QUANTITATIVE VERSION OF WEYL'S LAW

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Abstract. We prove a general estimate for the Weyl remainder of an elliptic, semiclassical pseudodifferential operator in terms of volumes of recurrence sets for the Hamilton flow of its principal symbol. This quantifies earlier results of Volovoy [23, 24]. Our result particularly improves Weyl remainder exponents for compact Lie groups and surfaces of revolution. And gives a quantitative estimate for Bérard's Weyl remainder in terms of the maximal expansion rate and topological entropy of the geodesic flow.

1. Introduction

Let X^n be a smooth, compact and n-dimensional manifold. Let $P \in \Psi^m_{h,\mathrm{cl}}(X)$ be a self-adjoint, semiclassical (h-)pseudodifferential operator (see Section 2). We assume that the total symbol of P has a classical expansion

$$P = p_h^W, \tag{1.1}$$

$$p_h \sim p_0 + hp_1 + \ldots, \tag{1.2}$$

for some smooth functions $p_j \in C^\infty(T^*X)$, $j = 0, 1, 2, \ldots$ on the cotangent space T^*X. Let $[a, b] \subset \mathbb{R}$ and assume that the principal symbol p_0 is elliptic in this interval, i.e. $p_0^{-1}[a, b] \Subset T^*X$ is compact. The spectrum of P_h in $[a, b]$ is then discrete for h sufficiently small. Denote by $N_h[a, b]$ the number of eigenvalues contained in this interval. Further suppose that a, b are noncritical values for the principal symbol p_0, whereby the energy levels $\Sigma_a := p_0^{-1}(a) \subset T^*X$, $\Sigma_b := p_0^{-1}(b) \subset T^*X$ are well-defined hypersurfaces and so is the Hamiltonian flow $e^{tH_{p_0}}$ for the principal symbol on them. We denote by the shorthand $e^{tH^a_{p_0}} := e^{tH_{p_0}}\big|_{\Sigma_a}$ the Hamilton flow restricted to the given energy level. And by $T^a_0 > 0$ the shortest period of the restricted Hamilton flow $e^{tH^a_{p_0}}$. These energy levels also carry the canonical Liouville volume form $d\nu$ such that $dp_0 \wedge d\nu$ is the canonical symplectic volume form in a neighborhood of these levels in T^*X.

Next, define the recurrence sets

$$S^a_{T,\varepsilon} := \left\{ (x, \xi) \subset \Sigma_a \middle| \exists t \in \left[\frac{1}{2}T^a_0, T\right] \text{ s.t. } d\left(e^{tH_{p_0}}(x, \xi), (x, \xi)\right) \leq \varepsilon \right\} \tag{1.3}$$

and $S^{a,e}_{T,\varepsilon} := \left\{ (x, \xi) \subset \Sigma_a | d\left((x, \xi), S^a_{T,\varepsilon}\right) \leq \varepsilon \right\}$ for each $T > \frac{1}{2}T^a_0$, $\varepsilon > 0$. These are defined with respect to a distance function d on the phase space that is equivalent to a manifold distance, although the dependence of the above (1.3) on d is suppressed in the notation.

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Next we further, for each $\delta \in \left[0, \frac{1}{2}\right)$ and $\ell > 0$, define the Ehrenfest time via

$$T_E^{\ell,a,\delta}(h) := \begin{cases} \frac{h^{-\frac{1}{C}\left(\frac{1}{2}-\delta-\ell\right)}}{\frac{1}{2}-\delta}; & \text{if } \exists C, \epsilon' > 0 \text{ s.t. } \left|\partial^\alpha e^{tH_{p_0}^{a'}}\right| \lesssim 1 + |t|^{C|\alpha|}, \ \forall \alpha \in \mathbb{N}_0^{2n-1}, |a - a'| \leq \epsilon', \\ \frac{|\ln h|}{\Lambda_{\max}^a + \ell}; & \text{otherwise.} \end{cases} \tag{1.4}$$

$$\text{Here} \quad \Lambda_{\max}^a := \limsup_{t\to\infty} t^{-1} \sup_{x\in\Sigma_a} \ln\left|de^{tH_{p_0}^a}(x)\right| \tag{1.5}$$

is the maximum expansion rate of the Hamilton flow $e^{tH_{p_0}^a}(x)$ on the energy level Σ_a defined with the help of its Jacobian $de^{tH_{p_0}^a}(x)$.

Our main result is the following general estimate on the Weyl remainder for P.

Theorem 1. *Let $P \in \Psi_{h,cl}^m(X)$ be a self-adjoint, h-pseudodifferential operator that is elliptic in the interval $[a, b]$ and whose principal symbol is noncritical at its endpoints. For each $\varepsilon = ch^\delta$, $\delta \in \left[0, \frac{1}{2}\right)$, $c, \ell > 0$ and $T \leq \left(\frac{1}{2} - \delta\right) . \max\left\{T_E^{\ell,a,\delta}(h), T_E^{\ell,b,\delta}(h)\right\}$, the Weyl counting function of the interval satisfies*

$$N_h[a, b] = (2\pi h)^{-n}\left[vol\ p_0^{-1}[a, b] + h\left(\int_{\Sigma_a} p_1 d\nu - \int_{\Sigma_b} p_1 d\nu\right) + hR_h\right] \tag{1.6}$$

$$\text{with} \quad |R_h| \leq (\nu(\Sigma_a) + \nu(\Sigma_b))\,T^{-1} + O\left(\nu\left(S_{T,\varepsilon}^{a,e}\right) + \nu\left(S_{T,\varepsilon}^{b,e}\right) + T^{-2} + h^{1-2\delta}\right) \tag{1.7}$$

as $h \to 0$.

Interesting specializations of the above arise depending on estimates for volumes of the recurrence sets. Firstly, it recovers the classical results of Hörmander [11] and Duistermaat-Guillemin [4], in their semiclassical form cf. [14, 19]. Namely, under the non-criticality assumption on the endpoints, one first obtains $N_h[a, b] = (2\pi h)^{-n}\left[\text{vol}\ p_0^{-1}[a, b] + O(h)\right]$ from (1.6), which is the semiclassical version of Hörmander's Weyl law. Secondly, and further assuming that the set of periodic Hamilton trajectories is of measure zero on the two energy levels Σ_a, Σ_b, one has $R_h = o(1)$ in (1.6). This is the semiclassical version of Duistermaat and Guillemin's Weyl law. This follows on letting ε, T be $h-$independent. Since the recurrence set volume approaches the measure of the set of periodic Hamilton trajectories as $\varepsilon \to 0$ and $T \to \infty$, one recovers Duistermaat-Guillemin.

Another interesting specialization of the general estimate is when the Hamilton flow is a contact Anosov flow. In this case the recurrence set volumes can be shown to satisfy exponential estimates in time $\nu\left(S_{T,\varepsilon}^{a,e}\right) = O\left(\varepsilon^{2n-1}e^{(2n-1)\Lambda_{\max}^a T}\right)$ with the exponent again being the maximal expansion rate (1.5) (see Section 4.1 below). By an appropriate choice of ε, T in this case, one obtains a logarithmic improvement in the Weyl law. A particular example of contact Anosov Hamiltonian flows are geodesic flows on negatively curved manifolds. Thus, if we particularly specialize to the case when $P_h = h^2\Delta_g$ is the semiclassical Laplacian for a negatively curved Riemannian metric (X, g), with $a < 0$, $b = 1$, we obtain the following quantitative version of Bérard's Weyl law.

Corollary 2. *Let (X^n, g) be a compact, negatively curved Riemannian manifold. The Weyl counting function for the semiclassical Laplacian $P_h := h^2\Delta_g$ satisfies the asymptotics*

$$N_h[0,1] = (2\pi h)^{-n}\, vol\ (S^*X)\,[1 + hR_h] \tag{1.8}$$

$$where \quad |R_h| \le 4\Lambda_{max}\,|\ln h|^{-1} + o\left(|\ln h|^{-1}\right) \tag{1.9}$$

$$\le \frac{16}{2n-1}\mathtt{h}_{top}\,|\ln h|^{-1} + o\left(|\ln h|^{-1}\right), \tag{1.10}$$

as $h \to 0$, in terms maximal expansion rate Λ_{max} and the topological entropy $\mathtt{h}_{top}$ (see Sec. 2.2) of the geodesic flow on the co-sphere bundle S^*X.

The next specialization concerns the semiclassical Laplacian $P_h = h^2\Delta_g$ corresponding to a bi-invariant metric on compact a Lie group G. The corresponding Hamiltonian flow is again the geodesic flow which is given simply by the group action. While the recurrence set volume for the geodesic flow on the unit co-sphere bundle can be shown to satisfy the bound $\nu\left(S^{1,e}_{T,\varepsilon}\right) = O\left(\varepsilon^{p-1}T^p\right)$, with $p = \text{rk}\ G$, being the rank of the Lie group (see Section 4.2). This gives the Weyl law below as a corollary.

Corollary 3. *Let G be a compact Lie group equipped with a bi-invariant Riemannian metric. The Weyl counting function for the semiclassical Laplacian $P_h := h^2\Delta_g$ satisfies the asymptotics*

$$N_h[0,1] = (2\pi h)^{-n}\, vol\ (S^*X)\left[1 + O\left(h^{1+\frac{p-1}{4p}-}\right)\right], \tag{1.11}$$

[1] *as $h \to 0$, where $p = rk\ G$ is the rank of the Lie group.*

Our final specialization concerns the semiclassical Laplacian $P_h = h^2\Delta_g$ on surfaces of revolution X. The corresponding Hamiltonian flow is again the geodesic flow. In this case, it has a simple description as an ordinary differential equation on the base surface (see 4.3 below). The Ehrenfest time (1.4) is infinite. Assuming the surface to be strictly convex, it has an equator; the unique rotationally invariant geodesic $\gamma_E \subset X$ of maximal length. For any point $x \in \gamma_E$ the equatorial return map $\theta : S^*_x X \to \gamma_E$ is well defined (see Figure 4.1 on page 18 in Section 4). This is the map sending a unit covector $\xi \in S^*_x X$ to the first return point $\theta(\xi)$ of the geodesic $\exp_x(t\xi)$ to the equator γ_E. The equatorial return map has an order of vanishing defined as

$$r_{\xi_0} := \text{ord}_{\xi_0}\left[\theta(\xi) - \theta(\xi_0)\right] = \min\left\{l | \partial^l_\xi\left[\theta(\xi) - \theta(\xi_0)\right](\xi_0) \ne 0\right\}, \quad r_{\xi_0} \ge 1,$$

for each covector $\xi_0 \in S^*_x X$. The function r_{ξ_0} is upper semi-continuous in ξ_0 and it is clearly independent of x by rotational invariance. We set

$$\begin{aligned} r &:= \sup_{\xi_0 \in S^*_x X} r_{\xi_0} \\ &= \sup_{\xi_0 \in S^*_x X} \text{ord}_\xi\left[\theta(\xi) - \theta(\xi_0)\right] \end{aligned} \tag{1.12}$$

to be its supremum over $\xi_0 \in S^*_x X$. There are standard formulas to calculate θ and r in terms of a given equation for the surface (see [2]). For a real analytic surface which is not Zoll we have $r < \infty$ is finite, while for a generic surface one has $r = 1$.

