: $(2,-1)$, Minimum: $(1,1)$.

4.

(a) 0
(b) $-\frac{9(t^2+9)}{(t^2-9)^2}$
(c) $-(2t+1)e^{-t^2-t}$
(d) $5t^4\cos(t^5)$.

Classification of critical points

(Second Derivative test)



Local maxima	Local minima	Saddle point
$\boxed{\begin{matrix} \det H_f > 0 \\ f_{xx} < 0 \end{matrix}}$	$\boxed{\begin{matrix} \det H_f > 0 \\ f_{xx} > 0 \end{matrix}}$	$\boxed{\det H_f < 0}$
eg. $f(x,y) = -x^2 - y^2$	eg. $f(x,y) = x^2 + y^2$	eg. $f(x,y) = x^2 - y^2$

Hessian: $H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

Figure 1.

18.02 Recitation 9

- The differential of a function $f(x, y, x)$ is the formal expression

$$df = f_x dx + f_y dy + f_z dz.$$

- Chain rule: Let $f(x, y, z)$ be a function of x, y and z. Let x, y and z in turn be functions of u and v. Thus the composite $w(u, v) = f(x(u, v), y(u, v), z(u, v))$ is a function of u and v. The partial derivative of the composite function w are given with the help of the chain rule via

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \end{aligned}$$

- Given a function $f(x, y, z)$ its gradient ∇f is the vector field

$$\nabla f = (f_x, f_y, f_z) = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}.$$

- The gradient vector field ∇f is normal to the level surfaces $f = c$. In other words, given $f(x_0, y_0, z_0) = c$ consider the tangent plane to the level surface $f(x, y) = c$ at (x_0, y_0). This plane has normal vector $\nabla f(x_0, y_0)$.
- Consider a unit vector $u = (u_1, u_2, u_3)$ and a function $f(x, y, z)$. The directional derivative of f in the direction of u at the point (x_0, y_0, z_0) is the limit

$$\left.\frac{df}{ds}\right|_u = \lim_{h\to 0} \frac{f(x_0 + u_1h, y_0 + u_2h, z_0 + u_3h) - f(x_0, y_0, z_0)}{h}.$$

It can be computed as the dot product of u with the gradient ∇f

$$\left.\frac{df}{ds}\right|_u = \nabla f.u$$

Problems

1. Use the chain rule to find $\frac{\partial f}{\partial t}$ for the composite function $f(x(t), y(t))$. Also check your answer by explicitly writing f as a function of t.

   (a) $f = \ln(x^2 + y^2), x = \sin(t), y = \cos(t)$
   (b) $f = \frac{3xy}{x^2-y^2}, x = t^2, y = 3t$
   (c) $f = e^{-x^2-y^2}, x = t, y = \sqrt{t}$
   (d) $\sin(xy), x = t, y = t^4$.

2. Find w_u for

   (a) $w = x^2y + y^2 + x, x = u^2v, y = uv^2$.
   (b) $w = e^{s+t}, s = uv, t = u + v$.
   (c) $w = \frac{x}{y}, x = u^2 - v^2, y = u^2 + v^2$.

3. (Notes 2D-3(c)) Find the tangent plane to the cone $x^2 + y^2 - z^2 = 0$ at the point (x_0, y_0, z_0).

4. (Notes 2D-1) Find the gradient of f and the directional derivative $\left.\frac{df}{ds}\right|_u$ in the direction u of the given vector at the given point for

(b) $f = \frac{xy}{z}, i + 2j - 2k, (2, -1, 1)$
(d) $f = \ln(2s + 3t), 4i - 3j, (-1, 1)$
(e) $f = (u + 2v + 3w)^2, -2i + 2j - k, (1, -1, 1)$.

18.02 Recitation 10

- **Lagrange Multipliers:** Consider functions $f(x,y,z)$ and $g(x,y,z)$. The maximum/minimum of f under the constraint $g(x,y,z)=0$ (i.e. over all points satisfying $g(x,y,z)=0$) occurs at a point (x_0,y_0,z_0) where

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)$$

for some scalar λ (called the *Lagrange multiplier*). Hence we have that

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0,y_0,z_0) &= \lambda\frac{\partial g}{\partial x}(x_0,y_0,z_0)\\ \frac{\partial f}{\partial y}(x_0,y_0,z_0) &= \lambda\frac{\partial g}{\partial y}(x_0,y_0,z_0)\\ \frac{\partial f}{\partial z}(x_0,y_0,z_0) &= \lambda\frac{\partial g}{\partial x}(x_0,y_0,z_0)\text{and}\\ g(x_0,y_0,z_0) &= 0.\end{aligned}$$

These four equations can now be solved for the four variables x_0,y_0,z_0 and λ.

- **Non independent variables:** Consider a function $f(x,y,z)$ where x,y and z are *non-independent variables* which satisfy the relation $g(x,y,z)=0$. Hence one of the variables x,y or z can be eliminated using $g=0$ to consider f as a function of the other two. The notation

$$\left(\frac{\partial f}{\partial x}\right)_y$$

denotes the partial derivative of f considered as a function of x and y (i.e. having eliminated z).

- This partial $\left(\frac{\partial f}{\partial x}\right)_y$ is computed using the method of differentials as follows. Since $g(x,y,z)=0$ we have

$$\begin{aligned}df &= f_x dx+f_y dy+f_z dz\\ dg &= g_x dx+g_y dy+g_z dz=0.\end{aligned}$$

Eliminating dz gives

$$df=\left(f_x-\frac{f_z g_x}{g_z}\right)dx+\left(f_y-\frac{f_z g_y}{g_z}\right)dy.$$

Hence $\left(\frac{\partial f}{\partial x}\right)_y=\left(f_x-\frac{f_z g_x}{g_z}\right)$.

