A GUTZWILLER TYPE TRACE FORMULA FOR THE MAGNETIC DIRAC OPERATOR

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Abstract. For manifolds including metric-contact manifolds with non-resonant Reeb flow, we prove a Gutzwiller type trace formula for the associated magnetic Dirac operator involving contributions from Reeb orbits on the base. As an application, we prove a semiclassical limit formula for the eta invariant.

1. Introduction

The trace formulas of Gutzwiller [19] and Duistermaat-Guillemin [13] are a clear statement of the semiclassical correspondence, expressing the spectrum of (h-) pseudo-differential operators in terms of periodic orbits of the underlying Hamiltonian dynamics as $h \to 0$. We refer to [7, 29] for a historical survey of trace formulas and the associated calculus of Fourier integral operators. For non-scalar pseudo-differential operators this calculus is often unavailable due to the non-diagonalizability of the principal symbol $\sigma(A)$. Indeed when the eigenvalues of $\sigma(A)$ are not smooth functions on the cotangent space, their corresponding Hamiltonian dynamics is not well-defined. The purpose of this article is to investigate the trace formula in one such case.

More precisely, let $\left(X, g^{TX}\right)$ be an oriented Riemannian manifold of odd dimension $n = 2m+1$ equipped with a spin structure. Let S be the corresponding spin bundle and let L be an auxiliary Hermitian line bundle. Fix a unitary connection A_0 on L and let $a \in \Omega^1(X;\mathbb{R})$ be a contact one form (i.e. one satisfying $a \wedge (da)^m > 0$). This gives a family of unitary connections on L via $\nabla^h = A_0 + \frac{i}{h}a$ and a corresponding family of coupled magnetic Dirac operators

$$D_h \coloneqq hD_{A_0} + ic(a) : C^\infty(S \otimes L) \to C^\infty(S \otimes L) \tag{1.1}$$

for $h \in (0,1]$.

Define the contact hyperplane $H = \ker(a) \subset TX$ as well as the Reeb vector field R via $i_R da = 0$, $i_R a = 1$. We shall now further assume that the Reeb flow of a is non-resonant. To state this assumption, let γ denote a Reeb orbit. For a fixed point $p \in \gamma$, the linearized Poincare return map $P_\gamma : T_pX \to T_pX$ has R_p as an eigenvector with eigenvalue 1 and restricts to a symplectic map on the contact hyperplane $P_\gamma^+ : H_p \to H_p$. We call the Reeb orbit γ non-degenerate if P_γ^+ has $n-1$ distinct eigenvalues not equal to 1. There now exists a symplectic basis for

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H_p in which P_γ^+ decomposes as

$$P_\gamma^+ = \left[\bigoplus_{j=1}^{N_e} P_{\gamma;\beta_j}^{+,e}\right] \oplus \left[\bigoplus_{j=1}^{N_h^+} P_{\gamma;\alpha_j^+}^{+,h}\right] \oplus \left[\bigoplus_{j=1}^{N_h^-} -P_{\gamma;\alpha_j^-}^{+,h}\right] \oplus \left[\bigoplus_{j=1}^{N_l} P_{\gamma;\alpha_j^0,\beta_j^0}^{+,l}\right] \tag{1.2}$$

for

$$P_{\gamma;\beta}^{+,e} = \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}, \quad \beta\in(0,2\pi) \tag{1.3}$$

$$P_{\gamma;\alpha}^{+,h} = \begin{bmatrix}e^{\alpha} & 0\\ 0 & e^{-\alpha}\end{bmatrix}, \quad \alpha>0 \tag{1.4}$$

$$P_{\gamma;\alpha^0,\beta^0}^{+,l} = \begin{bmatrix}e^{-\alpha^0}\cos\beta^0 & 0 & -e^{-\alpha^0}\sin\beta^0 & 0\\ 0 & e^{\alpha^0}\cos\beta^0 & 0 & -e^{\alpha^0}\sin\beta^0\\ e^{-\alpha^0}\sin\beta^0 & 0 & e^{-\alpha^0}\cos\beta^0 & 0\\ 0 & e^{\alpha^0}\sin\beta^0 & 0 & e^{\alpha^0}\cos\beta^0\end{bmatrix}, \quad \alpha^0>0, \beta^0\in(0,\pi). \tag{1.5}$$

