Bergman-Szegő kernel asymptotics in weakly pseudoconvex finite type cases

Chin-Yu Hsiao & Nikhil Savale

Universität zu Köln

Oct 9, 2020

WUSTL hypoelliptic laplacian and NCG seminar Seminar

Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

Naive higher dimensional generalization is false

Theorem (Poincare 1907)

In $\mathbb{C}^2$, the polydisk and disk $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$ (not biholomorphic).

Poincare's proof computes:
Aut $(\mathbb{D}^n) = PSU(n, 1) :=$

$$\left\{ T = \begin{bmatrix} A & b \\ c & d \end{bmatrix} | T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} T^* = \begin{bmatrix} I_n & \\ & -1 \end{bmatrix}, \det T = 1 \right\} / S^1 \text{ where } T.z = \frac{Az+b}{cz+d}$$

Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

Naive higher dimensional generalization is false

Theorem (Poincare 1907)

In $\mathbb{C}^2$, the polydisk and disk $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$ (not biholomorphic).

Poincare's proof computes:
$\mathrm{Aut}\,(\mathbb{D}^n) = PSU\,(n,1) :=$
$\left\{ T = \begin{bmatrix} A & b \\ c & d \end{bmatrix} | T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} T^* = \begin{bmatrix} I_n & \\ & -1 \end{bmatrix}, \det T = 1 \right\} / S^1$ where $T.z = \frac{Az+b}{cz+d}$
Questions:

-find generalization of RMT in higher dimensions?

-biholomorphism classification of domains in higher dimensions
-more robust biholomorphism invariants?

Bergman kernel

Let $U \subset \mathbb{C}^n$ open, connected (often smoothly bounded..)

Bergman kernel

Let $U \subset \mathbb{C}^n$ open, connected (often smoothly bounded..)

Define

$$L^2(U) := \left\{ f : U \to \mathbb{C}\,\text{measurable} \mid \|f\|^2_{L^2(U)} = \int_U |f|^2\, d\text{vol} < \infty \right\}$$

$$H^0_{(2)}(U) := \left\{ f \in L^2(U) \,|\, \bar{\partial} f = 0 \right\} \subset L^2(U)$$

Bergman projection: $\Pi_U : L^2(U) \to H^0_{(2)}(U) \subset L^2(U)$

Bergman kernel: $\Pi_U \in L^2(U \times U)$ satisfies

$$(\Pi_U f)(z) = \int_U \Pi_U(z, z') f(z')\, d\text{vol}(z')$$

Bergman kernel

Let $U \subset \mathbb{C}^n$ open, connected (often smoothly bounded..)

Define
$L^2(U) := \left\{ f : U \to \mathbb{C} \,\text{measurable} \,|\, \|f\|^2_{L^2(U)} = \int_U |f|^2 \, d\text{vol} < \infty \right\}$
$H^0_{(2)}(U) := \left\{ f \in L^2(U) \,|\, \bar{\partial} f = 0 \right\} \subset L^2(U)$

Bergman projection: $\Pi_U : L^2(U) \to H^0_{(2)}(U) \subset L^2(U)$
Bergman kernel: $\Pi_U \in L^2(U \times U)$ satisfies

$$(\Pi_U f)(z) = \int_U \Pi_U(z, z') \, f(z') \, d\text{vol}(z')$$

Properties:
1. $\Pi_U(z, z') = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z')}$, for $\{\varphi_j\}_{j=1}^{\infty}$ ONB of $H^0_{(2)}(U)$.
2. $\overline{\Pi_U(z, z')} = \Pi_U(z', z)$
3. $\Pi_U(z, z')$ hol./antihol. in z/z', smooth in the interior
4. (U bounded) $\Pi_U(z, z) > 0$ in the interior

Bergman metric

The Bergman metric on U is defined via

$$g_{ij}^{U} = \partial_{z_i} \partial_{\overline{z_i}} \left[\ln \Pi_U \left(z, z \right) \right]$$

Bergman metric

The Bergman metric on U is defined via

$$g_{ij}^{U} = \partial_{z_i}\partial_{\overline{z_i}}\left[\ln \Pi_U\left(z, z\right)\right]$$

Theorem (Bergman 1921)

Let $F : U_1 \to U_2$ be a biholomorphism between two domains. Then it is an isometry $F_* g^{U_1} = g^{U_2}$.