Problems

1. Use Lagrange multipliers to find the maximum values for the following functions under the given constraints

(a) $x+y+z$, given $\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{8}=4$.
(b) z, given $x^2+y^2+z^2=1$.
(c) xyz, given $xy+yz+zx=3$.

2. Use the method of Lagrange multipliers to show that the distance of the origin from the plane $ax + by + cz = d$ is given by $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$.

3. (Notes 2I-2) Find the point in the first octant on the surface $x^3y^2z = 6\sqrt{3}$ closest to the origin.

4. In each of these examples compute $\left(\frac{\partial x}{\partial u}\right)_v$

(a) $x = u^2 + v + w^3$, where $uvw = 1$.
(b) $x = u + v + w$, where $e^w = u + w$.
(c) $x = e^w$, where $w^2 - w = uv$.
(d) $x = \frac{uv}{w}$, where $uw^2 + \frac{v}{w} = 1$.

18.02 Recitation 11

- Consider non-independent variables satisfying the relation $g(x,y,z) = 0$. The partial $\left(\frac{\partial f}{\partial x}\right)_y$ is computed using the chain rule as follows. Since z is treated as a function of x and y differentiating gives

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_y &= f_x + f_z\frac{\partial z}{\partial x}\\ 0 &= g_x + g_z\frac{\partial z}{\partial x}.\end{aligned}$$

Plugging the value $\frac{\partial z}{\partial x} = -\frac{g_x}{g_z}$ from the second equation into the first gives

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(f_x - \frac{f_z g_x}{g_z}\right).$$

Problems

1. Use the method of Lagrange multipliers to show that the distance of the origin from the plane $ax + by + cz = d$ is given by $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$.

2. (Notes 2I-2) Find the point in the first octant on the surface $x^3y^2z = 6\sqrt{3}$ closest to the origin.

3. In each of these examples compute $\left(\frac{\partial x}{\partial u}\right)_v$

(a) $x = e^w$, where $w^2 - w = uv$.
(b) $x = \frac{uv}{w}$, where $uw^2 + \frac{v}{w} = 1$.
(c) $x = u^2 + vw$, where $\sin w + u = \frac{v}{w}$.

4. Find the equation of the tangent plane to the surface $z^2 = 11x^2 + 3xy + 2y^2$ at the point $(1, 2, 5)$.

5. Find the direction in which the directional derivative of $f(x,y) = \frac{2xy}{x^2+y^2}$ is maximized at the point $(1, 2)$ and find the value of this directional derivative.

18.02 Recitation 12

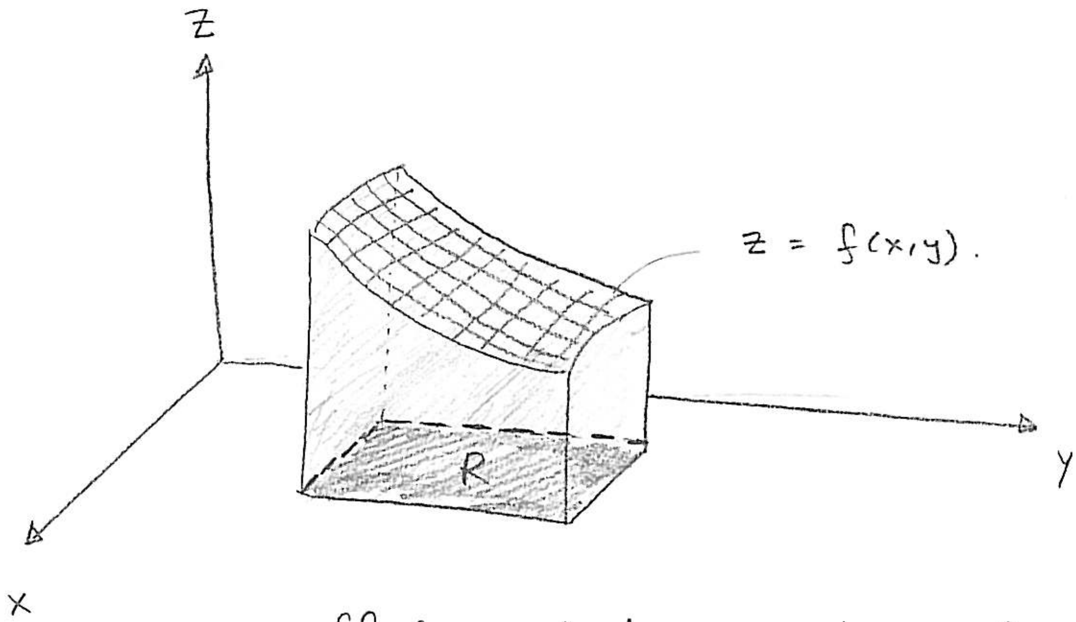
- A partial differential equation is an equation involving a function and its derivatives. For example the equation
$$f_t + (f_x)^3 + 6ff_x = 0$$
is a partial differential equation for the function of two variables $f(x,t)$.
- Partial Differential Equations in Physics -
Wave equation Consider the vertical diaplacement function $y = f(x,t)$ of a taut string. It satisfies the one dimensional equation wave equation
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t^2}.$$
The x, y and z components of the electric and magnetic field in vacuum also satisfy the three dimensional wave equation.
Heat equation Let $h(x,y,z,t)$ be the time-dependent temperature function in a room. It satisfies the equation heat equation
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{\partial h}{\partial t}.$$
Laplace's equation Assuming the temperature function h of a room is in a steady state (i.e. does not change with time) it will satisfy Laplace's equation
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0.$$
The gravitational/electrostatic potential functions in vacuum (i.e. a region free of mass/charge) also satisfy Laplace's equation.