Under some further assumptions stated below, the recurrence set volume for the geodesic flow on the unit cosphere bundle can be shown to satisfy $\nu\left(S^{1,e}_{T,\varepsilon}\right) = O\left(\varepsilon^{\frac{1}{r}}T^{1-\frac{1}{r}}\right)$ for ε, T^{-1} sufficiently small. This gives the Weyl law below as a corollary.

[1]Here $O\left(h^{\alpha-}\right)$ denotes a term that is $O\left(h^{\alpha-\ell}\right)$ for each $\ell > 0$.

Corollary 4. *Let (X^2, g) be a compact, strictly convex surface of revolution. The Weyl counting function for the semiclassical Laplacian $P_h := h^2\Delta_g$ satisfies the asymptotics*

$$N_h[0,1] = (2\pi h)^{-2} vol(S^*X)\left[1 + O\left(h^{1+\frac{1}{4r-1}}\right)\right], \tag{1.13}$$

as $h \to 0$. Here r (1.12) is the maximum order of vanishing of the equatorial return map.

The corollaries are all based on a judicious choice of ε, T in the main Theorem 1.

The general estimate in Theorem 1, or the arguments for it, are seemingly folklore. Less precise versions of it are explained in the articles of Volovoy [23, 24] as well as the recent book [14, Sec. 4.5.4] of Ivrii. However the identification of the various exponents and constants in our corollaries does not appear in them. Our own motivation came from the recent article of the author [22] where an analogous result is proved for the coupled Dirac operator. Our method is based on modifying standard arguments from the book of Dimassi-Sjöstrand [3, Ch. 11].

The first Corollary 2 quantifies Bérard's Weyl law [1] for negatively curved manifolds. Its identification of the maximal expansion rate (1.9) or topological entropy (1.10) appears to be new. It might however be of little interest since there are stronger conjectures. For instance it is conjectured that for a non-arithmetic or generic surface the remainder is $R_h = O(h^{2-})$. Although no better improvement is known since Bérard's article.

The second and third corollaries were also considered by Volovoy. He proved the remainder estimates $O\left(h^{1+\delta}\right)$, for some $\delta > 0$, in (1.11), (1.13). And our corollaries quantify the value of δ that can be chosen. In Corollary 3, the case of the torus $G = \mathbb{T}^n$ is particularly studied (see [7, Ch. 3]). In this case, the sharp Weyl remainder is conjecturally

$$R_h = \begin{cases} O\left(h^{\frac{3}{2}-}\right), & \text{for } n = 2, \quad \text{(Gauss circle problem)} \\ O\left(h^{2-}\right), & \text{for } n = 3, 4, \\ O\left(h^{2}\right), & \text{for } n \geq 5. \end{cases}$$

This is known in dimensions $n \geq 4$. While $R_h = O\left(h^{\frac{285}{208}-}\right), O\left(h^{\frac{5}{3}-}\right)$ are the best known results in low dimensions $n = 2, 3$ respectively [13]. Our Corollary 3 is worse than these known results for the torus. However, the estimate (1.11) for an arbitrary Lie group appears to be new. The problem of determining the sharp exponent in the Weyl is further unknown for general G.

Finally, we remark that although it is not our interest here, our method almost certainly gives similar improvements for local Weyl laws as well as L^2, L^∞ norms and averages of eigenfunctions etc.

The article is organized as follows. In Section 2 we begin with some preliminaries on semiclassical analysis and dynamical systems. In Section 3 we prove the general Weyl's law in our main Theorem 1 based on a modification of the argument in [3]. In Section 4 we consider recurrence sets for several dynamical systems including Anosov flows 4.1 as well as geodesic flows on Lie groups 4.2 and surfaces of revolution 4.3. The final Section 5 proves the three Corollaries 2, 3 and 4 based on the volume bounds from Section 4.

2. Preliminaries

In this section we review some preliminary notions from semiclassical analysis and dynamical systems that are used in the article.

2.1. Semiclassical analysis.

First we begin with some requisite facts from semi-classical analysis that shall be used in the paper, [3, 9] provide the standard references. The symbol space

$S^m(\mathbb{R}^{2n})$ is defined as the space of maps $a:(0,1]_h \to C^\infty\left(\mathbb{R}^{2n}_{x,\xi}\mathbb{C}\right)$ for which each semi-norm

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h}\langle\xi\rangle^{-m+|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi;h)\right|, \quad \alpha,\beta\in\mathbb{N}_0^n,$$

is finite. The more refined class $a \in S^m_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{C}^l\right)$ of classical symbols consists of those for which there exists an h-independent sequence a_k, $k=0,1,\ldots$, of symbols satisfying

$$a - \left(\sum_{k=0}^{N} h^k a_k\right) \in h^{N+1}S^m\left(\mathbb{R}^{2n}\right),\ \forall N. \tag{2.1}$$

Any given $a \in S^m(\mathbb{R}^{2n}), S^m_{\text{cl}}(\mathbb{R}^{2n})$ in one of the symbol classes above defines a one-parameter family of operators $a^W \in \Psi^m(\mathbb{R}^{2n}), \Psi^m_{\text{cl}}(\mathbb{R}^{2n})$ via Weyl quantization. The Schwartz kernel of the quantization is given by

$$a^W := \frac{1}{(2\pi h)^n}\int e^{i(x-y).\xi/h} a\left(\frac{x+y}{2},\xi;h\right)d\xi.$$

The above pseudodifferential classes of operators are closed under the usual operations of composition and formal-adjoint. Furthermore the classes are invariant under changes of coordinates. Thus one may invariantly define the classes of operators $\Psi^m(X), \Psi^m_{\text{cl}}(X)$ acting on smooth functions $C^\infty(X;\mathbb{C})$ on a smooth compact manifold X.

The principal symbol of a classical pseudodifferential operator $A\in\Psi^m_{\text{cl}}(X)$ is defined as an element in $\sigma(A)\in S^m(X)\subset C^\infty(T^*X;\mathbb{C})$. It is given by $\sigma(A)=a_0$, the leading term in the symbolic expansion (2.1) of its full Weyl symbol. While the second term a_1 is referred to as the sub-principal symbol of A. The principal symbol is multiplicative, commutes with adjoints and fits into a symbol exact sequence

$$\begin{aligned}\sigma(AB) &= \sigma(A)\sigma(B)\\ \sigma(A^*) &= \overline{\sigma(A)}\\ 0\to h\Psi^m_{\text{cl}}(X) \to &\Psi^m_{\text{cl}}(X) \xrightarrow{\sigma} S^m(X),\end{aligned} \tag{2.2}$$

with the formal adjoint defined with respect to an auxiliary density. The quantization map

$$\begin{aligned}\text{Op}: S^m(X) \to \Psi^m_{\text{cl}}(X) \quad &\text{satisfying}\\ \sigma(\text{Op}(a)) = a \in S^m(X)&\end{aligned} \tag{2.3}$$

gives an inverse to the principal symbol map. We sometimes use the alternate notation $\text{Op}(a) = a^W$. The quantization map above is however non-canonical and depends on the choice of a coordinate atlas as well as a subordinate partition of unity. From the multiplicative property of the symbol (2.2), it then follows that $\left[a^W,b^W\right]\in h\Psi^{m-1}_{\text{cl}}(X)$. Its principal symbol is given by the Poisson bracket

$$\frac{i}{h}\sigma\left(\left[a^W,b^W\right]\right) = \{a,b\}\in S^m(X).$$

Each $A\in\Psi^m_{\text{cl}}(X)$ has a wavefront set defined invariantly as a subset $WF(A)\subset\overline{T^*X}$ of the fibrewise radial compactification of the cotangent bundle T^*X. It is locally defined as follows, $(x_0,\xi_0)\notin WF(A)$, $A=a^W$, if and only if there exists an open neighborhood $(x_0,\xi_0;0)\in U\subset\overline{T^*X}\times(0,1]_h$ such that $a\in h^\infty\langle\xi\rangle^{-\infty}C^k\left(U;\mathbb{C}^l\right)$ for all k. The wavefront set satisfies the following basic properties under addition, multiplication and adjoints

$$\begin{aligned} WF(A+B) &\subset WF(A) \cup WF(B), \\ WF(AB) &\subset WF(A) \cap WF(B) \quad \text{and} \\ WF(A^*) &= WF(A). \end{aligned}$$

The wavefront set $WF(A) = \emptyset$ is empty if and only if $A \in h^{\infty}\Psi^{-\infty}(X)$ while we say that two operators $A = B$ microlocally on $U \subset \overline{T^*X}$ if $WF(A - B) \cap U = \emptyset$.

An operator $A \in \Psi^m_{\text{cl}}(X)$ is said to be elliptic if $\langle \xi \rangle^m \sigma(A)^{-1}$ exists and is uniformly bounded on T^*X. If $A \in \Psi^m_{\text{cl}}(X)$, $m > 0$, is formally self-adjoint, such that $A + i$ is elliptic, then it is essentially self-adjoint (with domain $C^{\infty}_c(X)$) as an unbounded operator on $L^2(X)$. Beals's lemma further implies that its resolvent $(A - z)^{-1} \in \Psi^{-m}_{\text{cl}}(X)$, $z \in \mathbb{C}$, $\text{Im}z \neq 0$, exists and is pseudodifferential. The Helffer-Sjöstrand formula now expresses the function $f(A)$, $f \in \mathcal{S}(\mathbb{R})$, of such an operator in terms of its resolvent

$$f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-1} dz d\bar{z},$$

with $\tilde{f}$ denoting an almost analytic continuation of f. One further has

$$WF(f(A)) \subset \Sigma^A_{\text{spt}(f)} \tag{2.4}$$

$$:= \bigcup_{\lambda \in \text{spt}(f)} \Sigma^A_{\lambda} \tag{2.5}$$

$$\text{where} \quad \Sigma^A_{\lambda} = \{(x, \xi) \in T^*X | \sigma(A)(x, \xi) = \lambda\} \tag{2.6}$$

is classical λ-energy level of A.

