New results and open problems about Bergman kernel asymptotics

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New trends and open problems in Geometry and Global Analysis, Castle Rauischholzhausen Marburg

Part I:

Survey of Bergman kernel asymptotics

Motivation

- Tian-Yau-Donaldson program
- Berezin-Toeplitz quantization
- Arithmetic geometry (asymptotics of the analytic torsion)
- Quantization of Chern-Simons theory
- Random Kähler geometry, quantum chaos
- Quantum Hall effect

Bergman Projection

- (X, Θ) Hermitian manifold, $\dim X = n$
- volume form $dv_X = \Theta^n/n!$
- $(L, h) \to X$ holomorphic Hermitian line bundle
- $L^2(X, L)$ = space of L^2 sections
- $(s, s')= \int_X \langle s(x), s'(x)\rangle_h \, dv_X(x)\,,\ s, s' \in L^2(X, L).$
- $H^0_{(2)}(X, L)$ = space of L^2 holomorphic sections
- $P : L^2(X, L) \to H^0_{(2)}(X, L)$ Bergman projection

Bergman Kernel

- $\{s_j : j = 1, \ldots, d_p\}$ ONB of $H^0_{(2)}(X, L)$.
- $P(\cdot,\cdot) : X \times X \to L \boxtimes L^*$
  $$P(x,y) = \sum_{j=1}^{d_p} s_j(x) \otimes s_j(y)^*$$ Bergman kernel
- $$P(x,x) = \sum_{j=1}^{d_p} |s_j(x)|^2_h$$ Bergman density function
- $(Ps)(x) = \int_X P(x,y)s(y)dv_X(y)$, $s \in L^2(X,L)$
- Bergman kernel does not depend on the choice of ONB

Example 1

- $X = \mathbb{D} \subset \mathbb{C}$, (L, h) trivial
- $H^0_{(2)}(X, L) = L^2(\mathbb{D}, d\lambda) \cap \mathcal{O}(\mathbb{D})$
- ONB: $\sqrt{\frac{j+1}{\pi}} z^j$, $j = 0, 1, \ldots \rightsquigarrow$ Bergman kernel

$$P(z, w) = \frac{1}{\pi} \sum_{j=0}^{\infty} (j+1) z^j \overline{w}^j = \frac{1}{\pi} \frac{1}{(1 - z\overline{w})^2}$$

Goal

- We are actually interested in a semiclassical limit!
- $L^p = L^{\otimes p}$
- P_p the Bergman projection on $H^0_{(2)}(X, L^p)$
- Asymptotics of $P_p(x, y)$ and $P_p(x, x)$ as $p \to \infty$

Example 2

- $X = \mathbb{P}^n$, Fubini-Study metric:

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \ldots + |z_n|^2)$$

- $L = \mathcal{O}(1)$, $h_{FS} = (|z_0|^2 + \ldots + |z_n|^2)^{-1}$
- $H^0_{(2)}(X, L^p) = H^0(X, L^p) =$ space of homogeneous polynomials in $n+1$ variables of degree p
- Basis $s_\alpha \sim z^\alpha$, $\alpha \in \mathbb{N}_0^{n+1}$, $|\alpha| = p$, $\|s_\alpha\|^2_{L^2} = \dfrac{\alpha!}{(n+p)!}$
- $P_p(x, y) = \displaystyle\sum_{|\alpha|=p} \frac{(n+p)!}{\alpha!} s_\alpha(x) \otimes s_\alpha(y)^*$
- $P_p(x, x) = \dfrac{(n+p)!}{p!} = p^n + b_{n-1}p^{n-1} + \ldots + b_n$