Problems

1. (Notes 2K-5) Find solutions to the one-dimensional heat equation $w_{xx} = w_t$ having the form
$$w = \sin(kx)e^{rt}$$
satifying the additional conditions $w(0,t) = w(1,t) = 0$ for all t. Interpret your solution physically. What happens to the temparature as $t \to \infty$?

2. (Notes 2K-3) Find all solutions to the two-dimensional Laplace equation of the form
$$h = ax^2 + bxy + cy^2$$
for some constants a, b and c. Show that they can be written in the form $c_1 f_1(x,y) + c_2 f_2(x,y)$ for certain fixed polynomials $f_1(x,y)$ and $f_2(x,y)$ with arbitrary constants c_1 and c_2.

3. For what constant c will the function
$$h = \frac{e^{-\frac{cx^2}{t}}}{\sqrt{t}}$$
satisfy the one-dimensional heat equation?



$$\iint_R f(x,y)\,dx\,dy = \text{volume of shaded region}$$

Figure 1.

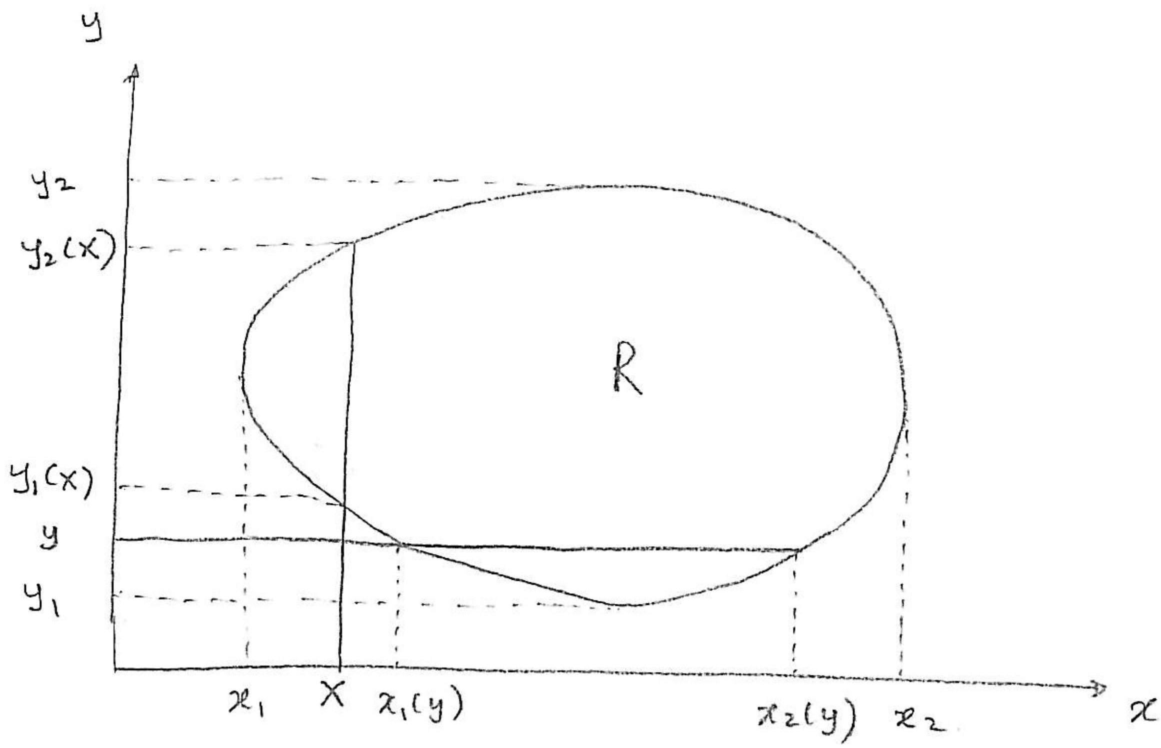


Figure 2.

18.02 Recitation 14

- Given a function of two variables $f(x,y)$ and a region R in the plane, the integral of f over R is expressed in polar coordinates by

$$\int\int_R f(x,y)dxdx = \int\int_R f(r\cos\theta, r\sin\theta)rdrd\theta.$$

- Consider a function of two variables $f(x,y)$ and a region R in the plane. Let $x(u,v), y(u,v)$ be written as functions of u, v. The integral of f can then be expressed with respect to the new variables u, v as

$$\int\int_R f(x,y)dxdy = \int\int_R f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

is the Jacobian of the change of variables.

Problems

1. Use polar coordinates to evaluate

   (a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dxdy$

   (b) $\int_0^1 \int_x^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy$

   (c) $\int_0^1 \int_x^1 x^2 dydx$

   (d) The area enclosed by the cardiod $r = 1 - \cos\theta$.

2. Find the center of mass of the part of the annulus $\{1 < x^2 + y^2 < b^2\}$ in the upper half plane. For what b does the center of mass lie outside the annulus itself?

3. Find the area of the region in the first quadrant bounded by the lines $y = x, y = 2x$ and the hyperbolas $xy = 1, xy = 2$.

4. Evaluate the integral $\int\int_T \sin\left(\frac{x+y}{x-2y}\right) dxdy$ where T is the triangle with vertices $(1,0), (4,0)$ and $(3,1)$.

