MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 1

1. Introduction

A differential equation is any equation involving a function and its derivatives. These are broadly classified into ordinary and partial differential equations. An ordinary differential equation (ODE for short) is one involving a function of a single variable (often thought of as time). The most general form of an ordinary differential equation involving the function $f(t)$ is

$$F\left(f, f', f'', \ldots, f^{(n)}, t\right) = 0$$

for some function F of several variables. The order of the differential equation is highest order of differentiation involved. Hence the above equation has order n. Ordinary differential equations can be further classified as being linear or nonlinear. A linear ODE is one which has the more restricted form

$$L[f] = a_0 f + a_1 f' + \ldots + a_n f^{(n)} = g(t)$$

for some coefficient functions $a_0, \ldots, a_n$. We shall often restrict to the case when these coefficient functions are constant in order to be able to solve the equation. If is happens that the right hand side $g(t) = 0$, then this equation is said to be homogeneous (and inhomogeneous othewise). Towards the end of the course we shall study some partial differential equations which are those involving a function of several variables.

2. Simple Examples

We begin today with some simple examples

Problem 1. Find the general solution to the differential equation

$$y' = x$$

and more generally to the equation

$$y' = f(x).$$

Solution. Both equations are solved by integrating the right hand side. The general solution to the first is

$$y = \frac{x^2}{2} + C.$$

While the general solution to the second is

$$y = \int f(x)\, dx + C.$$

Hence we see that integration is solving the simplest kind of differential equation.

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Problem 2. Find the general solution to the equation

$$y' = y.$$

Solution. This equation is solved using separation of variables. We have

$$\frac{dy}{dx} = y \implies \frac{dy}{y} = dx$$

which upon integration gives

$$\log(y) = x + c.$$

Hence we have

$$y = e^c e^x = Ce^x.$$

We now solve an inhomogeneous equation

Problem 3. Find the general solution to the equation

$$y' - y = x.$$

Solution. This problem is solved in two steps. The first step consists in finding the solution y_h to the corresponding homogeneous equation obtained by setting the right hand side to 0. Hence we must solve

$$y_h' - y_h = 0.$$

From the previous problem this gives

$$y_h = Ce^x.$$

The next step consists in finding one particular solution y_p to the problem . There isn't a systematic way to do this. In specific examples we can venture a guess. Here we try

$$y_p = Ax + B.$$

Putting this into our problem gives

$$y_p' - y_p = A - (Ax + B) = x.$$

This gives $A = -1, B = 1$ by comparing the coefficient of x and the constant coefficient. Hence $y_p = -x + 1$. The general solution y_{gen} to the problem is now the sum of the homogeneous and the particular solutions

$$\begin{aligned} y_{gen} &= y_h + y_p \\ &= Ce^x - x + 1. \end{aligned}$$

Since the solution involved determining the coefficients A and B of the linear polynomial $Ax + B$, it is called the **method of undetermined coefficients**.

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LECTURE 2

1. Method of Undetermined Coefficients

We shall continue with some more examples of the method of undetermined coefficients. This method is used to find the particular and general solution to a linear ODE with constant coefficients. We first start with a homogeneous example

Problem 1. Find the solution to the equation

$$y'' + 5y' + 6y = 0, \quad y(0) = 0, y'(0) = 2.$$

Solution. The recipe consists in first writing down the associated characteristic polynomial equation. This is obtained by replacing polynomials in place of derivatives. Here we have the characteristic equation

$$p(r) = r^2 + 5r + 6 = 0.$$

This has the roots $r = -2, -3$. The general solution is now the linear combination of exponentials

$$y = C_1 e^{-2x} + C_2 e^{-3x}.$$

To find the solution with the given initial values of $y(0), y'(0)$ we simply differentiate to obtain

$$\begin{aligned} y(0) &= \quad C_1 + C_2 &&= 0 \\ y'(0) &= \quad -2C_1 - 3C_2 &&= 2. \end{aligned}$$

This gives $C_1 = 2, C_2 = -2$. Hence we have

$$y(x) = 2e^{-2x} - 2e^{-3x}.$$

Problem 2. Find the general solution to

$$y'' - 2y' + 2 = 0.$$

Solution. Again we write the characteristic equation

$$p(r) = r^2 - 2r + 2 = 0.$$

However this now has imaginary roots

$$r = 1 \pm i.$$

The general solution still is

$$\begin{aligned} y &= C_1 e^{(1+i)x} + C_2 e^{(1-i)x} \\ &= C_1 e^x (\cos x + i \sin x) + C_2 e^x (\cos x - i \sin x) \\ &= C_1' e^x \cos x + C_2' e^x \sin x \end{aligned}$$

using the formula $e^{ix} = \cos x + i \sin x$.

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Problem 3. Find the general solution to the inhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2x}.$$

Solution. The roots of the characteristic equation $r^2 - 3r - 4 = 0$ are $r = -1, 4$. Hence the general solution to the associated homogeneous equation $y'' - 3y' - 4y = 0$ is

$$y_h = C_1 e^{-x} + C_2 e^{4x}.$$

We now need to find a particular solution to the problem. We venture the guess

$$y_p = Ae^{2x}.$$

On differentiating this gives

$$y_p'' - 3y_p' - 4y_p = -6Ae^{x} = 3e^{2x}.$$

Hence we have $A = -\frac{1}{2}$ and $y_p = -\frac{1}{2}e^{2x}$. The general solution is now a sum of the homogeneous and particular

$$\begin{aligned} y_{gen} &= y_h + y_p \\ &= C_1 e^{-x} + C_2 e^{4x} + 3e^{2x}. \end{aligned}$$

Problem 4. Find a particular solution to

$$y'' - y = x^2.$$

Solution. This time we guess

$$y_p = Ax^2 + Bx + C.$$

Plugging this into the equation gives

$$y_p'' - y_p = 2A - \left(Ax^2 + Bx + C\right) = x^2.$$

On comparing coefficients we see $A = -1, B = 0$ and $C = -2$. Hence the particular solution is

$$y_p = -x^2 - 2.$$

Problem 5. Find the particular solution to

$$y'' - 4y = e^{2t}.$$

Solution. As in problem 3 we may try $y_p = Ae^{2t}$. However substitution into our problem gives

$$y_p'' - 4y_p = A(4-4)e^{2t} = 0$$

which is always zero and cannot solve the equation. The reason this has happened is because the exponent 2 of e^{2t} is a root of the characteristic equation. As a second guess we try

$$y_p = Ate^{2t}.$$

This gives

$$y_p'' - 4y_p = 4Ae^{2t} = e^{2t}$$

which is now solvable. Hence $A = \frac{1}{4}$ and $y_p = \frac{1}{4}e^{2t}$.

All the above examples are of the form

$$ay'' + by' + cy = g(x).$$

The homogeneous solution is of the form

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

when r_1, r_2 are distinct roots of the characteristic polynomial $p(r) = ar^2 + br + c = 0$. But how does one guess the particular solution. The following table gives the form of the guess.

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$g(x)$	y_p
$e^{\alpha x}$	$x^s e^{\alpha x}$, where s is the smallest number such that $p^{(s)}(\alpha) \neq 0$
x^n	$x^s (A_0 + A_1 x + \ldots A_n x^n)$, where s is the smallest number such that $p^{(s)}(0) \neq 0$

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LECTURE 3

1. Method of Variation of Parameters

In this lecture we continue our study of linear inhomogenous ODE's. These are equations of the form

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t)\,.$$

In the last lecture we saw how to solve the above equation in the case where the coefficients $p(t), q(t)$ were constants. Moreover we also assumed that the function $g(t)$ was either a polynomial $(g(t) = A_0 + A_1 t + \ldots + A_n t^n)$ or a (possibly complex) exponential $g(t) = e^{\alpha t}, \alpha$ being real or complex. The key step involved guessing a right form for the particular solution y_p.

We now proceed to solve the equation in the general case. However our method requires that *we know the general solution to the homogeneous equation*

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t) \tag{1.1}$$

solving $L[y_h] = y'' + p(t)\,y' + q(t)\,y = 0$ (as is the case when $p(t), q(t)$ are constants). Hence we again have to find one particular solution y_p to the problem $L[y_p] = g(t)$. We try a solution of the form

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

where the constants c_1, c_2 in (1.1) now get replaced by functions $u_1(t), u_2(t)$. Differentiating this gives

$$\begin{aligned} y_p'(t) =& u_1(t)\,y_1'(t) + u_2(t)\,y_2'(t) \\ &+ u_1'(t)\,y_1(t) + u_2'(t)\,y_2(t)\,. \end{aligned}$$

We need to differentiate again to find $L[y_p]$. In order to simplify the computation of the second derivative let us try to find a solution where the second line above is zero

$$u_1'(t)\,y_1(t) + u_2'(t)\,y_2(t) = 0. \tag{1.2}$$

We then have

$$y_p'' = u_1'(t)\,y_1'(t) + u_2'(t)\,y_2'(t) + u_1(t)\,y_1''(t) + u_2(t)\,y_2''(t)\,.$$

Hence we may compute

$$\begin{aligned} L[y_p] = \quad & u_1(t)\,[y_1'' + p(t)\,y_1' + q(t)\,y_1] \\ + \quad & u_2(t)\,[y_2'' + p(t)\,y_2' + q(t)\,y_2] \\ & + u_1'(t)\,y_1'(t) + u_2'(t)\,y_2'(t) \qquad = g(t)\,. \end{aligned}$$

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Since y_1, y_2 were solutions to the homogeneous equation we must have

$$u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t) = g(t).$$

Combining the above with (1.2) we end up with the pair of equations

$$\begin{aligned} u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t) &= 0 && (1.3) \\ u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t) &= g(t). && (1.4) \end{aligned}$$

Cramer's rule then gives us

$$\boxed{u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}, \quad u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}.}$$

We call the determinant in the denominators $W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ the **Wronskian** of y_1, y_2. Finally we may integrate the above to get the particular solution

$$\boxed{y_p = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1, y_2)(s)} ds.}$$

We now do some examples of this method.

Problem 1. Find a particular solution of the equation

$$y'' + y = \tan(t).$$

Solution. The corresponding homogeneous equation is $y'' + y = 0$. This has the solutions $y_1 = \cos t$ and $y_2 = \sin t$. The Wronskian of these two solutions is

$$W(\sin t, \cos t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

Hence we have

$$\begin{aligned} u_1'(t) &= \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\frac{\sin^2 t}{\cos x} = \cos t - \sec t \\ u_2'(t) &= \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \sin t. \end{aligned}$$

On integration we have

$$\begin{aligned} u_1(t) &= \sin t - \ln|\sec t + \tan t| \\ u_2(t) &= -\cos t. \end{aligned}$$

Hence the particular solution is

$$y_p(t) = (\sin t - \ln|\sec t + \tan t|) \cos t + (-\cos t) \sin t.$$

Problem 2. Find a particular solution to the equation

$$y'' - 2y' + y = t^{-2} e^t.$$

Solution. The corresponding homogeneous equation $y'' - 2y' + y = 0$ has the solutions $y_1 = e^t, y_2 = te^t$. The Wronskian of these two is

$$W\left(e^t, te^t\right) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

This gives

$$\begin{aligned} u_1'(t) &= \frac{\begin{vmatrix} 0 & te^t \\ t^{-2}e^t & e^t + te^t \end{vmatrix}}{e^{2t}} = -\frac{1}{t} \\ u_2'(t) &= \frac{\begin{vmatrix} e^t & 0 \\ e^t & t^{-2}e^t \end{vmatrix}}{e^{2t}} = \frac{1}{t^2}. \end{aligned}$$

Hence $u_1(t) = -\ln t$, $u_2(t) = -\frac{1}{t}$. Hence the particular solution is

$$y_p(t) = (-\ln t)\, e^t - e^t.$$

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LECTURE 4

1. General Theory of nth order Linear ODE

In this lecture we shall discuss the general linear ODE of nth order. The general equation of this kind has the form

$$p_n(t)\,y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \ldots + p_0(t)\,y = g(t).$$

In this generality it is impossible to solve this equation explicitly. However one can still prove existence and uniqueness for the solution. In order to do this let us assume that the leading coefficient $p_n(t) \neq 0$ is non-zero on some interval $I = [a, b]$. We may then divide by $p_n(t)$ to obtain the equation

$$L[y] = y^{(n)} + q_{n-1}(t)\,y^{(n-1)} + \ldots + q_0(t)\,y = h(t). \tag{1.1}$$

for the new coefficient functions $q_{n-1}(t) = \frac{p_{n-1}(t)}{p_n(t)}, \ldots, q_0(t) = \frac{p_0(t)}{p_n(t)}$ and $h(t) = \frac{g(t)}{p_n(t)}$. The general theory of linear ODE's now tells us that a solution to (1.1) always exists for all times $t \in I = [a, b]$. Moreover let us consider the corresponding homogeneous equation

$$L[y] = y^{(n)} + q_{n-1}(t)\,y^{(n-1)} + \ldots + q_0(t)\,y = 0.$$

Let us say we have n solutions $y_1(t), \ldots, y_n(t)$ to the above homogeneous equation. Recall that these solutions are said to be *linearly dependent* if there exists constants $c_1, \ldots, c_n$ not all zero such that

$$c_1 y_1(t) + \ldots + c_n y_n(t) = 0,$$

and they are said to be *linearly independent* otherwise. The Wronskian of these n functions $y_1, \ldots, y_n$ is

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & \ddots & \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}.$$

If the Wronskian $W(y_1, \ldots, y_n)(t_0) \neq 0$ is non-zero for some point t_0 in the interval $t_0 \in I = [a, b]$ then these solutions are linearly independent on the interval I. In this case the general solution y_h to $L[y_h] = 0$ can be written as a linear combination of these

$$y_h = c_1 y_1 + \ldots + c_n y_n.$$

If y_p is a particular solution to the equation $L[y_p] = g(t)$, then the general solution to the equation $L[y] = g(t)$ is of the form

$$y = y_p + c_1 y_1 + \ldots + c_n y_n.$$

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Example 1. Consider the equation

$$\left(t^2-1\right)y^{(3)}+ty'+e^t=\cos t.$$

Find an interval of time t containing 0, for which the solution is sure to exist.

Problem 2. *Solution.* The leading coefficient t^2-1 is non-zero for $t\neq -1,1$. Hence the largest interval containing 0 of the complement $\mathbb{R}\setminus\{-1,1\}$ is $(-1,1)$. The solution is sure for this interval $(-1,1)$.

2. Higher Order Equations with Constant Coefficients

The general theory does not tell us how to solve the equations explicitly. In order to find explicit solutions we restrict attention to equations with constant coefficients. Hence we consider the equation of the form

$$L[y]=a_ny^{(n)}+\ldots+a_0y=g\left(t\right) \tag{2.1}$$

with $a_n,\ldots,a_0$ being constants. These can be solved in a very similar fashion to the equations of second order. Again we write the characteristic polynomial

$$p\left(r\right)=a_nr^n+a_{n-1}r^{n-1}+\ldots+a_0=0.$$

Let us assume that one has n distinct roots $r_1,r_2,\ldots,r_n$ to the above equation. Then the general solution to the a corresponding homogeneous equation $L\left[y\right]=0$ is of the form

$$y=c_1e^{r_1t}+\ldots+c_ne^{r_nt}.$$

If a certain root (say r_1) is repeated s times, then each of $e^{r_1t},te^{r_1t},\ldots,t^{s-1}e^{r_1t}$ is a solution to $L\left[y\right]=0$. The general solution now contains a linear combination of these.

Example 3. Find the general solution to the homogeneous equation

$$L[y]=y'''+2y''-y'-2y=0.$$

*Solution.*We try a solution of the form $y=e^{rt}$. This gives

$$L[y]=\underbrace{\left(r^3+2r^2-r-2\right)}_{p(r)=\text{ characteristic polynomial}}.e^{rt}=0.$$

The characteristic polynomial factorizes as $r^3+2r^2-r-2=\left(r^2-1\right)\left(r+2\right)$. Thus its roots are $r=-2,\pm1$. The general solution to the problem is

$$y=c_1e^{-2t}+c_2e^{-t}+c_3e^t.$$

Example 4. Find the general solution to the homogeneous equation

$$y'''-y=0.$$

Solution. The characteristic polynomial is r^3-1. This factorizes as $r^3-1=\left(r-1\right)\left(r^2+r+1\right)=0$. This now gives

$$r=1\text{ or }r=\frac{-1\pm i\sqrt{3}}{2}.$$

Hence the general solution is of the form

$$y=c_1e^t+c_2e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)+c_3e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Example 5. Find the general solution to the homogeneous equation

$$y^{(4)} + 2y'' + y = 0.$$

Solution. The characteristic polynomial factorizes $r^4 + 2r^2 + 1 = \left(r^2 + 1\right)^2$. Hence we have the roots

$$r = \pm i, \pm i$$

with them being repeated roots. Hence four solutions are $y_1 = \cos t$, $y_2 = \sin t$, $y_3 = t\cos t$, $y_4 = t\sin t$. The general solution is now a linear combination of these

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

Example 6. Find the general solution to the homogeneous equation

$$y^{(4)} - y = 0.$$

Solution. The characteristic polynomial factorizes as $r^4 - 1 = \left(r^2 + 1\right)\left(r^2 - 1\right)$. This has the roots $r = \pm 1, \pm i$. Hence the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

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LECTURE 5

1. The Method of Undetermined Coefficients in nth Order

The method of undetermined coefficients can also be used to solve an nth order linear ODE with constant coefficients. This is an equation of the form

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(t).$$

This method however requires that the right hand side $g(t)$ to be a (real or complex) exponential or a polynomial in t. In these cases one can find a particular solution to the equation in the same way as in the second order case. The table giving the form of the guess is the same as from lecture 2

$g(t)$	y_p
$e^{\alpha t}$	$t^s e^{\alpha t}$, where s is the smallest number such that $p^{(s)}(\alpha) \neq 0$
t^n	$t^s (A_0 + A_1 t + \ldots A_n t^n)$, where s is the smallest number such that $p^{(s)}(0) \neq 0$

The general solution now has the form

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous problem.

Exercise 1. Find the particular solution to the equation

$$L[y] = y''' + y = 1.$$

Solution. In this case it is easy to guess the particular solution $y_p = 1$. More systematically, the right hand side is a polynomial of degree 0. The characteristic equation is $r^3 + 1 = 0$, which does not have 0 as a root. Hence the form for the particular solution should be $y_p = A_0$ a constant. Plugging this into the equation gives $A_0 = 1$.

Exercise 2. Find the particular solution to the equation

$$y''' + y = t + 1. \tag{1.1}$$

Solution. Again it is easy to guess $y_p = t + 1$, since the third derivative of this $y_p''' = 0$. But we would like to solve it more systematically. Since the characteristic equation $r^3 + 1 = 0$ again does not have 0 as a root, the form for the solution should be a polynomial of the same degree as the right hand side of (1.1). Hence $y_p = At + B$. Plugging this into the equation (1.1) gives $y_p''' + y_p = At + B = t + 1$. Hence we have $y_p = t + 1$.

Exercise 3. Find the particular solution

$$y''' + y = t^3. \tag{1.2}$$

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Solution. Now this takes solving! There is no immediate guess. Our guess is now a degree three polynomial, again the same degree as the right hand side of (1.2). Hence $y_p = At^3 + Bt^2 + Ct + D$. We compute

$$y_p''' + y_p = 6A + At^3 + Bt^2 + Ct + D = t^3.$$

Hence by comparing coefficients $A = 1, B = C = 0, D = -6A = -6$. Hence

$$y_p = t^3 - 6.$$

Exercise 4. Find the particular solution to

$$L[y] = y''' + y' = 1. \tag{1.3}$$

Solution. In this example the characteristic polynomial is $r^3+r = r\left(r^2+1\right) = 0$. This has the roots $r = 0, \pm i$. Hence the general solution to homogeneous equation $L[y] = 0$ is

$$y_h = c_1 + c_2 \sin t + c_3 \cos t.$$

Although the right hand side of the inhomogeneous is degree zero, we see that guessing a constant solution $y_p = A$ does not work as this solution is part of the homogeneous (i.e. $L[y_p] = 0$). This happens because 0 is a root of the characteristic equation. Hence we multiply this guess by a t and obtain $y_p = At$. This gives

$$y_p''' + y_p' = A = 1.$$

Hence we have $y_p = t$ is the particular solution.

Exercise 5. Find the particular solution to

$$L[y] = y''' + y' = t.$$

Solution. Again our initial guess should be a polynomial of the same degree as the right hand side $y_p = At + B$. However one checks that this solution cannot work. This is again due to the fact that 0 is a root of the characteristic equation $\left(r^3 + r = 0\right)$ of multiplicity 1. Hence the initial guess needs to multiplies by $t^1$to get $y_p = t\,(At + B)$. This now gives

$$y_p''' + y_p' = 2At + B = t.$$

Hence $A = \frac{1}{2}, B = 0$ and $y_p = \frac{1}{2}t^2$.

Exercise 6. Find the particular solution to

$$L[y] = y''' + y'' = t.$$

Solution. The characteristic equation is now $r^3+r^2 = 0$. It has roots $r = 0, 0, -1$ with 0 now being a root of multiplicity 2. The initial guess of the particular solution is $y_p = At+B$, of the same degree as the right hand side. To keep on trying this and its successive multiples by t leads to utter exasperation. So one should know what to try from the beginning. The right multiple of the initial guess is $t^2\,(At + B)$, where the exponent 2 of t^2 is exactly the multiplicity of 0 as a root of the characteristic polynomial. This now gives

$$y''' + y'' = 6A + 6At + 2B = t.$$

Hence $A = \frac{1}{6}, B = -3A = -\frac{1}{2}$ and $y_p = t^2\left(\frac{1}{6}t - \frac{1}{2}\right)$.