Curvature

- Hermitian holomorphic line bundle $(L, h) \to X$

- Curvature form $c_1(L, h) = \frac{\sqrt{-1}}{2\pi}(\nabla^L)^2$, ($\nabla^L$ Chern connection)
- local holomorphic frame s of L on $U \subset X \rightsquigarrow$

$$|s(x)|_h^2 = e^{-2\varphi(x)}, \quad x \in U$$

  where $\varphi : U \to \mathbb{R}$ is smooth, called local weight
- $c_1(L, h)|_U = dd^c\varphi = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi$
- (L, h) positive :⇔ $c_1(L, h)$ positive ⇔ $\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\right)$ positive definite
- (L, h) semipositive :⇔ $c_1(L, h)$ semipositive ⇔ $\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\right)$ positive semidefinite

Asymptotic expansion

Theorem (Catlin 1998, Zelditch 1999)

(X, ω) compact Kähler, $(L, h) \to X$, with $c_1(L, h) = \omega$. Then

$$P_p(x, x) = b_0(x)p^n + b_1(x)p^{n-1} + \ldots = \sum_{j=0}^{\infty} b_j(x)p^{n-j}, p \to \infty$$

where $b_0 = 1$.

- Tian (1990): $P_p(x, x) = b_0(x)p^n + O(p^{n-1})$
- $b_1 = \frac{1}{8\pi} r(\omega)$, $r(\omega)$ = scalar curvature of ω (Z. Lu)
- $\pi^2 b_2 = -\frac{\Delta r(\omega)}{48} + \frac{1}{96}|R^{TX}|^2 - \frac{1}{24}|\mathrm{ric}_\omega|^2 + \frac{1}{128} r(\omega)^2$

  (Z. Lu, X. Wang)

Compatibility with Riemann-Roch-Hirzebruch

$$\int_X P_p(x,x)dv_X = \int_X \sum_j |S_j|_h^2 \, dv_X = \dim H^0(X, L^p)$$

$$\int_X P_p(x,x)dv_X = \int_X (b_0(x)p^n + b_1(x)p^{n-1} + b_2(x)p^{n-2} + O(p^{n-3}))\, dv_X$$

$$= p^n \int_X \frac{\omega^n}{n!} + p^{n-1} \int_X \frac{r(\omega)}{8\pi} \frac{\omega^n}{n!} + p^{n-2} \int_X b_2(x) \frac{\omega^n}{n!} + O(p^{n-3})$$

$$= p^n \int_X \frac{c_1(L)^n}{n!} + p^{n-1} \int_X \frac{c_1(X)}{2} \frac{c_1(L)^{n-1}}{(n-1)!} +$$

$$+ p^{n-2} \int_X \{\mathrm{td}\ (T^{(1,0)}X)\}^{(4)} \frac{c_1(L)^{n-2}}{(n-2)!}$$

$$+ O(p^{n-3})$$

Asymptotic expansion variation

Theorem

(X, Θ) *compact Hermitian,* $(L, h) \to X$ *positive,* $\omega = c_1(L, h)$. *Then*

$$P_p(x, x) = b_0(x)p^n + b_1(x)p^{n-1} + \ldots = \sum_{j=0}^{\infty} b_j(x)p^{n-j}, p \to \infty$$

where $b_0 = c_1(L, h)^n / \Theta^n$ *and*

$$b_1 = \frac{1}{8\pi} b_0 \left[r(\omega) - 2\Delta_\omega \log(\det b_0) \right]$$

$\alpha_1(x), \ldots, \alpha_n(x)$ eigenvalues of $c_1(L, h)$ w.r.t. $\Theta \rightsquigarrow$

$$b_0(x) = \alpha_1(x) \ldots \alpha_n(x)$$

Proof (local index theorem: Dai/Liu/Ma, Ma/–)