5. Use elliptical coordinates $x = ar\cos\theta$, $y = br\sin\theta$ to show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

18.02 Recitation 15

- A vector field $F(x,y)$ in the plane is a vector function of two variables

$$F(x,y) = (f(x,y), g(x,y)) = f(x,y)i + g(x,y)j.$$

- Given a curve C in the plane and a vector field $\vec{F} = fi + gj$ define the line integral to be

$$\int_C F.dr = \int_C (fdx + gdy) = \int_a^b \left(f(x(t),y(t))\frac{dx}{dt} + g(x(t),y(t))\frac{dy}{dt} \right) dt, \tag{1}$$

where $(x(t),y(t))$ is a parametrization for the curve C. The first two terms in the equality (1) are notations for the line integral while the last term is the definition.

Problems

1. (Notes 4A-3) Write down an expression for each of the following vector fields

(a) Each vector has the same direction and magnitude as $i + 2j$.
(b) The vector at (x,y) is directed radially towards the origin with magnitude r^2.
(c) The vector at (x,y) is tangent to the circle through (x,y) with center at the origin, clockwise direction, magnitude r^2.

2. (Notes 4B-1) For each of the vector fields F and curves C evaluate $\int_C F.dr$

(a) $F = (x^2 - y)i + 2xj$, C_1, C_2 both run from $(-1,0)$ to $(1,0)$ with C_1 along the x-axis and C_2 along the parabola $y = 1 - x^2$.
(b) $F = xyi - x^2j$, C: quarter circle running from $(0,1)$ to $(1,0)$.
(c) $F = yi - xj$, C: the triangle with vertices $(0,0), (0,1), (1,0)$ oriented clockwise.
(d) $F = yi$, C: ellipse $x = 2\cos t$, $y = \sin t$ oriented clockwise.
(e) $F = 6yi + xj$, C is the curve $x = t^2, y = t^3$ running from $(1,1)$ to $(4,8)$.
(f) $F = (x+y)i + xyj$, C is the broken line running from $(0,0$ to $(0,2)$ to $(1,2)$.

3. Calculate the work done by a space shuttle when it moves in the gravitational force field

$$F = -Gm_1m_2\frac{(xi + yj)}{(x^2 + y^2)^{\frac{3}{2}}}$$

along the trajectory $(x(t),y(t)) = (t\cos t, t\sin t)$ as t varies from $t = 2\pi$ to $t = 4\pi$.

4. (Notes 4B-3) For $F = i + j$ find a line segment C such that $\int_C F.dr$ is

(a) minimum (over all curves between the same endpoints as C)
(b) maximum
(c) zero.

18.02 Recitation 16

- Consider a vector field $F(x,y) = M(x,y)i + N(x,y)j$ in the plane. The following are equivalent
  (a) the vector field is the gradient $F = \nabla f$ of some function f
  (b) the line integral $\int_C F.dr = f(b) - f(a)$ for some function f on the plane and where b and a are the two endpoints of the curve C
  (c) the line integral $\int_C F.dr = 0$ for any closed curve C in the plane
  (d) $M_y = N_x$.
  We say that the vector field F is *conservative* in case any of the above is satisfied. The statement (b) above is referred to as the fundamental theorem of line integrals.
- Given a conservative vector field $F = Mi + Nj$, a function f satisfying $F = \nabla f$ is called a *potential function* for the vector field F. Such a potential function can be found in the following two ways
  (a) $f(x,y) = \int_C F.dr$ where C is any curve joining (x,y) to some fixed point (x_0, y_0)
  (b) setting $f_x = M$ we may solve
  $$f = \int_0^x M dx + g(y)$$
  for some function $g(y)$ which in turn can be found from $f_y = N$ via
  $$g(y) = C + \int_0^y \left(N - \int_0^x M_y dx \right) dy,$$
  for some constant C.

Problems

1. Use geometry to compute the line integrals $\int_C F.dr$ where $F = \frac{xi+yj}{x^2+y^2}$ and the curve C is

(a) the semicircle in the upper half plane joining $(2,0)$ to $(-2,0)$
(b) the straight line joining $(1,1)$ to $(4,4)$

2. (Notes 4C-1) Let $f = x^3y + y^3$ and C be the curve $y^2 = x$ from $(1,-1)$ to $(1,1)$. Calculate $F = \nabla f$. Then find $\int_C F.dr$ in three different ways

(a) directly
(b) using path independence to replace C by a simpler path.
(c) by using fundamental theorem of line integrals.

3. Find the value of a for which the following vector fields are conservative and find the corresponding potential functions

(a) $(y^2 + 2x)i + axyj$
(b) $e^{x+y}((x+a)i + xj)$
(c) $(axy + x^2)i + (x^2 + y^2)j$.

4. Check whether the gravitational vector field

$$F = -Gm_1m_2\frac{(xi + yj)}{(x^2 + y^2)^{\frac{3}{2}}}$$

is conservative and if so find its potential function.