Proof.

From defining property $\Pi_{U_1}\left(z, w\right) = \det\left(\frac{\partial F}{\partial z}\right)\left[\Pi_{U_2}\left(F\left(z\right), F\left(w\right)\right)\right]\overline{\det\left(\frac{\partial F}{\partial w}\right)}$.
Mixed partials of Jacobians are zero.

□

Computations of Π_U

1. $U = \mathbb{D}^n$ (disk)

ONB: $\sqrt{\frac{\binom{\alpha+n}{n}}{\pi^n}} z^\alpha$

$$\Pi_{\mathbb{D}^n}(z, z') = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ corresponds to Poincare disk model for hyperbolic space

Computations of Π_U

1. $U = \mathbb{D}^n$ (disk)

ONB: $\sqrt{\frac{\binom{\alpha+n}{n}}{\pi^n}} z^\alpha$

$$\Pi_{\mathbb{D}^n}(z, z') = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ corresponds to Poincare disk model for hyperbolic space

2. (D'Angelo '78)

$U = E_p := \left\{ |z_1|^2 + |z_2|^{2p} \leq 1 \right\}$ (ellipsoid)

$$\Pi_{E_p}(z, z') = \frac{2}{\pi^2} \frac{1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^3} + \frac{2}{\pi^2} \frac{p-1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^2}$$

Computations of Π_U

1. $U = \mathbb{D}^n$ (disk)

ONB: $\sqrt{\frac{\binom{\alpha+n}{n}}{\pi^n}} z^\alpha$

$$\Pi_{\mathbb{D}^n}(z, z') = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ corresponds to Poincare disk model for hyperbolic space

2. (D'Angelo '78)

$U = E_p := \left\{ |z_1|^2 + |z_2|^{2p} \leq 1 \right\}$ (ellipsoid)

$$\Pi_{E_p}(z, z') = \frac{2}{\pi^2} \frac{1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^3} + \frac{2}{\pi^2} \frac{p-1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^2}$$

Computing curvatures: $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$, $E_p \ncong E_{p'}$ for $p \neq p'$ (not biholomorphic).

Fefferman's theorem

Another strategy for $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$: look for behaviour of biholomorphisms near boundary (boundary of polydisk is non-smooth)

Conjecture: (cf. Krantz '13) Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded. Then any biholomorphism $F : U_1 \to U_2$ extends smoothly to the boundary.

Fefferman's theorem

Another strategy for $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$: look for behaviour of biholomorphisms near boundary (boundary of polydisk is non-smooth)

Conjecture: (cf. Krantz '13) Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded. Then any biholomorphism $F : U_1 \to U_2$ extends smoothly to the boundary.

.

Theorem (Fefferman '74)

Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded and strongly pseudoconvex. Then $U_1 \cong U_2$ (biholomorphic) $\iff$ ∃CR diffeomorphism $\partial U_1 \cong \partial U_2$.

Proof.

Uses Bergman kernel expansion. Given $U = \{\rho < 0\}$, $d\rho|_{\partial U} \neq 0$ (boundary defining function) , then

$$\begin{aligned} \Pi_U(z,z) &= a(z)\rho^{-n-1} + b(z)\ln(-\rho) \\ &\sim \sum_{j=0}^{\infty} a_j(x)\rho^{-n-1+j} + \sum_{j=0}^{\infty} b_j(x)\rho^j \ln(-\rho), \quad \text{as } \rho \to 0, \end{aligned}$$

for $a(z), b(z) \in C^\infty(\overline{U})$, $z = (x, \rho)$ local coord near boundary.Study geodesic flow for Bergman metric near boundary to obtain boundary extension.