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LECTURE 6

1. The Method of Variation of Parameters in nth Order

Consider a general nonhomogeneous equation of the form

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(t),$$

with constant coefficients $a_n, \ldots, a_0$ and $g(t)$ being a general function of time. Unless the function $g(t)$ is a polynomial multiple of an exponential, the method of undetermined coefficients does not apply. However one can use the method of variation of parameters as for second order equations. Again we let $y_1(t), y_2(t), \ldots, y_n(t)$ be n linearly independent solutions to the corresponding homogeneous equation $L[y] = 0$. Hence the general solution to the homogeneous problem is

$$y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

The Wronskian of these n functions is

$$W(t) = W(y_1, \ldots, y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & \ddots & \\ y_n & & & y_n^{(n-1)} \end{vmatrix}.$$

Next we write down the determinant of the matrix $W_m(t)$ obtained by replacing the m^{th} column by the vector $\begin{vmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{vmatrix}$. We then find the functions u_m satisfying

$$u_m'(t) = \frac{W_m(t)}{W(t)}.$$

The particular solution is then of the form

$$y_p(t) = u_1(t) y_1(t) + \ldots + u_n(t) y_n(t).$$

The general solution to the problem is now $y_{gen}(t) = y_p(t) + y_h(t)$ a sum of the homogeneous and the particular.

Example 1. Write the general solution to the equation

$$L[y] = y''' + y' = \cot t$$

using variation of parameters.

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Solution. The characteristic equation is $r^3 + r = r\left(r^2+1\right) = 0$ which has roots $r = 0, \pm i$. Hence the general solution to the homogeneous problem $L[y] = 0$ is

$$y_h = c_1 + c_2 \cos t + c_3 \sin t.$$

which is a linear combination of the three linearly independent solutions $y_1 = 1, y_2 = \cos t, y_3 = \sin t$. The Wronskian of these three functions is

$$W(1, \cos t, \sin t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1.$$

Hence the partial Wronskians are

$$\begin{aligned} W_1(t) &= \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \cot t & -\cos t & -\sin t \end{vmatrix} = \cot t \\ W_2(t) &= \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \cot t & -\sin t \end{vmatrix} = -\frac{\cos^2 t}{\sin t} \\ W_3(t) &= \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \cot t \end{vmatrix} = -\cos t. \end{aligned}$$

Next we have to integrate

$$\begin{aligned} u_1 &= \int \cot t dt = \ln|\sin t| \\ u_2 &= \int -\frac{\cos^2 t}{\sin t} dt = \int \frac{\sin^2 t - 1}{\sin t} dt \\ &= \int \sin t - \csc t = -\cos t - \ln|\csc t - \cot t| \\ u_3 &= \int -\cos t dt = -\sin t. \end{aligned}$$

Hence

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \ln|\sin t| + \cos t\left(-\cos t - \ln|\csc t - \cot t|\right) + \sin t\left(-\sin t\right) \\ &= \ln|\sin t| - \cos t\left(\ln|\csc t - \cot t|\right) - 1. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y_{gen} &= y_p + y_h \\ &= c_1 + c_2 \cos t + c_3 \sin t + \ln|\sin t| - \cos t\left(\ln|\csc t - \cot t|\right) \end{aligned}$$

where we have absorbed -1 from y_p into the constant c_1.

Exercise 2. Find the general solution to the equation

$$L[y] = y''' - y' = t.e^t$$

using both the method of undetermined coefficients as well as variation of parameters.

Solution. The characteristic equation is $r^3 - r = 0$. Hence we have $r = 0, \pm 1$. The general solution to the homogeneous equation is

$$y_h = c_1 + c_2 e^t + c_3 e^{-t}.$$

To find the particular solution via undetermined coefficients our guess should be of the form $y_p = (At + B)\, e^t$. However since zero is a root of the characteristic polynomial this needs to multiplied by t. Hence we guess

$$\begin{aligned} y_p &= t\,(At + B)\,.e^t \\ y_p' &= \left[At^2 + (2A + B)\,t + B\right] e^t \\ y_p'' &= \left[At^2 + (4A + B)\,t + 2A + 2B\right] e^t \\ y_p''' &= \left[At^2 + (6A + B)\,t + 6A + 3B\right] e^t. \end{aligned}$$

Hence $y_p''' - y_p' = [4At + (6A + 2B)] = t$ gives $A = \frac{1}{4}, B = -3A = -\frac{3}{4}$. We thus have $y_p = \left(\frac{1}{4}t^2 - \frac{3}{4}t\right) e^t$. The general solution is now

$$y_{gen} = \left(\frac{1}{4}t^2 - \frac{3}{4}t\right) e^t + c_1 + c_2 e^t + c_3 e^{-t}.$$

To solve the same problem via variation of parameters first consider the three linearly independent solutions to the homogeneous equation $y_1 = 1, y_2 = e^t, y_3 = e^{-t}$. They have the Wronskian

$$W\left(1, e^t, e^{-t}\right) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2.$$

The partial Wronskians are

$$\begin{aligned} W_1(t) &= \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ te^t & e^t & e^{-t} \end{vmatrix} = -2te^t \\ W_2(t) &= \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & te^t & e^{-t} \end{vmatrix} = t \\ W_3(t) &= \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & te^t \end{vmatrix} = te^{2t}. \end{aligned}$$

Hence we may integrate

$$\begin{aligned} u_1 &= \int \left(-te^t\right) dt = -te^t + \int e^t dt \\ &= -te^t + e^t \\ u_2 &= \frac{1}{2}\int t dt = \frac{t^2}{4} \\ u_3 &= \frac{1}{2}\int te^{2t} dt = \frac{1}{4}\left[te^{2t} - \int e^{2t} dt\right] \\ &= \left(\frac{t}{4} - \frac{1}{8}\right) e^{2t}. \end{aligned}$$

This gives

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3 \\ &= \left[-t + 1 + \frac{t^2}{4} + \frac{t}{4} - \frac{1}{8}\right] e^t \\ &= \left(\frac{1}{4}t^2 - \frac{3}{4}t\right) e^t - \frac{1}{8}e^t. \end{aligned}$$

The general solution is then

$$y_{gen} = \left(\frac{1}{4}t^2 - \frac{3}{4}t\right) e^t - \frac{1}{8}e^t + c_1 + c_2e^t + c_3e^{-t}.$$

This is the same as before since the $-\frac{1}{8}e^t$ can be absorbed inside the constant c_2.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 7

1. Improper Integrals

We begin by recalling that real valued function $f(t)$ of one variable is continuous at $t = a$ if the limit $\lim_{t\to a} f(t)$ exists and equals $f(a)$. Similarly, f is continous on $[\alpha, \beta]$ if it continous at each point in the interval $t \in [\alpha, \beta]$. Now f is said to be piecewise continuous on $[\alpha, \beta]$ if it continous at all except a finitely many points $t_1, \ldots, t_n \in [\alpha, \beta]$. An example is the function

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t \leq 1 \\ 0 & t > 0. \end{cases}$$

This is continous at all points except $t = 0, 1$. Hence it is piecewise continuous on the real line.

For a piecewise continous function $f(t)$ the integral

$$\int_a^b f(t)\, dt$$

always makes sense, for finite values of a and b. However here we are interested in the integral

$$\int_a^\infty f(t)\, dt := \lim_{b\to\infty} \int_a^b f(t)\, dt.$$

The right hand side is the defining equation for this integral. Such an integral is called an **improper integral**. However the limit may not exist (or the integral may not converge). It is known to converge under the assumptions of the following theorem.

Theorem 1. *Assume $f(t)$ is piecewise continous for $t \geq a$ and $|f(t)| \leq g(t)$ for $t \geq M$. Then if $\int_M^\infty g(t)\, dt$ converges so does $\int_a^\infty f(t)\, dt$.*

2. Laplace transform

Let $f(t)$ be a real valued function of one variable. Its Laplace trasform is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

However we again need some hypotheses to make sure the above integral converges for the Laplace trasform to make sense.

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Theorem 2. *Let $f(t)$ be piecewise continuous on the interval $[0, A]$ for any $A > 0$. Assume that there exists constants K, a, M such that*

$$|f(t)| \leq Ke^{at}$$

for all $t \geq M$. Then the Laplace trasform $F(s)$ exists for $s > a$.

Note that the Laplace transform only depends on the values of $f(t)$ for positive time $t > 0$. The Laplace transform is a **linear operator** i.e. for constants c_1, c_2 we haved

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}.$$

We now compute some examples of the Laplace trasform.

Example 3. Find $\mathcal{L}\{f(t)\}$ for $f(t) = 1$.

We compute

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} dt \\ &= \lim_{A \to \infty} \left[-\frac{e^{-st}}{s} \right]_{t=0}^{A} \\ &= \lim_{A \to \infty} \left[\frac{1}{s} - \frac{e^{-sA}}{s} \right] \\ &= \frac{1}{s}, \qquad \text{for } s > 0. \end{aligned}$$

Example 4. Find $\mathcal{L}\{f(t)\}$ for $f(t) = e^{at}$.

We compute

$$\begin{aligned} F(s) &= \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt \\ &= \frac{1}{s-a}, \qquad \text{for } s > a. \end{aligned}$$

Example 5. Find $\mathcal{L}\{f(t)\}$ for $f(t) = \sin(at)$.

We first write $\sin(t) = \frac{e^{iat} + e^{-iat}}{2i}$. Using the linearity of the Laplace transform we have

$$\begin{aligned} F(s) &= \frac{1}{2i} \left\{ \frac{1}{s+ia} + \frac{1}{s-ia} \right\} \\ &= \frac{a}{s^2 + a^2}. \end{aligned}$$

Example 6. Find $\mathcal{L}\{f(t)\}$ for $f(t) = t$.

We compute

$$\begin{aligned} F(s) &= \int_0^\infty te^{-st}dt \\ &= \lim_{A\to\infty}\int_0^A te^{-st}dt \\ &= \lim_{A\to\infty}\left\{\left[-\frac{t.e^{-st}}{s}\right]_{t=0}^A - \int_0^A \left(-\frac{e^{-st}}{s}\right)dt\right\} \\ &= \lim_{A\to\infty}\left[-\frac{t.e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_{t=0}^A \\ &= \frac{1}{s^2}. \end{aligned}$$

A similar integration by parts gives $\mathcal{L}\{f(t)\} = F(s) = \frac{n!}{s^{n+1}}$ for $f(t) = t^n$.

Example 7. Find $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1. \end{cases}$

We compute

$$\begin{aligned} F(s) &= \int_0^1 e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_{t=0}^1 \\ &= \left[\frac{1}{s} - \frac{e^{-s}}{s}\right]. \end{aligned}$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 8

1. Properties and Further examples of Laplace Tranforms

One of the main applications of the Laplace transform is that it allows us to solve initial values problems. We will see what these are and how to solve them shortly, but we first shall need some further properties of the Laplace transform. An important one is the calculation of the Laplace transform of a derivative.

Theorem 1. *Let $f(t)$ be a function of atmost exponential growth $|f(t)| \leq Ke^{at}$ for some a, K. Then the Laplace transform of $F(s) = \mathcal{L}\{f'(t)\}$ exists for $s > a$ and is given by*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. The proof is again integration by parts. From the definition of the Laplace trasform we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t)\, e^{-st} dt \\ &= \left[f(t)\, e^{-st}\right]_0^\infty + s\int_0^\infty f(t)\, e^{-st} dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}.\end{aligned}$$

□

If one iterates the above formula one arrives at

$$\boxed{\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).} \tag{1.1}$$

We may now use the above formula to compute further examples of the Laplace transform.

Example 2. Find $\mathcal{L}\{f(t)\}$ for $f(t) = t^n$.

We put $f(t) = t^n$ in the formula (1.1). Since $f^{(i)}(t) = n.(n-1)\ldots(n-i+1)\, t^{n-i}$ we have $f^{(n-i)}(0) = 0$ for $i < n$. Also $\mathcal{L}\left\{f^{(n)}(t)\right\} = \mathcal{L}\{n!\} = \frac{n!}{s}$. Hence this gives

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

.

Example 3. Find $\mathcal{L}\{f(t)\}$ for $f(t) = \cos(at)$.

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This can be done in a similar fashion (using complex exponentials) as the Laplace transform of $\sin(at)$ from last lecture. However lets do it differently now.

$$\begin{aligned}\mathcal{L}\{\cos(at)\} &= \frac{1}{a}\mathcal{L}\left\{[\sin(at)]'\right\} \\ &= \frac{s}{a}\mathcal{L}\{\sin(at)\} - \frac{1}{a}\sin(0) \\ &= \frac{s}{a}\cdot\frac{a}{s^2+a^2} \\ &= \frac{s}{s^2+a^2}.\end{aligned}$$

Using the computation of the Laplace transform of $\sin(at)$ from the last lecture.

Now we come to another property of the Laplace transform.

Theorem 4. *If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$.*

Proof. This follows immedietly from the definition.

$$\begin{aligned}\mathcal{L}\left\{e^{ct}f(t)\right\} &= \int_0^\infty f(t)\, e^{ct}e^{-st}dt \\ &= \int_0^\infty f(t)\, e^{-(s-c)t}dt \\ &= F(s-c).\end{aligned}$$

□

This now immedietly gives the following transforms

$$\begin{aligned}\mathcal{L}\left\{e^{ct}\sin(at)\right\} &= \frac{a}{(s-c)^2+a^2} \\ \mathcal{L}\left\{e^{ct}\cos(at)\right\} &= \frac{s-c}{(s-c)^2+a^2} \\ \mathcal{L}\left\{t^n e^{ct}\right\} &= \frac{n!}{(s-c)^{n+1}}.\end{aligned}$$

2. Solutions To Initial Value Problems

The Laplace transform can be used to solve initial value problems. Below is an example.

Example 5. Solve the differential equation

$$y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0.$$

Solution. First let us do this using the a method we know: via the characteristic equation. The characteristic equation is $r^2 - r - 2r = (r-2)(r+1) = 0$ and hence has roots $r = -1, 2$. Hence the general solution is

$$y(t) = c_1e^{-t} + c_2e^{2t}.$$

The initial conditions now give $c_1 + c_2 = 1$, $2c_2 - c_1 = 0$. Hence $c_1 = \frac{2}{3}, c_2 = \frac{1}{3}$ and we have $y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$.

Let us now learn how to solve the same problem via Laplace transforms! First we transform both sides of the differential equation

$$\begin{aligned} \mathcal{L}\left\{y''(t) - y'(t) - 2y(t)\right\} &= 0 \\ s^2\mathcal{L}\left\{y(t)\right\} - sy(0) - s^2y'(0) - \left[s\mathcal{L}\left\{y(t)\right\} - y(0)\right] - 2\mathcal{L}\left\{y(t)\right\} &= 0. \end{aligned}$$

Some algebra now gives

$$\mathcal{L}\left\{y(t)\right\} = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Using partial fractions we now get

$$\mathcal{L}\left\{y(t)\right\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

But now the table of Laplace transforms now gives us

$$y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

Example 6. Find the solution to the differential equation

$$y'' + y = \sin(2t), \quad y(0) = 2, y'(0) = 1.$$

Solution. Let $Y(s) = \mathcal{L}\left\{y(t)\right\}$. Again we transform the equation to get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}.$$

Using the initial conditions we now get

$$\begin{aligned} Y(s) &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \\ &= \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}. \end{aligned}$$

Hence the table of Laplace transforms now gives

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 9

1. Solutions To Initial Value Problems

The Laplace transform can be used to solve initial value problems. Below is an example.

Example 1. Solve the differential equation

$$y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0.$$

Solution. First let us do this using the a method we know: via the characteristic equation. The characteristic equation is $r^2 - r - 2r = (r-2)(r+1) = 0$ and hence has roots $r = -1, 2$. Hence the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{2t}.$$

The initial conditions now give $c_1 + c_2 = 1$, $2c_2 - c_1 = 0$. Hence $c_1 = \frac{2}{3}, c_2 = \frac{1}{3}$ and we have $y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$.

Let us now learn how to solve the same problem via Laplace transforms! First we transform both sides of the differential equation

$$\begin{aligned} \mathcal{L}\{y''(t) - y'(t) - 2y(t)\} &= 0 \\ s^2\mathcal{L}\{y(t)\} - sy(0) - s^2y'(0) - [s\mathcal{L}\{y(t)\} - y(0)] - 2\mathcal{L}\{y(t)\} &= 0. \end{aligned}$$

Let us plug in the initial values of $y(0) = 1, y'(0) = 0$ and denote $\mathcal{L}\{y(t)\} = Y(s)$. We hence have

$$Y(s)\left(s^2 - s - 2\right) - s + 1 = 0.$$

Some algebra (yes just algebra!) now gives

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Using partial fractions we now get

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

But now the table of Laplace transforms now gives us

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

Example 2. Find the solution to the differential equation

$$y'' + y = \sin(2t), \quad y(0) = 2, y'(0) = 1.$$

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Solution. Let $Y(s) = \mathcal{L}\{y(t)\}$. Again we transform the equation to get

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}.$$

Using the initial conditions we now get

$$\begin{aligned} Y(s) &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \\ &= \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}. \end{aligned}$$

Hence the table of Laplace transforms now gives

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t).$$

2. Partial Fractions and the Inverse Laplace Transform

In each of the two problems above the crucial step was finding the inverse Laplace transform. Often the inverse Laplace transform that we require is that of a rational function (a ratio of two polynomials). The theory of partial fractions now comes in handy.

Let $p(s), q(s)$ be two polynomials such that $\deg p(s) < \deg q(s)$. Then we would like a partial fractions decomposition of $\frac{p(s)}{q(s)}$. We first factorize

$$q(s) = (s-a_1)^{\alpha_1}(s-a_2)^{\alpha_2}\dots\left(s^2+b_1 s+c_1\right)^{\beta_1}\left(s^2+b_2 s+c_2\right)^{\beta_2}\dots$$

into its linear and quadratic (corresponding to complex roots) factors. The theory of partial fractions now says that it is possible to write the rational function as a sum

$$\begin{aligned} \frac{p(s)}{q(s)} &= \frac{c_{11}}{(s-a_1)} + \frac{c_{12}}{(s-a_1)^2} + \dots + \frac{c_{1\alpha_1}}{(s-a_1)^{\alpha_1}} \\ &+ \frac{c_{21}}{(s-a_2)} + \frac{c_{22}}{(s-a_2)^2} + \dots + \frac{c_{2\alpha_2}}{(s-a_2)^{\alpha_2}} \\ &+ \dots \\ &+ \frac{d_{11}s+e_{11}}{(s^2+b_1 s+c_1)} + \frac{d_{12}s+e_{12}}{(s^2+b_1 s+c_1)^2} + \dots + \frac{d_{1\beta_1}s+e_{1\beta_1}}{(s^2+b_1 s+c_1)^{\beta_1}} \\ &+ \frac{d_{21}s+e_{21}}{(s^2+b_2 s+c_1)} + \frac{d_{22}s+e_{22}}{(s^2+b_2 s+c_2)^2} + \dots + \frac{d_{2\beta_2}s+e_{2\beta_2}}{(s^2+b_2 s+c_2)^{\beta_2}} \\ &+ \dots \end{aligned}$$

The Laplace transform table is now used to find the inverse transform of each term appearing on the right hand side above. Let us see how this works out in examples.

Example 3. Find the inverse Laplace transforms for

(1) $F(s) = \frac{1}{2s+1}$
(2) $F(s) = \frac{s+1}{s^2+4}$
(3) $F(s) = \frac{1}{s^2+2s+5}$
(4) $F(s) = \frac{1}{s^3+s}$
(5) $F(s) = \frac{1}{(s-1)^4}$
(6) $F(s) = \frac{s^2+1}{(s-1)^3}$

Solution.

(1) We write

$$F(s) = \frac{1}{2s+1} = \frac{1}{2}.\frac{1}{\left(s+\frac{1}{2}\right)}$$

Hence $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-\frac{t}{2}}$.

(2) We write

$$\begin{aligned} F(s) &= \frac{s+1}{s^2+4} \\ &= \frac{s}{s^2+4} + \frac{1}{s^2+4} \\ &= \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4} \end{aligned}$$

Hence $\mathcal{L}^{-1}\{F(s)\} = \cos(2t) + \frac{1}{2}\sin(2t)$.

(3) We write

$$\begin{aligned} F(s) &= \frac{1}{s^2+2s+5} \\ &= \frac{1}{(s+1)^2+4} \\ &= \frac{1}{2}\mathcal{L}\left\{e^{-t}\sin(2t)\right\}. \end{aligned}$$

Hence the $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-t}\sin(2t)$.

(4) The denominator can be factorized $s^3+s = s\left(s^2+1\right) = s(s+i)(s-i)$. Hence by the theory of partial fractions

$$\begin{aligned} F(s) &= \frac{1}{s(s+i)(s-i)} \\ &= \frac{A}{s} + \frac{B}{s+i} + \frac{C}{s-i} \end{aligned}$$

The coefficients can be figured out to be $A=1, B=-\frac{1}{2}, C=-\frac{1}{2}$. This gives

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{1}{2}\left\{\frac{1}{s+i} + \frac{1}{s-i}\right\} \\ &= \frac{1}{s} - \frac{s}{s^2+1} \\ &= \mathcal{L}\{1-\cos(t)\}. \end{aligned}$$

Hence the $\mathcal{L}^{-1}\{F(s)\} = 1-\cos(t)$.

(5) We write

$$F(s) = \frac{1}{(s-1)^4} = \frac{1}{3!}\frac{3!}{(s-1)^4}$$

Hence $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}t^3e^t$.

(6) Set $s-1=u$. Then we have

$$\begin{aligned} F(s)=\frac{s^2+1}{(s-1)^3} &= \frac{(u+1)^2+1}{u^3} \\ &= \frac{u^2+2u+2}{u^3} \\ &= \frac{1}{u}+\frac{2}{u^2}+\frac{2}{u^3} \\ &= \frac{1}{s-1}+\frac{2}{(s-1)^2}+\frac{2}{(s-1)^3}. \end{aligned}$$

Hence we have $\mathcal{L}^{-1}\{F(s)\}=e^t+2te^t+t^2e^t$.

3. Step Functions

In many physical problems it is important to consider discontinuous functions. This is the case when one is trying to model a force or signal which turns on or turns off at given points in time. The most important prototype of a discontinuous function is the **unit step function** also known as the **Heaviside funcition.** This is defined by the formula

$$u_c(t)=\begin{cases}0 & t<c\\ 1 & t\geq c.\end{cases}$$

Let us find the Laplace transform of this function.

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st}u_c(t)\,dt \\ &= \int_c^\infty e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_c^\infty \\ &= \frac{e^{-cs}}{s}. \end{aligned}$$

In general we have.