- Localization uses spectral gap of the Kodaira Laplacian $\Box_p = \overline{\partial}^* \overline{\partial} : \Omega^{0,0}(X, L^p) \to \Omega^{0,0}(X, L^p)$
- $\mathrm{Spec}\,(\Box_p) \subset \{0\} \cup [C_0 p - C_1, \infty),\ C_0 = \inf_{x \in X} \alpha_1(x)$
- $f \in C_0^\infty(\mathbb{R}),\ F = \widehat{f} \rightsquigarrow$

$$\left|F(\Box_p)(x,y) - P_p(x,y)\right|_{C^l} = O(p^{-\infty})$$

- $F(\Box_p)(x,y)$ depends only on geometric data on $B(x,\varepsilon)$
- $\rightsquigarrow P_p(x,y)$ depends only on local data
- Work on $B^{TX}(0,\varepsilon) \equiv B(x,\varepsilon)$ with a local model Laplacian
- Rescale coordinates and develop the rescaled operator in Taylor series

Interpretation of the first term b_0

- $P_p(x,x) = b_0(x)p^n + O(p^{n-1}) \rightsquigarrow$ holomorphic sections are spread everywhere over X ($b_0(x) \neq 0$).
- They concentrate where the curvature is strong
- $P_p(x,y)$ localizes near each fixed point x and equals approximatively a "peak section" which is close to a Gaussian $p^n \exp(-|x-y|^2/\sqrt{p})$.
- To prove this kind of localization is a key point.
- Can put peak sections near every point and they decay quickly enough that they nearly don't overlap each other.
- Heuristically then, in the limit we can find an L^2-orthonormal basis of sections parametrized by the points of X, each point x corresponding to a section localized entirely at x.

Interpretation of the second term b_1

- $P_p(x,x) = b_0(x)p^n + b_1(x)p^{n-1} + O(p^{n-2})$
- $b_1(x)$ is essentially the scalar curvature of the metric $\omega = c_1(L,h)$
- Scalar curvature measures the difference in volume of a small geodesic ball compared with the volume of a Euclidean ball of the same radius.
- So the scalar curvature tells us how closely we can push together the peaked sections making up our L^2-orthonormal basis from above.

Tian's approximation

Theorem

$(L, h) \to X$ *positive, Kodaira map:*
$\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^*)$, $x \mapsto \{S \in H^0(X, L^p) : S(x) = 0\}$ *Then*

$$\left| \frac{1}{p}\Phi_p^*(\omega_{FS}) - c_1(L,h) \right|_{C^\ell} \leq \frac{C_\ell}{p}$$

Proof.

.

$$\frac{1}{p}\Phi_p^*(\omega_{FS}) = c_1(L,h) + \frac{\sqrt{-1}}{2\pi p}\partial\bar{\partial}\log P_p(x,x)$$

$$c_1(L,h) + \frac{\sqrt{-1}}{2\pi p}\partial\bar{\partial}\log p^n\big(b_0(x) + O(\frac{1}{p})\big)$$

□

Generalizations

- Semipositive bundles (asymptotics on the positive part)
- Noncompact manifolds (asymptotics on compact sets)
- Orbifolds
- Symplectic manifolds
- Singular metrics

Open questions

- Semipositive bundles (asymptotics near degenerate points)
- Noncompact manifolds (uniform asymptotics near infinity)
- Complex spaces (asymptotics near singularities)
- Manifolds with boundary (asymptotics near the boundary)
- Partial Bergman kernels

Part II:

Semipositive Bergman kernels & sub-Riemannian geometry

Sub-Riemannian (sR) geometry

Sub-Riemannian (sR) geometry is the study of metric distributions $(Y, E \subset TY, g^E)$ inside the tangent space.
Subbundle E is assumed to be *bracket-generating*.
Peculiar phenomena (Hausdorff dimension & abnormal geodesics..) arise.

References:

- R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. 2002.
- M. Gromov, *Carnot-Carathéodory spaces seen from within*, 1996, (in Bellaïche & Risler, *Sub-Riemannian geometry*)

Bracket-generating distributions

$E \subset TY$ bracket generating: $C^\infty(E)$ generates $C^\infty(TY)$ under Lie bracket $[,]$.