2.1.1. *The class* $\Psi^m_{\delta}(X)$. We shall also need a more exotic class of scalar symbols $S^m_{\delta}(\mathbb{R}^{2n}; \mathbb{C})$ defined for each $0 \leq \delta < \frac{1}{2}$. A function $a : (0, 1]_h \to C^{\infty}\left(\mathbb{R}^{2n}_{x,\xi}; \mathbb{C}\right)$ is said to be in this class if and only if

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} h^{(|\alpha|+|\beta|)\delta} \left| \partial^{\alpha}_x \partial^{\beta}_{\xi} a(x, \xi; h) \right| \tag{2.7}$$

is finite $\forall \alpha, \beta \in \mathbb{N}^n_0$. This class of operators is also closed under the standard operations of composition, adjoint and changes of coordinates; allowing for the definition of the same exotic pseudodifferential algebra $\Psi^m_{\delta}(X)$ on a compact manifold. The class $S^m_{\delta}(X)$ is a family of functions $a : (0, 1]_h \to C^{\infty}(T^*X; \mathbb{C})$ satisfying the estimates (2.7) in every coordinate chart and induced trivialization. Such a family can be quantized to $a^W \in \Psi^m_{\delta}(X)$ satisfying $a^W b^W = (ab)^W + h^{1-2\delta} \Psi^{m+m'-1}_{\delta}(X)$ for another $b \in S^{m'}_{\delta}(X)$. The operators in $\Psi^0_{\delta}(X)$ are uniformly bounded on $L^2(X)$. Finally, the wavefront an operator $A \in \Psi^m_{\delta}(X)$ is similarly defined and satisfies the same basic properties as before.

2.2. **Dynamical invariants.** Here we state some facts on dynamical systems and their invariants that are used in the article. Their proofs can be found in the texts [15, 16].

Let $V \in C^{\infty}(TY)$ be a smooth vector field on a compact Riemannian manifold $\left(Y, g^{TY}\right)$ of dimension m. A point $y \in Y$ is said to be a *regular point* for the flow of V if there is a sequence of numbers $\lambda_1(y) > \ldots > \lambda_k(y)$ and decomposition for the tangent space $T_yY = \oplus^k_{j=1} E_j(y)$ such that

$$\limsup_{t \to \infty} t^{-1} \ln \left| de^{tV}(y) u \right| = \lambda_j(y), \tag{2.8}$$

$\forall u \in E_j(y), 1 \leq j \leq k$. The set of regular points $R \subset Y$ is a Borel set of full measure. The numbers $\lambda_j(y)$ and subspaces $E_j(y)$ are referred to as the *Lyapunov exponents* and *eigenspaces* of the flow at $y \in Y$ respectively. The maximal expansion rate of the flow is the supremum of the maximum of these

$$\begin{aligned}\Lambda_{\max}(V) &:= \limsup_{t\to\infty} t^{-1} \sup_{y\in Y} \ln \left| de^{tV}(y) \right| \\ &= \sup_{y\in R\subset Y} \max_{j=1,\dots,k} \lambda_j(y).\end{aligned} \tag{2.9}$$

It is well known that $\Lambda_{\max}(V)$ is an upper semi-continuous function of V.

Additionally, it is useful to define the sum of the positive Lyapunov exponents as the Borel function

$$\begin{aligned}\chi &: R \to \mathbb{R} \\ \chi(y) &:= \sum_{\lambda_j(y)>0} \lambda_j(y) \dim E_j(y).\end{aligned} \tag{2.10}$$

In case there is no positive Lyapunov exponent, we set $\chi(y) := 0$.

Next, let μ_Y be a Borel probability measure on Y invariant by the flow of V. The measure theoretic entropy of a partition $\mathcal{P} = \{P_1, \dots, P_N\}$ of Y into measurable subsets is defined to be $H(\mathcal{P}) := -\sum_{P\in\mathcal{P}} \mu_Y(P) \ln \mu_Y(P)$. The measure theoretic entropy of the flow with respect to the partition is set to be

$$H(\mathcal{P}, V) := \lim_{N\to\infty} N^{-1} H\left(\bigvee_{j=0}^{N} e^{-jV}\mathcal{P} \right).$$

Here $\mathcal{P}_1 \bigvee \mathcal{P}_2$ above denotes the minimal common refinement of two partitions $\mathcal{P}_1, \mathcal{P}_2$. The supremum of the above over all finite measurable partitions is the measure theoretic entropy of the flow

$$h_{\mu_Y}(V) = \sup_{\mathcal{P}} H(\mathcal{P}, V). \tag{2.11}$$

While the topological entropy is the supremum of the above over all the set of all V-invariant Borel probability measures $\mathcal{M}_V$

$$h_{\text{top}}(V) := \sup_{\mu_Y \in \mathcal{M}_V} h_{\mu_Y}(V). \tag{2.12}$$

The relation between the measure theoretic entropy and the Lyapunov exponents is given by the Margulis-Ruelle inequality [16, Thm. 10.2]

$$h_{\mu_Y}(V) \leq \int_Y \chi(y)\, d\mu_Y. \tag{2.13}$$

In case μ_Y is absolutely continuous with respect to a smooth measure then Pesin's formula says that one has equality in the above. Following the definitions (2.9), (2.10) and (2.12) it is easy to see that the Margulis-Ruelle inequality particularly implies

$$h_{\text{top}}(V) \leq m.\Lambda_{\max}. \tag{2.14}$$

3. Weyl Law

We now give a proof for Theorem 1 based on a modification of standard arguments in [3, Ch. 11] using wave trace asymptotics. The task being to make the arguments therein quantitative.

3.1. **Wave trace asymptotics.** First, consider the energy band $\Sigma_{[a-\epsilon',a+\epsilon']}$, for each $\epsilon'>0$. Denote by the shorthand $e^{tH_{p_0}^{[a-\epsilon',a+\epsilon']}}:=e^{tH_{p_0}}|_{\Sigma_{[a-\epsilon',a+\epsilon']}}$ the Hamilton flow restricted to the given energy band. And by $T_0^{[a-\epsilon',a+\epsilon']}>0$ the shortest period of the restricted Hamilton flow $e^{tH_{p_0}^{[a-\epsilon',a+\epsilon']}}$. In similar vein as (1.3) and (1.4), the recurrence sets for the energy band are defined via

(3.1)
$$S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon']}:=\left\{(x,\xi)\subset\Sigma_{[a-\epsilon',a+\epsilon']}|\exists t\in\left[\frac{1}{2}T_0^{[a-\epsilon',a+\epsilon']},T\right]\text{ s.t. }d\left(e^{tH_{p_0}}(x,\xi),(x,\xi)\right)\leq\varepsilon\right\},$$

and $S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],e}:=\left\{(x,\xi)\subset\Sigma_{[a-\epsilon',a+\epsilon']}|d\left((x,\xi),S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon']}\right)\leq\varepsilon\right\}$. Below it shall also be useful to define the intermediate recurrence set

$$S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],\frac{1}{2}}:=\left\{(x,\xi)\subset\Sigma_{[a-\epsilon',a+\epsilon']}|d\left((x,\xi),S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon']}\right)\leq\frac{\varepsilon}{2}\right\}.$$

Again d denotes a phase space distance on the band that is equivalent to some Riemannian distance on it.

Furthermore

(3.2)
$$T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h):=\begin{cases}\frac{h^{-\frac{1}{C}\left(\frac{1}{2}-\delta-\ell\right)}}{\frac{1}{2}-\delta}; & \text{if }\exists C>0\text{ s.t. }\left|\partial^\alpha e^{tH_{p_0}^{[a-\epsilon',a+\epsilon']}}\right|\lesssim|t|^{C|\alpha|},\ \forall\alpha\in\mathbb{N}_0^{2n-1},\\ \frac{|\ln h|}{\Lambda_{\max}^{[a-\epsilon',a+\epsilon']}+\ell}; & \text{otherwise,}\end{cases}$$

(3.3)
$$\text{and}\quad\Lambda_{\max}^{[a-\epsilon',a+\epsilon']}:=\limsup_{t\to\infty}t^{-1}\sup_{x\in\Sigma_{[a-\epsilon',a+\epsilon']}}\ln\left|de^{tH_{p_0}}(x)\right|$$

are the Ehrenfest time and the maximal expansion rate respectively of the energy band.

Next, we choose a square microlocal partition of unity $\left\{A_j=a_j^W\right\}_{j=0}^{M_h}\in\Psi_{h,\delta}^0(X)$, $\delta\in\left[0,\frac{1}{2}\right)$, $M_h=O\left(h^{-2n\delta}\right)$, $\sum_{j=0}^{M_h}A_j^2=1$ on $\Sigma_{[a-\epsilon',a+\epsilon']}\subset T^*X$, as follows. The first among these is chosen such that $a_0\in C^\infty(T^*X;[0,1])$, with $A_0=a_0^W$ self-adjoint and

$$a_0=\begin{cases}1, & \text{on }S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],\frac{1}{2}}\\ 0, & \text{on }\left(S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],e}\right)^c.\end{cases}$$

One satisfying correct symbolic estimates in $\Psi_{h,\delta}^0(X)$ is found by an application of the Whitney extension theorem [12, Sec. 2.3]. For points p in the complement of $S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],\frac{1}{2}}$, one has $d\left(e^{tH_{p_0}}p',p'\right)\geq\varepsilon$ for each $d(p,p')\leq\frac{\varepsilon}{4}$. It is hence covered by open radius $\frac{\varepsilon}{4}$-balls U_j such that $d\left(e^{tH_{p_0}}U_j,U_j\right)\geq\frac{\varepsilon}{4}$ for $t\in\left[\frac{1}{2}T_0^{[a-\epsilon',a+\epsilon']},T\right]$. We now apply Egorov's theorem to Ehrenfest time [17, Sec. 3.4], the modification of this to the first case of (3.2) is an easy exercise. This gives that the rest of the pseudodifferential operators $A_1,\dots,A_M\in\Psi_{h,\delta}^0(X)$, $\varepsilon=ch^\delta$, in the partition of unity covering $\left(S_{T,\varepsilon}^{[a-\epsilon',a+\epsilon'],e}\right)^c$ maybe chosen such that

$$A_{j,t} := e^{-\frac{it}{h}P_h} A_j e^{\frac{it}{h}P_h} \in \Psi^0_{h,\delta'}(X)$$

(3.4)

$$\text{for} \quad \delta' = \begin{cases} \frac{1}{2} - \ell; & \text{if } \exists C > 0 \text{ s.t. } \left| \partial^\alpha e^{tH_{p_0}^{[a-\epsilon',a+\epsilon']}} \right| \lesssim |t|^{C|\alpha|}, \ \forall \alpha \in \mathbb{N}_0^{2n-1}, \\ \frac{1}{2} - \left(\frac{1}{2} - \delta\right) \frac{\ell}{\Lambda_{\max}^{[a-\epsilon',a+\epsilon']} + \ell}; & \text{otherwise,} \end{cases}$$

(3.5)

with $WF(A_j) \cap WF(A_{j,t}) = \emptyset$,

$j = 1, \ldots, M$ and $t \in \left[\frac{1}{2} T_0^{[a-\epsilon',a+\epsilon']}, \left(\frac{1}{2} - \delta\right) T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)\right]$.