Theorem 4. *If the Laplace transform $\mathcal{L}\{f(t)\}=F(s)$. Then $\mathcal{L}\{u_c(t)f(t-c)\}=e^{-cs}F(s)$.*

Proof. This is just changing variables of integration. We have

$$\begin{aligned} \mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt \\ &= \int_0^\infty e^{-s(t'+c)}f(t')\,dt',\ t=t'+c \\ &= e^{-cs}\int_0^\infty e^{-st'}f(t')\,dt' \\ &= e^{-cs}F(s). \end{aligned}$$

□

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LECTURE 10

1. Step Functions

In many physical problems it is important to consider discontinuous functions. This is the case when one is trying to model a force or signal which turns on or turns off at given points in time. The most important prototype of a discontinuous function is the **unit step function** also known as the **Heaviside funcition.** This is defined by the formula

$$u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & c \le t. \end{cases}$$

More generally we may have a discontinuous function of an arbitrary shape

$$g(t) = \begin{cases} 0 & 0 \le t < c \\ f(t-c) & 0 \le t. \end{cases}$$

The last function can be written as $g(t) = u_c(t) f(t-c)$.

These functions serve as the building block for a lot of piecewise discontinuous functions. As example consider the function

$$f(t) = \begin{cases} 1 & 0 \le t < 4 \\ 3 & 4 \le t < 5. \end{cases}$$

This function can be written as a linear combination of unit step functions

$$f(t) = 1 + 2u_4(t).$$

Note that the coefficients 2 of $u_4(t)$ represents the change in the value of the function at $t = 4$. As another example we have

$$f(t) = \begin{cases} -1 & 0 \le t < 1 \\ 4 & 1 \le t < 7 \\ -2 & 7 \le t. \end{cases}$$

This time the function is a linear combination of

$$f(t) = -1 + 5u_1(t) - 6u_7(t).$$

More generally we may have a piecewise discontinuous function of arbitrary shape.

$$f(t) = \begin{cases} t^2 & 0 \le t < 5 \\ 2 & 5 \le t. \end{cases}$$

This function can be written as $f(t) = t^2 + u_5(t)\left(2 - t^2\right)$. Again the coefficient function $\left(2 - t^2\right)$ of $u_5(t)$ is the difference of the two values near $t = 5$.

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Let us find the Laplace transform of unit step function.

$$\begin{aligned}\mathcal{L}\left\{u_c(t)\right\} &= \int_0^\infty e^{-st}u_c(t)\,dt \\ &= \int_c^\infty e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_c^\infty \\ &= \frac{e^{-cs}}{s}.\end{aligned}$$

More generally we have.

Theorem 1. *If the Laplace transform* $\mathcal{L}\left\{f(t)\right\} = F(s)$. *Then* $\mathcal{L}\left\{u_c(t)\,f(t-c)\right\} = e^{-cs}F(s)$.

Proof. This is just changing variables of integration. We have

$$\begin{aligned}\mathcal{L}\left\{u_c(t)\,f(t-c)\right\} &= \int_0^\infty e^{-st}u_c(t)\,f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt \\ &= \int_0^\infty e^{-s\left(t'+c\right)}f(t')\,dt',\ t = t'+c \\ &= e^{-cs}\int_0^\infty e^{-st'}f(t')\,dt' \\ &= e^{-cs}F(s).\end{aligned}$$

□

The above theorem can now be used to find Laplace transforms of arbitrary piecewise discontinuous functions. Below is an example.

Exercise 2. Find the Laplace transform of

$$g(t) = \begin{cases} 1 & 0 \le t < 2 \\ t^2 & 2 \le t. \end{cases}$$

Solution. First write the function as

$$g(t) = 1 + \left(t^2 - 1\right)u_2(t).$$

The transform of the the first summand is simply $\mathcal{L}\left\{1\right\} = \frac{1}{s}$. To find the transform of the next summand we need to write in the form $u_2(t)\,f(t-2)$. It is clear that

$$\begin{aligned}\left(t^2-1\right)u_2(t) &= \left[\left((t-2)+2\right)^2 - 1\right]u_2(t) \\ &= f(t-2)\,u_2(t)\end{aligned}$$

for $f(t) = (t+2)^2 - 1 = t^2 + 4t + 3$. Now the we may use our theorem to compute the Laplace transform

$$\begin{aligned}\mathcal{L}\left\{\left(t^2-1\right)u_2(t)\right\} &= e^{-2s}\mathcal{L}\left\{t^2+4t+3\right\} \\ &= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right).\end{aligned}$$

Hence the final answer is

$$\mathcal{L}\{g(t)\} = \frac{1}{s} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right).$$

Similarly we may find more inverse transforms using our theorem.

Exercise 3. Find the inverse Laplace transform of

(1) $F(s) = \frac{1}{(s-1)^4}$
(2) $F(s) = \frac{s^2+1}{(s-1)^3}$
(3) $F(s) = \frac{e^{-2s}}{s^2+3s-4}$
(4) $F(s) = \frac{e^{-s}}{s^2+2s+2}$

Solution.

(1) We write

$$F(s) = \frac{1}{(s-1)^4} = \frac{1}{3!}\frac{3!}{(s-1)^4}$$

Hence $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}t^3e^t$.

(2) Set $s-1=u$. Then we have

$$\begin{aligned} F(s) = \frac{s^2+1}{(s-1)^3} &= \frac{(u+1)^2+1}{u^3} \\ &= \frac{u^2+2u+2}{u^3} \\ &= \frac{1}{u} + \frac{2}{u^2} + \frac{2}{u^3} \\ &= \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{(s-1)^3}. \end{aligned}$$

Hence we have $\mathcal{L}^{-1}\{F(s)\} = e^t + 2te^t + t^2e^t$.

(3) First find the Laplace inverse of $\frac{1}{s^2+3s-4}$.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{1}{s+4}\right\} \\ &= \frac{1}{5}e^t - \frac{1}{5}e^{-4t}. \end{aligned}$$

Now it remains to use our theorem to shift this computation

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+3s-4}\right\} = \left[\frac{1}{5}e^{(t-2)} - \frac{1}{5}e^{-4(t-2)}\right]u_2(t).$$

(4) First find the Laplace inverse of $\frac{1}{s^2+2s+2}$.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= e^{-t}\sin(t) \end{aligned}$$

Now it remains to use our theorem to shift this computation

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+3s-4}\right\} = \left[e^{-(t-1)}\sin(t-1)\right]u_1(t).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 11

1. Step Functions

In the last lecture we saw the theorem.

Theorem 1. *If the Laplace transform* $\mathcal{L}\{f(t)\} = F(s)$. *Then*

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s).$$

Conversely, if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *then*

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u_c(t) f(t-c).$$

Let us see further examples of computing Laplace transforms using this.

Exercise 2. Find the Laplace transform of

$$g(t) = \begin{cases} 2 & 0 \le t < 2 \\ t+1 & 2 \le t < 5 \\ 3t & 5 \le t. \end{cases}$$

Solution. First write the function as

$$g(t) = 2 + (t-1)u_2(t) + (2t-1)u_5(t).$$

Let us Laplace transform each term above one by one. First $\mathcal{L}\{2\} = \frac{2}{s}$. Next

$$\begin{aligned} \mathcal{L}\{(t-1)u_2(t)\} &= \mathcal{L}\{(t-2+1)u_2(t)\} \\ &= \mathcal{L}\{(t-2)u_2(t)\} + \mathcal{L}\{u_2(t)\} \\ &= \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}. \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}\{(2t-1)u_5(t)\} &= \mathcal{L}\{[2(t-5+5)-1]u_5(t)\} \\ &= \mathcal{L}\{2(t-5)u_5(t)\} + \mathcal{L}\{9u_5(t)\} \\ &= \frac{2e^{-5s}}{s^2} + \frac{9e^{-5s}}{s}. \end{aligned}$$

Hence the final answer is

$$\mathcal{L}\{g(t)\} = \frac{2}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s} + \frac{2e^{-5s}}{s^2} + \frac{9e^{-5s}}{s}.$$

Similarly we may find more inverse transforms using our theorem.

Exercise 3. Find the inverse Laplace transform of

(1) $F(s) = \frac{1}{(s-1)^4}$

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(2) $F(s) = \frac{s^2+1}{(s-1)^3}$
(3) $F(s) = \frac{e^{-2s}}{s^2+3s-4}$
(4) $F(s) = \frac{e^{-s}}{s^2+2s+2}$

Solution.

(1) We write

$$F(s) = \frac{1}{(s-1)^4} = \frac{1}{3!}\frac{3!}{(s-1)^4}$$

Hence $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}t^3e^t$.

(2) Set $s-1=u$. Then we have

$$\begin{aligned} F(s) = \frac{s^2+1}{(s-1)^3} &= \frac{(u+1)^2+1}{u^3} \\ &= \frac{u^2+2u+2}{u^3} \\ &= \frac{1}{u} + \frac{2}{u^2} + \frac{2}{u^3} \\ &= \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{(s-1)^3}. \end{aligned}$$

Hence we have $\mathcal{L}^{-1}\{F(s)\} = e^t + 2te^t + t^2e^t$.

(3) First find the Laplace inverse of $\frac{1}{s^2+3s-4}$.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{1}{s+4}\right\} \\ &= \frac{1}{5}e^t - \frac{1}{5}e^{-4t}. \end{aligned}$$

Now it remains to use our theorem to shift this computation

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+3s-4}\right\} = \left[\frac{1}{5}e^{(t-2)} - \frac{1}{5}e^{-4(t-2)}\right]u_2(t).$$

(4) First find the Laplace inverse of $\frac{1}{s^2+2s+2}$.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= e^{-t}\sin(t) \end{aligned}$$

Now it remains to use our theorem to shift this computation

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+3s-4}\right\} = \left[e^{-(t-1)}\sin(t-1)\right]u_1(t).$$

2. Differential equations with discontinuous forcing functions

Now we learn how to solve a linear inhomogeneous ordinary differential equation with a discontinuous forcing function. This is the equation of the form

$$L[y] := a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = g(t),$$

where $g(t)$ is now a piecewise continuous function of time. Let us do this through examples.

Example 4. Find the solution to the equation

$$y'' + 4y = g(t) = \begin{cases} 1 & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t, \end{cases}$$

given the initial conditions $y(0) = y'(0) = 0$.

Solution. The function $g(t) = u_\pi(t) - u_{2\pi}(t)$. We may then Laplace transform both sides of the equation to get

$$\left(s^2 + 4\right) Y(s) = \mathcal{L}\{g(t)\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

Hence we have

$$\begin{aligned} Y(s) &= \frac{\left(e^{-\pi s} - e^{-2\pi s}\right)}{s\left(s^2+4\right)} \\ &= \left(e^{-\pi s} - e^{-2\pi s}\right)\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right] \\ &= \frac{e^{-\pi s}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - \frac{e^{-2\pi s}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]. \end{aligned}$$

Knowing the Laplace inverse

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} = 1 - \frac{1}{2}\sin(2t),$$

it remains to shift this to get

$$y(t) = \frac{1}{4}u_\pi(t)\left[1 - \frac{1}{2}\sin(2(t-\pi))\right] + \frac{1}{4}u_{2\pi}(t)\left[1 - \frac{1}{2}\sin(2(t-2\pi))\right].$$

For large times we see that

$$y(t) = \frac{1}{2} - \frac{1}{4}\sin(2t), \qquad 2\pi \leq t.$$

This problem models a pendulum that starts oscillating once it is pushed gently for a short amount of time.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 12

1. Differential equations with discontinuous forcing functions

Let us do another example of a differential equation with a discontinuous forcing function.

Example 1. Find the solution to the equation

$$y'' + y = g_a(t) = \begin{cases} \frac{1}{a} & 0 \le t < a \\ 0 & a \le t, \end{cases} \tag{1.1}$$

given the initial conditions $y(0) = y'(0) = 0$.

Solution. This is very similar to the equation of last lecture. First we write $g_a(t) = \frac{1}{a} - \frac{1}{a}u_a(t)$ in terms of step functions. Now we Laplace transform to get

$$\left(s^2+1\right)Y(s) = \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right].$$

Hence we have

$$Y(s) = \frac{1}{a}\left[1 - e^{-as}\right]\frac{1}{s\left(s^2+1\right)}.$$

Partial fractions now give

$$\begin{aligned} Y(s) &= \frac{1}{a}\left[1 - e^{-as}\right]\left[\frac{1}{s} - \frac{s}{s^2+1}\right] \\ &= \frac{1}{a}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - \frac{e^{-as}}{a}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]. \end{aligned}$$

Now our solution is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. The Laplace transform of the first term above is easy from the table for the second term one need to use the exponential shift rule to get

$$y_a(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{a}\left[1 - \cos t\right] - \frac{1}{a}\left[1 - \cos(t-a)\right]u_a(t).$$

For times $t \ge a$, this solution equals $y_a(t) = \frac{\cos(t-a) - \cos t}{a}$.

2. Impulse functions

Now let us make some further observations regarding the solution to the final problem. Note that the solution to the last problem is

$$y_a(t) = \begin{cases} \frac{1}{a}\left[1 - \cos t\right] & 0 \le t < a \\ \frac{\cos(t-a) - \cos t}{a} & a \le t. \end{cases}$$

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Now let us observe what happens to the solution as $a \to 0$. In the limit we get the function

$$y_0(t) = \lim_{a\to 0} \frac{\cos(t-a) - \cos t}{a} = \sin(t).$$

This is a perfectly well-defined (and enough loved) function. However what happens to our original forcing function $g_a(t)$ (of equation (1.1)) as $a \to 0$? Things get rather spooky! It is perhaps best visualized by its graph which is non-zero on a smaller and smaller interval $[0, a]$ as $a \to 0$. We also note that the total integral $\int_{-\infty}^{\infty} g_a(t)\, dt = 1$ independently of a. Let us denote

$$\lim_{a\to 0} g_a(t) = \delta(t). \tag{2.1}$$

The limiting object is not quite a function, however it has a lot of the same properties of being one. Its true nature will be left undefined here. It is much the Leprechuan of this course! We will see it but won't get to know it.

We will call $\delta(t)$ the **unit impulse function** or the **Dirac delta function**. The main property it has is that

$$\begin{aligned} \delta(t) = 0 \quad &\text{for} \quad t \neq 0, \\ \int_{-\infty}^{\infty} \delta(t)\, dt \quad &= \quad 1. \end{aligned}$$

It models a unit impulse (a instantaneous force of unit intensity) applied at the instant of time $t = 0$. One may similarly define the impulse function $\delta(t - t_0)$ corresponding to a unit impulse applied at time $t = t_0$. This now has the property

$$\begin{aligned} \delta(t - t_0) = 0 \quad &\text{for} \quad t \neq t_0, \\ \int_{-\infty}^{\infty} \delta(t - t_0)\, dt \quad &= \quad 1. \end{aligned}$$

We can use (2.1) to figure out much about this function. First lets compute its Laplace transform

$$\begin{aligned} \mathcal{L}\{\delta(t)\} &= \lim_{a\to 0} \mathcal{L}\{g_a(t)\} \\ &= \lim_{a\to 0} \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right] \\ &= \lim_{a\to 0} \frac{se^{-as}}{s}, \quad \text{(by L'hospitals rule)} \\ &= 1. \end{aligned}$$

Another important property it has is

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,\delta(t) &= \lim_{a\to 0} \int_{-\infty}^{\infty} f(t)\, g_a(t) \\ &= \lim_{a\to 0} \frac{1}{a} \int_0^a f(t)\, dt \\ &= \lim_{a\to 0} f(a), \quad \text{(by L'hospitals rule)} \\ &= f(0). \end{aligned}$$

Similar properties of $\delta(t-t_0)$, which can be figured out by simply changing variables, are

$$\begin{aligned} \mathcal{L}\{\delta(t-t_0)\} &= e^{-st_0} \\ \int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) &= f(t_0). \end{aligned}$$

Now we can solve a forcing problem involving impulses. Below is an example.

Example 2. Find the solution to

$$y'' + 9y = \delta(t-1),$$

with the initial conditions $y(0) = y'(0) = 0$.

Solution. First Laplace transform both sides

$$\left(s^2+9\right) Y(s) = e^{-s}.$$

Then we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\}.$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$. We have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\} \\ &= u_1(t) \quad \frac{1}{3}\sin(3(t-1)). \end{aligned}$$

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LECTURE 13

1. Impulse functions

In the last lecture we introduced $\delta(t)$ the **unit impulse function** or the **Dirac delta function**. This is not a function in the usual sense but in a generalized sense. Mathematically, this means that it doesn't have values (especially not at zero) but integrals in which it appears still make sense. The main property it has is that

$$\begin{aligned} \delta(t) = 0 \quad \text{for} \quad & t \neq 0, \\ \int_{-\infty}^{\infty} \delta(t)\, dt \quad = \quad & 1. \end{aligned}$$

In physical problems, it models a unit impulse (a instantaneous force of unit total intensity) applied at the instant of time $t = 0$. Think of a football kick, golf shot or any kind of jerk. It may be useful to think of it as the limit

$$\begin{aligned} \delta(t) &= \lim_{a \to 0} g_a(t) \qquad (1.1) \\ g_a(t) = \frac{1}{a} - \frac{1}{a} u_a(t) &= \begin{cases} \frac{1}{a} & 0 \leq t < a \\ 0 & a \leq t. \end{cases} \end{aligned}$$

The function $g_a(t)$ models a force of unit total intensity ($\int_{-\infty}^{\infty} g_a(t)\, dt = 1$) applied over an interval of a units in time.

One may similarly define the impulse function $\delta(t - t_0)$ corresponding to a unit impulse applied at time $t = t_0$. This now has the property

$$\begin{aligned} \delta(t - t_0) = 0 \quad \text{for} \quad & t \neq t_0, \\ \int_{-\infty}^{\infty} \delta(t - t_0)\, dt \quad = \quad & 1. \end{aligned}$$

We can use (1.1) to figure out much about this function. First lets compute its Laplace transform

$$\begin{aligned} \mathcal{L}\{\delta(t)\} &= \lim_{a \to 0} \mathcal{L}\{g_a(t)\} \\ &= \lim_{a \to 0} \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right] \\ &= \lim_{a \to 0} \frac{se^{-as}}{s}, \quad \text{(by L'hospitals rule)} \\ &= 1. \end{aligned}$$

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Another important property it has is

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,\delta(t) &= \lim_{a\to 0}\int_{-\infty}^{\infty} f(t)\,g_a(t)\\ &= \lim_{a\to 0}\frac{1}{a}\int_{0}^{a} f(t)\,dt\\ &= \lim_{a\to 0} f(a), \quad \text{(by L'hospitals rule)}\\ &= f(0).\end{aligned}$$

Similar properties of $\delta(t-t_0)$, which can be figured out by simply changing variables, are

$$\begin{aligned}\mathcal{L}\{\delta(t-t_0)\} &= e^{-st_0}\\ \int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) &= f(t_0).\end{aligned}$$

Now we can solve a forcing problem involving impulses. Below is an example.

Example 1. Find the solution to

$$y'' + 9y = \delta(t-1),$$

with the initial conditions $y(0) = y'(0) = 0$.

Solution. First Laplace transform both sides

$$\left(s^2+9\right)Y(s) = e^{-s}.$$

Then we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\}.$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$. We have

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\}\\ &= u_1(t)\ \frac{1}{3}\sin(3(t-1)).\end{aligned}$$

.

Example 2. Find the solution to

$$y'' + 5y' = \delta(t-2)$$

with initial conditions $y(0) = y'(0) = 0$.

Solution. Laplace transforming both sides of the solution we get

$$\left(s^2+5s\right)Y(s) = e^{-2s}.$$

Hence

$$\begin{aligned}Y(s) &= \frac{e^{-2s}}{s^2+5s}\\ &= \frac{e^{-2s}}{5}\left\{\frac{1}{s} - \frac{1}{s+5}\right\}.\end{aligned}$$

To inverse transform the above first inverse transform $\mathcal{L}^{-1}\left\{\frac{1}{5}\left(\frac{1}{s}-\frac{1}{s+5}\right)\right\}=\frac{1-e^{-5t}}{5}$. It remains to shift this computation to get

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{5}\left[1-e^{-5(t-2)}\right]u_2(t).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 14

1. The Convolution Integral

So far, you have probably been convinced that finding the inverse Laplace transform is crucial to solving a differential equation. Finding the inverse Laplace is perhaps also more difficult than the Laplace transform itself since there is no easy formula for it. Often it is important to find the inverse Laplace transform of a product of functions. The product operation is the counterpart (on the transform side) of a more exotic operation (on the function side) called convolution. Given functions $f(t)$ and $g(t)$ of time, define their **convolution** as

$$(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau.$$

Theorem 1. *If* $\mathcal{L}\{f(t)\} = F(s)$ *and* $\mathcal{L}\{g(t)\} = G(s)$*, then*

$$\mathcal{L}\{(f * g)(t)\} = F(s)\, G(s).$$

In other words, if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$*, then*

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = (f * g)(t).$$

Proof. By definition we have

$$\begin{aligned} F(s) &= \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \\ G(s) &= \int_0^\infty e^{-s\tau} g(\tau)\, \tau. \end{aligned}$$

Hence

$$\begin{aligned} F(s)\, G(s) &= \left(\int_0^\infty e^{-s\xi} f(\xi)\, d\xi\right)\left(\int_0^\infty e^{-s\tau} g(\tau)\, d\tau\right) \\ &= \int_0^\infty g(\tau)\left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\, d\xi\right] d\tau. \end{aligned}$$

Now we perform the change of variables $t = \xi + \tau$ and $\tau = \tau$. The Jacobian of the change of variables is 1. After calculating the limits of integration we get

$$\begin{aligned} F(s)\, G(s) &= \int_0^\infty \left[\int_0^t e^{-st} g(\tau)\, f(t-\tau)\, d\tau\right] dt \\ &= \int_0^\infty e^{-st}\left[\int_0^t g(\tau)\, f(t-\tau)\, d\tau\right] dt \\ &= \mathcal{L}\{(f * g)(t)\}. \end{aligned}$$

□

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The convolution also satisfies the following basic properties.

$$\begin{aligned} f * g &= g * f \\ f * (g_1 + g_2) &= f * g_1 + f * g_2 \\ (f * g) * h &= f * (g * h) \\ f * 0 &= 0 * f. \end{aligned}$$

Example 2. Find the convolutions and their Laplace transforms

(1) $1 * t$
(2) $t * \sin t$
(3) $\sin t * \cos t$
(4) $\delta(t) * \cos t$
(5) $\delta(t) * f(t)$
(6) $\left[\sin t \delta\left(t - \frac{\pi}{2}\right)\right] * t$

Solution.
1. $1 * t = t * 1 = \int_0^t \tau d\tau = \frac{t^2}{2}$. Hence the Laplace transform is $\mathcal{L}\{1 * t\} = \frac{1}{s^3}$.
2.

$$\begin{aligned} t * \sin(t) &= \int_0^t \tau \sin(t - \tau) d\tau \\ &= [\tau \cos(t - \tau)]_0^t - \int_0^t \cos(t - \tau) d\tau \\ &= t - [-\sin(t - \tau)]_0^t \\ &= t - \sin t. \end{aligned}$$

Also, the Laplace transform is $\mathcal{L}\{t * \sin t\} = \mathcal{L}\{t\} \mathcal{L}\{\sin t\} = \frac{1}{s^2(s^2+1)}$.
3.

$$\begin{aligned} \sin t * \cos t &= \int_0^t \sin \tau \cos(t - \tau) d\tau \\ &= \int_0^t \frac{1}{2} [\sin(t) + \sin(2\tau - t)] d\tau \\ &= \frac{t}{2} \sin(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\sin t * \cos t\} = \mathcal{L}\{\sin t\} \mathcal{L}\{\cos t\} = \frac{s}{(s^2+1)^2}$.
4.

$$\begin{aligned} \delta(t) * \cos t &= \int_0^t \delta(\tau) \cos(t - \tau) d\tau \\ &= \cos(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\delta(t) * \cos t\} = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.
5.

$$\begin{aligned} \delta(t) * f(t) &= \int_0^t \delta(\tau) f(t - \tau) d\tau \\ &= f(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\delta(t) * f(t)\} = \mathcal{L}\{f(t)\} = F(s)$.