Examples:
1. Contact case: $E^{2m} = \ker\alpha \subset TY^{2m+1}$; rank $d\alpha|_E = 2m$.
Normal form (Darboux): $\alpha = dy_3 - y_2 dy_1$; $E = \mathbb{R}[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3}]$
Generation (1 step):$[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3}] = \partial_{y_3}$

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3. Martinet case: $E^2 = \ker\alpha \subset TY^3$, $\underbrace{Z^2}_{\text{union of hypersurfaces}} = \{\alpha \wedge d\alpha = 0\} \subset Y$
with $TY \pitchfork E$.
Normal form: $\alpha = dy_3 - y_2^2 dy_1$; $E = \mathbb{R}\left[\partial_{y_2}, \partial_{y_1} + y_2^2\partial_{y_3}\right]$
Generation (2 step):$\left[\partial_{y_2}, \partial_{y_1} + y_2^2\partial_{y_3}\right] = 2y_2\partial_{y_3}$, $\left[\partial_{y_2}, \left[\partial_{y_2}, \partial_{y_1} + y_2^2\partial_{y_3}\right]\right] = 2\partial_{y_3}$

Flag, metric & dimension

In general defines canonical flag:

$$\{0\} = E_0 \subset \underbrace{E_1}_{=E} \subset \ldots \subset E_{r(y)} = TY$$

by $E_{j+1} = [E, E_j]$, $j \geq 1$.
Step$= r\left(y\right),$ Growth vector $= k^E\left(y\right) = \left(\mathrm{dim}E_0, \mathrm{dim}E_1, \ldots, \mathrm{dim}E_{r(y)}\right)$.

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(Chow-Rashevsky '37) E bracket-generating $\implies$ any two $y_1, y_2 \in Y$ connected by *horizontal* path $\gamma \in C^{0,1}\left([0,1]_t ; Y\right)$,$\dot{\gamma}(t) \in E_{\gamma(t)}$ a.e.

$\left(Y, d^E\right)$ is a metric space with $d^E = \inf_{\gamma \text{ horizontal}} \int_0^1 dt \, |\dot{\gamma}(t)|$.

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(Ball-Box Thm)

$$\underbrace{Q(y)}_{\text{Hausdorff dimension}} := \lim_{\varepsilon \to 0} \frac{\ln \text{vol} B_\varepsilon(y)}{\ln \varepsilon} = \sum_{j=1}^{r(x)} j\left[k_j(y) - k_{j-1}(y)\right] > n$$

Geodesics

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Not all minimizers obtained this way!

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Example. $Y = \mathbb{R}^3$, $E = \ker\left[dy_3 - y_2^2 dy_1\right]$, has vanishing hypersurface
$Z = \{\alpha \wedge d\alpha = 0\} = \{y_2 = 0\}$.
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Open question: Are abnormal minimizers smooth?

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Well understood in cases:

- Contact case: none.
- Quasi contact case: Integral curves of $L^E := \ker d\alpha|_E$ (topological)
- Martinet case: Integral curves of $\ker \alpha|_Z =: L^E \to Z$ (topological)

sR Laplacian

Let $(Y, E \subset TY, g^E)$ sR manifold.
Choose an auxiliary density μ to define

$$\text{sR Laplacian}: \quad \Delta_{g^E,\mu} := \left(\nabla^{g^E}\right)^*_\mu \circ \nabla^{g^E}$$

where $g^E\left(\nabla^{g^E} f, U\right) = U(f)$, $\forall U \in C^\infty(E)$ is sR-gradient.

Changing the density: $\Delta_{g^E,h\mu} = h^{-1/2}\Delta_{g^E,\mu}h^{1/2} + h^{-1/2}\left(\Delta_{g^E,\mu}h^{1/2}\right)$

Characteristic variety: $\Sigma = \left\{\sigma\left(\Delta_{g^E,\mu}\right) = H^E = 0\right\} = E^\perp$.

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(Rothschild & Stein '76) $\|f\|^2_{H^{1/r}} \lesssim \left\langle \Delta_{g^E,\mu} f, f\right\rangle + \|f\|^2_{L^2}$ where $r = \max_{x\in X}\ r(x)$.