18.02 Recitation 17

- Given a vector field $F = M(x,y)i + N(x,y)j$ its curl is defined as the function
$$\text{curl}F = N_x - M_y.$$
The divergence of the vector field is defined to be the function
$$\text{div}F = M_x + N_y.$$
- Given a vector field $F = M(x,y)i + N(x,y)j$ and a curve C the flux of F across C is defined to be
$$\int_C F.nds = \int_C Mdy - Ndx.$$
- **Green's Theorem in Tangential form**: Consider a vector field $F = M(x,y)i + N(x,y)j$. If R is a region with boundary being curve C then
$$\int_C Mdx + Ndy = \int\int_R (N_x - M_y)dA. \tag{1}$$
Here the curve C is traversed so that the region R is on the right. The equation (1) can also be read as
$$\int_C F.dr = \int\int_R \text{curl}F dA.$$
- **Green's Theorem in Normal form**: Consider a vector field $F = M(x,y)i + N(x,y)j$. If R is a region with boundary being curve C then
$$\int_C Mdy - Ndx = \int\int_R (M_x + N_y)dA. \tag{2}$$
Here the curve C is traversed so that the region R is on the right. The equation (2) can also be read as
$$\int_C F.nds = \int\int_R \text{div}F dA.$$

Problems

1. (Notes 4D-1) For each of the vector fields F and curves C evaluate $\int_C F.dr$ both directly and using Green's theorem

(a) $F = 2yi + xj$, $C : x^2 + y^2 = 1$
(b) $F = x^2(i + j)$, C : rectangle joining $(0,0), (2,0), (0,1)$ and $(2,1)$
(c) $F = xyi + y^2j$, $C : y = x^2$ and $y = x$, $0 \le x \le 1$.

2. Use Green's theorem to evaluate $\int_C Pdx + Qdy$ where

(a) $P = 2y + \sqrt{9 + x^3}, Q = 5x + e^{\arctan y}$, C is the positively oriented circle $x^2 + y^2 = 4$

(b) $P = -y^2 + exp(e^x), Q = \arctan y$, C is the boundary of the region between the parabolas $y = x^2$ and $x = y^2$.

3. (Notes 4D-4) Show that $\int_C -y^3 dx + x^3 dy$ is positive along any simple closed curve C directed counterclockwise.

4. (Notes 4D-5) Show that the value of the integral $\int_C xy^2 dx + (x^2 y + 2x) dy$ around any square C in the xy plane only depends on the size of the square and not upon its position.

5. (Notes 4F-3) Verify Green's theorem in normal form for the vector field $F = xi + yj$ where C is the closed curve formed by the upper half of the unit circle and the x axis interval $[-1, 1]$.

18.02 Recitation 18

- Given a function $f(x,y)$ and a curve C the line integral of f with respect to arclength is defined to be
$$\int_C f\,ds = \int_a^b f(x(t),y(t))\,\sqrt{x'(t)^2+y'(t)^2}dt$$
for some parametrization $(x(t),y(t))$ of C.
- Given a vector field $F = M(x,y)i + N(x,y)j$ and a curve C the flux of F across C is
$$\int_C F.n\,ds = \int_C M dy - N dx.$$
Here n is the unit normal vector to the curve C obtained by rotating the unit tangent vector clockwise by angle $\frac{\pi}{2}$.
- Given a vector field $F = M(x,y)i+N(x,y)j$ its divergence is defined to be the function
$$\text{div}F = M_x + N_y.$$
- **Green's Theorem in Normal form**: Consider a vector field $F = M(x,y)i+ N(x,y)j$. If R is a region with boundary being curve C then
$$\int_C M dy - N dx = \int\int_R (M_x + N_y) dA. \tag{1}$$
Here the curve C is traversed so that the region R is on the right. The equation (1) can also be read as
$$\int_C F.nds = \int\int_R \text{div}F dA.$$

Problems

1. Calculate curl and divergence for the vector fields

   (a) $x^3 i + y^3 j$
   (b) $2xi + 3yj$
   (c) $xi - yj$.

Calculate the flux of each of the above vector fields across the positively unit circle $x^2+y^2 = 1$. Now verify Green's theorem in normal form for the above vector fields.

2. (Notes 4E-1) Let $F = -yi + xj$. Evaluate $\int_C F.n\,ds$ geometrically where

   (a) C is the circle of radius a centered at the origin, directed counterclockwise.
   (c) C is the line running from $(0,0)$ to $(1,0)$.

3. (Notes 4E-5) Let F be defined everywhere except the origin so that the direction of F is radially outwards and its magnitude is $|F| = r^m$ where m is an integer. Evaluate the flux of F accross a circle of radius a. For what value of m will this flux be independent of a?

4. (Notes 4F-2) Let $F = \omega(-yi + xj)$

(a) Calculate $\text{div}F$ and $\text{curl}F$.

(b) Using physical interpretations explain why it is resonable that $\text{div}F = 0$.

(c) Using physical interpretations explain why it is resonable that $\text{curl}F = 2\omega$ at the origin.

18.02 Recitation 19

- Given a function $f(x, y)$ and a curve C the line integral of f with respect to arclength is defined to be
$$\int_C f\, ds = \int_a^b f(x(t), y(t))\, \sqrt{x'(t)^2 + y'(t)^2} dt$$
for some parametrization $(x(t), y(t))$ of C.
- Given a vector field $F = M(x, y)i + N(x, y)j$ and a curve C the flux of F across C is
$$\int_C F.n\, ds = \int_C M dy - N dx.$$
Here n is the unit normal vector to the curve C obtained by rotating the unit tangent vector clockwise by angle $\frac{\pi}{2}$.
- Given a vector field $F = M(x, y)i + N(x, y)j$ its divergence is defined to be the function
$$\mathrm{div} F = M_x + N_y.$$
- **Green's Theorem in Normal form**: Consider a vector field $F = M(x, y)i+ N(x, y)j$. If R is a region with boundary being curve C then
$$\int_C M dy - N dx = \iint_R (M_x + N_y) dA. \tag{1}$$
Here the curve C is traversed so that the region R is on the right. The equation (1) can also be read as
$$\int_C F.n ds = \iint_R \mathrm{div} F dA.$$
- **Extended Green's theorem:** Consider a region R with boundary being the union of curves $C_1, \ldots, C_m$. We orient the curves C_i such that the region R lies on the right while traversing C_i in the positively oriented direction. Then the generalized Green's theorem says
$$\int_{C_1} F.dr + \ldots + \int_{C_m} F.dr = \iint_R \mathrm{curl} F\, dA$$
for any vector field F.
- A simply connected region is a region R consisting of one piece and with the property: for any simple closed curve C contained in R the interior of C lies in R.
- For a continously differentiable vector field F defined on a simple connected region R, the following are equivalent:
  (a) there exits a continuously differentiable function f defined on R such that $\nabla f = F$
  (b) $\mathrm{curl} F = 0$ for all points in R.