6. Firstly note that $\sin(t)\,\delta\left(t-\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)\delta\left(t-\frac{\pi}{2}\right)=\delta\left(t-\frac{\pi}{2}\right)$. Hence

$$\begin{aligned}\left[\sin(t)\,\delta\left(t-\frac{\pi}{2}\right)\right]*t &= \delta\left(t-\frac{\pi}{2}\right)*t\\ &= \int_0^t \delta\left(\tau-\frac{\pi}{2}\right)(t-\tau)\,d\tau\\ &= \left(t-\frac{\pi}{2}\right)u_{\frac{\pi}{2}}(t).\end{aligned}$$

Also the Laplace transform is $\mathcal{L}\left\{\left[\sin(t)\,\delta\left(t-\frac{\pi}{2}\right)\right]*t\right\}=\mathcal{L}\left\{\left(t-\frac{\pi}{2}\right)u_{\frac{\pi}{2}}(t)\right\}=\frac{e^{-\frac{\pi}{2}s}}{s^2}$.

Example 3. Find the solution to the initial value problem

$$y''+3y'+2y=g(t),$$

$y(0)=1, y'(0)=0$, in terms of a convolution integral.

Solution. Laplace transform both sides of the equation to obtain

$$Y(s)\left(s^2+3s+2\right)-s-1=G(s).$$

Hence we have

$$\begin{aligned}Y(s) &= \frac{s+1}{s^2+3s+2}+\frac{G(s)}{s^2+3s+2}\\ &= \frac{1}{s+2}+G(s)\left[\frac{1}{s+1}-\frac{1}{s+2}\right].\end{aligned}$$

Hence

$$y(t)=e^{-2t}+g(t)*\left[e^{-t}-e^{-2t}\right].$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 15

1. Convolution Integral

Last time we discussed the convolution integral. To recall the definition, given functions $f(t)$ and $g(t)$ of time, their **convolution** is

$$(f*g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

Let us now do some more examples of computing convolutions.

Example 1. Find the convolutions and their Laplace transforms

(1) $1*t$
(2) $t*\sin t$
(3) $\sin t*\cos t$
(4) $\delta(t)*\cos t$
(5) $\delta(t)*f(t)$
(6) $\left[\sin t\delta\left(t-\frac{\pi}{2}\right)\right]*t$

Solution.

1. $1*t=t*1=\int_0^t\tau d\tau=\frac{t^2}{2}$. Hence the Laplace transform is $\mathcal{L}\{1*t\}=\frac{1}{s^3}$.

2.

$$\begin{aligned} t*\sin(t) &= \int_0^t \tau\sin(t-\tau)\,d\tau \\ &= [\tau\cos(t-\tau)]_0^t-\int_0^t\cos(t-\tau)\,d\tau \\ &= t-[-\sin(t-\tau)]_0^t \\ &= t-\sin t. \end{aligned}$$

Also, the Laplace transform is $\mathcal{L}\{t*\sin t\}=\mathcal{L}\{t\}\mathcal{L}\{\sin t\}=\frac{1}{s^2(s^2+1)}$.

3.

$$\begin{aligned} \sin t*\cos t &= \int_0^t \sin\tau\cos(t-\tau)\,d\tau \\ &= \int_0^t \frac{1}{2}[\sin(t)+\sin(2\tau-t)]\,d\tau \\ &= \frac{t}{2}\sin(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\sin t*\cos t\}=\mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\}=\frac{s}{(s^2+1)^2}$.

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4.

$$\begin{aligned} \delta(t) * \cos t &= \int_0^t \delta(\tau)\cos(t-\tau)\,d\tau \\ &= \cos(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\delta(t) * \cos t\} = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.

5.

$$\begin{aligned} \delta(t) * f(t) &= \int_0^t \delta(\tau) f(t-\tau)\,d\tau \\ &= f(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\{\delta(t) * f(t)\} = \mathcal{L}\{f(t)\} = F(s)$.

6. Firstly note that $\sin(t)\,\delta\left(t-\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\delta\left(t-\frac{\pi}{2}\right) = \delta\left(t-\frac{\pi}{2}\right)$. Hence y

$$\begin{aligned} \left[\sin(t)\,\delta\left(t-\frac{\pi}{2}\right)\right] * t &= \delta\left(t-\frac{\pi}{2}\right) * t \\ &= \int_0^t \delta\left(\tau-\frac{\pi}{2}\right)(t-\tau)\,d\tau \\ &= \left(t-\frac{\pi}{2}\right) u_{\frac{\pi}{2}}(t). \end{aligned}$$

Also the Laplace transform is $\mathcal{L}\left\{\left[\sin(t)\,\delta\left(t-\frac{\pi}{2}\right)\right] * t\right\} = \mathcal{L}\left\{\left(t-\frac{\pi}{2}\right) u_{\frac{\pi}{2}}(t)\right\} = \frac{e^{-\frac{\pi}{2}s}}{s^2}$.

Now we do an example of computing inverse Laplace transforms using convolutions.

Example 2. Compute inverse Laplace transforms for the following using convolutions.

(1) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+9)}\right\}$
(2) $\mathcal{L}^{-1}\left\{\frac{1}{s^{11}(s^2+1)}\right\}$

Solution.

(1) By the convolution theorem we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+9)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}.\frac{s}{(s^2+9)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}.\right\} * \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)}\right\} \\ &= \sin(t) * \cos(3t) \\ &= \int_0^t \sin(t-\tau)\cos(3\tau)\,d\tau \\ &= \frac{1}{2}\int_0^t [\sin(t+2\tau) + \sin(t-4\tau)]\,d\tau \\ &= \frac{1}{4}[\cos t - \cos(3t)] + \frac{1}{8}[\cos(3t) - \cos t] \\ &= \frac{1}{8}[\cos t - \cos(3t)]. \end{aligned}$$

(2) By the convolution theorem we have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^{11}\left(s^2+1\right)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^{11}}\cdot\frac{1}{\left(s^2+1\right)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^{11}}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)}\right\} \\
&= \frac{t^{10}}{10!} * \sin(t) \\
&= \frac{1}{10!}\int_0^t \tau^{10}\sin(t-\tau)\,d\tau \\
&= \frac{1}{10!}\left[\sin(t)\int_0^t \tau^{10}\cos(\tau)\,d\tau - \cos(t)\int_0^t \tau^{10}\sin(\tau)\,d\tau\right].
\end{aligned}$$

It remains to evaluate the integrals $\int_0^t \tau^{10}\cos(\tau)\,d\tau$ and $\int_0^t \tau^{10}\sin(\tau)\,d\tau$. Let us define

$$\begin{aligned}
I_1(a) &= \int_0^t \cos(a\tau)\,d\tau = \frac{\sin(at)}{a} \\
I_2(a) &= \int_0^t \sin(a\tau)\,d\tau = \frac{[1-\cos(at)]}{a}.
\end{aligned}$$

Clearly

$$\begin{aligned}
\left(\frac{\partial}{\partial a}\right)^{10}[I_1(a)] &= \int_0^t \tau^{10}\cos(a\tau)\,d\tau \\
\left(\frac{\partial}{\partial a}\right)^{10}[I_2(a)] &= \int_0^t \tau^{10}\sin(a\tau)\,d\tau.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^{11}\left(s^2+1\right)}\right\} &= \frac{1}{10!}\sin(t)\left[\left(\frac{\partial}{\partial a}\right)^{10}\left(\frac{\sin(at)}{a}\right)\right]_{a=1} \\
&\quad -\frac{1}{10!}\cos(t)\left[\left(\frac{\partial}{\partial a}\right)^{10}\left(\frac{1-\cos(at)}{a}\right)\right]_{a=1}.
\end{aligned}$$

2. Fundamental Solution or Impulse Response

One of the main applications of convolution is that it helps with solving a linear constant coefficient ODE with arbitrary forcing function. For example consider

$$L[y] := ay'' + by' + cy = g(t),$$

where the right hand side is arbitrary. Let us now find a particular solution to the above with the initial conditions $y(0) = y'(0) = 0$. Taking the Laplace transform on both sides gives

$$\left(as^2 + bs + c\right)Y(s) = G(s)$$

where $Y(s) = \mathcal{L}\{y(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Hence

$$\begin{aligned} Y(s) &= \frac{G(s)}{(as^2+bs+c)}, \quad \text{and} \\ y(t) &= \mathcal{L}^{-1}\left\{\frac{G(s)}{(as^2+bs+c)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(as^2+bs+c)}\right\} * g(t). \end{aligned}$$

Let us define $E(t) = \mathcal{L}^{-1}\left\{\frac{1}{(as^2+bs+c)}\right\}$ where as^2+bs+c is obviously the characteristic polynomial. It is not hard to see (by setting $g(t) = \delta(t)$ for instance) that $E(t)$ satisfies

$$L[E(t)] = \delta(t).$$

The function $E(t)$ is hence known as the **impulse response** or the **fundamental solution** (my preferred terminology). The particular solution to the general forcing problem is then a covolution

$$y(t) = E(t) * g(t).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 16

1. Systems of first order ordinary differential equations

So far, we have been studying differential equations involving just a single function of one variable. Following Chapter 7 of the textbook we shall now consider systems of differential ordinary differential equations involving **several functions** of one variable (often time). These are still ordinary since there is still just a single variable involved. The most general system of ordinary differential equations is of the form

$$\begin{aligned} y_1' &= F_1(t, y_1, \ldots, y_n) \\ y_2' &= F_2(t, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(t, y_1, \ldots, y_n). \end{aligned} \tag{1.1}$$

One might be inclined to think that this is not the most general system since it is only of first order. However a higher order system can be turned into a system of first order equations. As an example let us consider the n^{th} order equation

$$y^{(n)} = F\left(t, y, y', \ldots, y^{(n-1)}\right). \tag{1.2}$$

This can be turned into a system of n first order equations in the functions $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. Then (1.2) becomes the system of equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= F(t, y_1, y_2, \ldots, y_n). \end{aligned}$$

A special case is when the system is **linear**. In this case the the functions $F_1, \ldots, F_n$ are linear and hence (1.2) takes the form

$$\begin{aligned} y_1' &= p_{11}(t)\, y_1 + \ldots + p_{1n}(t)\, y_n + g_1(t) \\ y_2' &= p_{21}(t)\, y_1 + \ldots + p_{2n}(t)\, y_n + g_2(t) \\ &\vdots \\ y_n' &= p_{n1}(t)\, y_1 + \ldots + p_{nn}(t)\, y_n + g_n(t). \end{aligned} \tag{1.3}$$

If the functions $g_1(t) = \ldots = g_n(t) = 0$ then the system is **homogeneous** and otherwise **inhomogeneous** (I still insist on using this word!).

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Example 1. Consider the second order differential equation

$$y'' + y' + \frac{1}{4}y = 0,$$

with initial conditions $y(0) = 0, y'(0) = 1$. Write the above as a system first order equations.

Solution. Define $y_1 = y, y_2 = y'$. We then have $y_2' = y'' = -y' - \frac{1}{4}y = -y_2 - \frac{1}{4}y_1$. This gives the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_2 - \frac{1}{4}y_1. \end{aligned}$$

With the initial conditions $y_1(0) = y(0) = 0, y_2(0) = y'(0) = 1$.

Conversely, a system of ordinary differential equations can sometimes be written as a single equation of higher order. Below is an example.

Example 2. Solve the system of ordinary differential equations

$$\begin{aligned} y_1' &= 4y_1 - 3y_2 \\ y_2' &= 2y_1 - y_2 \end{aligned}$$

with the initial conditions $y_1(0) = y_2(0) = 1$.

Solution. Begin by using the first equation to write y_2 in terms of y_1. We then have

$$\begin{aligned} y_2 &= \frac{1}{3}[4y_1 - y_1'] \quad \text{and hence} \\ y_2' &= \frac{1}{3}[4y_1' - y_1'']. \end{aligned}$$

This gives

$$\begin{aligned} 2y_1 - y_2 &= \frac{1}{3}[4y_1' - y_1''] \quad \text{or} \\ 2y_1 - \frac{1}{3}[4y_1 - y_1'] &= \frac{1}{3}[4y_1' - y_1'']. \end{aligned}$$

The last one is an equation for y_1 and gives

$$y_1'' - 3y_1' + 2y_1 = 0.$$

The general solution is

$$y_1(t) = c_1 e^t + c_2 e^{2t}.$$

The initial conditions are $y_1(0) = 1, y_1'(0) = 4y_1(0) - 3y_2(0) = 1$. This gives $c_1 = 1, c_2 = 0$ and hence we have

$$\begin{aligned} y_1(t) &= e^t \\ y_2(t) &= \frac{1}{3}[4y_1 - y_1'] = e^t. \end{aligned}$$

1.1. **Existence and Uniqueness of the Solution.** The system in the example is rather simple. What can be said for the solution of the general system? In general it is impossible to solve explicitly for the solution of (1.1) or even (1.2) (we will have to make further restrictions). However, it is still possible to claim existence and uniqueness.

Theorem 3. *For the system* (1.1) *assume that* $F_1, \ldots, F_n$ *are all continous functions for* $(t, y_1, \ldots, y_n) \in R = [\alpha, \beta] \times [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]$ *and let* $\left(t_0, y_1^0, \ldots, y_n^0\right) \in R$. *Then there is a unique solution* $(y_1(t), \ldots, y_n(t))$ *to* (1.1) *with the initial conditions* $(y_1(t_0), \ldots, y_n(t_0)) = \left(y_1^0, \ldots, y_n^0\right)$*for a small positive interval in time* $t \in [t_0 - h, t_0 + h], h > 0$.

For the linear system (1.3) one can say slightly better.

Theorem 4. *For the system* (1.3) *assume that* $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ *are all continuous functions of time on the interval* $[\alpha, \beta]$ *and let* $t_0 \in [\alpha, \beta]$. *Then there is a unique solution* $(y_1(t), \ldots, y_n(t))$ *to* (1.1) *with the initial conditions* $(y_1(t_0), \ldots, y_n(t_0)) = \left(y_1^0, \ldots, y_n^0\right)$ *for all* $t \in [\alpha, \beta]$.

Notice that Theorem 4 says more than Theorem 3 about the linear case since the initial conditions $\left(y_1^0, \ldots, y_n^0\right)$ are now arbitrary and the solution now exists throughout the interval $t \in [\alpha, \beta]$.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 17

1. Matrices

Since a system of differential equations is quite efficiently written and solved in matrix notation it will be handy to review matrices. An $m \times n$ matrix is a rectangular array of numbers consisting of m rows and n columns such as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We often also write $A = (a_{ij})$ and call $m \times n$ its order. Below are a few important properties of matrices. Below we let $A_{m\times n} = (a_{ij})$ and $B_{n\times r} = (b_{ij})$ denote two $m \times n$ and $n \times r$ matrices.

(1) **Equality.** The matrices $A = B$ are equal if and only if the have the same order and $a_{ij} = b_{ij}$ for each i, j.
(2) **Zero.** The symbol 0 also denotes the matrix with all entries equal to 0.
(3) **Addition.** The sum of matrices is

$$A + B = (a_{ij} + b_{ij}).$$

(4) **Multiplication by a constant.** Given a real or complex number α we have

$$\alpha A = (\alpha a_{ij}).$$

(5) **Subtraction.** The difference of the two matrices is

$$A - B = (a_{ij} - b_{ij}).$$

(6) **Multiplication.** The multiplication of the two matrices is the $m\times r$ matrix given by

$$\begin{aligned} AB &= (c_{ij}), \quad \text{where} \\ c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj}. \end{aligned}$$

A key feature (evoking ambivalent reactions) of matrix multiplication is that it is non-commutative i.e.

$$AB \neq BA$$

in general.

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(7) **Transpose and Adjoint.** The transpose and adjoint of the matrix are

$$\begin{aligned} A^T_{n\times m} &= (a_{ji}) \quad \text{and} \\ A^*_{n\times m} &= (\bar{a}_{ji}) \end{aligned}$$

where $\bar{z}$ denotes the complex conjugate of z.

(8) **Multiplication of vectors.** An $n \times 1$ matrix is often called an n-vector. Given two n-vectors x, y define their dot product as

$$x^T y = y^T x = \sum_{i=1}^{n} x_i y_i,$$

while their inner product is defined by

$$(x, y) = x^T \bar{y} = \sum_{i=1}^{n} x_i \bar{y}_i.$$

(9) **Identity.** The $n \times n$ identity matrix is

$$I_n = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}}_{n \text{ columns}}.$$

(10) **Inverse.** The inverse of an $n \times n$ matrix A is another matrix (denoted by A^{-1}) satisfying

$$AA^{-1} = A^{-1}A = I_n.$$

The inverse may not always exist. It exists if and only if $\det A \neq 0$. If invertible the inverse is ongiven by **Cramer's rule.** To find this first consider the minor M_{ij} obtained by deleting the i^{th} row and j^{th} column. Let $d_{ij} = \det M_{ij}$ and

$$c_{ij} = (-1)^{i+j} d_{ij}.$$

The inverse is now given by

$$A^{-1} = \frac{1}{\det A} (c_{ij}).$$

The inverse may also be more efficiently found via Gaussian elimination.

2. Matrix Functions

By a matrix valued function $A(t)$ we mean a function which assigns to each values in time t a matrix

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \dots & a_{mn}(t) \end{bmatrix}.$$

Hence its entries $a_{ij}(t)$ are all functions of time. These may be differentiated and integrated entrywise

$$\begin{aligned} \frac{dA}{dt} &= \left(\frac{da_{ij}}{dt}\right) \quad \text{and} \\ \int_a^b A(t)\,dt &= \left(\int_a^b a_{ij}dt\right). \end{aligned}$$

These satisfy the same rules for differentiation/integration of sums and products

$$\begin{aligned} \frac{d(A+B)}{dt} &= \frac{dA}{dt} + \frac{dB}{dt} \\ \frac{d(AB)}{dt} &= \left(\frac{dA}{dt}\right)B + A\left(\frac{dB}{dt}\right) \\ \int_a^b (A+B)\,dt &= \int_a^b Adt + \int_a^b Bdt. \end{aligned}$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 18

1. Determinants and Inverses

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}. \tag{1.1}$$

We now recall the definition of its **determinant**. It is best to define it inductively by assuming that it is already defined for $(n-1) \times (n-1)$ matrices. Now we first define the minor matrix M_{ij} which is the $(n-1) \times (n-1)$ matrix defined by deleting the i^{th} row and j^{th} column. Let

$$m_{ij} = \det M_{ij}. \tag{1.2}$$

The determinant of A is now the sum

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} m_{ij}.$$

This way we have expanded the determinant by its j^{th} column. The answer does not depend on which column we use. We may also expand by any (say the i^{th}) row and hence the determinant also equals

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} m_{ij}.$$

The **inverse** of a matrix denoted A^{-1} is another matrix which satisfies

$$AA^{-1} = A^{-1}A = I.$$

Such a matrix may or may not exist. If it exist we say that A is **invertible** and if it does not we say that A is **singular**. The column vectors of A are the $n \times 1$ matrices

$$C_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \dots, C_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

while its row vectors are the $1 \times n$ matrices

$$\begin{aligned} R_1 &= \begin{bmatrix} a_{11} & \dots & a_{1n} \end{bmatrix}, \\ &\vdots \\ R_n &= \begin{bmatrix} a_{n1} & \dots & a_{nn} \end{bmatrix}. \end{aligned}$$

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Recall that we call $C_1, \ldots, C_n$ **linearly dependent** if and only if there exists constants $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 C_1 + \ldots + \alpha_n C_n = 0.$$

And we say that they are **linearly independent** otherwise. We now have the following important theorem of linear algebra.

Theorem 1. *(Fundamental Theorem of Linear Algebra) Given an $n \times n$ matrix A, the following are equivalent.*

(1) *the inverse A^{-1} exists,*
(2) $\det A \neq 0$,
(3) *the column vectors $C_1, \ldots, C_n$ of A are linearly independent,*
(4) *the row vectors $R_1, \ldots, R_n$ of A are linearly independent,*
(5) *for any $n \times 1$ matrix X we have $AX = 0 \iff X = 0$.*

.