Next choose $h-$independent functions $f, \theta \in C_c^\infty(\mathbb{R})$ with $f, \check{\theta} = \mathcal{F}^{-1}\theta \geq 0$ and $\text{spt}(\theta)$ contained in a sufficiently small neighborhood of the origin. Here $\mathcal{F}^{-1}$, $\mathcal{F}_h^{-1}$ denote the classical and semi-classical inverse Fourier transforms respectively. The trace norm and trace of a positive self-adjoint operator, computed with respect to the same auxiliary density on the manifold, being equal one has

$$\left\| A_0 f(P) \left(\mathcal{F}_h^{-1}\theta\right)(\lambda - P) A_0 \right\|_{\text{tr}} = \text{tr } \left[A_0^2 f(P) \left(\mathcal{F}_h^{-1}\theta\right)(\lambda - P)\right] \tag{3.6}$$

$$= f(\lambda)\theta(0)(2\pi h)^{-n} \left[\int_{\Sigma_\lambda} a_0^2 d\nu + O\left(h^{1-2\delta}\right)\right] \tag{3.7}$$

$$\leq f(\lambda)\theta(0)(2\pi h)^{-n} \left[\nu\left(S_{T,\varepsilon}^{\lambda,e}\right) + O\left(h^{1-2\delta}\right)\right] \tag{3.8}$$

for each $\lambda \in \mathbb{R}$. Here the asymptotics of the trace in the line (3.7) above are evaluated by a standard FIO parametrix and application of stationary phase formula as in [3, Ch. 10]. The exponent in $h^{1-2\delta}$ arises due to the presence of two derivatives of the amplitude $a_0^2 \in S_\delta^0$ with each h-term in the stationary phase formula. The last equation (3.8) can be claimed for arbitrary $\theta \in C_c^\infty(\mathbb{R})$ of small support. The condition $\check{\theta} \geq 0$ can be removed as in [3, eq. 11.5].

Next, for $\theta_c(t) := \theta(t - c)$ one has $\mathcal{F}_h^{-1}\theta_c(x) = e^{i\frac{xc}{h}} \mathcal{F}_h^{-1}\theta(x)$. Furthermore $e^{ic(\lambda - P)}$ being a unitary operator, the trace norm on the left hand side of (3.6) is hence unchanged under translation of θ. Writing $\theta \in C_c^\infty(-T, T)$, of possibly h-dependent compact support $T = T(h)$, as a sum of $O(T)$ translates of functions with h-independent compact support near the origin we obtain

$$\left\| A_0 f(P) \left(\mathcal{F}_h^{-1}\theta\right)(\lambda - P) A_0 \right\|_{\text{tr}} = f(\lambda) \|\theta\|_{C^0} (2\pi h)^{-n} O\left(T\left[\nu\left(S_{T,\varepsilon}^{\lambda,e}\right) + O\left(h^{1-2\delta}\right)\right]\right).$$

Furthermore, Egorov theorem to Ehrenfest time gives

$$\text{tr } \left[A_j^2 e^{\frac{it}{h}P}\right] = \text{tr } \left[e^{\frac{it}{h}P} A_{j,t} A_j\right] = O\left(h^\infty\right) \tag{3.9}$$

for $\theta \in C_c^\infty\left(\frac{1}{2}T_0^{[a-\epsilon',a+\epsilon']}, T\right)$, $T \leq \left(\frac{1}{2} - \delta\right) T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$ following (3.5). Now note that A_j's were chosen to comprise of a square partition of unity on the band $\Sigma_{[a-\epsilon',a+\epsilon']}$. While $WF(f(P)) \subset \Sigma_{[a-\epsilon',a+\epsilon']}$ for $f \in C_c^\infty(a - \epsilon', a + \epsilon')$ by (2.4). Thus the last two equations combine to give

$$\left|\text{tr } \left[f(P)\left(\mathcal{F}_h^{-1}\theta\right)(\lambda - P)\right]\right| = f(\lambda) \|\theta\|_{C^0} (2\pi h)^{-n} O\left(T\left[\nu\left(S_{T,\varepsilon}^{\lambda,e}\right) + O\left(h^{1-2\delta}\right)\right]\right) \tag{3.10}$$

for $f \in C_c^\infty(a - \epsilon', a + \epsilon')$ and $\theta \in C_c^\infty(-T, T)$ with $T \leq \left(\frac{1}{2} - \delta\right) T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$.

Finally writing $\theta \in C_c^\infty(-T,T)$ as a sum $\theta = \theta_1 + \theta_2$ of an h-independent function θ_1 supported sufficiently near the origin and θ_2 supported away from the origin. Applying the estimate (3.10) for θ_2 while using again the FIO parametrix for the h-independent function θ_1 gives

(3.11)
$$\operatorname{tr}\left[f(P)\left(\mathcal{F}_h^{-1}\theta\right)(\lambda-P)\right] = f(\lambda)\,\theta(0)\,(2\pi h)^{-n}\left[\int_{\Sigma_\lambda} d\nu + \|\theta\|_{C^0}\,O\left(T\nu\left(S_{T,\varepsilon}^{\lambda,e}\right)+Th^{1-2\delta}\right)\right]$$

for $f \in C_c^\infty(a-\epsilon', a+\epsilon')$ and $\theta \in C_c^\infty(-T,T)$ with $T \leq \left(\frac{1}{2}-\delta\right)T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$.

3.2. **Tauberian argument.** Following the last equation (3.11), the rest of the proof of Theorem 1 follows a standard Tauberian argument as in [3, Ch. 11] or [21, Appx. B]. We provide the details below for completeness.

First choose an even, $h-$independent Schwartz function $\theta \in \mathcal{S}(\mathbb{R})$ such that $\check{\theta} \geq \frac{1}{1+\epsilon}$, $\epsilon > 0$, on $[0,1]$ and $1 = \theta(0) = \int d\xi\check{\theta}(\xi)$. Setting $\theta_T(x) = \theta(T^{-1}x)$, satisfying $\check{\theta_T}(\xi) = T\check{\theta}(T\xi)$, and choosing $f \in C_c^\infty(a-\epsilon', a+\epsilon')$ with $f(\lambda) = 1$, the trace asymptotics (3.11) now give

$$\begin{aligned}\frac{T}{(1+\epsilon)h}N\left(\lambda,\lambda+T^{-1}h\right) \leq& \operatorname{tr}\left[f(P)\left(\mathcal{F}_h^{-1}\theta_T\right)(\lambda-P)\right]\\ =&(2\pi h)^{-n}\left[\int_{\Sigma_\lambda} d\nu + O\left(T\nu\left(S_{T,\varepsilon}^{\lambda,e}\right)+Th^{1-2\delta}\right)\right]\end{aligned}$$

for each $\epsilon > 0$, $\lambda \in [a-\epsilon', a+\epsilon']$ and $T \leq \left(\frac{1}{2}-\delta\right)T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$. Hence

$$N\left(\lambda,\lambda+T^{-1}h\right) \leq (2\pi h)^{-n}\left[T^{-1}\int_{\Sigma_\lambda} d\nu + O\left(\nu\left(S_{T,\varepsilon}^{\lambda,e}\right)+h^{1-2\delta}\right)\right] \tag{3.12}$$

for each $\ell > 0$, $\lambda \in [a-\epsilon', a+\epsilon']$ and $T \leq \left(\frac{1}{2}-\delta\right)T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$.

Next, the spectral measure for P is defined as $\mu_f(\lambda') := \sum_{\lambda\in\operatorname{Spec}(P)} f(\lambda)\,\delta(\lambda-\lambda')$. Now choose a different even function $\theta \in \mathcal{S}(\mathbb{R})$ such that its transform satisfies $\operatorname{spt}\left(\check{\theta}\right) \subset [-1,1]$, $1 \geq \check{\theta}(\xi) \geq 0$ and $\int \check{\theta}(\xi)\,d\xi = 1$. The two term asymptotics of the wave trace from [3, Ch. 10] now give

$$\mu_f * \left(\mathcal{F}_h^{-1}\theta\right)(\lambda) = f(\lambda)\,\theta(0)\,(2\pi h)^{-n}\left[\underbrace{\int_{\Sigma_\lambda} d\nu}_{=:c_0(\lambda)} + h\underbrace{\int_{\Sigma_\lambda} p_1 d\nu}_{=:c_1(\lambda)} + O\left(h^2\right)\right],$$

$\forall\lambda \in \mathbb{R}$, with the second term involving the sub-principal symbol p_1 of the operator P.

Both sides above involving Schwartz functions in λ, the remainder above can be replaced by $O\left(\frac{h^2}{\langle\lambda\rangle^2}\right)$. Integrating further gives

$$\int_{-\infty}^{a} d\lambda \int d\lambda' \left(\mathcal{F}_h^{-1}\theta\right)(\lambda-\lambda')\,\mu_f(\lambda') \tag{3.13}$$

$$= \int_{-\infty}^{0} d\lambda \int d\lambda'' \left(\mathcal{F}_h^{-1}\theta\right)(\lambda-\lambda'')\,\mu_f(\lambda''+a) \tag{3.14}$$

$$=\theta(0)\,(2\pi h)^{-n}\left[\int_{-\infty}^{a} d\lambda f(\lambda)\,c_0(\lambda) + h\int_{-\infty}^{a} d\lambda f(\lambda)\,c_1(\lambda) + O\left(h^2\right)\right].$$

Now note that

$$\int_{-\infty}^{0} d\lambda \left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda-\lambda''\right) = 1_{(-\infty,0]}\left(\lambda''\right) + \phi_0\left(\frac{\lambda''}{h}\right) \tag{3.15}$$

where $\phi_0\left(x\right) := \int_{-\infty}^{0} dt\check{\theta}\left(t-x\right) - 1_{(-\infty,0]}\left(x\right)$ is a function that is rapidly decaying with all derivatives, odd, smooth on $\mathbb{R}_x\setminus\{0\}$ and satisfies $\phi_0'\left(x\right) = \check{\theta}\left(-x\right)$ for $x\neq 0$.