  The above are *not* necessarily equivalent for a non-simply connected region.

Problems

1. Calculate curl and divergence for the vector fields

   (a) $x^3i + y^3j$
   (b) $2xi + 3yj$
   (c) $xi - yj$.

Calculate the flux of each of the above vector fields across the positively unit circle $x^2 + y^2 = 1$. Now verify Green's theorem in normal form for the above vector fields.

2. (Notes 4E-1) Let $F = -yi + xj$. Evaluate $\int_C F.n\, ds$ geometrically where

   (a) C is the circle of radius a centered at the origin, directed counterclockwise.
   (c) C is the line running from $(0, 0)$ to $(1, 0)$.

3. (Notes 4E-5) Let F be defined everywhere except the origin so that the direction of F is radially outwards and its magnitude is $|F| = r^m$ where m is an integer. Evaluate the flux of F accross a circle of radius a. For what value of m will this flux be independent of a?

4. (Notes 4F-2) Let $F = \omega(-yi + xj)$

   (a) Calculate div$F$ and curl$F$.
   (b) Using physical interpretations explain why it is resonable that div$F = 0$.
   (c) Using physical interpretations explain why it is resonable that curl$F = 2\omega$ at the origin.

5. Which of the following regions are simply connected?

   (a) R=the unit disk $\{(x, y)|x^2 + y^2 \leq 1\}$
   (b) R=the upper half plane $\{(x, y)|y \geq 0\}$.
   (c) R= all points in the plane except the origin
   (d) R= all points in the plane except the first quadrant.

Limits of triple integrals

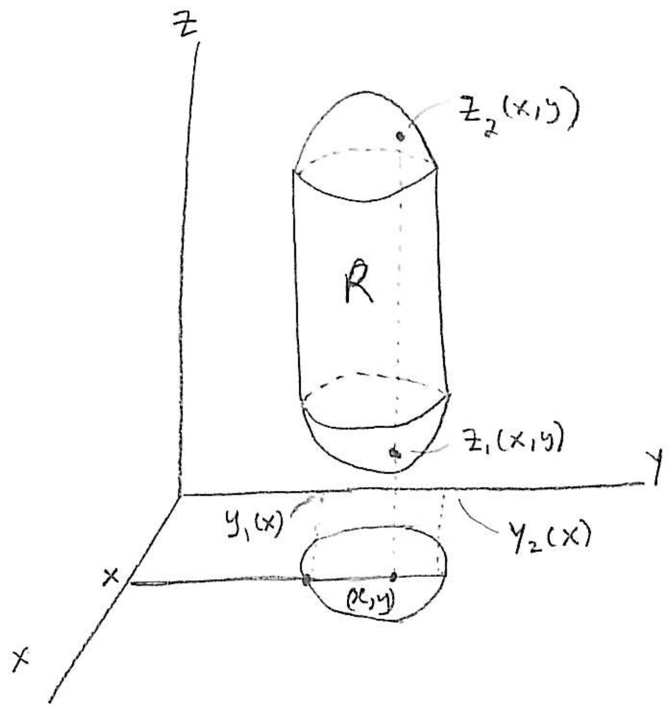
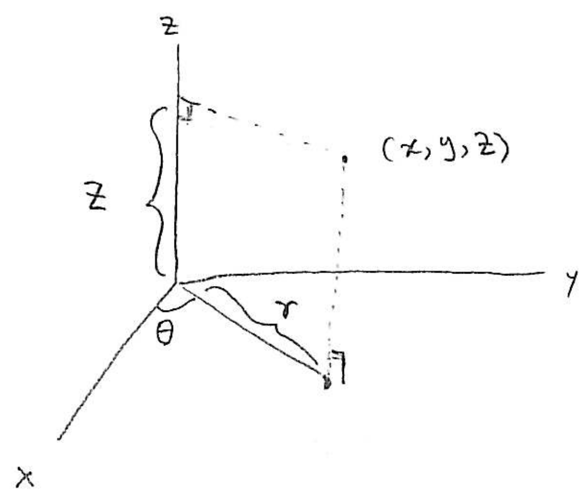


Figure 1

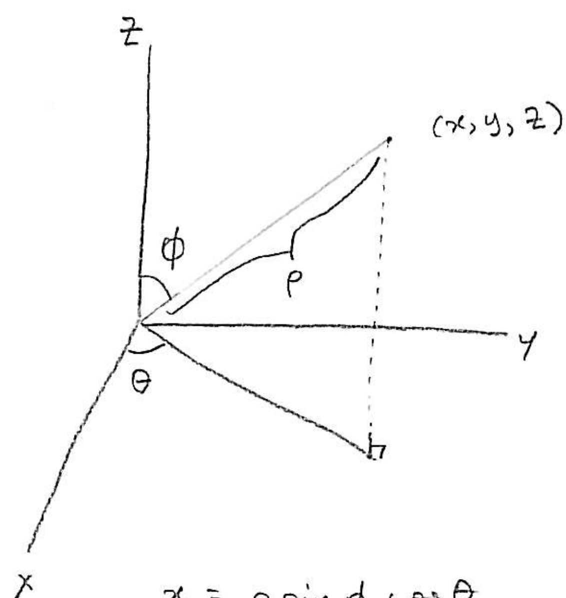
Cylindrical coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Figure 2.