If the inverse exists how does one compute it? There are two ways. One by **Gaussian elimination** and the other using **Cramer's** rule. For Cramers rule the formula is simple. Recall that m_{ij} was defined in (1.2) to be the determinant of the minor. Let $b_{ij} = (-1)^{i+j} m_{ij}$. The inverse of the matrix is now given by

$$A^{-1} = \frac{1}{\det A} (b_{ij}).$$

For Gaussian elimination one performs row (or column) operations until one reduces A to the identity matrix. Let us do this by example.

Example 2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}$$

by Gaussian elimination.

Solution. For Gaussian elimination we first write the equation A^{-1} needs to solve

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now perform simultaneous row operations to both sides until we reduce the left hand side matrix to the identity. To do this reduction we try to eliminate all other entries in the first column except $a_{11} = 1$. Hence let us subtract row 1 from row 2

(performing $R_2 - R_1$)

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 2 & 7 & 7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(performing } R_2 - R_1\text{)}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{(performing } R_3 - 2R_1\text{)}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{(performing } R_1 - 3R_3\text{)}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{(performing } R_3 - R_1\text{)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{(performing } R_1 - 3R_3\text{)}.$$

Now having reduced the matrix on the left to the identity the corresponding matrix on the right is the required inverse

$$A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

One can check that one gets the same anwer by Cramer's rule.

2. Systems of Linear Equations

Consider the system of n linear equations in n variables $x_1, \ldots, x_n$ given by

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

This system can be written in matrix notation as

$$AX = B \tag{2.1}$$

where A is the matrix in (1.1) while $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. If the inverse exists (that is either one of the conditions in theorem Theorem 1 are true) then the solution to the above system or the matrix equation (2.1) is unique and is given by

$$X = A^{-1}B.$$

However if the matrix A is sigular then the solution may not exist or may not be unique and things can be tricky. This again need to be figured out by Gaussian elimination. Let us do this by example again.

Example 3. Find the solutions of the system

$$\begin{aligned} x_1 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 0 \\ 3x_1 + 9x_3 &= 3. \end{aligned}$$

Solution. Again this is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & 9 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Performing $R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that we can row reduce no further. Hence we are left with the system of equations

$$\begin{aligned} x_1 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

These are only **2** equations in **3** variables since the last equation is redundant. Hence we may choose $x_3 = c$ an arbitrary constant and this gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 3c \\ -2c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

This is a **one dimensional** space of solutions.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 19

1. Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. You are probably familiar that it can be viewed as a linear transformation of n dimensional Euclidean space

$$\begin{aligned} A : \mathbb{R}^n &\to \mathbb{R}^n, \\ X &\mapsto AX. \end{aligned}$$

In general the matrix A when acting on a column vector can change its direction. It is often a natural question to ask whether there are column vectors X whose direction is unchanged by the matrix A. Hence these column vectors are only changed in length by multiplication by a scalar. Hence we are looking for a non-trivial vector X such that

$$AX = \lambda X \quad \text{for some number } \lambda. \tag{1.1}$$

This is clearly equivalent to solving the matrix equation

$$(A - \lambda I)\, X = 0.$$

By the fundamental theorem of algebra of last class we know that for the above equation to have a non-trivial solution, $A - \lambda I$ must be singular or

$$\det(A - \lambda I) = 0.$$

This is a degree n polynomial equation in λ. It is called the **characteristic equation** of A. A root of this equation is called an **eigenvalue**[1] of the matrix A. A vector which solves the equation (1.1) is called an **eigenvector** with eigenvalue λ.

For a 2×2 matrix the characteristic equation is easily written down. It is

$$\begin{aligned} \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} &= 0 \quad \text{or} \\ \lambda^2 - \underbrace{(a+d)}_{\text{tr}A}\lambda + \underbrace{(ad-bc)}_{\det A} &= 0. \end{aligned}$$

Here $\text{tr}A$ is the **trace** of the matrix A given by the sum of the diagonal entries of the matrix.

Example 1. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

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[1]This is anglicised german. In plain words it means "*my very own* value", although its translation is a controversial subject.

Solution. The characteristic equation is

$$\begin{aligned} \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0. \end{aligned}$$

This clearly has the roots $\lambda_1, \lambda_2 = 2, 3$. These are the eigenvalues. To find the eigenvector corresponding to the eigenvalue $\lambda_1 = 2$ we have to solve

$$\begin{bmatrix} 2-\lambda_1 & 1 \\ 0 & 3-\lambda_1 \end{bmatrix} X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} X_1 = 0.$$

Hence we must have

$$X_1 = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for any $c \neq 0$. Similarly an eigenvector with eigenvalue $\lambda_2 = 3$ needs to solve

$$\begin{bmatrix} 2-\lambda_2 & 1 \\ 0 & 3-\lambda_2 \end{bmatrix} X_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} X_2 = 0.$$

This clearly gives

$$X_2 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $c \neq 0$.

The characteristic equation $\det(A - \lambda I) = 0$ will often have n distinct roots. However sometimes the roots may be repeated. The **multiplicity** of λ as an eigenvalue of A is its multiplicity as a root of the characteristic polynomial. If the eigenvalue has multiplicity 1 then it is said to be **simple**. Let us do an example of multiple eigenvalues.

Example 2. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. The eigenvalue equation is

$$\begin{aligned} \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} &= -\lambda^3 + 3\lambda + 2 = 0 \\ &= (\lambda+1)(\lambda+1)(-\lambda+2). \end{aligned}$$

Hence the eigenvalues are $\lambda_1, \lambda_2, \lambda_3 = 2, -1, -1$. Hence the eigenvalue -1 is repeated. To find the eigenvector corresponding to $\lambda_1 = 2$ we have to solve the equation

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} X_1 = 0.$$

On row reduction this system is reduced to

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} X_1 = 0.$$

This gives

$$X_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the second and third eigenvalues $\lambda_2, \lambda_3 = -1$ we have to solve the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} X = 0.$$

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then this gives the single equation $x_1 + x_2 + x_3 = 0$. Hence letting $x_1 = c_2, x_2 = c_3$ we get

$$X = c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Choosing $c_2 = 1, c_3 = 0$ we have

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

while choosing $c_2 = 0, c_3 = 1$ gives

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

In this example we found three linearly independent eigenvectors although we had only two eigenvalues. This may not always happen. For instance if the matrix is

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then one may check that there are repeated eigenvalues $\lambda_1, \lambda_2 = 0, 0$. However the eigenvector solves the equation

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Hence $x_2 = 0$ is the only constraint and we have just one linearly independent eigenvector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We will look at the repeated eigenvalues again in section 7.8.

There is however one situation in which one does have n linearly independent eigenvalues. If the matrix is **Hermitian** (which means $A^* = A$) then its eigenvalues are all real and it always has n linearly independent, orthonormal eigenvectors. If the matrix has real entries then being Hermitian is the same as being **self-adjoint**. The proofs of these statement are challenge problems #32 and #33 in Section 7.3 of your textbook.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 20

1. Systems of First Order Linear ODE

We now come to our study of systems of linear first order ordinary differential equations. This is a system that has the form

$$
\begin{aligned}
x_1' &= a_{11}(t)\,x_1 + \ldots + a_{1n}(t)\,x_n + g_1(t) \\
x_2' &= a_{21}(t)\,x_1 + \ldots + a_{2n}(t)\,x_n + g_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)\,x_1 + \ldots + a_{nn}(t)\,x_n + g_n(t)\,.
\end{aligned}
\tag{1.1}
$$

In matrix notation this has the form

$$X'(t) = A(t)\,X(t) + G(t)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & & & \\ \vdots & & & \\ a_{n1}(t) & \ldots & & a_{nn}(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}.$$

We first consider the corresponding homogeneous system

$$X'(t) = A(t)\,X(t)\,. \tag{1.2}$$

The above system satisfies the **principle of superposition**: If $X_1(t), X_2(t)$ are solutions to (1.2) then so is $c_1 X_1(t) + c_2 X_2(t)$.

Let

$$X_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, X_n(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

be n solutions to (1.2). Consider the matrix

$$\mathbb{X}(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & & & \\ \vdots & & & \\ x_{n1}(t) & \ldots & & x_{nn}(t) \end{bmatrix}.$$

The **Wronskian** of $X_1(t), \ldots, X_n(t)$ is defined to be

$$W[X_1(t), \ldots, X_n(t)] = \det \mathbb{X}(t)\,.$$

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It can be shown (challenge! section 7.4 #2) that the Wronskian satisfies the differential equation

$$\frac{dW}{dt} = \underbrace{\left[a_{11}(t) + a_{22}(t) + \ldots + a_{nn}(t)\right]}_{\mathrm{tr}A} W.$$

Here tr$A$ denotes the **trace** of a matrix A defined as the sum of its diagonal entries. Hence we may now solve the above equation to get

$$W(t) = c \exp\left\{\int \left[a_{11}(t) + a_{22}(t) + \ldots + a_{nn}(t)\right] dt\right\}.$$

This shows that if $X_1(t), \ldots, X_n(t)$ define solutions of our system on the interval $t \in [\alpha, \beta]$, then either $W \equiv 0$ or $W(t) \neq 0$ for $t \in [\alpha, \beta]$.

2. Systems with constant coefficients

A general system of the form (1.2) is still too difficult to solve. To find a solution explicitly, we restrict to the case when the coefficient matrix $A(t) = A$ is a constant matrix and independent of time t. Hence we are left with the constant coefficient system

$$X' = AX. \tag{2.1}$$

Some solutions of the above system can easily be guessed. Firstly let X_1 be an eigenvector of A with eigenvalue λ_1 then we can check that

$$X_1(t) = e^{\lambda_1 t} X_1$$

is a solution to the above system. Now if $X_2, \ldots, X_n$ are further eigenvectors with eigenvalues $\lambda_2, \ldots, \lambda_n$. Then by the principle of superposition

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \ldots + c_n e^{\lambda_n t} X_n \tag{2.2}$$

is another solution. In the case when we are able to find n linearly independent eigenvectors for the matrix A, (2.2) denotes the general solution to the system (2.1). Below is an example.

Problem 1. Find the general solution of the system

$$X' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X.$$

Solution. From the last lecture we know that the eigenvectors of the above matrix are

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$ respectively. Hence the general solution to the above system is

$$X(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Let us now do an example of solving an initial value problem for a system.

Problem 2. Find the solution to the following system

$$X' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution. From last lecture we again know that the eigenvectors of the above matrix are

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 2, \lambda_2 = 3$ respectively. Hence the general solution to the equation is

$$X(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find the constants c_1, c_2 we use the initial condition to get

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives $c_1 = -1, c_2 = 1$ and hence

$$X(t) = -e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 21

Discussed Practice Problems, hence break from taking lecture notes. Whew! See 'Solutions to Practice Problems Exam I.pdf ' instead.

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MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 22

Discussed Exam Problems, hence break from taking lecture notes again. We should have more of these. Another WHEW!

Date: 10/16.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 23

1. Systems of First Order Linear ODE: Complex Eigenvalues

In the last lecture (or in the one preceding that nerve-racking test) we were studying system of linear first order equations with constant coefficients. These are equations of the form

$$X' = AX$$

with A being a matrix independent of time. The eigenvalues of the matrix A played an important role in solving the above equation. However the eigenvalues in the examples that we considered previously were always real. In this lecture we consider examples where the eigenvalues are imaginary and see how that affects the behaviour of the solution.

Example 1. Find the general solution to the differential equation

$$X' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X.$$

Solution. First find the eigenvalues and eigenvector of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The characteristic equation is

$$\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0.$$

Hence $\lambda = \pm i$. To find the eigenvector for $\lambda_1 = i$ we need to solve

$$\begin{bmatrix} -\lambda_1 & -1 \\ 1 & -\lambda_1 \end{bmatrix} X_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} X_1 = 0.$$

This gives $X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Similarly we find the the eigenvector for $\lambda_2 = -i$ to be $X_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$. Hence two solutions to the equation are

$$\begin{aligned} X_1(t) &= \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \\ X_2(t) &= \begin{bmatrix} i \\ -1 \end{bmatrix} e^{it} = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} + i \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

using the ever sublime Euler's formula ($e^{it} = \cos t + i \sin t$). Hence any linear combination of $X_1(t), X_2(t)$ is also a linear combination of $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ and $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$. Notice that these last two functions are the real and imaginary parts of $X_1(t)$

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or $X_2(t)$. This is always the case when we have a real matrix with imaginary eigenvalues. The general solution is now a linear combination

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Note that this solution represents a circle in the $(x(t), y(t))$ plane and hence stays bounded.

.

Example 2. Find the general solution to the differential equation

$$X' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X.$$

Solution. As before we first find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The characteristic equation is

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2 = 0.$$

This has the roots $\lambda = 1 \pm i$. To find the eigenvector for $\lambda_1 = 1 + i$ we need to solve

$$\begin{bmatrix} 1-\lambda_1 & -1 \\ 1 & 1-\lambda_1 \end{bmatrix} X_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} X_1 = 0.$$

This gives $X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Similarly we find the the eigenvector for $\lambda_2 = 1 - i$ to be $X_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$. Hence two solutions to the equation are

$$\begin{aligned} X_1(t) &= \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+i)t} = e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + ie^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \\ X_2(t) &= \begin{bmatrix} i \\ -1 \end{bmatrix} e^{(1-i)t} = e^t \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} + ie^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \end{aligned}$$

Hence any linear combination of $X_1(t), X_2(t)$ is also a linear combination of $e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ and $e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$. Notice that these last two functions are the real and imaginary parts of $X_1(t)$ or $X_2(t)$. This is always the case when we have a real matrix with imaginary eigenvalues. The general solution is now a linear combination

$$X(t) = c_1 e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

As $t \to \infty$ this solution $X(t)$ is an outgoing spiral while as $t \to -\infty$ we have $X(t) \to 0$.

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Example 3. Find the general solution to the differential equation

$$X' = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} X.$$

Solution. First we find the eigenvalues. The characteristic polynomial is

$$\det\begin{bmatrix}-1-\lambda & 0 & 1\\ 1 & -2-\lambda & 0\\ 0 & 2 & -1-\lambda\end{bmatrix} = -\lambda^3 - 4\lambda^2 - 5\lambda = 0$$

and has roots $\lambda = 0, -2 \pm i$. The eigenvector for $\lambda_1 = 0$ is found by row reduction to be $X_1 = \begin{bmatrix}2\\1\\2\end{bmatrix}$. Hence one solution is

$$X_1(t) = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

The eigenvector for $\lambda_2 = -2 - i$ is found by row reduction to be $X_2 = \begin{bmatrix}1\\i\\-1-i\end{bmatrix}$.

Hence another solution is

$$X_2(t) = \begin{bmatrix}1\\i\\-1-i\end{bmatrix} e^{(-2-i)t} = e^{-2t}\begin{bmatrix}\cos t\\ \sin t\\ -\cos t - \sin t\end{bmatrix} + ie^{-2t}\begin{bmatrix}-\sin t\\ \cos t\\ -\cos t + \sin t\end{bmatrix}.$$

Taking the real and imaginary parts we get three linearly independent real solutions

$$\begin{bmatrix}2\\1\\2\end{bmatrix}, e^{-2t}\begin{bmatrix}\cos t\\ \sin t\\ -\cos t - \sin t\end{bmatrix}, \text{ and } e^{-2t}\begin{bmatrix}-\sin t\\ \cos t\\ -\cos t + \sin t\end{bmatrix}.$$

Hence the general solution is

$$X(t) = c_1\begin{bmatrix}2\\1\\2\end{bmatrix} + c_2e^{-2t}\begin{bmatrix}\cos t\\ \sin t\\ -\cos t - \sin t\end{bmatrix} + c_3e^{-2t}\begin{bmatrix}-\sin t\\ \cos t\\ -\cos t + \sin t\end{bmatrix}.$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 24

1. Fundamental Matrices

So far in this chapter we have been studying the first order constant coefficient systems of the type

$$X' = AX \tag{1.1}$$

with A being a matrix independent of time. A time dependent matrix

$$\Phi(t) = \begin{bmatrix} x_1^1(t) & \dots & x_1^n(t) \\ \vdots & \ddots & \vdots \\ x_n^1(t) & \dots & x_n^n(t) \end{bmatrix}$$

is said to be a **fundamental matrix** for the system (1.1) if the columns are linearly independent and each column vector of $\Phi(t)$ is a solution to (1.1). The matrix itself solved the equation

$$\Phi' = A\Phi.$$

Hence finding fundamental matrices is the same as finding linearly independent solutions to (1.1). A fundamental solution is unique once an initial condition $\Phi(0) = \Phi_0$ is specified. The fundamental solution satisfying the initial condition $\Phi(0) = I$ has special significance. It is given by the **matrix exponential** as an infinite series

$$\begin{aligned} \Phi(t) &= e^{At} \\ &:= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \end{aligned}$$

The above series always converges. The solution to the initial value problem

$$X' = AX, \quad X(0) = X_0$$

is then given in terms of the matrix exponential as $X(t) = e^{At}X_0$.

So then how does one compute the matrix exponential e^{At}? The case when A is diagonalizable is easy. In this case there exists a matrix T such that

$$T^{-1}AT = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

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is a diagonal matrix. Rewriting thi is the same as $TDT^{-1} = A$. The matrix exponential is then calculated to be

$$\begin{aligned} e^{At} &= Te^{Dt}T^{-1} \\ &= T \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} T^{-1}. \end{aligned}$$

What about the case when A is not diagonalizable? Umm.... we'll leave that for the next lecture. For now let us do some examples.

Example 1. Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ to

$$X' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X.$$

Example. *Solution.* From lecture 20, the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ are $\lambda_1 = 2, \lambda_2 = 3$ with corresponding eigenvectors $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence the diagonalizing or transformation matrix is

$$\begin{aligned} T &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ with} \\ T^{-1} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence the matrix exponential is

$$\begin{aligned} e^{At} &= T \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

.

Example 2. Use the fundamental matrix of the above example to solve the initial value problem

$$X' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution. The solution is simply given in terms of the fundamental matrix by

$$\begin{aligned} X(t) &= e^{At} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} + e^{2t} \\ e^{3t} \end{bmatrix}. \end{aligned}$$

.

Example 3. Find a fundamental matrix $\Phi(t)$ satisfying to the system

$$X' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X.$$

Solution. The eigenvalues were again found in lecture 20 as $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$ with eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Hence a fundamental solution is given by

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{bmatrix}.$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 25

1. First Order Sysytems: Repeated Eigenvalues

In the last lecture we figured how to compute matrix exponentials e^{tA} in the case when the matrix A was diagonalizable. This in particular allows us to write down the solution to the initial value problem

$$X' = AX, \quad X(0) = X_0$$

which is simply given by $e^{tA}X_0$. Diagonalizability of A was the important hypothesis in solving the system. The matrix A is diagonalizable in particular when it has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ $(\lambda_i \neq \lambda_j)$.

Now what happens if the eigenvalues of A are not distinct (i.e. some of them are repeated)? Let us say $\lambda_1 = \lambda_2 (= \lambda)$. If we are still able to find two linearly independent eigenvectors ξ_1, ξ_2. Then we still in good shape as we have two linearly independent solutions

$$\begin{aligned} X_1(t) &= e^{\lambda t}\xi_1 \quad \text{and} \\ X_2(t) &= e^{\lambda t}\xi_2. \end{aligned}$$

However we may not be able to find two such eigenvectors! We will always find one by solving $(A - \lambda I)\xi = 0$ and hence have one solution $X_1(t) = e^{\lambda t}\xi$. But we might be stumped while looking for a second if there is a unique solution to $(A - \lambda I)\xi = 0$. In this case we have to look for a **generalized eigenvector** η. This is a vector which solves the equation

$$\boxed{(A - \lambda I)\eta = \xi.}$$

Notice that the right hand side is now not zero but instead equals the first eigenvector ξ. This generalized eigenvector also solves the generalized eigenvalue equation

$$\boxed{(A - \lambda I)^2 \eta = 0.}$$

The second linearly independent solution to our equation is now

$$\boxed{X_2(t) = te^{\lambda t}\xi + e^{\lambda t}\eta.}$$

Let us now see this in an exemplary problem.

Example 1. Find the general solution to the equation

$$X' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} X.$$

Date: 10/30.

Solution. First to find the eigenvalues the characteristic equation is

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Which gives $\lambda = 2$ as a repeated eigenvalue with multiplicity 2. To find the eigenvector $\xi = \begin{bmatrix} a \\ b \end{bmatrix}$ we need to solve

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \xi = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

This gives $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an eigenvector (hence $X_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$as a solution). Also notice that this is the one and only linearly independent eigenvector. Hence for the second solution, we next look for a generalized eigenvector $\eta = \begin{bmatrix} c \\ d \end{bmatrix}$ which solves

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \eta = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This gives $c+d = -1$ and hence $\eta = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ as a generalized eigenvector. The second solution to the equation is now given by

$$\begin{aligned} X_2(t) &= te^{\lambda t}\xi + e^{\lambda t}\eta \\ &= te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned}$$

The general solution is now a linear combination of these

$$\begin{aligned} X(t) &= c_1 X_1(t) + c_2 X_2(t) \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right). \end{aligned}$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 26

1. First Order Sysytems: Repeated Eigenvalues

In the last lecture we figured how to compute matrix exponentials e^{tA} in the case when the matrix A was diagonalizable. This in particular allows us to write down the solution to the initial value problem

$$X' = AX, \quad X(0) = X_0$$

which is simply given by $e^{tA}X_0$. Diagonalizability of A was the important hypothesis in solving the system. The matrix A is diagonalizable in particular when it has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ $(\lambda_i \neq \lambda_j)$.

Now what happens if the eigenvalues of A are not distinct (i.e. some of them are repeated)? Let us say $\lambda_1 = \lambda_2 (= \lambda)$. If we are still able to find two linearly independent eigenvectors ξ_1, ξ_2. Then we still in good shape as we have two linearly independent solutions

$$\begin{aligned} X_1(t) &= e^{\lambda t}\xi_1 \quad \text{and} \\ X_2(t) &= e^{\lambda t}\xi_2. \end{aligned}$$

However we may not be able to find two such eigenvectors! We will always find one by solving $(A - \lambda I)\xi = 0$ and hence have one solution $X_1(t) = e^{\lambda t}\xi$. But we might be stumped while looking for a second if there is a unique solution to $(A - \lambda I)\xi = 0$. In this case we have to look for a **generalized eigenvector** η. This is a vector which solves the equation

$$\boxed{(A - \lambda I)\eta = \xi.}$$

Notice that the right hand side is now not zero but instead equals the first eigenvector ξ. This generalized eigenvector also solves the generalized eigenvalue equation

$$\boxed{(A - \lambda I)^2\eta = 0.}$$

The second linearly independent solution to our equation is now

$$\boxed{X_2(t) = te^{\lambda t}\xi + e^{\lambda t}\eta.}$$

Let us now see this in an exemplary problem.