CR manifolds

A CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$
-codim: $\dim_{\mathbb{C}} T^{1,0}X = n$
-non-degeneracy: $T^{1,0}X \cap \overline{T^{1,0}X} = \emptyset$
-integrability: $\left[T^{1,0}X, T^{1,0}X\right] \subset T^{1,0}X$

$f : X_1 \to X_2$ is a CR map iff $f_* T^{1,0}X_1 \subset T^{1,0}X_2$

CR manifolds

A CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$
-codim: $\dim_{\mathbb{C}} T^{1,0}X = n$
-non-degeneracy: $T^{1,0}X \cap \overline{T^{1,0}X} = \emptyset$
-integrability: $\left[T^{1,0}X, T^{1,0}X\right] \subset T^{1,0}X$

$f : X_1 \to X_2$ is a CR map iff $f_* T^{1,0}X_1 \subset T^{1,0}X_2$

Examples:
1. $X = \partial U$, $U \subset \mathbb{C}^n$smoothly bounded domain.
Where $T^{1,0}X = T^{1,0}\mathbb{C}^n \cap T_{\mathbb{C}}X$.

2. Y cpx manifold. $\left(L, h^L\right)$ Hermitian holomorphic.
CR manifold $X = S^1 L$, HX =Chern connection, $T^{1,0}X = \ker\left(J - i\right) \subset HX \otimes \mathbb{C}$

Pseudoconvexity, Finite type

CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$

Levi distribution: $HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$
Levi form: $\mathscr{L} \in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX)$
$\mathscr{L}(U,V) := [[U,V]] \in T_xX/H_xX$

Pseudoconvexity, Finite type

CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$

Levi distribution: $HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$
Levi form: $\mathscr{L} \in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX)$
$\mathscr{L}(U,V) := [[U,V]] \in T_xX/H_xX$

$x \in X$ is weakly/ strongly pseudoconvex $\iff \mathscr{L}_x(., J_x.)$ is positive definite/semi-definite
$x \in X$ is finite type $\iff HX$ is bracket generating at x

type of a point $x \in X$ is $r(x) = 1 + \min \#$ brackets in HX necessary to generate TX
type of strongly pseudoconvex point $x \in X$ is $r(x) = 2$.

Tangential CR complex & Szegő kernel

Tangential CR operator: $\bar{\partial}_b^q : C^\infty\left(\Lambda^q T^{0,1*}X\right) \to C^\infty\left(\Lambda^{q+1}T^{0,1*}X\right)$,
Defined by Leibniz rule following:$\left(\bar{\partial}_b^0 f\right)\left(\bar{Z}\right) = \bar{Z}\left(f\right)$, $f \in C^\infty\left(X\right)$, $\overline{Z} \in T^{0,1}X$.

$\bar{\partial}_b \circ \bar{\partial}_b = 0$ (uses integrability)
Kohn-Rossi cohomology: $H^{0,q}\left(X\right) := \ker\left(\bar{\partial}_b^q\right) / \mathrm{Im}\left(\bar{\partial}_b^{q-1}\right)$

Tangential CR complex & Szegő kernel

Tangential CR operator: $\bar{\partial}_b^q : C^\infty\left(\Lambda^q T^{0,1*}X\right) \to C^\infty\left(\Lambda^{q+1}T^{0,1*}X\right)$,
Defined by Leibniz rule following: $\left(\bar{\partial}_b^0 f\right)\left(\bar{Z}\right) = \bar{Z}\left(f\right)$, $f \in C^\infty\left(X\right)$, $\overline{Z} \in T^{0,1}X$.

$\bar{\partial}_b \circ \bar{\partial}_b = 0$ (uses integrability)
Kohn-Rossi cohomology: $H^{0,q}\left(X\right) := \ker\left(\bar{\partial}_b^q\right)/\mathrm{Im}\left(\bar{\partial}_b^{q-1}\right)$

Choosing Hermitian metric on $T^{1,0}X$ & volume form μ.
Kohn Laplacian: $\Box_b^q := \bar{\partial}_b^{q-1}\left(\bar{\partial}_b^{q-1}\right)^* + \left(\bar{\partial}_b^q\right)^*\bar{\partial}_b^q$

Szegő projector: $\Pi_b^q : L^2\left(\Lambda^q T^{0,1*}X\right) \to \ker\ \left(\bar{\partial}_b^q\right)$

Hodge theory

Kohn described Hodge theory when Levi form non-degenerate.