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Discrete spectrum (φ_j, λ_j); $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, on a compact manifold.
Spectral asymptotics questions: Weyl law, trace formula, propagation, ergodicity ...
(mostly open)

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Do Hausdorff dimension, abnormal geodesics play a role?

sR heat trace

Does the spectrum see the Hausdorff dimension?

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Theorem (Ben Arous 1989, Léandre 1992... Barilari 2011, Colin de Verdiere-Hillairet-Trélat)

There exist $a_j(y) \in C^\infty(Y)$, $j = 0, 1, \ldots$,

$$e^{-t\Delta_{g^E,\mu}}(y,y) \sim t^{-Q(y)/2}\left[\sum_{j=0}^{\infty} a_j(y)\, t^j\right].$$

The expansion is in general <u>not</u> uniform in $x \in X$. Does not yield trace asymptotics.

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Theorem (Métivier 1976, Colin De Verdiere-Hillairet-Trélat)

If E *is equiregular*

$$N(\lambda) \sim \underbrace{\frac{\lambda^{Q/2}}{\Gamma(Q/2+1)}\int_X a_0}_{=vol\{H^E \leq \lambda\}}.$$

Spectrum and dynamics

Does the spectrum see the abnormals?

Theorem (Melrose 1984)

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Theorem (S.)

(X^4, E^3) 4D quasi-contact. Suppose any invariant subset of characteristic foliation $L^E \subset TX$ is of zero or full measure (and $L_Z \mu_{Popp} = 0$). Then one has quantum ergodicity (QE).

Circle bundles

Natural place for sR-structures: $\left(\underbrace{Y^n}_{S^1 L}, \underbrace{E^{n-1}}_{HX\,\text{horizontal}}, \underbrace{g^E}_{\pi^* g^{TX}}\right)$ with $\left(L, h^L, \nabla^L\right) \to \left(X^{n-1}, g^{TX}\right)$ is a Hermitian line bundle with connection.
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Decomposition: $X = \cup_{j=2}^r X_j$; $X_j = \{x | r_x = j\}$

Bochner Laplacian

Fourier: $C^\infty(Y) = \oplus_{p=-\infty}^{\infty} C^\infty(X, L^p)$; $\underbrace{\Delta_{g^E,\mu_Y}}_{\text{sR Laplacian}} = \oplus_{p=-\infty}^{\infty} \underbrace{\Delta_p}_{=(\nabla^{L^p})^* \nabla^{L^p} \text{Bochner}}$

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Theorem (Marinescu-S.)

Assume X_r submanifold with R^L non-degenerate

$$\lambda_0^p \sim p^{2/r}\left[c_0 + c_1 p^{-1/r} + c_2 p^{-2/r} + \ldots\right]$$
$$N\left(a p^{2/r}, b p^{2/r}\right) \sim p^{dim(X_r)} C_{a,b}$$

R. Montgomery '95 ($\dim Y = 2$, $r = 3$), Helffer-Mohamed '96, Helffer-Kordyukov '09 (Y_r hypersurface of transverse vanishing).

Bergman kernel

If X cpx. and L holomorphic one has Kodaira Laplacian $\square_p : \Omega^{0,*}(X; L^p) \to \Omega^{0,*}(X; L^p)$.
2D Weitzenbock formula:

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Local index theory technique of Bismut-Lebeau '91, Dai-Liu-Ma '06 gives

Theorem (Marinescu-S.)

For $dim X = 2$ & R^L semi-positive of finite order

$$P_p(x,x) \sim p^{2/r_x}\left[\sum_{j=0}^{N} b_j(x)\, p^{-j/r_x}\right]$$

where $r_x - 2 = ord\left(R^L_x\right)$.

R. Berman 2009 (on positive part away from base locus), Hsiao-Marinescu 2014 (on positive part when twisted by canonical).

Thank you.