Spherical coordinates



$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Figure 3.

18.02 Recitation 22

- Given a surface S and a paramentrization $S = \{(x(u,v), y(u,v), z(u,v)) | (u,v) \in D \subset \mathbb{R}^2\}$ the surface area element for S is

$$dS = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, dudv.$$

- The integral with respect to surface area of a function $f(x,y,z)$ is given in terms of this parametrization via

$$\iint_S f(x,y,z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, dudv.$$

- Given a vector field $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ in space. The flux of F across the surface S is defined as

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_S \vec{F}.d\vec{S} = \iint_D \left(P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)}\right) dudv,$$

where $\vec{n}$ is the unit normal vector to the surface at the point (x,y,z) and $(x(u,v), y(u,v), z(u,v))$ is a parametrization for S.
- Given a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ in space its divergence is defined to be the function

$$\vec{\nabla}.\vec{F} = P_x + Q_y + R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

- **Divergence theorem**: Given a closed surface S which bounds a solid region R in space one has

$$\iint_S F.dS = \iiint_R \nabla.F\, dV$$

where the left hand side denotes the flux out of the region R.

Problems

1. Find parametrizations for the surfaces

   (a) The hemisphere $x^2 + y^2 + z^2 = 4$, $x \geq 0$
   (b) Part of the cone $y^2 = x^2 + z^2$ with $0 \leq y \leq 2$
   (c) Part of cylinder $x^2 + y^2 = 1$ between the $z = 2$ and $z = 4$ planes
   (d) Part of the paraboloid $z = x^2 + y^2$ below the $z = 1$ plane.

2. Find the the flux of the vector fields $F = i$, $F = xi$ and $F = xi + yj + zk$ for each of the surfaces in problem 1.

3. Find the surface areas of each of the parts in problem 1.

4. (Notes 6B-9) Find the center of mass of a uniform density $\delta = 1$ hemispherical shell of radius a which has its base on the xy-plane.

5. Using divergence theorem, find the flux of the vector field F out of the closed surface S where

(a) $F = x^3i + y^3j + z^3k$, and S is the surface of the cylinder $x^2 + y^2 = 9$ between $z = -1$ and $z = 4$

(b) $F = (x + \cos y)i + (y + \sin z)j + (z + e^x)k$ and S is the boundary of the region bounded by the planes $z = 0, y = 0, y = 2$ and $z = 1 - x^2$

(c) $F = xi + yj + 3k$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

6. Show that the flux of the radial vector field $F = \frac{1}{3}(xi + yj + zk)$ out through the boundary of any solid region equals the volume of the region.

7. Using divergence theorem find the flux of the vector field $F = e^{x+z}j$ through the *non-closed* upper hemisphere given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

18.02 Recitation 23

- Consider the vector field $\mathbf{F} = Pi + Qj + Rk$ in space. The *divergence* of $\mathbf{F}$ is the function
$$\nabla.\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$
The *curl* of $\mathbf{F}$ is the vector field
$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k. \end{aligned}$$
- **Divergence theorem**: Given a closed surface S which bounds a solid region R in space one has
$$\iint_S F.dS = \iiint_R \nabla.F\, dV$$
where the left hand side denotes the flux out of the region R.
- Consider a vector field $\mathbf{F}$ in space. The following are equivalent
  (a) the vector field is the gradient $\mathbf{F} = \nabla f$ of some function f
  (b) the line integral $\int_C \mathbf{F}.dr = 0$ for any closed curve C in space
  (c) there exists some function f in space such that the line integral $\int_C \mathbf{F}.dr = f(b) - f(a)$ for all curves C where b and a are the two endpoints of the curve C
  (d) the curl $\nabla \times \mathbf{F} = 0$.

  We say that the vector field F is *conservative* in case any of the above is satisfied. The statement (c) above is referred to as the fundamental theorem of line integrals and the function f is called the potential function.

Problems

1. Verify divergence theorem in the following cases

   (a) $\mathbf{F} = xi + yj + zk$ and S is the spherical surface $x^2 + y^2 + z^2 = 1$.
   (b) $\mathbf{F} = (y+z)i + (z+x)j + (x+y)k$ and S is the surface of the tetrahedron formed by the coordinate planes and the plane $x + y + z = 1$.
   (c) $\mathbf{F} = y^2 j + yzk$ and S is formed by the cylinder $y^2 + z^2 = 1$ and the coordinates planes $x = 0$ and $x = 1$.

2. Using divergence theorem find the flux of the vector field $F = e^{x+z} j$ through the *non-closed* upper hemisphere given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

3. (Notes 6D-1) Find the line integrals $\int_C \mathbf{F}.dr$ for

   (a) $\mathbf{F} = yi + zj - xk$ and C is the twisted cubic $(x, y, z) = (t, t^2, t^3)$ running from $(0, 0, 0)$ to $(1, 1, 1)$.
   (b) $\mathbf{F} = yi + zj - xk$ and C is the line running from $(0, 0, 0)$ to $(1, 1, 1)$.

(c) $\mathbf{F} = yi + zj - xk$ and C is the broken line segment running from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

(d) $\mathbf{F} = zxi + zyj + xk$ and C is the helix $(x,y,z) = (\cos t, \sin t, t)$ from $(1,0,0)$ to $(1,0,2\pi)$.