Theorem (Kohn '65)

Let $(X^{2n+1}, T^{1,0}X)$ *CR manifold. Assume* $\mathscr{L}$ *is non-degenerate of constant signature* (n_-, n_+), $n = n_- + n_+$. *Then for each* $q \neq n_-, n_+$ *(i.e.* $Y(q)$ *condition)* $\Box_b^q$ *is hypoelliptic and with a corresponding Hodge theorem* $H^{0,q}(X) = ker \ (\Box_b^q)$.

$(X^{2n+1}, T^{1,0}X)$ strongly pseudoconvex $\implies Y(q)$ for $q \neq 0, n$.
When $Y(q)$ fails $\ker \ (\Box_b^q)$ can be infinite dimensional.

Hodge theory

Kohn described Hodge theory when Levi form non-degenerate.

Theorem (Kohn '65)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ CR manifold. Assume $\mathscr{L}$ is non-degenerate of constant signature $\left(n_-, n_+\right)$, $n = n_- + n_+$. Then for each $q \neq n_-, n_+$ (i.e. $Y(q)$ condition) $\Box_b^q$ is hypoelliptic and with a corresponding Hodge theorem $H^{0,q}(X) = ker\ \left(\Box_b^q\right)$.

$\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex $\implies Y(q)$ for $q \neq 0, n$.
When $Y(q)$ fails $\ker\ \left(\Box_b^q\right)$ can be infinite dimensional.

Theorem (Kohn '65)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ such that $Y(q)$ condition holds. Furthermore assuming $\bar{\partial}_b^q$ has closed range, $\exists G : H^s\left(\Lambda^{q+1}T^{0,1*}X\right) \to H^{s+\frac{1}{2}}\left(\Lambda^q T^{0,1*}X\right)$ such that $\Pi_b^0 = I - G\bar{\partial}_b^q$

Embedding question

Can $\left(X^{2n+1}, T^{1,0}X\right)$ be embedded into $\mathbb{C}^N$ by CR functions?

Theorem (Boutet de Monvel '75)

If $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex and $2n+1 \geq 5$ then yes.

In dimension three counterexamples of Grauert '62, Rossi '64, Andreotti-Siu '70 ...

Kohn embeddability of $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex $\iff$ $\bar{\partial}_b^q$ has closed range, .$\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex $\implies Y\left(q\right)$ for $q \neq 0, n$.

Embedding question

Can $\left(X^{2n+1}, T^{1,0}X\right)$ be embedded into $\mathbb{C}^N$ by CR functions?

Theorem (Boutet de Monvel '75)

If $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex and $2n+1 \geq 5$ then yes.

In dimension three counterexamples of Grauert '62, Rossi '64, Andreotti-Siu '70 ...

Kohn embeddability of $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex $\iff$ $\bar{\partial}_b^q$ has closed range, .$\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex $\implies Y(q)$ for $q \neq 0, n$.

Theorem (Lempert '92)

Let $\left(X^3, T^{1,0}X\right)$ strongly pseudoconvex three dimensional CR manifold admitting a transversal, CR circle action. Then it is embeddable into $\mathbb{C}^N$.

Szegő parametrix

Interesting to describe the singularities of its Schwartz kernel.

Theorem (Boutet de Monvel-Sjöstrand '75)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ *be strongly pseudoconvex CR manifold. Assume* ∂_b *has closed range. Then near each* $x \in X$ *there exist coordinates* $\Big(\underbrace{x_1 \ldots, x_{2n}}_{=x'}, x_{2n+1}\Big)$ *such that*

$$\begin{aligned}\Pi_b^0 (x, y) &= \int_0^\infty dt e^{it\Psi(x,y)} a (t; x, y) \\ &= \int_0^\infty dt e^{it(x_{2n+1} - y_{2n+1})} \underbrace{e^{it\Phi(x',y')} a (t; x, y)}_{=b(t;x,y)}\end{aligned}$$

Phase: $\overline{\Phi (x', y')} = \Phi (y', x')$, $Im\Phi (x', y') \geq C |x' - y'|^2$, $x' = y' \iff \Phi = 0$.
Amplitude: $a (t; x, y) \in S^n_{t,cl}$, $a \sim \sum_{j=0}^\infty t^{n-j} a_j (x, y)$

Szegő parametrix

Interesting to describe the singularities of its Schwartz kernel.