Next with $x\geq 0$ we compute

$$\begin{aligned}
&\left|\phi_0\left(x\right)-\phi_0*\check{\theta}_T\left(x\right)\right| \\
=&\left|\int dy\left[\phi_0\left(x\right)-\phi_0\left(x-T^{-1}y\right)\right]\check{\theta}\left(y\right)\right| \\
\leq&\int_{y\leq xT} dy\left|\phi_0'\left(c\left(x,y\right)\right)\right| T^{-1}\left|y\right|\check{\theta}\left(y\right) + 2\int_{y\geq xT} dy\check{\theta}\left(y\right) \\
\leq& T^{-1}\underbrace{\int_{-\infty}^{xT} dy\left|y\right|\check{\theta}\left(y\right)}_{=\theta_1(xT)} + 2\underbrace{\int_{y\geq xT} dy\check{\theta}\left(y\right)}_{=\theta_2(xT)},
\end{aligned} \tag{3.16}$$

where $c\left(x,y\right)\in\left[x-T^{-1}y,x\right]$. A similar estimate holds for $x\leq 0$.

Now pairing the second term of (3.16) with $\mu_f\left(\lambda''+a\right)$ gives

$$\int d\lambda''\theta_2\left(\frac{\lambda'' T}{h}\right)\mu_f\left(\lambda''+a\right) \leq \left(2\pi h\right)^{-n}\left[T^{-1}\left\|f\right\|_{C^0}\left(\int_{\Sigma_a} d\nu\right) + O\left(\nu\left(S_{T,\varepsilon}^{a,e}\right)+h^{1-2\delta}\right)\right] \tag{3.17}$$

on covering $\mathbb{R}_{\lambda''}$ with intervals of size $O\left(T^{-1}h\right)$ and using the Weyl estimate (3.12). A similar estimate

$$\int d\lambda'' T^{-1}\theta_1\left(\frac{\lambda'' T}{h}\right)\mu_f\left(\lambda''+a\right) = O\left(h^{-n}T^{-1}\left[T^{-1}+\nu\left(S_{T,\varepsilon}^{a,e}\right)+h^{1-2\delta}\right]\right) \tag{3.18}$$

then gives

$$\begin{aligned}
&\int d\lambda''\left[\phi_0\left(\frac{\lambda''}{h}\right)-\phi_0*\check{\theta}_T\left(\frac{\lambda''}{h}\right)\right]\mu_f\left(\lambda''+a\right) \\
&\leq\left(2\pi h\right)^{-n}\left[T^{-1}\left\|f\right\|_{C^0}\left(\int_{\Sigma_a} d\nu\right)+O\left(T^{-2}+\nu\left(S_{T,\varepsilon}^{a,e}\right)+h^{1-2\delta}\right)\right].
\end{aligned} \tag{3.19}$$

on combining(3.16), (3.17) and (3.18).

The second term above (3.19) is estimated on integrating (3.11) against ϕ_0 as

$$\begin{aligned}
\int d\lambda''\phi_0*\check{\theta}_T\left(\frac{\lambda''}{h}\right)\mu_f\left(\lambda''+a\right) &= \left(2\pi h\right)^{-n}\left[\int d\lambda\phi_0\left(\lambda\right)f\left(0\right)\theta\left(0\right)c_0\left(0\right)+O\left(\nu\left(S_{T,\varepsilon}^{\lambda,e}\right)+h^{1-2\delta}\right)\right] \\
&= O\left(\nu\left(S_{T,\varepsilon}^{\lambda,e}\right)h^{-n}+h^{-n+1-2\delta}\right).
\end{aligned} \tag{3.20}$$

since ϕ_0 is an odd function.

Finally, set $f_a^- = 1_{(-\infty,a]}(x) f(x)$. Then combining (3.13), (3.15), (3.19) and (3.20) gives

$$\begin{aligned}
\operatorname{tr} f_a^-(P) &= \int d\lambda' 1_{(-\infty,a]}(\lambda')\,\mu_f(\lambda') \\
&= (2\pi h)^{-n}\left(\int_{-\infty}^{a} d\lambda f(\lambda)\,c_0(\lambda) + h\int_{-\infty}^{a} d\lambda f(\lambda)\,c_1(\lambda) + O\left(h^2\right)\right) \\
&\quad + \int d\lambda'' \phi_0\left(\frac{\lambda''}{\sqrt{h}}\right)\mu_f(\lambda''+a) \\
&= (2\pi h)^{-n}\left(\int_{-\infty}^{a} d\lambda f(\lambda)\,c_0(\lambda) + h\int_{-\infty}^{a} d\lambda f(\lambda)\,c_1(\lambda)\right) + R(h),
\end{aligned}$$

$$\text{with}\quad R(h) \leq (2\pi h)^{-n+1}\left[T^{-1}\|f\|_{C^0}\left(\int_{\Sigma_a} d\nu\right) + O\left(T^{-2} + \nu\left(S_{T,\varepsilon}^{a,e}\right) + h^{1-2\delta}\right)\right],$$

for each $\ell > 0$ and $T \leq \left(\frac{1}{2} - \delta\right) T_E^{\ell,[a-\epsilon',a+\epsilon'],\delta}(h)$. Letting $\epsilon' \to 0$ and using the upper semi-continuity of $\Lambda_{\max}$, one obtains the above for each $\ell > 0$ and $T \leq \left(\frac{1}{2} - \delta\right) T_E^{\ell,a,\delta}(h)$.

A similar estimate can be proved for the functional trace of $f_b^+(x) = 1_{[b,\infty)}(x) f(x)$. The Weyl law of Theorem 1 now follows on writing the counting function as the difference of two such functional traces.

4. Examples of recurrence

In this section we give estimates on the volumes of recurrence sets of various flows. These shall be used in the next Section 5 to prove the corollaries stated in the introduction.

4.1. **Anosov flows.**

First we shall consider the recurrence set of an Anosov vector field V on a compact manifold Y^m of dimension m. To recall, a vector field V is said to be Anosov if there exists a constant $c_1 > 0$ and a continuous V-invariant splitting

$$TY = \mathbb{R}[V] \oplus E^u \oplus E^s \quad \text{such that}$$

$$\begin{aligned}
\left\| e^{tV}\big|_{E^s} \right\| &\leq e^{-c_1 t}, \\
\left\| e^{-tV}\big|_{E^u} \right\| &\leq e^{-c_1 t}, \qquad (4.1)
\end{aligned}$$

$\forall t > 0$. Here the norm is taken with respect to some Riemannian metric g^{TY} on the manifold. For such a vector field its recurrence set $S_{T,\varepsilon}$ (1.3), Lyapunov exponents $\lambda_j(y)$ (2.8), maximal expansion rate $\Lambda_{\max}$ (2.9) and topological entropy $\mathtt{h}_{\text{top}}$ (2.12) can be analogously defined.

Let $\lambda > \Lambda_{\max}$ be greater than the maximal expansion rate. By definition of the maximal expansion rate, and the semi-group property of the flow, there exists $c > 0$ such that

$$\begin{aligned}
\left\| \left(e^{tV}\right)^* f \right\|_{C^2} &\leq c e^{\lambda t} \|f\|_{C^2} \quad \text{and} \\
d^{g^{TY}}\left(e^{tV}x_1, e^{tV}x_2\right) &\leq c e^{\lambda t} d^{g^{TY}}(x_1, x_2) \qquad (4.2)
\end{aligned}$$

for all $t > 0$, $x_1, x_2 \in Y$ and $f \in C^\infty(Y)$. From here, the following exponential bounds on the volume of the recurrence set

$$\nu\left(S_{T,\varepsilon}\right) = O\left(\varepsilon^m e^{m\lambda T}\right) \qquad (4.3)$$

$$\nu\left(S_{T,\varepsilon}^e\right) = O\left(\varepsilon^m e^{m\lambda T}\right), \qquad (4.4)$$

can be proved following an argument given in [5]. Namely, the recurrence set above has an obvious lift

$$\tilde{S}_{T,\varepsilon} := \left\{ (y,t) \,|\, t \in \left[\frac{1}{2}T_0, T\right] \text{ s.t. } d^{g^{TY}}\left(e^{tV}x, x\right) \leq \varepsilon \right\} \subset Y \times \mathbb{R}_t \tag{4.5}$$

satisfying $\pi_Y\left(\tilde{S}_{T,\varepsilon}\right) = S_{T,\varepsilon}$ under the projection onto the first Y factor.

The volume bounds (4.3), (4.4) then follow from the following proposition.

Proposition 5. *For each $\lambda > \Lambda_{max}$, the lift $\tilde{S}_{T,\varepsilon}$ (4.5) satisfies the volume estimate*

$$\nu_{Y\times\mathbb{R}}\left(\tilde{S}_{T,\varepsilon}\right) = O\left(\varepsilon^m e^{m\lambda T}\right) \tag{4.6}$$

with respect to the Riemannian product measure on $Y \times \mathbb{R}$.

Proof. First we claim that there exist $C, \delta > 0$ of the following significance: for each $\varepsilon > 0$ and each pair $(x,t), (x',t') \in \tilde{S}_{T,\varepsilon}$ satisfying $|t-t'| \leq \delta$, $d^{g^{TY}}(x,x') \leq \delta e^{-\lambda t}$ one has

$$|t-t'| \leq C\varepsilon, \quad d^{g^{TY}}\left(x, \cup_{t\in[-1,1]} e^{tV}x'\right) \leq C\varepsilon. \tag{4.7}$$

By choosing δ sufficiently small and using (4.2) we may work in a sufficiently small geodesic coordinate chart. The V-direction and $E^u \oplus E^s$ being transverse, we may replace x' by $e^{tV}(x')$, $t \in [-1,1]$, to arrange $x - x' \in E^u(x) \oplus E^s(x)$. Using (4.2) and a Taylor expansion in x, t we obtain

$$\begin{aligned}\left| e^{tV}(x) - e^{t'V}(x') - de^{tV}(x)(x-x') - V\left(e^{tV}(x')\right)(t-t') \right| &\leq c_3 e^{\lambda t}|x-x'|^2 + c_3|t-t'|^2 \\ &\leq c_3\delta|x-x'| + c_3\delta|t-t'|.\end{aligned}$$

Since $(x,t), (x',t') \in \tilde{S}_{T,\varepsilon}$ the above gives

$$\begin{aligned}c_3\delta|x-x'| + c_3\delta|t-t'| + 2\varepsilon &\geq \left| \left(I - de^{tV}(x)\right)(x-x') - V\left(e^{tV}(x')\right)(t-t') \right| \\ &\geq c_4|x-x'| + c_4|t-t'|\end{aligned}$$

with the second line above following from the Anosov property. It then remains to choose δ sufficiently small in relation to c_3, c_4 to obtain (4.7).