Example 1. Find the general solution to the equation

$$X' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} X.$$

Date: 11/1.

Solution. First to find the eigenvalues the characteristic equation is

$$\det\begin{bmatrix}1-\lambda & -1\\ 1 & 3-\lambda\end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0.$$

Which gives $\lambda = 2$ as a repeated eigenvalue with multiplicity 2. To find the eigenvector $\xi = \begin{bmatrix}a\\ b\end{bmatrix}$ we need to solve

$$\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\xi = \begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 0.$$

This gives $\xi = \begin{bmatrix}1\\ -1\end{bmatrix}$ as an eigenvector (hence $X_1(t) = \begin{bmatrix}1\\ -1\end{bmatrix}e^{2t}$as a solution). Also notice that this is the one and only linearly independent eigenvector. Hence for the second solution, we next look for a generalized eigenvector $\eta = \begin{bmatrix}c\\ d\end{bmatrix}$ which solves

$$\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\eta = \begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}.$$

This gives $c+d = -1$ and hence $\eta = \begin{bmatrix}0\\ -1\end{bmatrix}$ as a generalized eigenvector. The second solution to the equation is now given by

$$\begin{aligned}X_2(t) &= te^{\lambda t}\xi + e^{\lambda t}\eta\\ &= te^{2t}\begin{bmatrix}1\\ -1\end{bmatrix} + e^{2t}\begin{bmatrix}0\\ -1\end{bmatrix}.\end{aligned}$$

The general solution is now a linear combination of these

$$\begin{aligned}X(t) &= c_1X_1(t) + c_2X_2(t)\\ &= c_1\begin{bmatrix}1\\ -1\end{bmatrix}e^{2t} + c_2\left(te^{2t}\begin{bmatrix}1\\ -1\end{bmatrix} + e^{2t}\begin{bmatrix}0\\ -1\end{bmatrix}\right).\end{aligned}$$

Let us now do an example of a 3×3 system with repeated eigenvalues.

Example 2. Find the general solution to the system

$$X' = \begin{bmatrix}2 & -1 & 2\\ 0 & 0 & 2\\ 0 & -1 & 3\end{bmatrix}X.$$

Solution. First let us find the eigenvalues. The characteristic equation is

$$\det\begin{bmatrix}2-\lambda & -1 & 2\\ 0 & -\lambda & 2\\ 0 & -1 & 3-\lambda\end{bmatrix} = -\lambda^3 + 5\lambda^2 - 6\lambda + 2 = -(\lambda-1)(\lambda-2)^2.$$

Hence the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ (repeated with multiplicity 2). To find the eigenvector for $\lambda_1 = 1$ we need to solve

$$\begin{bmatrix}2-\lambda_1 & -1 & 2\\ 0 & -\lambda_1 & 2\\ 0 & -1 & 3-\lambda_1\end{bmatrix}\xi_1 = \begin{bmatrix}1 & -1 & 2\\ 0 & -1 & 2\\ 0 & -1 & 2\end{bmatrix}\xi_1 = 0.$$

An eigenvector is clearly foudn to be $\xi_1 = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}$.

To find an eigenvector for $\lambda_2 = 2$ we need to solve

$$\begin{bmatrix} 2-\lambda_2 & -1 & 2 \\ 0 & -\lambda_2 & 2 \\ 0 & -1 & 3-\lambda_2 \end{bmatrix} \xi_2 = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xi_2 = 0.$$

The only eigenvector possible here is (a scalar multiple of) $\xi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So we have only two eigenvectors for a 3×3 matrix. Yikes! Fortunately we just learnt the secret to success in this case. We need to look for a generalized eigenvector which solves the generalized eigenvalue equation

$$\begin{aligned} \begin{bmatrix} 2-\lambda_2 & -1 & 2 \\ 0 & -\lambda_2 & 2 \\ 0 & -1 & 3-\lambda_2 \end{bmatrix} \eta_2 &= \xi_2 \quad \text{or} \\ \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \eta_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

On row reduction (R1-R3, R2-2R3) this gives

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \eta_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $\eta_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a generalized eigenvector. There are more but one is enough. Hence we now have our general solution as

$$X(t) = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \left(te^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

In, both examples above the relevant matrices were not diagonalizable. For example in the first example we had the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

There isn't any transformation matrix T such that $T^{-1}AT$ is diagonal (such a matrix is usual comprised of a full set of eigenvectors). However what happens when we take T to have the eigenvector ξ and the generalized eigenvector η as column vectors? Hence in Example 1

$$T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

We may now compute

$$T^{-1}AT = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

So the result is not quite diagonal. But almost! If only it weren't for that pesky1 in the top right corner.

Similarly in the second example

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$$

and the matrix consisting of generalized eigenvectors is

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We then compute

$$T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

So again the result is not quite diagonal but has a pesky 1 ever so close to the diagonal.

In general the matrix of generalized eigenvectors transforms the matrix into one comprised of **Jordan blocks**. A Jordan block is a matrix of the form

$$J_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

There might be more than one Jordan block asscociated with a repeated eigenvalue. The final result is that for any matrix A there is a transformation matrix T such that

$$T^{-1}AT = \begin{bmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_k} \end{bmatrix}$$

is comprised of Jordan blocks and is in **Jordan canonical form**. Proving this in full and glorious generality is all intense linear algebra. Anyone up for the challenge?

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 27

1. Inhomogeneous First Order Sysytems

So far we have been studying homogeneous first order systems $X' = AX$. Now we shall move on to inhomogeneous ones of the kind

$$X' = AX + G(t),$$

where $G(t)$ is the forcing function. Again we shall assume the matrix A is independent of time.

1.1. **Diagonalizable case.**

Again the easy case is when the matrix A is diagonalizable case. Hence we have

$$\begin{aligned} T^{-1}AT &= D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{or} \\ TDT^{-1} &= A. \end{aligned}$$

Let us define

$$Y := T^{-1}X, \quad H(t) := T^{-1}G(t)$$

We then have that Y satisfies the equation

$$Y' = DY + H(t).$$

If $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, $H(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix}$. This gives the equation

$$y_j'(t) = \lambda_j y_j(t) + h_j(t) \quad \text{for } 1 \leq j \leq n.$$

This can be solved using the method of section 2.1 with the help of our long lost friend the **integrating factor** ($e^{\lambda_j t}$ in this case). A particular solution is

$$\boxed{y_j^p(t) = e^{\lambda_j t} \int e^{-\lambda_j s} h_j(s)\, ds}$$

and the general solution is

$$\boxed{y_j(t) = e^{\lambda_j t} \int e^{-\lambda_j s} h_j(s)\, ds + c_j e^{\lambda_j t}.}$$

Date: 11/4.

1.2. **Method of undetermined coefficients.** This method works in the case when the forcing function has the special form

$$G(t) = t^k e^{\alpha t} w$$

or any linear combination of these. We also allow α to be real or complex (and in the complex case we see cosines and sines in $G(t)$). The form of the particular solution is now guessed to be

$$\boxed{y_p(t) = \left(t^{k+s} v_{k+s} + \ldots + t v_1 + v_0\right) e^{\alpha t}}$$

where $v_0, \ldots, v_k$ are constant vectors and

$$s = \text{multiplicity of } \alpha \text{as an eigenvalue of } A.$$

Plugging this into our equation and comparing coefficients will give a set of equations for $v_{k+s}, \ldots, v_0$. Let us see and example of using both these methods in an example.

Example 1. Find a particular solution to the equation

$$X' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

Solution. The eigenvalues of this matrix were found in lecture 20 $\lambda_1 = 2, \lambda_2 = 3$ with eigenvectors

$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and transformation matrix is given by

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$H(t) = T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} 2e^{-t} - 3t \\ 3t \end{bmatrix}.$$

Hence we get

$$Y_p(t) = \begin{bmatrix} -\frac{2}{3}e^{-t} + \frac{3}{2}t + \frac{3}{4} \\ -\left(t + \frac{1}{3}\right) \end{bmatrix}.$$

Hence

$$X_p(t) = TX = \begin{bmatrix} -\frac{2}{3}e^{-t} + \frac{1}{2}t + \frac{5}{12} \\ -\left(t + \frac{1}{3}\right) \end{bmatrix}$$

This is the solution via the diagonalization formula.

Now let us use vundetermined coefficients. We find particular solutions to

$$X_1' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X_1 + \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$$
$$X_2' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X_1 + \begin{bmatrix} 0 \\ 3t \end{bmatrix}$$

separately and add them $X_p(t) = X_1(t) + X_2(t)$. From the method undeteremined coefficients formula we have

$$X_1(t) = e^{-t} v.$$

On substitution this gives the equation

$$\begin{aligned} -e^{-t}v &= e^{-t}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} v + e^{-t}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{hence} \\ v &= \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} \\ X_1 \quad (t) &= \begin{bmatrix} -\frac{2}{3} \\ 0 \end{bmatrix} e^{-t}. \end{aligned}$$

Similarly

$$X_2\left(t\right) = v_1 t + v_0$$

gives

$$\begin{aligned} v_1 &= -A^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} \\ v_0 &= A^{-1}v_1 = \begin{bmatrix} \frac{5}{12} \\ -\frac{1}{3} \end{bmatrix}. \end{aligned}$$

Thus

$$X_p\left(t\right) = \begin{bmatrix} -\frac{2}{3}e^{-t} + \frac{1}{2}t + \frac{5}{12} \\ -\left(t + \frac{1}{3}\right) \end{bmatrix}$$

is the same as before.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 28

1. Inhomogeneous First Order Sysytems

In the last lecture we began studying inhomogeneous first order systems of the kind

$$X' = A_{n\times n}X + G(t), \quad X(0) = X_0 \tag{1.1}$$

where $G(t)$ is the forcing function. We saw the method of undetermined coefficients for solving the above equation when $G(t)$ was of a special kind. Now we see two further methods the variation of parameters and the diagonalization method.

1.1. **Variation of parameters.** In this method we assume that we know the fundamental matrix $\Phi(t)$ to the homogeneous equation. Recall that this is the $n \times n$ matrix which solves $\Phi'(t) = A\Phi(t)$, $\Phi(0) = I$. The solution to the inhomogeneous equation is then given by **Duhamel's formula**

$$\boxed{X(t) = \Phi(t)X_0 + \Phi(t)\int_0^t \Phi(s)^{-1}G(s)\,ds.}$$

Voila!

Assuming A to be independent of time, since we have $\Phi(t) = e^{tA}$, the above formula can be written in terms of the matrix exponential

$$\boxed{X(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A}G(s)\,ds.}$$

1.2. **Diagonalizable case.** Another method of solving the inhomogeneous equation (1.1) is by diagonalization. Hence we assume that the matrix A is diagonalizable (otherwise we need to bring it into and use the Jordan canonical form) i.e.

$$\begin{aligned} T^{-1}AT &= D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{or} \\ TDT^{-1} &= A. \end{aligned}$$

Let us define

$$Y := T^{-1}X, \quad H(t) := T^{-1}G(t)$$

We then have that Y satisfies the equation

$$Y' = DY + H(t).$$

Date: 11/6.

If $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, $H(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix}$. This gives the equation

$$y_j'(t) = \lambda_j y_j(t) + h_j(t) \quad \text{for } 1 \leq j \leq n.$$

This can be solved using the method of section 2.1 with the help of our long lost friend the **integrating factor** ($e^{\lambda_j t}$ in this case). A particular solution is

$$\boxed{y_j^p(t) = e^{\lambda_j t} \int e^{-\lambda_j s} h_j(s)\, ds}$$

and the general solution is

$$\boxed{y_j(t) = e^{\lambda_j t} \int e^{-\lambda_j s} h_j(s)\, ds + c_j e^{\lambda_j t}.}$$

This gives $Y_p(t)$ and hence $X_p(t) = TY_p(t)$.

Let us now see examples of using these.

Example 1. Find a particular solution to the equation

$$X' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} X + \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}.$$

Solution. First lets do this via diagonalization. The eigenvalues of this matrix were found in lecture 20 $\lambda_1 = 2, \lambda_2 = 3$ with eigenvectors

$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and transformation matrix is given by

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$H(t) = T^{-1} \begin{bmatrix} 2t \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}.$$

Hence we get

$$Y_p(t) = \begin{bmatrix} e^{2t} \int_0^t 2e^{-s} ds \\ 0 \end{bmatrix} = 2 \begin{bmatrix} e^{2t} - e^t \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Hence

$$X_p(t) = TX = e^{2t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

and the general solution is

$$X(t) = e^t \begin{bmatrix} -2 \\ 0 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now let us use variation of parameters. First compute the matrix exponential $e^{tA}$for $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. For this particular matrix this was done in lecture 24 giving

$$e^{tA} = \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix}.$$

Hence the general solution to our problem is

$$\begin{aligned} X(t) &= \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{2(t-s)} & e^{3(t-s)} - e^{2(t-s)} \\ 0 & e^{3(t-s)} \end{bmatrix} \begin{bmatrix} 2e^s \\ 0 \end{bmatrix} ds \\ &= (c_1 - c_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + e^{2t} \int_0^t \begin{bmatrix} 2e^{-s} \\ 0 \end{bmatrix} ds \\ &= (c_1 - c_2 + 2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + e^t \begin{bmatrix} -2 \\ 0 \end{bmatrix}. \end{aligned}$$

Which is the same as before but with different constants.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 29

1. Nonlinear Differential Equations: The phase place for linear systems

In today's lecture we begin with Chapter 9. This chapter is concerned with the study of nonlinear differential equations. Recall that a linear equation is one for which the linear combination of any two solutions is a solution. A prototype of a nonlinear system of equations is

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned}$$

where F, G are nonlinear functions of x, y. In general it is impossible to solve for the solution to such a system. So there is nothing to learn?? (!!)

However much can be said about the qualitative behaviour of the solution to the equation in these cases. Before we get to qualitative behaviour of the solution to nonlinear equations, let us take a step back and study the qualitative behaviour of linear equations. Linear equations we have been studying so far have been of the type

$$X' = AX \tag{1.1}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a constant matrix and $X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. An **equilibrium solution** or a **critical point** is a point (x_0, y_0) in the plane which is also a solution to the differential equation. In the case of equation (1.1) if $\det A \neq 0$, the only equilibrium point is the origin.

A critical point is called **stable** if any solution which is close to (x_0, y_0) at time $t = 0$ stays close to it for all time $t > 0$. Otherwise it is said to be **unstable.**

We call the xy-plane the **phase plane** and a plot of the trajectories in this plane is called the **phase portrait.** The qualitative behaviour of the phase portrait depends a lot on the eigenvalues λ_1, λ_2 of the matrix A. The cases to consider are the following

1.1. **Real unequal eigenvalue of the same sign.**

In this case the solution is of the form

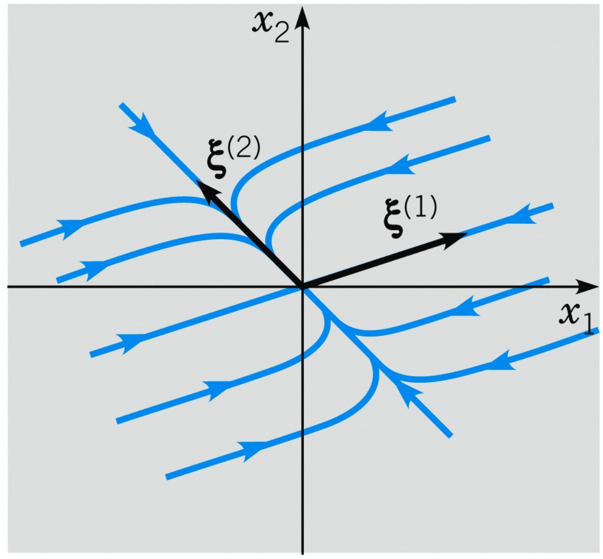
$$X(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2.$$

Assuming $\lambda_1 < \lambda_2 < 0$ the phase portrait is given in Figure 1.1.

Rewriting the solution as

$$X(t) = e^{\lambda_2 t} \left[c_1 e^{(\lambda_1 - \lambda_2)t} \xi_1 + c_2 \xi_2 \right]$$

Date: 11/8.



(a)

Figure 9.1.1a

FIGURE 1.1. $\lambda_1 < \lambda_2 < 0$

it is clear that the solution approches the origin along the eigenvector ξ_2 as $t \to \infty$. This type of critical point is called a **node** or **nodal sink** (Stable).

In the case where $\lambda_1 > \lambda_2 > 0$ the phase portrait is similar but now a **nodal source** (Unstable).

1.2. **Real eigenvalues of opposite sign.** The general solution is again of the form

$$X(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2.$$

The phase portrait is given in Figure 1.2.

Assuming $\lambda_1 > 0 > \lambda_2$ any solution is asymptotic to ξ_1, ξ_2 as $t \to \infty, -\infty$ respectively. This type of critical point is called a **saddle** (Unstable).

1.3. **Equal eigenvalues: two independent eigenvectors.** Let us say the common eigenvalue is λ. The general solution is again of the form

$$X(t) = c_1 e^{\lambda t}\xi_1 + c_2 e^{\lambda t}\xi_2.$$

The phase portrait is given in Figure 1.3. This type of critical point is called a **proper node** or **star point.** It is stable or unstable for $\lambda < 0$ or $\lambda > 0$ respectively.

1.4. **Equal eigenvalues: one independent eigenvector.** Let us say the common eigenvalue is λ. The general solution is again of the form

$$X(t) = c_1 e^{\lambda t}\xi + c_2 e^{\lambda t}(\xi t + \eta).$$

The phase portrait is given in Figure 1.4.

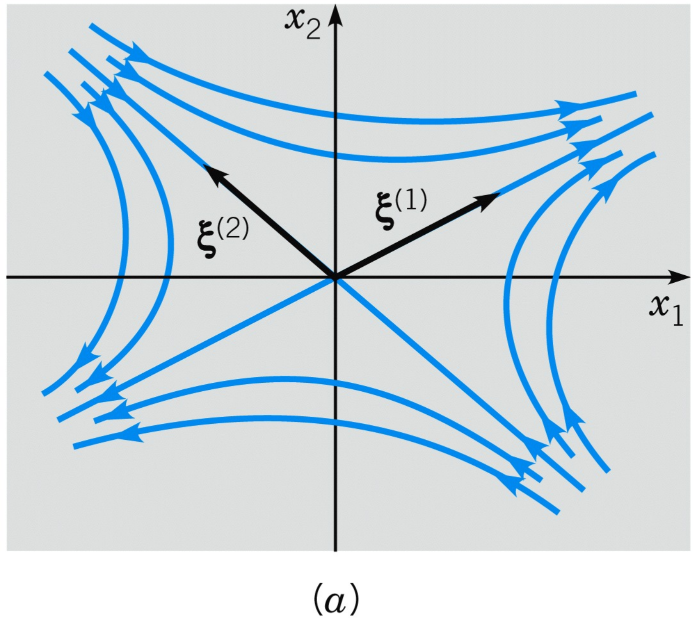


Figure 9.1.2a

FIGURE 1.2. $\lambda_1 > 0 > \lambda_2$

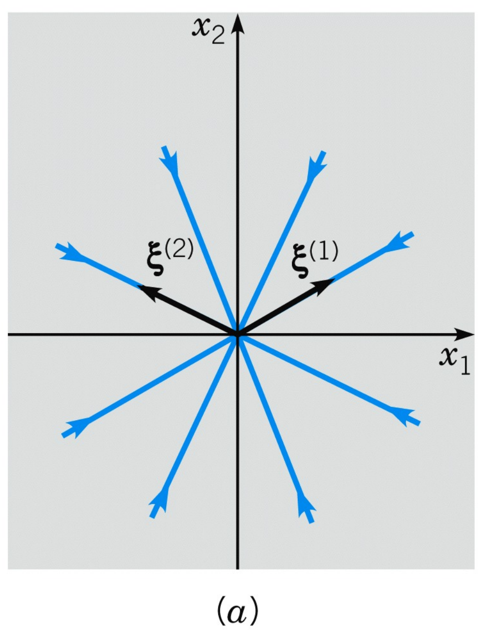


Figure 9.1.3a

FIGURE 1.3. $\lambda_1 = \lambda_2 < 0$ with two eigenvectors

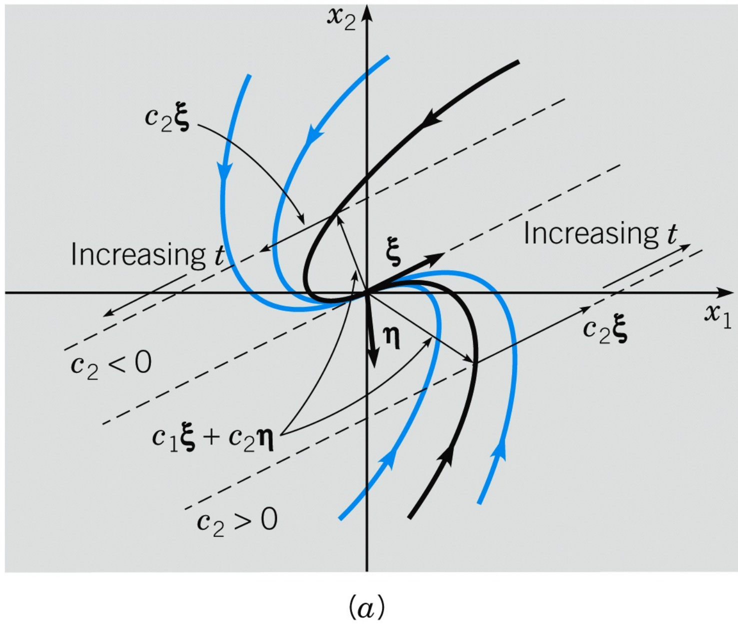


Figure 9.1.4a

FIGURE 1.4. $\lambda_1 = \lambda_2 < 0$ with one eigenvector

As $t \to -\infty$ the solution is asymptotic to the eigenvector ξ. This type of critical point is called an **improper node** or **degenerate node.** It is stable or unstable for $\lambda < 0$ or $\lambda > 0$ respectively.