Theorem (Boutet de Monvel-Sjöstrand '75)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ be strongly pseudoconvex CR manifold. Assume ∂_b has closed range. Then near each $x \in X$ there exist coordinates $\Big(\underbrace{x_1 \ldots, x_{2n}}_{=x'}, x_{2n+1}\Big)$ such that

$$\begin{aligned}\Pi_b^0(x,y) &= \int_0^\infty dt e^{it\Psi(x,y)} a(t;x,y)\\ &= \int_0^\infty dt e^{it(x_{2n+1}-y_{2n+1})} \underbrace{e^{it\Phi(x',y')} a(t;x,y)}_{=b(t;x,y)}\end{aligned}$$

Phase: $\overline{\Phi(x',y')} = \Phi(y',x')$, *Im*$\Phi(x',y') \geq C|x'-y'|^2$, $x'=y' \iff \Phi = 0$.
Amplitude: $a(t;x,y) \in S^n_{t,cl}$, $a \sim \sum_{j=0}^{\infty} t^{n-j} a_j(x,y)$

Note: regrouped amplitude $b(t;x,y) \in S^n_{t,\frac{1}{2},\mathrm{cl}}$(lies in a more general class; $\partial_x^\alpha \partial_y^\beta \partial_t^\gamma b = O\left(t^{n+\frac{1}{2}(|\alpha|+|\beta|)-\gamma}\right)$)

Szegő parametrix

More generally when $Y(q)$ condition fails..

Theorem (Hsiao '10, Hsiao-Marinescu '17)

Let $(X^{2n+1}, T^{1,0}X)$ be CR manifold. Let $\mathscr{L}$ be non-degenerate of constant signature $(q, n-q)$ (i.e. $Y(q)$ condition fails). Assuming $\bar{\partial}_b^q$ has closed range, there exists similar description for $\Pi_q(x, y)$.

Bergman asymptotics

Specialize to $X = \partial U$.
Consider Poisson operator

$$P : C^{\infty}(X) \to C^{-\infty}(U)$$
$$\Box_U P = 0, \gamma P = I.$$

Theorem (Boutet de Monvel '71)

The Poisson operator maps $P : H^s(X) \to H^{s+\frac{1}{2}}(U)$. *Furthermore* $P^* P : C^{\infty}(X) \to C^{\infty}(X)$ *is elliptic, injective pseudodifferential of order* -1 *with* $\sigma(P^* P)^{-1} = \sigma(\Delta_X)$.

Furthermore P approximately relates the Bergman-Szegő projectors $\Pi_U" = "P (P^* P)^{-1} \Pi_b^0 P$ (at highest order)

Using the above recover/refine Fefferman's theorem

Bergman semiclassics

Second specialization $X = S^1 L$.
Where Y cpx manifold. (L, h^L) Hermitian holomorphic.
HX =Chern connection, $T^{1,0}X = \ker(J - i) \subset HX \otimes \mathbb{C}$

X strongly pseudoconvex iff (L, h^L) positive
kth Fourier component of $\Pi_b^0 = \Pi_k : L^2(X, L^k) \to H^0(X, L^k)$

This gives

Theorem (Tian '90, Catlin '97, Zelditch '98)

LetY^{2n} cpx manifold. (L, h^L) positive, Hermitian, holomorphic. The Bergman kernel admits on-diagonal expansion

$$\Pi_k(y, y) \sim k^n [b_0(y) + b_1(y) k + \ldots]$$

valid in C^α for all α.

Different more geometric method: Dai-Liu-Ma '06, Ma-Marinescu '07 (without Boutet de Monvel -Sjöstrand . Based on local index theory of Bismut-Lebeau '91).

Greens function, pointwise bounds

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type.
Let maximum number of brackets be $r := \max_{x \in X} r(x)$.

Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ *weakly pseudoconvex, finite type. Assume* $\bar{\partial}_b$ *has closed range. There exists (microlocal)* $G : H^s(X) \to H^{s+\frac{1}{r}}(X)$ *such that* $\Pi_b^0 = I - G\bar{\partial}_b$.

Greens function, pointwise bounds

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type.
Let maximum number of brackets be $r := \max_{x \in X} r\left(x\right)$.

Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\overline{\partial}_b$ has closed range. There exists (microlocal) $G : H^s\left(X\right) \to H^{s+\frac{1}{r}}\left(X\right)$ such that $\Pi_b^0 = I - G\overline{\partial}_b$.

Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\overline{\partial}_b$ has closed range. Near any point $x' \in X$ of type $r\left(x'\right)$ there exists coordinates $\left(x_1, x_2, x_3\right)$ centered at x' such that

$$\left|\partial_x^\alpha \Pi\left(x,0\right)\right| \leq C_\alpha \left[d^H\left(x,0\right)\right]^{-2-r-\alpha_1-\alpha_2-r\alpha_3}$$

$$d^H\left(x,0\right) = \left|x_1\right| + \left|x_2\right| + \left|x_3\right|^{1/r\left(x'\right)} .$$

($d^H\left(x,0\right)$ is equivalent to the sub-Riemannian CC distance between $x, 0$).

Similar bounds for boundaries of weakly pseudoconvex finite type domains in $\mathbb{C}^2$: McNeal '89, Nagel-Rosay-Stein-Wainger '89

Szegő parametrix

Consider $(X^3, T^{1,0}X)$ weakly pseudoconvex, finite type ($HX = \mathrm{Re}\ [T^{1,0}X \oplus T^{0,1}X]$ bracket generating).
Let maximum number of brackets be $r := \max_{x\in X} r(x)$.

Theorem (Hsiao-S. '20)

Let $(X^3, T^{1,0}X)$ weakly pseudoconvex, finite type. Assume $\bar{\partial}_b$ has closed range. Near any point $x' \in X$ of type $r(x')$ there exists coordinates (x_1, x_2, x_3) centered at x' such that

$$\Pi(x,0) = \int_0^\infty dt\, e^{itx_3} b(x;t) + C^\infty(X)$$

where $b \sim t^{\frac{2}{r}} \left[\sum_{j=0}^\infty t^{-\frac{j}{r}} b_j\left(t^{\frac{1}{r}}x_1, t^{\frac{1}{r}}x_2\right)\right] \in S^{\frac{2}{r}}_{\frac{1}{r},cl}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$, $b_j \in \mathcal{S}\left(\mathbb{R}^2\right)$.

Christ estimates are equivalent to $b \in S^{\frac{2}{r}}_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$ (i.e. $\partial_t^k \partial_x^\alpha b = O\left(t^{m-k+\frac{|\alpha|}{r}}\right)$ without classical expansion).

Bergman asymptotics

Specializations:

Theorem (Hsiao-S. '20)

Let $X^3 = \partial U$ *be boundary of weakly pseudoconvex, finite type domain* $U = \{\rho < 0\} \subset \mathbb{C}^2$. *For any point* $x' \in X = \partial U$ *on the boundary, of type* $r = r(x')$, *the Bergman kernel satisfies the asymptotics*

$$\Pi_U(z,z) \sim \sum_{j=0}^{\infty} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{\infty} b_j (-\rho)^{j} \log(-\rho),$$

as $z \to x'$ *for some set of reals* a_j, b_j *with* $a_0 > 0$.

Fefferman '74 (strongly pseudoconvex case), D'Angelo '78 (ellipsoids), Boas-Straube-Yu '95 (h-extendible/semiregular domains), Kamimoto '98, '04 (tube domains, toric domains)...

Weakly pseudoconvex embedding

Theorem (Marinescu-S. '18)

Let Y^2 Riemann surface. (L, h^L) Hermitian, holomorphic, semi-positive. Assume that R^L vanishes to finite order at each point. The Bergman kernel admits on-diagonal expansion

$$\Pi_k(y,y) \sim k^{\frac{2}{r}} \left[b_0(y) + b_1(y) k^{-\frac{2}{r}} + b_2(y) k^{-\frac{4}{r}} \dots \right]$$

where $r = r(y) = 2 + ord_y(R^L)$.