Finally, let x_j, $j = 1, \ldots N$, be a maximal set of points such that $d^{g^{TY}}(x_i, x_j) \geq \delta e^{-\lambda T}$. As the balls $\left\{ B_{\frac{\delta e^{-\lambda T}}{2}}(x_j) \right\}_{j=1}^N$ centered at these points are disjoint, the bound $N \leq c_5 e^{m\lambda T}$ follows by a computation of the total volume. Furthermore the sets

$$\begin{aligned}B_{j,k} &:= B_{2\delta e^{-\lambda T}}(x_j) \times \left[\frac{1}{2}T_0 + k\delta, \frac{1}{2}T_0 + (k+1)\delta\right], \\ S_{j,k} &:= \tilde{S}_{T,\varepsilon} \cap B_{j,k}, \qquad j = 1, \ldots N, \quad k = 0, \ldots, 1 + \left[\delta^{-1}T\right],\end{aligned}$$

cover $Y \times \left[\frac{1}{2}T_0, T\right]$ and $\tilde{S}_{T,\varepsilon}$ respectively. By (4.7), small $O(\varepsilon)$ size neighborhoods of the orbits

$$\left(\underbrace{\frac{1}{2}T_0 + \left(k + \frac{1}{2}\right)\delta}_{=:t_k}, \cup_{t\in[-1,1]} e^{(t_k+t)V}(x_j) \right)$$

of volume $O(\varepsilon^m)$, then cover $\tilde{S}_{T,\varepsilon}$ proving (4.6). □

4.1.1. Λ_{max} *vs* $\mathtt{h}_{top}$. In section 2.2 we stated how the inequality $h_{\text{top}}(V) \leq m.\Lambda_{\max}$ (2.14) follows from the Margulis-Ruelle formula for a general flow of a vector field V on a compact manifold m-dimensional manifold Y. In this section, we show that a reverse inequality holds between the two invariants when the vector field is further assumed to be Anosov. Namely, we shall prove the following.

Theorem 6. *For an Anosov vector field V on a compact m-dimensional manifold Y one has*

$$\frac{m}{4}.\Lambda_{max} \leq h_{top}(V). \tag{4.8}$$

We shall prove the above at the end of this subsection following some preparation. Namely, to prove the above we shall use the equivalent Bowen-Margulis definition of topological entropy. For each $T > 0$ one defines the Bowen distance on Y via

$$d_T^{g^{TY}}(y_1, y_2) := \sup_{t\in[0,T]} d^{g^{TY}}\left(e^{tR}y_1, e^{tR}y_2\right),$$

where $d^{g^{TY}}$ denotes the Riemannian distance corresponding to some Riemannian metric g^{TY} on Y. A (T, ϵ) separated subset $S \subset Y$ is a finite set in which any two distinct points are at least distance ϵ apart with respect to the above d_T. Denote by $N(T, \epsilon)$ the maximum cardinality of a (T, ϵ) separated set in Y. The topological entropy of the flow (2.12) is now equivalently defined by

$$\mathtt{h}_{\text{top}} = \mathtt{h}_{\text{top}}(V) := \lim_{\epsilon\to 0}\left(\limsup_{T\to\infty}\frac{\ln N(T,\epsilon)}{T}\right). \tag{4.9}$$

Next for each $s \in (0, 1]$, let $\mathcal{D}_s(Y)$, $\mathcal{D}_{s-}(Y)$ respectively be the set of compatible distorted distance functions d on the manifold satisfying

$$d^{g^{TY}} \lesssim d \lesssim \left(d^{g^{TY}}\right)^s$$
$$d^{g^{TY}} \lesssim d \lesssim \left(d^{g^{TY}}\right)^{s-\epsilon}, \quad \text{for some } \epsilon > 0,$$

respectively. The following inclusions are clear

$$\mathcal{D}_s(Y) \subset \mathcal{D}_{s'}(Y),$$
$$\mathcal{D}_{s-}(Y) \subset \mathcal{D}_{s'-}(Y), \quad s' < s.$$

Furthermore, all distances in $\mathcal{D}_s(Y)$, $\mathcal{D}_{s-}(Y)$ define the same manifold topology while $\mathcal{D}_1(Y)$ is the set of all distances equivalent to the $d^{g^{TY}}$ and hence includes all Riemannian distances. To each distance $d \in \mathcal{D}_s(Y)$, $\mathcal{D}_{s-}(Y)$ we can associate the Lipschitz constant of its time one flow

$$L_d = L_d\left(e^V\right) := \sup_{y_1\neq y_2}\frac{d\left(e^V y_1, e^V y_2\right)}{d(y_1, y_2)}. \tag{4.10}$$

The following notion of the local skewness of the time one map shall also be useful. It is defined as

$$SL_d\left(e^V\right) := \sup_{\varepsilon>0}\inf_{0<d(y_1,y_2)<\varepsilon}\frac{d\left(e^V y_1, e^V y_2\right)}{d(y_1, y_2)}.$$

We now have the following inequalities for topological entropy.

Lemma 7. *The topological entropy* (2.12) *of an Anosov vector field satisfies the inequalities*

$$\frac{m}{2}\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(Y)}\ln L_d\right) \leq \mathtt{h}_{top} \leq m\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(Y)}\ln L_d\right)$$

in relation to the infimum of the log Lipschitz constants (4.10) *in* $\mathcal{D}_{\frac{1}{2}-}(Y)$.

Proof. Let HD (d) denote the Hausdorff dimension of the manifold with respect to the distance $d \in \mathcal{D}_{\frac{1}{2}-}(Y)$. The inequalities

$$\mathrm{HD}(d)\ln SL_d \leq \mathtt{h}_{\mathrm{top}} \leq \mathrm{HD}(d)\ln L_d \tag{4.11}$$

are fairly well known (see [6, 20] or [25, Thm. 7.15]). Furthermore $\frac{m}{2} \leq \mathrm{HD}(d) \leq m$ follows from the definition. This proves one half of the lemma

$$\mathtt{h}_{\mathrm{top}} \leq m\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(Y)} \ln L_d\right). \tag{4.12}$$

One is now left with constructing a sequence of distances $d_k \in \mathcal{D}_{\frac{1}{2}-}(Y)$, $k = 1, 2, \ldots$ such that $m \ln L_{d_k}$ approaches $2\mathtt{h}_{\mathrm{top}}$ as $k \to \infty$. Such a sequence of distances d_k can be constructed for expansive maps [6, 20], cf. also the construction of the Hamenstädt distance [10]. The time one map e^V is however not expansive in the flow direction. This lack of expansiveness can nonetheless be replaced with the following instability property that is satisfied by the flow e^V [18]: there is a positive constant $c_2 > 0$ for which one has the following implication

$$y \neq e^{tV}x,\ \forall t \in \mathbb{R} \implies d^{g^{TY}}\left(e^{jV}x, e^{jV}y\right) > c_2 \text{ for some } j \in \mathbb{Z}. \tag{4.13}$$

In fact the proof of the above therein gives the following stronger statement: for any $\epsilon > 0$ there exist positive constants $c > 0$, $\alpha > 1$ such that for $\alpha_\epsilon := \alpha + \epsilon$ one has the stronger implication

$$\begin{aligned} &y \neq e^{tV}x,\ \forall t \in \mathbb{R} \quad \text{and} \quad d^{g^{TY}}(x, y) < c_2, \\ \implies &\alpha d^{g^{TY}}(x, y) \leq \max\left\{d^{g^{TY}}\left(e^{V}x, e^{V}y\right), d^{g^{TY}}\left(e^{-V}x, e^{-V}y\right)\right\} \leq \alpha_\epsilon d^{g^{TY}}(x, y). \end{aligned}$$

The constant α can be related to the exponent c_1 in the definition (4.1) of the Anosov condition.

We now define

$$N(x, y) := \begin{cases} \infty, & x = y, \\ \inf\left\{N \in \mathbb{N}_0 \middle| \max_{j\in[-N,N]} d^{g^{TY}}\left(e^{jV}x, e^{jV}y\right) > c_2\alpha^{-|j|}\right\}, & x \neq y. \end{cases}$$

The following bounds are straightforward

$$\max\left\{0, \frac{\ln \frac{c_2}{d(x,y)}}{\ln \alpha\alpha_\epsilon}\right\} \leq N(x, y) \leq \max\left\{0, \frac{\ln \frac{c_2}{d(x,y)}}{\ln \alpha}\right\}. \tag{4.14}$$

And we now set

$$\rho(x, y) := \alpha^{-N(x,y)}, \quad \text{satisfying} \tag{4.15}$$

$$\frac{d^{g^{TY}}(x, y)}{c_2} \leq \rho(x, y) \leq \left[\frac{d^{g^{TY}}(x, y)}{c_2}\right]^{\ln\alpha/\ln(\alpha\alpha_\epsilon)} \qquad \text{for } d^{g^{TY}}(x, y) \leq c_2. \tag{4.16}$$

It hence follows that ρ defines the same manifold topology as $d^{g^{TY}}$, but it does not quite define a distance. The inequalities (4.14) further give $d^{g^{TY}}(x, y) \geq \frac{c_2}{2} \implies N(x, y) \leq \frac{\ln 2}{\ln \alpha} \implies \alpha^N \leq \alpha^{\frac{\ln 2}{\ln \alpha}} = 2$. And applying the triangle inequality for $d^{g^{TY}}$ one obtains

$$\begin{aligned} &\min\left\{N(x, y), N(y, z)\right\} \leq M + N(x, z) \qquad \text{and} \\ &\qquad\qquad \rho(x, z) \leq 2\max\left\{\rho(x, y), \rho(y, z)\right\} \quad \forall x, y, z \in Y \end{aligned}$$

as a weaker version of the triangle inequality for ρ. One now applies Frink's metrization theorem [8] to obtain the existence of a metric D on Y satisfying

$$D(x,y) \le \rho(x,y) \le 4D(x,y). \tag{4.17}$$

Thus D defines the same topology as $d^{g^{TY}}$. And furthermore we have $D \in \mathcal{D}_{\frac{1}{2}-}(Y)$ on account of (4.16) and (4.17).