1.5. **Complex eigenvalues with nonzero real part.** Let us say the eigenvalues are $\lambda \pm i\mu$. If $\lambda > 0$, the solution approaches $\infty, 0$ as $t \to \infty, -\infty$. The phase portrait is also as given in Figure 1.5.

This type of critical point is called a **spiral sink** (Stable)**.** If the real part $\lambda > 0$, the the solution approaches $\infty, 0$ as $t \to -\infty, \infty$ and we have a **spiral source** (Unstable)**.**

1.6. **Purely Imaginary Eigenvalues.** In this case the eigenvalues are $\pm i\mu$. In this cases the trajectories will form ellipses around the origin. The phase portrait is as shown in Figure 1.6.

This type of critical point is known as a **center** (Stable).

2. Summary of types

Finally we summarize all possibilities in the following table.

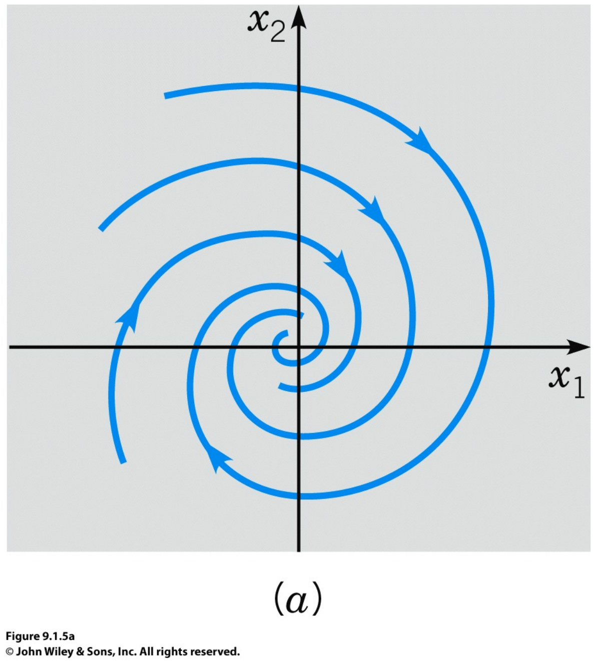


FIGURE 1.5. $\lambda \pm i\mu$ with $\lambda > 0$

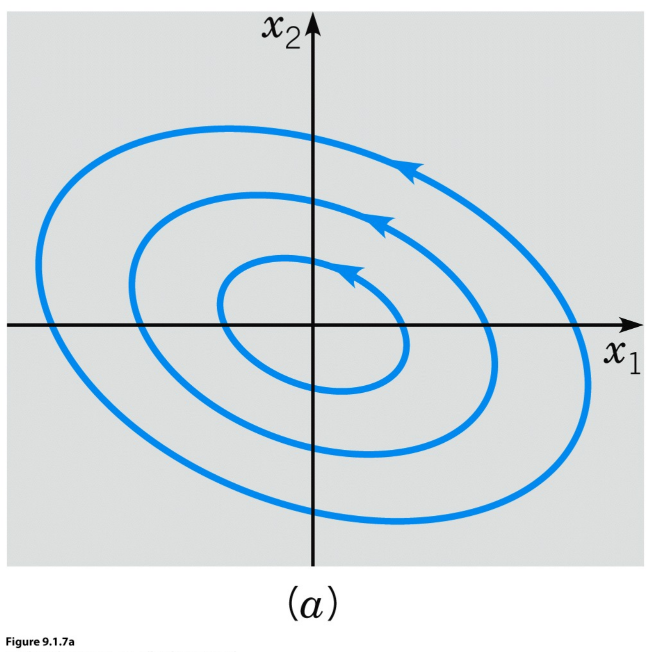


FIGURE 1.6. purely imaginary eigenvalues $\pm i\mu$

Eigenvalues	Type of Critical point	Stability
$\lambda_1 > \lambda_2 > 0$	Nodal source	Unstable
$\lambda_1 < \lambda_2 < 0$	Nodal sink	Stable
$\lambda_1 > 0 > \lambda_2$	Saddle	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper node	Stable
$\lambda \pm i\mu,\ \mu > 0$	Spiral source	Unstable
$\lambda \pm i\mu,\ \mu < 0$	Spiral sink	Stable
$\pm i\mu$	Center	Stable

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 30

1. Autonomous systems and stability

In this lecture we continue our study of nonlinear differential equations. We shall mostly be concerned with equations that are of the form

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \tag{1.1}$$

where the right hand side consists of functions F, G that are independent of time. Such a system is called an **autonomous**. A critical point of this system (x_0, y_0) is said to be **asymptotically stable** if any trajectory or solution curve $(x(t), y(t))$ which originates close to (x_0, y_0) eventually converges to it $(x(t), y(t)) \to (x_0, y_0)$. The **basin of attraction** B is the region of the plane near an asymptotically stable critical point such that for any trajectory $(x(t), y(t))$ originating in B, we have convergence $(x(t), y(t)) \to (x_0, y_0)$. A trajectory that bounds a basin of attraction is called a **separatrix**.

An autonomous equation can sometimes be solved using methods (or perhaps reminisces) of Chapter 2. This can be done for instance by rewriting (1.1) in the form

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \quad \text{or} \quad F(x,y)\,dy = G(x,y)\,dx.$$

The last equation can be solved when it is exact (i.e. $F_x = -G_y$). In this case we can find a function $H(x,y)$ which satisfies

$$\begin{aligned} H_y &= F \\ H_x &= -G \end{aligned}$$

and the equation is hence solved implicitly by

$$H(x,y) = c.$$

Below is an example.

Example 1. Find the equation of the form $H(x,y) = c$ satisfied by the trajectories of the autonomous system

$$\begin{aligned} x' &= y \\ y' &= -x + \frac{x^3}{6}. \end{aligned}$$

Date: 11/11.

Solution. Since $(y)_x = 0 = \left(-x + \frac{x^3}{6}\right)_y$ this system is exact. Look for a function H such that $H_y = y$. Hence $H = \frac{y^2}{2} + f(x)$ and then

$$H_x = f'(x) = -x + \frac{x^3}{6}$$

and $f(x) = -\frac{x^2}{2} + \frac{x^4}{24} + c$. Hence the trajectories are along the level curves

$$\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{24} = c.$$

.

Example 2. Find all critical points of the system

$$\begin{aligned} x' &= 2x - x^2 - xy \\ y' &= 3y - 2y^2 - 3xy \end{aligned}$$

and determine their stability and type from the phase plot.

Solution. For the critical points we need to solve

$$\begin{aligned} 2x - x^2 - xy = x(2 - x - y) &= 0 \\ 3y - 2y^2 - 3xy = y(3 - 2y - 3x) &= 0. \end{aligned}$$

The first equatioon gives $x = 0$ or $x + y = 2$. While the second equation gives $y = 0$ or $2y + 3x = 3$. This gives four possible system of equations and four critical points. One of them (corresponding to $x = 0$ and $y = 0$) is the criical point $(0,0)$(unstable, nodal source). Another corresponding to the the possibility ($x + y = 2$ and $2y + 3x = 3$) is the critical point $(-1,3)$(aymptotically stable, nodal sink). The other two critical points are $\left(0, \frac{3}{2}\right)$ (unstable, saddle) and $(2,0)$ (asymptotically stable, nodal sink). Here the stability is determined from the phase plots.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 30

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MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 32

1. Locally Linear systems

In this lecture we continue our study of nonlinear differential equations. To remind ourselves these are equations that are of the form

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \tag{1.1}$$

where the right hand side consists of nonlinear functions F, G. We now focus attention on **locally linear** systems. Assume that we are at a critical point of the system Let us take this critical point to be say $(0,0)$. The system is said to be **locally linear** near the critical point $(0,0)$ if it can be written in the form

$$\begin{aligned} x' &= ax + by + f(x,y) \\ y' &= cx + dy + g(x,y) \end{aligned}$$

such that

$$\frac{f(x,y)}{r} \to 0 \quad \text{and} \quad \frac{g(x,y)}{r} \to 0 \text{ as } r = \sqrt{x^2+y^2} \to 0.$$

In this case we say that

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{1.2}$$

is the corresponding linear system near $(0,0)$.

In order to classify the type and stability of a locally linear system we find the eigenvalues of the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for the corresponding linear system. The following table now gives the type and stablity of the critical point of this locally linear system

Eigenvalues	Type of Critical point	Stability
$\lambda_1 > \lambda_2 > 0$	Nodal source	Unstable
$\lambda_1 < \lambda_2 < 0$	Nodal sink	Asmptotically Stable
$\lambda_1 > 0 > \lambda_2$	Saddle	Unstable
$\lambda_1 = \lambda_2 > 0$	**Node or Spiral Source**	Unstable
$\lambda_1 = \lambda_2 < 0$	**Node or Spiral Sink**	Asmptotically Stable
$\lambda \pm i\mu,\ \mu > 0$	Spiral source	Unstable
$\lambda \pm i\mu,\ \mu < 0$	Spiral sink	Asmptotically Stable
$\pm i\mu$	Center	**Indeterminate**

Date: 11/15.

The only differences from the classification of critical points of linear systems are marked in **bold.** These occur in the cases where we have purely imaginary eigenvalues (center) or we have equal eigenvalues (proper or improper node).

Example 1. Show that the following system is locally linear and find the type and stability of $(0,0)$ as a critical point

$$\begin{aligned} x' &= x + y^2 \\ y' &= 2y - xy. \end{aligned}$$

Solution. To show this system is locally linear we check

$$\begin{aligned} \frac{y^2}{r} &= \frac{r^2 (\sin\theta)^2}{r} = r (\sin\theta)^2 \to 0 \\ \frac{-xy}{r} &= \frac{-r^2 (\cos\theta)(\sin\theta)}{r} = -r (\cos\theta)(\sin\theta) \to 0 \end{aligned}$$

as $r \to 0$. The correspndng linear system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This has eigenvalues $1, 2$ and is hence an unstable nodal source.

.

Example 2. Find the critical points of the nonlinear system

$$\begin{aligned} x' &= 1 + y \\ y' &= x^2 - y^2. \end{aligned}$$

Then find the linear system near each critical point and discuss its type and stability.

Solution. To find the critical points we set $1 + y = 0$, $x^2 - y^2 = 0$. This gives $y = -1$ and $x = \pm 1$. Hence the critical points are $(1, -1)$ and $(-1, -1)$. To find the linear system near $(1, -1)$ we make the change of variables $u = x - 1, v = y + 1$. we then get the system

$$\begin{aligned} u' &= v \\ v' &= 2u + 2v + \left(u^2 - v^2\right). \end{aligned}$$

The linear system near this critical point is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are $\lambda = 1 \pm \sqrt{3}$. Hence we have an unstable saddle.

To find the linear system near $(-1, -1)$ we make the change of variables $u = x + 1, v = y + 1$. we then get the system

$$\begin{aligned} u' &= v \\ v' &= 2u + 2v + \left(u^2 - v^2\right). \end{aligned}$$

The linear system near this critical point is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are $\lambda = 1 \pm i$. Hence we have an unstable spiral source.

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 32

1. Prey Predator Model

In this lecture, we shall discuss an important physical/biological example that is modelled on a non-linear ordinary differential equation. Let $x(t), y(t)$ denote the population of prey and predators respectively at any given time. To be specific, think of rabbits and foxes in a forest. Or think of gazelles and cheetah in the african savannah. Or think of redear and bass in a lake. Or think of aphids and ladybugs on tomato plants.

In describing the model governing the evolution of the two populations $x(t), y(t)$ we assume

(1) $\frac{dx}{dt} \propto x$, this is because the more prey we have the more they will reproduce.
(2) $\frac{dy}{dt} \propto -y$, this because the more predators we have the more they will compete amongst each other for food.
(3) $\frac{dx}{dt} \propto -xy$ and $\frac{dy}{dt} \propto xy$, this because xy is the number of encounters between prey and predators. Each encounter is harmful to the population of the prey and is beneficial to the population of the predators.

Following these assumptions we are led to the following nonlinear system describing the evolution of the two populations

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= -by + \beta xy. \end{aligned} \tag{1.1}$$

Here a, b, α, β are the (positive) constants of proportionality given by our assumptions and observed in nature. The critical points of this system are easily found to be $(x, y) = (0, 0)$ and $(x, y) = \left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$. Let us now try to figure out the type and stability of these two critical points.

Let us start with $(x, y) = (0, 0)$. Assuming $f(x, y) = ax - \alpha xy,\ g = -by + \beta xy$ the linearization of the system is given by the **Jacobian matrix**

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} a - \alpha y & -\alpha x \\ \beta y & -b + \beta x \end{bmatrix}.$$

Evaluated at the point $(0, 0)$ this gives the system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The above matrix has one positive and one negative eigenvalue and hence $(0, 0)$ is an unstable saddle.

Date: 11/20.

Now let us consider the critical point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$. Again evaluating the Jacobian at this point gives the linearized system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\alpha b}{\beta} \\ \frac{a\beta}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues of the above matrix are purely imaginary $\lambda = \pm i\sqrt{ab}$. Hence this is a center. Its stability is now a conundrum since the stability of the center in the nonlinear case it indeterminate.

To solve the problem of stablity, let's so back to equation (1.1) and observe that it is solvable by separation! Dividing the two equations in (1.1) we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{-by + \beta xy}{ax - \alpha xy} \\ \left(\frac{a - \alpha y}{y}\right) dy &= \left(\frac{-b + \beta x}{x}\right) dx \\ a \ln y - \alpha y &= -b \ln x + \beta x + C \\ H(x, y) := \alpha y + \beta x - b \ln x - a \ln y &= C. \end{aligned}$$

Next we show that the point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$ is a local minimum. To recall how to do this from multivariable calculus, we have to consider the Hessian matrix of H defined by

$$\begin{aligned} \text{Hessian}(H) &:= \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b}{x^2} & 0 \\ 0 & \frac{a}{y^2} \end{bmatrix}. \end{aligned}$$

Evaluated at the point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$ this gives the matrix

$$\begin{bmatrix} \frac{\beta^2}{b} & 0 \\ 0 & \frac{\alpha^2}{a} \end{bmatrix}$$

which is positive definite (has both positive eigenvalues). This tells us that the point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$ is a local minimum for $H(x, y)$. Now the trajectories of our system 1.1 (being level curves of H) have to be closed curves around this local minimum. Hence the point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$ is stable. This is clearly seen in the plots below in Figures 1.1, 1.2 and 1.3. We hence see cyclic changes in the population of prey and predators. On your homework you will be asked to draw and analyze such plots for various values of a, b, α, β.

The above method works more generally for exact systems as stated below.

Theorem 1. *Consider the nonlinear system*

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y). \end{aligned}$$

Assume that it is exact ($f_x = -g_y$), so that there exists an energy function $H(x, y)$ with $H_y = f, H_x = -g$. Then any critical point (x_0, y_0) which is also a local minimum (or local maximum) of H is stable.

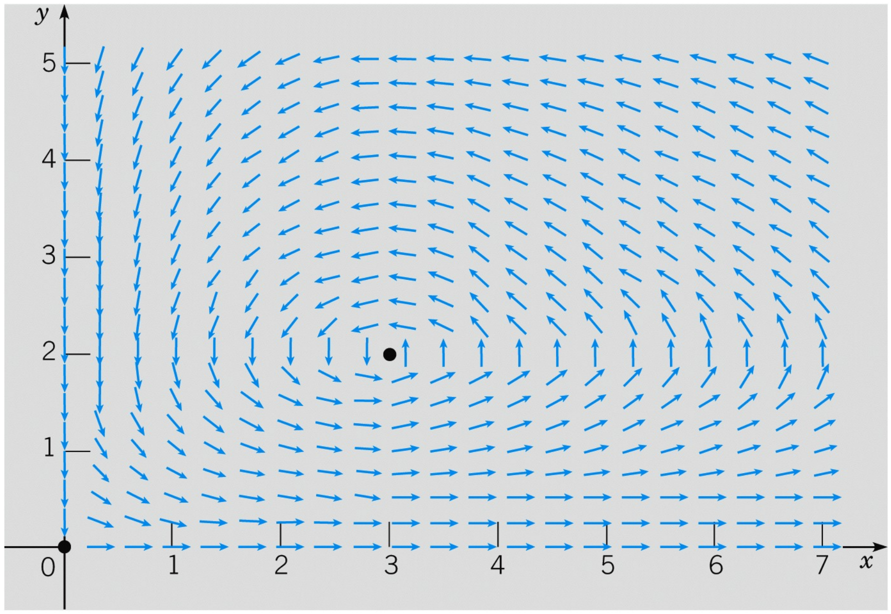

Figure 9.5.1

FIGURE 1.1. Direction Field

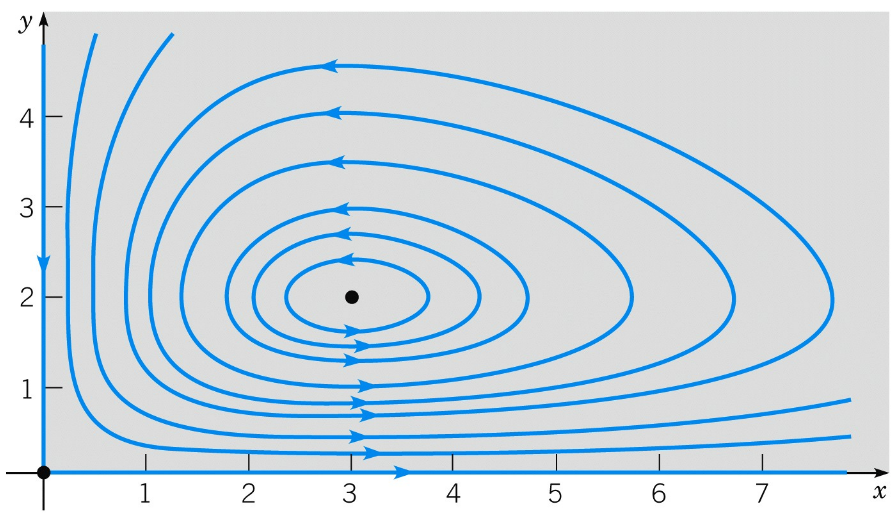

Figure 9.5.2

FIGURE 1.2. Solution Curves

What then happens when we have a critical point (center) of a system that is not exact. Another method is the use of **Liapunov functions** (generalizations of energy functions) from section 9.6. A beautiful method which alas we plan to skip! Another challenging occupation for the deligent student.

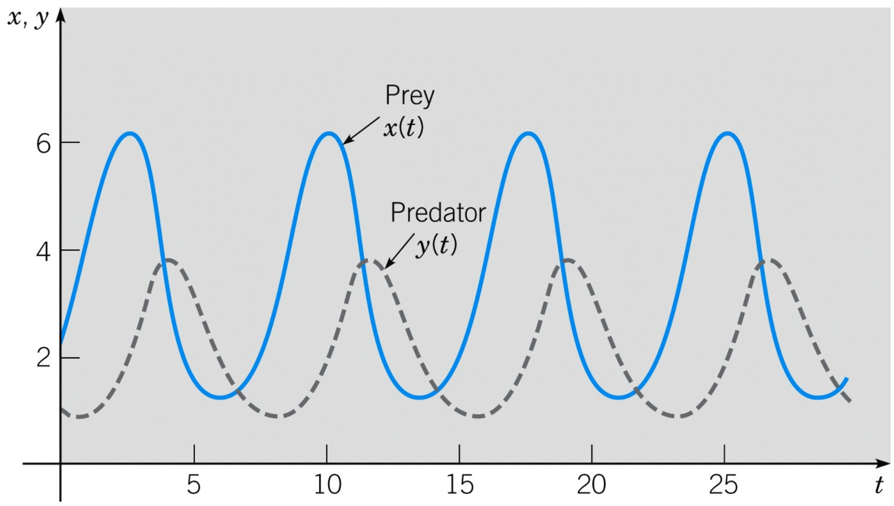

Figure 9.5.3

FIGURE 1.3. Some plots of $x(t)$ vs $y(t)$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 34

1. Prey Predator Model

In this lecture, we shall discuss an important physical/biological example that is modelled on a non-linear ordinary differential equation. Let $x(t), y(t)$ denote the population of prey and predators respectively at any given time. To be specific, think of rabbits and foxes in a forest. Or think of gazelles and cheetah in the african savannah. Or think of redear and bass in a lake. Or think of aphids and ladybugs on tomato plants.

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Date: 11/22.

