A Gutzwiller type trace formula for the magnetic Dirac operator

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Workshop on Geometric Quantization

Coupled/Magnetic Dirac Operator

$\left(Y^{2n+1}, g^{TY}\right)$ oriented, spin.
$\left(L, h^{L}\right)$ Hermitian line bundle with unitary connection A_0.
$a \in \Omega^1(Y)$ gives family of connections $A_r = A_0 + ira$, $r \in \mathbb{R}$.
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Equivalently study $D_h = hD_{A_0} + ic(a)$ as $h = \frac{1}{r} \to 0$.

Quantization analog

X^{2n} complex, Hermitian. $L \to X$ Hermitian holomorphic line bundle

$$D_p = \sqrt{2}\left(\bar{\partial}_{L^p} + \bar{\partial}^*_{L^p}\right) : \Omega^{0,*}\left(L^p\right) \to \Omega^{0,*}\left(L^p\right).$$

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(Bordemann-Meinrenken -Schlichenmaier '94) $\|T_{f,p}\| := \|\Pi_p f\| \sim \|f\|_\infty$, .

$\left[T_f, T_g\right] = \frac{i}{p} T_{\{f,g\}} + O\left(p^{-1}\right)$

Quantization analog

$\left(X^{2n}, J, \omega\right)$ almost Kahler (i.e. $g = \omega\left(., J.\right) > 0$)
$\left(L, \nabla^{L}\right) \rightarrow X$ prequantum
Renormalized Laplacian:

$$\Delta_p := \left(\nabla^{L^p}\right)^{*} \nabla^{L^p} - np$$

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$\mathrm{Spec}(\Delta_p) \subset [-M, M] \cup [C_1 p - C_2)\,;\; N_p = \underbrace{\#\mathrm{Spec}(\Delta_p) \cap (-M, M)}_{\sim \mathrm{vol}(X) p^n}$

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$$\rho(x) = \frac{1}{24} |\nabla J|^2\,; \quad \text{(Spectral density function)}$$

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Proved using rescaling/local index theory arguments (cf. Ma-Marinescu '07).

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Same rescaling technique gives

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(Taubes '09) $\eta (D_{A_r}) = o\left(r^{n+\frac{1}{2}+\varepsilon}\right)$. (Tsai '16) in 3D
Used in proof of 3D Weinstein conjecture via Seiberg-Witten equations.

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Is $\eta\left(D_{A_r}\right)=o\left(r^{n+\frac{1}{2}}\right)$ sharp?
On S^1: $\eta\left(i\partial_\theta+r\right)=\{r\}=O(1)$
More generally;

$$\begin{array}{l} Y^{2m+1} = \text{unit circle bundle of positive } \mathcal{L} \\ \downarrow \\ X^{2m} \; (\text{cpx}) \end{array}$$

Chern connection gives $TY=TS^1\oplus\pi^*TX$; choose adiabatic metrics $g_\varepsilon^{TY}=g^{TS^1}\oplus\frac{g^{TX}}{\varepsilon}$, $\varepsilon>0$.
$L=\mathbb{C}$ with $a=e^*$ =dual to S^1 generator (contact with periodic Reeb flow)

Circle bundle

Fourier decomp. along S^1 writes $\mathrm{Spec}\left(D_{A_r}\right)$ in terms of Dolbeault Laplacian $\Delta_p^q : \Omega^{0,q}\left(X, \mathcal{L}^p\right) \to \Omega^{0,q}\left(X, \mathcal{L}^p\right)$.

Type 1.

$$\lambda_r = (-1)^q \left[p + \varepsilon\left(q - \frac{m}{2}\right) - r\right]$$

multiplicity $= \dim\, H^q\left(X, \mathcal{K} \otimes \mathcal{L}^p\right)$.

Type 2.

$$\lambda_r^{\pm} = \frac{(-1)^{q+1} \pm \sqrt{\left[2p + \varepsilon\left(2q - m\right) - r + 1\right]^2 + 4\mu^2\varepsilon}}{2}$$

$0 < \frac{1}{2}\mu^2 \in \mathrm{Spec}\left(\Delta_k^p\right)$.

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For $\varepsilon \ll 1$ small can compute (Bismut-Cheeger '89, Dai '91)

$$\eta_r = \sum_{j=0}^{m} \left\{ \left(\frac{r^{j+1}}{(j+1)!} - \sum_{k=1}^{\left[r+\frac{\varepsilon m}{2}\right]} \frac{k^j}{j!} \right) \int_X c_1(\mathcal{L})^j \left[\mathrm{ch}(\mathcal{K})\mathrm{td}(X)\right]^{m-j} \right\} + O(1)$$

Here $\eta_r = O\left(r^n\right)$. No leading term.