It is now an exercise to show that $\rho\left(e^{jV}x, e^{jV}y\right) \le \alpha^j \rho(x,y)$ with equality on some neighborhood $V_j \subset Y \times Y$ of the diagonal in the product. From (4.17) this gives

$$\begin{aligned} D\left(e^{jV}x, e^{jV}y\right) &\le 4\alpha^j D(x,y), \quad \forall x, y \in Y, \\ D\left(e^{jV}x, e^{jV}y\right) &\ge \frac{1}{4}\alpha^j D(x,y), \quad \forall (x,y) \in V_j. \end{aligned} \tag{4.18}$$

And thus

$$L_D\left(e^{jV}\right) \le 4\alpha^j \le 16 S L_D\left(e^{jV}\right). \tag{4.19}$$

Finally we define the following sequence of distances

$$d_k(x,y) := \max_{0\le j\le k-1} \frac{D\left(e^{jV}x, e^{jV}y\right)}{L_D^{j/n}} \tag{4.20}$$

$k \in \mathbb{N}$, which are all equivalent to D. Their Lipschitz constants $L_{d_k}\left(e^V\right) = \left[L_D\left(e^{kV}\right)\right]^{1/k}$ are seen to be given in terms of the D-Lipschitz constants of the time k map. Using (4.11), (4.19) and $\mathtt{h}_{\text{top}}\left(e^{kV}\right) = k\mathtt{h}_{\text{top}}\left(e^V\right)$ one now obtains

$$\begin{aligned} m\left(\frac{\ln\alpha}{\ln(\alpha\alpha_\epsilon)}\right)\ln L_{d_k}\left(e^V\right) &\le \mathrm{HD}(d_k)\ln L_{d_k}\left(e^V\right) \\ &\le \frac{\mathrm{HD}(d_k)}{k}\ln L_D\left(e^{kV}\right) \\ &\le \frac{\mathrm{HD}(d_k)}{k}\left[\ln 16 + \ln S L_D\left(e^{kV}\right)\right] \\ &\le \frac{\mathrm{HD}(d_k)}{k}\ln 16 + \frac{1}{k}\mathtt{h}_{\text{top}}\left(e^{kV}\right) \\ &= \frac{\mathrm{HD}(d_k)}{k}\ln 16 + \mathtt{h}_{\text{top}}\left(e^V\right) \\ &\le \frac{m}{k}\ln 16 + \mathtt{h}_{\text{top}}\left(e^V\right). \end{aligned}$$

Letting $k \to \infty$, and noting that $\left(\frac{\ln\alpha}{\ln(\alpha\alpha_\epsilon)}\right) \to \frac{1}{2}$ as $\epsilon \to 0$, one obtains the theorem. □

The above lemma now implies the main result of this subsection Theorem 6.

Proof of Theorem 6. As the previous Lemma 7 shows, for each $\lambda > \frac{2}{m}\mathtt{h}_{\text{top}}$ we have

$$L_d \le e^\lambda, \quad \text{for some } d \in \mathcal{D}_{\frac{1}{2}-}(Y).$$

Using the semi-group property of the flow one obtains a positive constant $c > 0$ such that

$$d\left(e^{tV}x_1, e^{tV}x_2\right) \le c e^{\lambda t} d(x_1, x_2).$$

From $d^{g^{TY}} \lesssim d \lesssim \left(d^{g^{TY}}\right)^{\frac{1}{2}-\epsilon}$ this further gives

$$d^{g^{TY}}\left(e^{tV}x_1, e^{tV}x_2\right) \leq ce^{2\lambda t} d^{g^{TY}}\left(x_1, x_2\right)$$
$$\left\|\left(e^{tV}\right)^* f\right\|_{C^2} \leq ce^{2\lambda t}\left\|f\right\|_{C^2} \tag{4.21}$$

$\forall x_1, x_2 \in Y,\ f \in C^2(Y)$. The inequality $\Lambda_{\max} \leq 2\lambda$ now follows easily from the last equation and the definition of the maximal expansion rate. □

4.2. **Compact Lie Groups.** Next we consider geodesic flows associated to bi-invariant metrics on compact Lie groups. In this case the volume bound on the recurrence set is the one given below. It was essentially proved by Volovoy in [24, Prop. 4] and we refine the bound while following his outline.

Theorem 8. *Let G be a compact Lie group equipped with a bi-invariant metric g. The recurrence sets for its geodesic flow satisfies the volume bounds*

$$\nu\left(S^1_{T,\varepsilon}\right) = O\left(\varepsilon^{p-1}T^p\right) \tag{4.22}$$
$$\nu\left(S^{1,e}_{T,\varepsilon}\right) = O\left(\varepsilon^{p-1}T^p\right), \tag{4.23}$$

where $p = rk\ G$ is the rank of the Lie group.

Proof. With $\mathfrak{g}$ being its Lie algebra, let $\exp : \mathfrak{g} \to G$ denote the exponential mapping of the Lie group. For bi-invariant metrics, the path $A \exp tb$, $b \in \mathfrak{g} = T_I G$, gives the geodesic through the point $A \in G$ in the direction $dL_A(b) \in T_A G$ (here L_A denotes left multiplication by A).

From the compactness of the Lie group, one finds a uniform Lipschitz constant $C > 0$ such that

$$d^g\left(A \exp tb, A\right) \leq C d^g\left(\exp tb, I\right), \quad \forall A \in G,\ b \in \mathfrak{g},\ t > 0. \tag{4.24}$$

This gives a constant C_1 such that

$$\nu\left(S^1_{T,\varepsilon}\right) \leq C_1 \nu_I \underbrace{\left\{a \in S^*_I G | \exists t \in \left[\frac{1}{2}T_0, T\right] \text{ s.t. } d^g\left(\exp ta, I\right) \leq \varepsilon\right\}}_{=S^1_{T,\varepsilon,I}} \tag{4.25}$$

$$\nu\left(S^{1,e}_{T,\varepsilon}\right) \leq C_1 \nu_I \underbrace{\left\{a \in S^*_I G | d^g\left(a, S^1_{T,\varepsilon,I}\right) \leq \varepsilon\right\}}_{=S^{1,e}_{T,\varepsilon,I}} \tag{4.26}$$

for each $\varepsilon > 0$ and $T > \frac{1}{2}T_0$. Here ν_I denote the induced measure on the unit sphere inside the dual Lie algebra $\mathfrak{g}^* = S^*_I G$. It thus suffices to estimate the measure of the recurrence set based at the identity $S^1_{T,\varepsilon,I}$ on the right hand side above (4.25).

Now let $H = \mathbb{T}^p \subset G$, $p = \text{rk}\ G$, be a maximal torus. Similar recurrence sets $S^{1,H}_{T,\varepsilon,I}$, $S^{1,e,H}_{T,\varepsilon,I}$ as (4.25), (4.26) can be defined that is based at the identity $I \in H$ in the maximal torus. In [24, Prop. 3] the measure bound $\nu\left(S^{1,H}_{T,\varepsilon,I}\right) = O\left(\varepsilon^{p-1}T^p\right)$ for the based recurrence set inside an arbitrary torus was proved. In fact, [24, Cor. 3] showed that $S^{1,H}_{T,\varepsilon,I} \subset S^*_I H$, and thus $S^{1,e,H}_{T,\varepsilon,I}$ too, could be covered with a collection of radius ε- balls $\left\{B_\varepsilon\left(h_j\right) | h_j \in S^*_I H\right\}_{j=1}^M$, where $M = O\left(T^p\right)$. For a general group, any element $a \in S^1_{T,\varepsilon,I} \subset S^*_I G$ is conjugate to an element in the torus $\exp\left(t\text{ad}_P a\right) = P\left(\exp ta\right)P^{-1} \in H$ for some $P \in G$. It follows from (4.24) that the conjugates $\text{ad}_P a \in S^{1,H}_{T,C\varepsilon,I}$, $\text{ad}_P b \in S^{1,e,H}_{T,C\varepsilon,I}$ are elements of the based recurrence sets of the torus, for $a, b \in S^1_{T,\varepsilon,I}, S^{1,e}_{T,\varepsilon,I}$ respectively, and hence in one of the $M = O\left(T^p\right)$ balls $B_\varepsilon\left(h_j\right)$ of radius

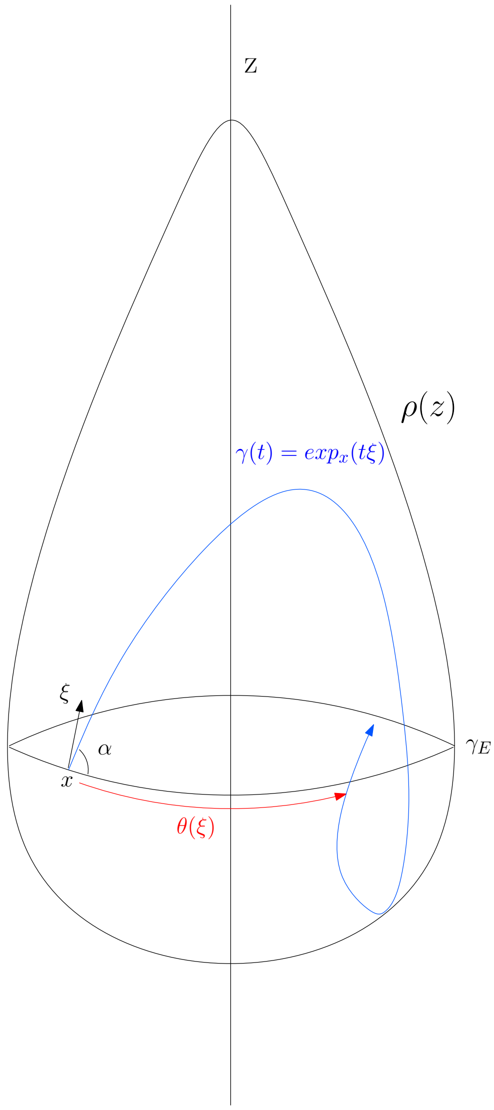

FIGURE 4.1. Surface of revolution

ε. It thus suffices to prove the estimate $\nu(S_j) = O(\varepsilon^{p-1})$ on the volumes of the conjugates $S_j \coloneqq G.B_\varepsilon(h_j).G^{-1}$, $j = 1, 2, \ldots, M$ of these balls. This was also done by Volovoy in [24, pgs. 134-135]. □

4.3. **Surface of revolution.** We now consider geodesic flows on compact surfaces of revolution.

Namely, the manifold is now given as

$$X = \{(\rho(z)\cos\phi, \rho(z)\sin\phi, z) \,|\, \phi \in [0, 2\pi], z \in [a_-, a_+]\}$$
$$\subset \mathbb{R}^3. \tag{4.27}$$

Here $\rho : (a_-, a_+) \to (0, \infty)$ is a smooth function satisfying $\lim_{z\to a_\pm} \rho(z) = 0$, $\lim_{z\to a_\pm} \rho'(z) = \mp\infty$. The surface (4.27) is thus obtained by rotating the curve $(\rho(z), 0, z)$, $a \le z \le b$, around the z-axis (see Figure 4.1 on page 18). We shall further assume that the surface is strictly convex. That is, the function $-\rho$ is strictly convex satisfying $-\rho''(z) > 0$. Thus ρ is maximized at a unique $z_0 \in [a_-, a_+]$. The curve $\gamma_E \coloneqq \{(x, y, z) \in X | z = z_0\}$ shall be referred to as the *equator*. The points $(0, 0, a_\pm)$ are referred to as the north and south poles respectively.