Now let us consider the critical point $\left(\frac{b}{\beta}, \frac{a}{\alpha}\right)$. Again evaluating the Jacobian at this point gives the linearized system

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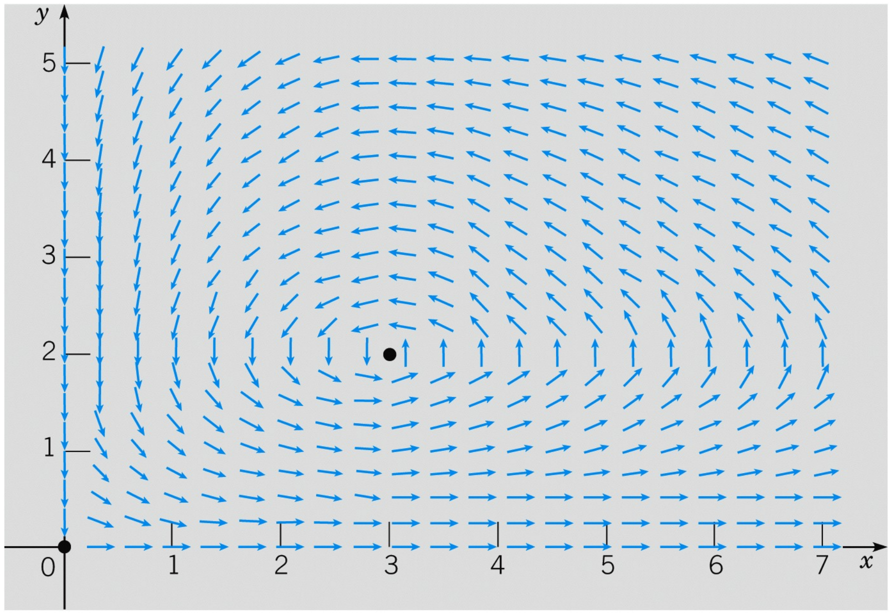

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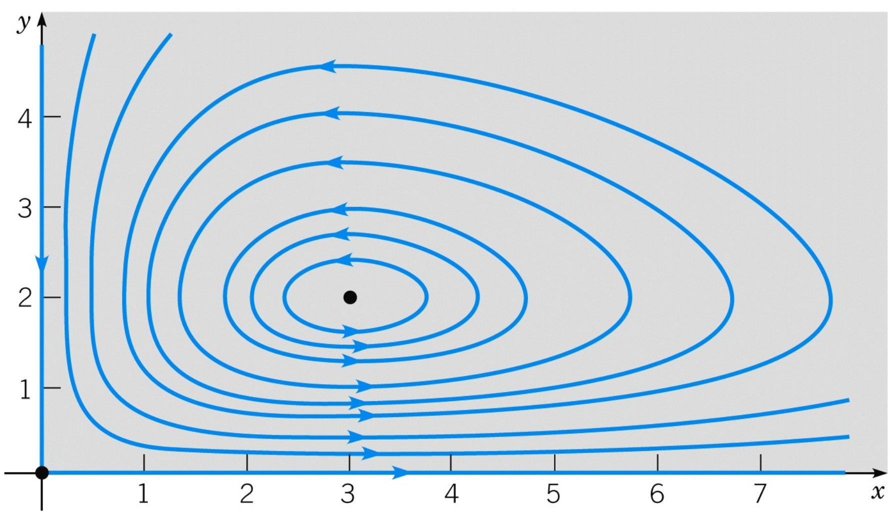

Figure 9.5.2

FIGURE 1.2. Solution Curves

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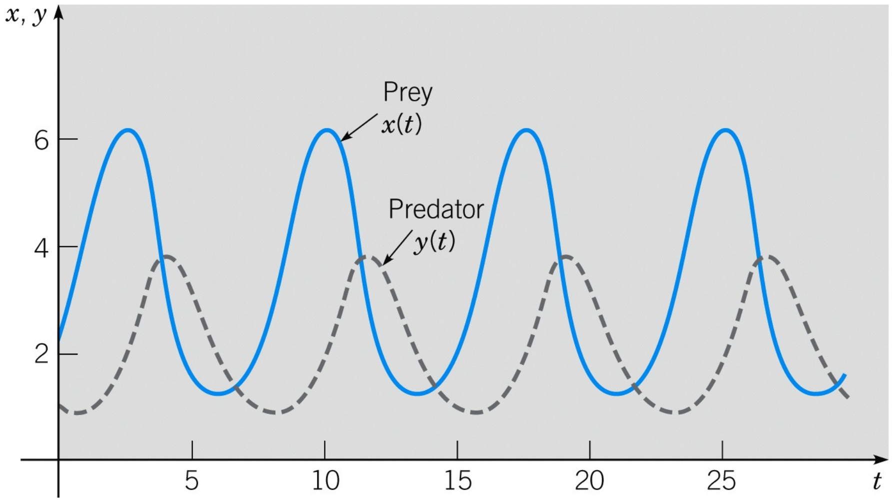

FIGURE 1.3. Some plots of $x(t)$ vs $y(t)$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 34

1. Two point boundary value problems

Today we shall begin with the new chapter 10. This chapter is really about partial differential equations or PDE's (differential equations for functions of more than one variable). However, before we delve into PDE's we shall find useful the study of so called **boundary value problems** for Ordinary differential equations ODE's. To describe what one means by a boundary value problem, recall that thus far we have studied initial value problems for ODE's of the sort

$$y'' + ay' + by = 0, \quad y(0) = y_0, y'(0) = y_1.$$

Here two initial conditions for $y(t)$ were prescribed at the same point $t = 0$. In contrast, a boundary value problem has two conditions prescribed at two different points in time

$$y'' + ay' + by = 0, \quad y(\alpha) = y_0, y'(\beta) = y_1.$$

Thus above the boundary conditions are prescribed at $t = \alpha, t = \beta$ (the boundary points of the interval $[\alpha, \beta]$). Let us now see an example.

Exercise 1. Solve the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 1, y'\left(\frac{\pi}{4}\right) = -1.$$

Solution. The characteristic polynomial is $r^2 + 4 = 0$ with roots $r = \pm 2i$. We know, by heart at this point, that the general solution to the equation is

$$y(t) = c_1 \sin(2t) + c_2 \cos(2t).$$

The boundary conditions now give

$$\begin{aligned} y(0) = c_2 &= 1 \\ y'\left(\frac{\pi}{4}\right) = c_1 &= -1. \end{aligned}$$

Hence the solution is

$$y(t) = \cos(2t) - \sin(2t).$$

Let us now see another strakly contrasting example.

Exercise 2. Solve the boundary value problem

$$y'' + y = 0, \quad y(0) = 1, y'(\pi) = a.$$

Solution. The characteristic polynomial is $r^2 + 1 = 0$ with roots $r = \pm i$. Hence, the general solution to the equation is

$$y(t) = c_1 \sin(t) + c_2 \cos(t).$$

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The boundary conditions now give

$$\begin{aligned} y(0) = c_2 &= 1 \\ y'(\pi) = -c_2 &= a. \end{aligned}$$

Hence for a solution to exist the boundary condition must satisfy $a = -1$. Thus there is no solution for $a \neq -1$ and for $a = -1$ there are infinitely many solutions of the form

$$y(t) = c_1 \sin(t) + \cos(t).$$

Hence we observe that whilst the solution to a initial value problem always uniquely exists, the solution to a boundary value problem might not exist or might not be unique. Getting a general existence and uniquenedd result is hence not possible.

1.1. **Eigenvalue problems.** We now turn to eigenvalue problems for ordinary differential equations. An example of this sort is given by the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, y(\pi) = 0. \tag{1.1}$$

Formally this equation is similar to an eigenvalues problem. To see this note that for a matrix A (sending vectors to vectors) the eigenvalue equation is $AX = \lambda X$. In similar vein the operator $\frac{d^2}{dt^2}$ of taking second derivatives (sending functions to functions) has an **eigenvalue equation**

$$\frac{d^2}{dt^2} y = -\lambda y.$$

Again λ is called an eigenvalue while y is now called an eigenfunction.

Let us now solve the eigenvalue problem (1.1) to find the eigenvalues. First assume $\lambda > 0$ and hence $\lambda = \mu^2$,$\mu > 0$. We then have the characteristic polynomial $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$. Hence the general solution is

$$y(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t).$$

The initial conditions give $y(0) = c_1 = 0$ and $y(\pi) = c_2 \sin(\mu\pi) = 0$. Hence to have a non-trivial solution we must have $\sin(\mu\pi) = 0$ or $\mu = n$ a positive integer. Hence we get all the positive eigenvalues

$$\lambda = n^2.$$

The corresponding eigenfunctions are

$$y_n(t) = \sin(n\pi).$$

For $\lambda < 0$, we have $\lambda = -\mu^2$. This gives the characteristic polynomial $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$. Hence the general solution is

$$y(t) = c_1 e^{\mu t} + c_2 e^{-\mu t}.$$

The initial conditions give $y(0) = c_1 + c_2 = 0$ and $y(\pi) = c_1 e^{\mu\pi} + c_2 e^{-\mu\pi} = 0$. This gives $c_1 = c_2 = 0$. Hence there is no nontrivial solution and no negative eigenvalues. Finally let us check if $\lambda = 0$ is a possible eigenvalue. In this case we get

$$y(t) = c_1 t + c_2.$$

The initial conditions again give $c_1 = c_2 = 0$ and there are no nontrivial solutions. Thus the eigenvalues and eigenfunction of (1.1) are

$$\lambda_n = n^2, \quad y_n(t) = \sin(n\pi), \quad n = 1, 2, \ldots$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 36

1. Fourier Series

In this lecture we discuss Fourier series. These will be of crucial importance in solving partial differential equations introduced in the last lecture. A Fourier series is defined for a **periodic function** $f : \mathbb{R} \to \mathbb{R}$. To say that f is periodic, with period $2L$, is to say that

$$f(x + 2L) = f(x)$$

for all x. Notice that if $2L$ is a period for f then so are $4L, 6L, 8L, \ldots$. If there is no smaller number than $2L$ which is a period then we say that $2L$ is the **fundamental period** of f. Given a function on the interval $f : [-L, L] \to \mathbb{R}$, we can form the periodic function of period $2L$ given by its periodic extension $f(x + 2L) = f(x)$. Some examples of periodic functions of period L are furnished by sines and cosines

$$\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{3\pi x}{L}\right)\ldots$$

$$\cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{3\pi x}{L}\right)\ldots$$

Notice that a linear combination of periodic functions of the same period is also periodic. Hence the function

$$\sin\left(\frac{2\pi x}{L}\right) + 3\sin\left(\frac{6\pi x}{L}\right) - \cos\left(\frac{4\pi x}{L}\right)$$

is periodic of period $2L$.

In general given f any piecewise differentiable periodic function, of period L, it can be written as an infinite linear combination of these sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right). \tag{1.1}$$

An infinite sum as above is called a **Fourier series** and the coefficients a_m, b_m are called the **Fourier coefficients.** So given a function f, how does one find its Fourier series or its Fourier coefficients. The trick is to use the orthogonality

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relations

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= 0 \\
\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \\
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx &= \begin{cases} 0 & m \neq n \\ L & m = n \neq 0 \end{cases}.
\end{aligned}$$

These orthogonality relations immedietly give us

$$\begin{aligned}
a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx \\
b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx.
\end{aligned}$$

Let us find the Fourier series in an example.

Example 1. Find the Fourier series of the periodic function given by

$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x & 0 \leq x < 2 \end{cases}$$

and $f(x+4) = f(x)$.

Solution. The Fourier coefficients are

$$a_0 = \frac{1}{2} \int_0^2 x dx - \frac{1}{2} \int_{-2}^{0} x dx = 2$$

and for $m > 1$

$$\begin{aligned}
a_m &= \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{m\pi x}{2}\right) dx \\
&= \frac{1}{2} \int_0^2 x \cos\left(\frac{m\pi x}{2}\right) dx - \frac{1}{2} \int_{-2}^{0} x \cos\left(\frac{m\pi x}{2}\right) dx \\
&= \int_0^2 x \cos\left(\frac{m\pi x}{2}\right) dx \\
&= \left[\frac{2}{m\pi} x \sin\left(\frac{m\pi x}{2}\right)\right]_0^2 - \int_0^2 \frac{2}{m\pi} \sin\left(\frac{m\pi x}{2}\right) dx \\
&= \left[\left(\frac{2}{m\pi}\right)^2 \cos\left(\frac{m\pi x}{2}\right)\right]_0^2 \\
&= \frac{4}{(m\pi)^2} \left[\cos(m\pi) - 1\right] \\
&= \begin{cases} -\frac{8}{(m\pi)^2}, & m \text{ odd} \\ 0 & m \text{ even.} \end{cases}
\end{aligned}$$

On the other hand

$$b_m = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = 0.$$

Hence we have

$$f\left(x\right)=1-\frac{8}{\pi^{2}}\left(\cos\left(\frac{\pi x}{2}\right)+\frac{1}{3^{2}}\cos\left(\frac{3\pi x}{2}\right)+\frac{1}{5^{2}}\cos\left(\frac{5\pi x}{2}\right)+\dots\right).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 37

1. Fourier Series

In this lecture we discuss further examples of Fourier series. Recalll that any piecewise differentiable periodic function f, of period L, can be written as an infinite linear combination of sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right). \tag{1.1}$$

Here the **Fourier coefficients** a_m, b_m are given by

$$\begin{aligned} a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx. \end{aligned}$$

Let us find the Fourier series in an example.

Example 1. Find the Fourier series of the periodic function given by

$$f(x) = \begin{cases} -1, & -2 \le x < 0 \\ 1 & 0 \le x < 2 \end{cases}$$

and $f(x+4) = f(x)$.

Solution. The Fourier coefficients are

$$a_0 = \frac{1}{2}\int_0^2 dx - \frac{1}{2}\int_{-2}^0 dx = 0.$$

For $m \ge 1$ we have

$$\begin{aligned} a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi x}{2}\right) dx \\ &= \frac{1}{2}\int_0^2 \cos\left(\frac{m\pi x}{2}\right) dx - \frac{1}{2}\int_{-2}^0 \cos\left(\frac{m\pi x}{2}\right) dx \\ &= \frac{1}{2}\left[\frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)\right]_0^2 - \frac{1}{2}\left[\frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)\right]_{-2}^0 \\ &= 0. \end{aligned}$$

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Next we have

$$\begin{aligned} b_m &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{m\pi x}{2}\right)dx \\ &= \frac{1}{2}\int_{0}^{2} \sin\left(\frac{m\pi x}{2}\right)dx - \frac{1}{2}\int_{-2}^{0} \sin\left(\frac{m\pi x}{2}\right)dx \\ =\tfrac{1}{2} &\left[\frac{2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\right]_0^2 - \frac{1}{2}\left[\frac{2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\right]_{-2}^0 \\ &= \frac{1}{m\pi}\left[2-\cos(m\pi)-\cos(-m\pi)\right]_0^2 \\ &= \begin{cases} \frac{4}{m\pi}, & m \text{ odd} \\ 0 & m \text{ even.} \end{cases} \end{aligned}$$

Hence we have the Fourier series

$$f(x) = \frac{4}{\pi}\left\{\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \dots\right\}.$$

.

Example 2. Find the Fourier series of the periodic function given by

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$$

and $f(x+4) = f(x)$.

Solution. The Fourier coefficients are

$$a_0 = \frac{1}{2}\int_0^2 x\,dx - \frac{1}{2}\int_{-2}^0 x\,dx = 2$$

and for $m \ge 1$

$$\begin{aligned} a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi x}{2}\right)dx \\ &= \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{m\pi x}{2}\right)dx - \frac{1}{2}\int_{-2}^{0} x\cos\left(\frac{m\pi x}{2}\right)dx \\ &= \int_0^2 x\cos\left(\frac{m\pi x}{2}\right)dx \\ &= \left[\frac{2}{m\pi}x\sin\left(\frac{m\pi x}{2}\right)\right]_0^2 - \int_0^2 \frac{2}{m\pi}\sin\left(\frac{m\pi x}{2}\right)dx \\ &= \left[\left(\frac{2}{m\pi}\right)^2\cos\left(\frac{m\pi x}{2}\right)\right]_0^2 \\ &= \frac{4}{(m\pi)^2}\left[\cos(m\pi)-1\right] \\ &= \begin{cases} -\frac{8}{(m\pi)^2}, & m \text{ odd} \\ 0 & m \text{ even.} \end{cases} \end{aligned}$$

On the other hand

$$b_m = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$$

Hence we have

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \dots\right).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 38

1. Odd and Even functions

In the last lecture we discussed examples of computing Fourier series. Some computations can be simplified if one makes observations such as even of oddness of the function. Recall that a function is **even** if $f(-x) = f(x)$ for all x while a function is **odd** if $f(-x) = -f(x)$ for all x. Examples of even functions are $\cos(x), \cos(2x), 1, x^2$ etc. while examples of odd functions are $\sin(x), \sin(2x), x, x^3$. The sum or product of two even functions is even. The sum of two odd functions is odd but the product of two odd functions is even. The product of an even and an odd function is odd.

Given an odd periodic function $f(x)$, of period $2L$, the cosine coefficients in its Fourier series are zero

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = 0.$$

Hence its Fourier series

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$$

is purely a **sine series**.

On the other hand, given an even periodic function $f(x)$, of period $2L$, the sine coefficients in its Fourier series are zero

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

and hence its Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right)$$

is a purely **cosine series**. Let us see an example of this.

Example 1. Find the Fourier series for the periodic function

$$f(x) = x, \quad -L < x < L$$

and $f(x+2L) = f(x)$, $f(L) = 0$.

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Solution. Clearly $f(x)$ is an odd function. Hence its cosine coefficients $a_m = 0$. To find the sine coefficients

$$\begin{aligned} b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left[\sin\left(\frac{m\pi x}{L}\right) - \frac{m\pi x}{L}\cos\left(\frac{m\pi x}{L}\right)\right]_0^L \\ &= \frac{2L}{m\pi}(-1)^{m+1}. \end{aligned}$$

Hence the Fourier series is

$$f(x) = \frac{2L}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m\pi}\sin\left(\frac{m\pi x}{L}\right).$$

MATHEMATICS 30650 - DIFFERENTIAL EQUATIONS

LECTURE 39

1. Heat Equation

In this lecture we shall discuss how to solve the heat equation with applications to reducing your utility bill. The heat equation models heat conduction in a metal bar. We consider the ideal situation where it is of a uniform composition or density and its thickness is much smaller in comparison with its length. Let L be the length of the bar and $u(x,t)$ be the temparture at a point $0 \leq x \leq L$ on the bar at time $t \geq 0$. The equation governing the evolution of $u(x,t)$ is

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(x,0) &= f(x) \\ u(0,t) = 0, &\qquad u(L,t) = 0. \end{aligned} \tag{1.1}$$

The constant α is the thermal diffusivity and depends on the material bar. Here $f(x)$ denote the initial temperature at time $t = 0$. Also $u(0,t)$ & $u(L,t)$ are the temperatures at the two ends of the bar. In this idealization we assume that they are at temperature 0 (say being in contact with ice). In the next lecture we will consider the more realistic situation when the ends are at room temperature.

First let us solve the heat equation without the initial condition (i.e. find the general solution). For this purpose we need enough linearly independent solutions. Some linearly independent solutions are given by

$$u_n(x,t) = e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, \ldots.$$

Hence the general solution is a linear combination of these and we must have

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

for some constants. Plugging in this into the initial condition gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

Hence the constants are simply the coefficients in the Fourier sine series of f

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right).$$

Let us now do an example.

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Example 1. Find the solution to the heat equation

$$
\begin{aligned}
u_t &= u_{xx} \\
u(x,0) &= \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1. \end{cases} \\
u(0,t) = 0, &\qquad u(1,t) = 0.
\end{aligned}
\tag{1.2}
$$

Solution. Clearly $\alpha = L = 1$ in this example. Let us find the Fourier sine series of $u(x,0)$. The Fourier coefficients are given by

$$
\begin{aligned}
c_n &= 2\int_0^1 x \sin(n\pi x) \\
&= \frac{2}{n\pi}(-1)^{n+1}
\end{aligned}
$$

from last class. Hence

$$u(x,0) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x), \quad 0 < x < 1,$$

and the solution to the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t} \sin(n\pi x).$$

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LECTURE 40

1. Heat Equation with other boundary conditions

In the last lecture we learnt how to solve the heat equation governing the temperature of a metal bar when both ends of the bar were maintained at 0 temperature. We now study the more general problem when both ends of the bar are at room (or any non-zero temperature). The heat equation governing this termperature evolution is

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(x,0) &= f(x) \\ u(0,t) = T_1, \qquad & u(L,t) = T_2. \end{aligned} \tag{1.1}$$

Again the constant α is the thermal diffusivity and depends on the material bar and $f(x)$ denote the initial temperature at time $t = 0$. But now $u(0,t) = T_1$ & $u(L,t) = T_2$, the temperatures at the two ends of the bar, are maintained at some non-zero level. To solve this heat equation we first ask what happens to the temperature at large time. We expect the temperature to stablize to some **steady state temperature** $u(x,t) \to v(x)$ as $t \to 0$. The steady state is thus a time independent solution to the heat equation and hence plugging $v(x)$ into the heat equation gives $v''(x) = 0$. With the boundary conditions $v(0) = T_1, v(L) = T_2$ this gives

$$v(x) = T_1 + \frac{x}{L}(T_2 - T_1).$$

We now consider the difference $w(x,t) = u(x,t) - v(x)$ between the temperature and its steady state (aka the transient temperature). This difference clearly satisfies the heat equation

$$\begin{aligned} w_t &= \alpha^2 w_{xx} \\ w(x,0) &= f(x) - T_1 - \frac{x}{L}(T_2 - T_1) \\ w(0,t) = 0, \qquad & w(L,t) = 0, \end{aligned} \tag{1.2}$$

where we are now back to 0 boundary conditions as before. Hence the solution to the heat equation (1.1) is

$$u(x,t) = T_1 + \frac{x}{L}(T_2 - T_1) + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

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where the constants are the coefficients in the Fourier sine series of $f(x) - T_1 - \frac{x}{L}(T_2 - T_1)$

$$c_n = \frac{2}{L}\int_0^L \left\{ f(x) - T_1 - \frac{x}{L}(T_2 - T_1) \right\} \sin\left(\frac{n\pi x}{L}\right) dx.$$

Let us now do an example.

Example 1. Find the solution to the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= 0, \quad 0 \le x < 1 \\ u(0,t) = 0, & \quad u(1,t) = 1. \end{aligned} \tag{1.3}$$

Solution. Clearly $\alpha = L = 1$, $T_1 = 0, T_2 = 1$ in this example. The steady state temperature is

$$v(x) = x$$

We need to find the Fourier sine series of $-x$. The Fourier coefficients are given by

$$\begin{aligned} c_n &= 2\int_0^1 -x \sin(n\pi x)\, dx \\ &= \frac{2}{n\pi}(-1)^n \end{aligned}$$

from last class. Hence

$$u(x,0) - x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^n \sin(n\pi x), \quad 0 < x < 1,$$

and the solution to the heat equation is

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t} \sin(n\pi x).$$

2. Bar with insulated ends

Now we come to a different heat conduction problem. In this case with consider a bar whose ends are insulated (i.e. no heat is allowed to escape or enter from the ends). Mathematically this corresponds to the boundary conditions $u_x(0,t) = u_x(0,t) = 0$ and we now have the heat equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ u(x,0) &= f(x) \\ u_x(0,t) = 0, & \quad u_x(L,t) = 0. \end{aligned} \tag{2.1}$$

Separation of variables now gives the solution

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right)$$

where the constants are now given by the cosine series of f

$$c_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Observe that the steady state temperature is now

$$v(x) := \lim_{t\to\infty} u(x,t) = \frac{c_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx$$

and hence given by the average on the initial temperature.