4. (Notes 6E-3) For each vector field below find its curl and find a potential function if the curl is zero

(a) $xi + yj + zk$
(b) $(2xy + z)i + x^2 j + xk$
(c) $(y^2z^2)i + (x^2z^2)j + (x^2y^2)k$
(d) $yzi + xzj + xyk$.

18.02 Recitation 24

- **Stokes' theorem**: Consider a vector field $\mathbf{F}$ in space. If S is a surface with boundary C then
$$\int_C F.dr = \iint_S curl\mathbf{F}.dS,$$
where the orintations of S and C are compatible via the right hand rule.

Problems

1. Verify Stokes' theorem when S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and

   (a) $\mathbf{F} = xi + yj + zk$
   (b) $\mathbf{F} = yi - xj + zk$

2. Use Stokes theorem to compute $\int_C \mathbf{F}.dr$ where

   (a) $\mathbf{F} = 2zi + xj + 3yk$ and C is the intersection of the plane $z = y$ with the cylinder $x^2 + y^2 = 4$ orineted counterclockwise when viewed from above.
   (b) $\mathbf{F} = (y - x, x - z, x - y)$ and C is the boundary of the part of the plane $x + 2y + z = 2$ that lies in the first octant oriented counterclockwise when viewed from above.

3. Use Stokes theorem to evaluate $\iint_S curl\mathbf{F}.dS$ where $\mathbf{F} = 2yi + 3xj + e^z k$ and S is part of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$ with normal vector pointing upwards.

4. (Notes 6F-4) Show by direct calculation that $div(curl\mathbf{F}) = 0$ for any vector field $\mathbf{F}$. Now show that
$$\iint_S curl\mathbf{F}.dS = 0$$
for any closed surface S using both Divergence and Stokes' theorems.

18.02 Recitation 25

- **Vector Calculus**: Any successive composition in the following diagram is zero

$$\{\text{functions}\} \xrightarrow[\nabla]{\text{grad}} \{\text{vector fields}\} \xrightarrow[\nabla\times]{\text{curl}} \{\text{vector fields}\} \xrightarrow[\nabla\cdot]{\text{div}} \{\text{functions}\}.$$

In other words we have

$$\begin{aligned} \text{curl}(\text{grad} f) &= 0 \qquad (\nabla \times (\nabla f) = 0) \qquad \text{and} \\ \text{div}(\text{curl} F) &= 0 \qquad (\nabla \cdot (\nabla \times \mathbf{F}) = 0). \end{aligned}$$

The non-successive composition is the Laplacian of a function

$$\nabla^2 f = \nabla \cdot (\nabla f) = \text{div}(\text{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Problems

1. Prove the identities $\text{curl}(\text{grad} f) = 0$, $\text{div}(\text{curl} F) = 0$ and

$$\text{div}(\text{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

2. Show that for any closed surface S

$$\iint_S curl\mathbf{F}.dS = 0$$

using both Divergence and Stokes' theorems.

3. (Notes 6H-3) Prove that for any scalar function ϕ and vector field $\mathbf{F}$ one has

   (a) $\nabla \cdot (\phi \mathbf{F}) = \phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \phi$
   (b) $\nabla \times (\phi \mathbf{F}) = \phi \nabla \times \mathbf{F} + (\nabla \phi) \times \mathbf{F}$
   (c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

18.02 Recitation 26

- **Vector Calculus**: Any successive composition in the following diagram is zero

$$\{\text{functions}\} \xrightarrow[\nabla]{\text{grad}} \{\text{vector fields}\} \xrightarrow[\nabla\times]{\text{curl}} \{\text{vector fields}\} \xrightarrow[\nabla\cdot]{\text{div}} \{\text{functions}\}.$$

In other words we have

$$\begin{aligned} \text{curl}(\text{grad} f) &= 0 \qquad (\nabla \times (\nabla f) = 0) \qquad \text{and} \\ \text{div}(\text{curl} F) &= 0 \qquad (\nabla \cdot (\nabla \times \mathbf{F}) = 0). \end{aligned}$$

The non-successive composition is the Laplacian of a function

$$\nabla^2 f = \nabla \cdot (\nabla f) = \text{div}(\text{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

- **Maxwell's equations**:

Law	Differential form	Integral form
Gauss' Law	$\nabla \cdot E = \rho$	$\iint_S E \cdot dS = Q$
Gauss' Law for Magnetism	$\nabla \cdot B = 0$	$\iint_S B \cdot dS = 0$
Faraday's Law	$\nabla \times E = -B_t$	$\int_C E \cdot dr = -\frac{\partial}{\partial t} \iint_S B \cdot dS$
Ampere's Law	$\nabla \times B = j + E_t$	$\int_C B \cdot dr = I + \frac{\partial}{\partial t} \iint_S E \cdot dS$

E and B are the electric and magenetic fields while Q, I, ρ and j are total charge, total current, charge density and current density respectively.

Problems

1. Prove the identities $\text{curl}(\text{grad} f) = 0$, $\text{div}(\text{curl} F) = 0$ and
$$\text{div}(\text{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

2. Show that for any closed surface S
$$\iint_S \text{curl}\mathbf{F} \cdot dS = 0$$
using both Divergence and Stokes' theorems.

3. (Notes 6H-3) Prove that for scalar functions f, g and vector fields $\mathbf{F}, \mathbf{G}$ one has

   (a) $\nabla(fg) = f\nabla g + g\nabla f$.
   (b) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
   (c) $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}$
   (d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

4. Show how to go back and forth between the integral and differential forms of Maxwell's equations. Do the same with the equations for *charge conservation*
$$\rho_t = -\nabla \cdot j \quad \text{and} \quad Q_t = -I.$$