Sharp results

To prove $\eta_r = O\left(r^n\right)$
Set $\mathfrak{J} \in C^\infty\left(i\mathfrak{u}\left(TX\right)\right)$, via $g^{TY}\left(.,\mathfrak{J}.\right) = da\left(.,.\right)$

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Theorem

(S '17) If further Reeb flow e^{tR} of a is non-resonant

$$lim_{r\to\infty} r^{-n}\eta_r = -\frac{1}{2}\frac{1}{(2\pi)^{n+1}}\frac{1}{n!}\int_Y \left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)\right]$$

where $|\mathfrak{J}| = \sqrt{-\mathfrak{J}^2}$ (restricted to $R^\perp$).

Dynamical assumption excludes circle bundle example.

Trace and η

Semi-classically: $\eta_r = \eta_h = \eta\left(hD_{A_0} + c\left(a\right)\right)$, $h = \frac{1}{r} \to 0$.
Using Bismut-Freed

$$\begin{aligned}\eta_h &= \int_0^\infty \frac{1}{\sqrt{\pi t h}} \mathrm{tr}\left(D_h e^{-\frac{t}{h}D_h{}^2}\right) dt \\ &= \int_0^1 \frac{1}{\sqrt{\pi t h}} \mathrm{tr}\left(D_h e^{-\frac{t}{h}D_h{}^2}\right) dt + \mathrm{tr}\ E\left(\frac{1}{\sqrt{h}}D_h\right)\end{aligned}$$

Second term $E\left(x\right) = \mathrm{sign}\left(x\right) \mathrm{erfc}\left(|x|\right)$ has non-local trace.

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By Tauberian argument η_h asymptotics follow from

Theorem

(Gutzwiller trace) For $f, \theta \in C_c^\infty(\mathbb{R})$*;* $\lambda \in \mathbb{R}$*,*

$$tr\left[f\left(\frac{D_h}{\sqrt{h}}\right)\check{\theta}\left(\frac{\lambda\sqrt{h} - D_h}{h}\right)\right] \sim h^{-m}\left(\sum_{j=0}^\infty c_j(\lambda)\, h^{j/2}\right)$$

$$+ \sum_\gamma e^{\frac{i}{h}T_\gamma} e^{i\frac{\pi}{2}\mathfrak{m}_\gamma} \sum_{j=1}^\infty h^{j/2} \sum_{k=0}^j \lambda^k A_{\gamma,j,k}\theta(L_\gamma)$$

where $T_\gamma = period$*,* $L_\gamma = length$*,* $\mathfrak{m}_\gamma = Maslov\ index$ *of Reeb orbit.*

Trace formula

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Here $\sigma(D_h) = c(\xi + a)$ with eigenvalues $\pm|\xi + a|$.
Characteristic variety: $\Sigma = \{\xi = -a\}$.

Normal form

Proof uses microlocal normal forms.
Contact 3D model:

$$\mathtt{D}_0 = \begin{bmatrix} i & \\ & -i \end{bmatrix} h\partial_z + \underbrace{\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} (h\partial_x + iy) + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} (h\partial_y - ix)}_{\mathtt{D}_{00}} .$$

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More generally

$$D_h = \gamma^j w_j^k \left(h\partial_k + ia_k\right) + O\left(h\right)$$

where $g_j^i = w_j^k w_k^i$ (diagonalizes metric).

$$\underbrace{UDU^*}_{\text{FIO conjugation}} = \begin{bmatrix} i & \\ & -i \end{bmatrix} h\partial_z + \mathtt{D}_{00} + \underbrace{R}_{\text{commutes with } \mathtt{D}_{00}^2} + O\left(h^\infty\right).$$

Study trace on the level.

Koszul differentials

$U = e^{\frac{it}{h} f^W} e^{itc_0^W(b)}$ with
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Conjugation

$$e^{\frac{i}{h} f} \mathtt{D}_0 e^{-\frac{i}{h} f} = \mathtt{D}_0 + c_0 \Big(\underbrace{\tilde{w}_\partial}_{\text{Koszul}} f \Big) + \dots$$

$$e^{ic_0(b)} \mathtt{D}_0 e^{-ic_0(b)} = \mathtt{D}_0 + (-1)^{|a|} 2c_0 \Big(\underbrace{i_x}_{\text{Koszul}} b \Big) + hc_0 \Big(\underbrace{\tilde{w}_\partial}_{\text{Koszul}} b \Big) + \dots$$

$i_x = x i_{e_1} + \xi i_{e_2}$; $\tilde{w}_\partial = \partial_x e_2 \wedge - \partial_\xi e_1 \wedge$
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Generalizes similar normal form of Ivrii.
Large time trace fomula: similar normal form near (lifts to Σ of) Reeb orbits + weak propagation lemma (Dimassi-Sjostrand).

Further questions

Still to do:

1. Relax assumptions (contact, on$\mathrm{Spec}\mathfrak{J}$..)
2. Asymptotics of $\log\det D_h^2$ (expansion of low lying eingenvalues)
3. Understand propagator $e^{it\frac{D}{h}}$ & propagation of singularities
4. Apply technique to complex/CR analogs
5. Study microlocal analog (magnetic Laplacian $\rightarrow$subRiemannian Laplacian)

Thank you.