The metric on X is chosen to be the one induced from the Euclidean embedding. The geodesic flow on surfaces of revolution is well understood. Firstly, the Hamiltonian function can be computed

$$H\left(\xi_z,\xi_\phi\right)=\frac{1}{2}\left[\frac{\xi_\phi^2}{\rho^2}+\frac{\xi_z^2}{1+\rho_z^2}\right]$$

in terms of the cylindrical coordinates (z,ϕ). The angular momentum function ξ_ϕ Poisson commutes with the above Hamiltonian and is hence preserved under the geodesic flow. This conservation law can be rewritten in a more explicit form on the base. The geodesic $\gamma=(z\left(t\right),\phi\left(t\right))$ is the projection of the Hamilton trajectory

$$\left(z\left(t\right),\phi\left(t\right);\underbrace{\xi_z\left(t\right)}_{=(1+\rho_z^2)\dot{z}},\underbrace{\xi_\phi\left(t\right)}_{=\rho^2\dot{\phi}}\right),$$

and thus the function $\rho^2\dot{\phi}=\xi_\phi\left(t\right)=c$ is constant along the flow. Denote by α the angle between the velocity vector $\dot{\gamma}\left(t\right)$ at a point $\gamma\left(t\right)$ of the geodesic with the parallel through the point. It is easy to compute $\rho\dot{\phi}=\left|\dot{\gamma}\right|\cos\alpha$. The Hamiltonian/length $\frac{1}{2}\left|\dot{\gamma}\right|^2=H\left(\gamma\left(t\right)\right)$ is preserved along the flow. Thus the conservation law of ξ_φ can thus be restated by saying that

$$\rho\cos\alpha=c\;(\text{constant }) \tag{4.28}$$

along the flow. It is easy to integrate the above equation given the initial point and velocity vector. Conversely, and from the uniqueness existence of of geodesics, any unit speed (non-parallel) curve along which the above relation (4.28) holds is a geodesic. While the only parallel geodesic is the equator γ_E. The last three lines constitute the statement of *Clairaut's theorem.*

Theorem 9. *Consider* (X^2,g) *a compact, strictly convex surface of revolution. Then volumes of the recurrence sets of the geodesic flow* (1.3) *satisfy the estimates*

$$\nu\left(S^1_{T,\varepsilon}\right)=O\left(\varepsilon^{\frac{1}{r}}T^{1-\frac{1}{r}}\right) \tag{4.29}$$

$$\nu\left(S^{1,e}_{T,\varepsilon}\right)=O\left(\varepsilon^{\frac{1}{r}}T^{1-\frac{1}{r}}\right), \tag{4.30}$$

for ε,T *sufficiently small. Here* r (1.12) *is the maximum order of vanishing of the equatorial return map.*

Proof. It follows easily from the Clairaut relation (4.28), that every geodesic necessarily intersects the equator γ_E in a uniformly finite time. By compactness, it thus suffices to estimate the measure of the based recurrence sets

$$S^1_{T,\varepsilon,x}:=\left\{\xi\in S^*_xX|\exists t\in\left[\frac{1}{2}T_0,T\right]\text{ s.t. }d\left(\exp_x\left(t\xi\right),x\right)\leq\varepsilon,\ \exp_x\left(t\xi\right)\in\gamma_E\right\}$$
$$S^{1,e}_{T,\varepsilon,x}:=\left\{\xi\in S^*_xX|d\left(\xi,S^1_{T,\varepsilon,x}\right)\leq\varepsilon\right\}$$

for $x\in\gamma_E$.

Next for each $x\in\gamma_E$ on the equator, we define

$$\tau:S^*_xX\to\mathbb{R}_{>0} \tag{4.31}$$

$$\theta:S^*_xX\to S^1=[0,2\pi] \tag{4.32}$$

as functions defined for elements in the cosphere $\xi\in S^*_xX$ above x. The first maps ξ to the time of first return $\tau\left(\xi\right)$ of the geodesic $\gamma\left(t\right)=\exp_x\left(t\xi\right)$ to the equator. While the second maps ξ to the angle of rotation $\theta\left(\xi\right)$ of the equator needed to take x to the point of first return

$\exp_x(\tau(\xi)\xi)$ (see Figure 4.1 on page 18). There are standard formulas for θ in terms of the defining function ρ for the surface (cf. [2, Ch. 4B]). Now by Clairaut's equation (4.28) the angle α between the geodesic and the equator will be the same at $\exp_x(\tau(\xi)\xi)$. Thus by rotational symmetry, we have the relations

$$\gamma(t+\tau(\xi)) = R_{\theta(\xi)}\gamma(t)$$

and the subsequent times of return will be $2\tau(\xi), 3\tau(\xi), \dots$ respectively. From the continuity of τ one can thus find a positive $C>0$ such that for each $t>0$ there exists $|s|\leq C$ and $p\in\mathbb{Z}$, $|p|\leq Ct$, satisfying $\exp_x(t\xi) = R_{p\theta(\xi)}\gamma(s)$.

One is thus reduced to estimating the measure of the set

$$\tilde{S}_{T,\varepsilon,x} = \left\{\xi\in S_x^*X \middle| \exists p\in\mathbb{Z}\setminus\{0\}, q\in\mathbb{Z},\ |p|,|q|\leq CT \text{ s.t. } \left|2\pi\theta(\xi)-\frac{q}{p}\right| \leq \frac{\varepsilon}{p}\right\}. \tag{4.33}$$

From the definition of r (1.12) as the maximum vanishing order, one has $|\theta(\xi)-\theta(\xi_0)| \geq C|\xi-\xi_0|^r$ for each pair $\xi,\xi_0\in S_x^*X$ sufficiently close. The volume of the set above is now easily estimated as

$$\nu_x\left(\tilde{S}_{T,\varepsilon,x}\right) = O\left(\sum_{p=1}^{CT}\left(\frac{\varepsilon}{p}\right)^{\frac{1}{r}}\right) = O\left(\varepsilon^{\frac{1}{r}}T^{1-\frac{1}{r}}\right)$$

as required. □

5. Proofs of the Corollaries

In this section we prove the three corollaries of our main Theorem 1. They shall be based on the volume bounds on the recurrence sets from Section 4.

Proofs of the Corollaries 2, 3, 4. The pseudodifferential operator in all corollaries is the semi-classical Laplacian $P_h = h^2\Delta_g$ and the interval to be $[a,1]$ with $a<0$. The principal symbol of the Laplacian is $p_0 = |\xi|^2 \in C^\infty(T^*X)$ is the norm square function on the cotangent bundle. While the sub-principal symbol is zero $p_1=0$. Its relevant energy level is $\Sigma_1 = S^*X$ the unit cosphere bundle of the manifold. This carries the contact form $\alpha_g\in\Omega^1(S^*X)$ which is the restriction $\alpha_g = \alpha|_{S^*X}$ of the tautological one form on the cotangent space. It is then well known that the Hamilton vector field of the principal symbol $R_g = H_{|\xi|^2}$ is the Reeb vector field of this contact form.

For Corollary 2, the manifold is taken to be a negatively curved Riemannian manifold. In this case, the geodesic flow is known to be an Anosov Reeb flow. Hence the recurrence set volume bounds (4.3), (4.4) apply. We may now set $\varepsilon = h^\delta$, with $\delta=\frac{1}{4}$, and $T = \frac{1}{4}T_E^{\ell,1}(h) = \frac{1}{4}\frac{|\ln h|}{\Lambda^1_{\max}+\ell}$ in Theorem 1. Then (1.7) becomes

$$|R_h| \leq \text{vol}(S^*X)\,4\left(\Lambda^1_{\max}+\ell\right)|\ln h|^{-1} + O\left(h^{\frac{1}{4}.(2n-1).\frac{\ell}{\Lambda^1_{\max}+\ell}} + |\ln h|^{-2} + h^{\frac{1}{2}}\right)$$

using the volume bounds (4.3), (4.4). Since $\ell>0$ is arbitrary, the equation (1.9) is proved. The second estimate (1.10) is now a consequence of the inequality $\Lambda^1_{\max}\leq\frac{4}{n}h_{\text{top}}$ from Theorem 6.

For Corollary 3, the manifold $X=G$ is a compact Lie group equipped with a bi-invariant Riemannian metric. From [24, Lem. 2], the Jacobian satisfies the bounds in (1.4) with $C=1$. We may now set $\varepsilon=h^\delta$, with $\left[0,\frac{1}{2}\right)\ni\delta = \begin{cases}0, & p=1,\\ \frac{p+1}{4p}, & p>1,\end{cases}$ and $T = h^{-\left(\frac{1}{2}-\delta-\ell\right)} = h^{-\left(\frac{p-1}{4p}-\ell\right)}$ in

Theorem 1. Then (1.7) becomes

$$\begin{aligned} R_h &= O\left(T^{-1} + \varepsilon^{p-1}T^p + h^{1-2\delta}\right) \\ &= O\left(h^{\frac{p-1}{4p}-\ell}\right), \quad \forall \ell > 0, \end{aligned}$$

for $p > 1$ using the volume bounds (4.22), (4.23) as required.

Finally for the last Corollary 4, the manifold is a surface of revolution. From [24, Lem. 3], the Jacobian satisfies the bounds in (1.4) with $C = 0$. And consequently the Ehrenfest time is infinite. Here we set $\varepsilon = h^\delta$, with $\delta = \frac{2r-1}{4r-1} \in \left[0, \frac{1}{2}\right)$ and $T = h^{-\frac{1}{4r-1}}$ in Theorem 1. Then (1.7) becomes

$$\begin{aligned} R_h &= O\left(\varepsilon^{\frac{1}{r}}T^{1-\frac{1}{r}} + T^{-1} + h^{1-2\delta}\right) \\ &= O\left(h^{\frac{1}{4r-1}}\right) \end{aligned}$$

using (4.29) as required.

We remark that our choices of ε and T are optimal based on the corresponding bounds for recurrence set volumes in each case. □

6. Declarations

6.1. Ethics approval and consent to participate. Informed consent was obtained from all individual participants included in the study.

6.2. Consent for publication. I have read and understood the publishing policy, and submit this manuscript in accordance with this policy.

6.3. Availability of data and materials. No, all of the material is owned by the authors and/or no permissions are required.

6.4. Competing interests. No, I declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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