More generally Hsiao-S. '20 expansion of Fourier modes of Szegő kernel under locally-free, transversal CR circle action.

Theorem (Hsiao-S. '20)

Let X be a compact weakly pseudoconvex three dimensional CR manifold of finite type admitting a transversal, CR circle action. Then it has an equivariant CR embedding into some $\mathbb{C}^N$.

Lempert '92 (strongly pseudoconvex case), Christ '89 (X weakly pseudoconvex, finite type with $\bar{\partial}_b$ closed range).

Proof Sketch

Step 1. Let $Z = a_j(x)\,\partial_{x_j}$ be CR vector field.
$\tilde{Z} = \tilde{a_j}(z)\,\partial_{z_j} + \tilde{b_j}(z)\,\partial_{\bar{z}_j}$ almost analytic extension (i.e. $\tilde{a}$, $\tilde{b}$ almost analytically extend $a, 0$)
Construct almost analytic coordinates $w_j(z) = p_j(z) + iq_j(z)$ on $\mathbb{C}^3$ such that
$\tilde{Z} = \frac{1}{2}(\partial_{w_1} + i\partial_{w_2}) - \frac{i}{2}(\partial_{w_1}\varphi + i\partial_{w_2}\varphi)\,\partial_{w_3}$, for

$$\varphi(w_1, w_2) = \underbrace{\varphi_0(w_1, w_2)}_{\text{homogeneous, real coefficients}} + O\left(|w|^{r+1}\right)$$

Proof Sketch

Step 1. Let $Z = a_j(x)\,\partial_{x_j}$ be CR vector field.
$\tilde{Z} = \tilde{a_j}(z)\,\partial_{z_j} + \tilde{b_j}(z)\,\partial_{\bar{z}_j}$ almost analytic extension (i.e. $\tilde{a}$, $\tilde{b}$ almost analytically extend $a, 0$)
Construct almost analytic coordinates $w_j(z) = p_j(z) + iq_j(z)$ on $\mathbb{C}^3$ such that
$\tilde{Z} = \frac{1}{2}(\partial_{w_1} + i\partial_{w_2}) - \frac{i}{2}(\partial_{w_1}\varphi + i\partial_{w_2}\varphi)\,\partial_{w_3}$, for

$$\varphi(w_1, w_2) = \underbrace{\varphi_0(w_1, w_2)}_{\text{homogeneous, real coefficients}} + O\left(|w|^{r+1}\right)$$

Step 2. Local Bergman projector $B_t : L^2\left(\mathbb{R}^2_p\right) \to \ker(\partial_t)$ on $q = 0$, where
$\partial_t = \frac{1}{2}(\partial_{p_1} + i\partial_{p_2}) + \frac{1}{2}t(\partial_{p_1}\varphi + i\partial_{p_2}\varphi)$.
Show $B_t(p, p') \in \hat{S}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}}$.

Proof sketch

Construct appropriate symbol spaces $a \in \hat{S}^{m,k}_{\frac{1}{r}}$: satisfy estimates
$\partial_p^\alpha \partial_{p'}^\beta \partial_t^\gamma a(p,p',p_3,p_3',t) = O\left(t^{m-\gamma+\frac{1}{r}(|\alpha|+|\beta|)+\alpha_3+\beta_3}\right)$ on compact subsets.
Quantize to $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ via Fourier transform.

Proof sketch

Construct appropriate symbol spaces $a \in \hat{S}^{m,k}_{\frac{1}{r}}$: satisfy estimates
$\partial_p^\alpha \partial_{p'}^\beta \partial_t^\gamma a(p, p', p_3, p'_3, t) = O\left(t^{m-\gamma+\frac{1}{r}(|\alpha|+|\beta|)+\alpha_3+\beta_3}\right)$ on compact subsets.
Quantize to $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ via Fourier transform.

Step 3. Define almost analytic continuations of kernels in $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$.
Relate almost analytic extension of B_t to original Szegő kernel $\Pi_b(x, 0)$ on $\mathbb{R}^3_x$.
This part uses Christ estimates and Green's function.

Thank you.