We note that the summands in the decomposition (1.2) each correspond to: a pair of elliptic eigenvalues $e^{\pm i\beta}$ (of $P_{\gamma;\beta}^{+,e}$), a pair of positive/negative hyperbolic eigenvalues $\pm e^{\pm\alpha}$ (of $\pm P_{\gamma;\alpha}^{+,h}$) and a quartet of loxodromic eigenvalues $e^{\pm\alpha^0\pm i\beta^0}$ (of $P_{\gamma;\alpha^0,\beta^0}^{+,l}$). We call the Reeb orbit γ non-resonant if the two sets

$$\left\{\alpha_j^+\right\}_{j=1}^{N_h^+}\cup\left\{\alpha_j^-\right\}_{j=1}^{N_h^-}\cup\left\{\alpha_j^0\right\}_{j=1}^{N_l} \qquad \text{and}$$
$$\{2\pi\}\cup\left\{\beta_j\right\}_{j=1}^{N_e}\cup\left\{\beta_j^0\right\}_{j=1}^{N_l}$$

are rationally ($\mathbb{Q}-$) independent. We call the Reeb flow of a non-resonant if all its Reeb orbits are non-resonant.

Next, we shall assume that the metric g is *strongly suitable* to the contact form a. To define this, consider the contracted endomorphism $\mathfrak{J}: T_xX\to T_xX$ defined at each point $x\in X$ via

$$da\left(v_1,v_2\right) = g^{TX}\left(v_1,\mathfrak{J}v_2\right), \quad \forall v_1,v_2\in T_xX. \tag{1.6}$$

The contact assumption on the one form a implies that $\mathfrak{J}$ has a one dimensional kernel spanned by the Reeb vector field R. The endomorphism $\mathfrak{J}$ is clearly anti-symmetric with respect to the metric

$$g^{TX}\left(v_1,\mathfrak{J}v_2\right) = -g^{TX}\left(\mathfrak{J}v_1,v_2\right)$$

and hence its non-zero eigenvalues come in purely imaginary pairs $\pm i\mu$; $\mu>0$. We now say that the metric is *strongly suitable* to the contact form a if the spectrum of $\mathfrak{J}_x$ is independent of x: there exist positive constants $0<\mu_1\leq\mu_2\leq\ldots\leq\mu_m$ such that

$$\mathrm{Spec}\left(\mathfrak{J}_x\right) = \left\{0,\pm i\mu_1,\pm i\mu_2,\ldots,\pm i\mu_m\right\}, \quad \forall x\in X. \tag{1.7}$$

We note that this is a slight strengthening of the suitability assumption from [26] wherein $\mathrm{Spec}\left(\mathfrak{J}_x\right)$ was allowed to vary in x with one single function $\nu(x)\in C^\infty(X)$. Here are two examples of strongly suitable suitable metrics.

(1) The dimension of the manifold dim $X=3$. In this case a metric g^{TX} is strongly suitable if the magnetic field $|da|=\mu_1$ has constant strength.
(2) There is a smooth endomorphism $J:TX\to TX$, such that $\left(X^{2m+1},a,g^{TX},J\right)$ is a metric contact manifold. That is, we have

$$
\begin{aligned}
J^2 v_1 &= -v_1 + a\left(v_1\right)R, \\
g^{TX}\left(v_1, Jv_2\right) &= da\left(v_1,v_2\right), \quad \forall v_1,v_2\in T_xX.
\end{aligned} \tag{1.8}
$$

In this case the nonzero eigenvalues of $\mathfrak{J}_x=J_x$ are $\pm i$ (each with multiplicity m). For any given contact form a there exists an infinite dimensional space of $\left(g^{TX},J\right)$ satisfying (1.8). This case in particular includes all strictly pseudo-convex CR manifolds.

Our first result is now a Gutzwiller type trace formula for the magnetic Dirac operator (1.1). To state it precisely choose $f\in C_c^\infty\left(-\sqrt{2\mu_1},\sqrt{2\mu_1}\right)$. Let $\theta\in C_c^\infty\left(\mathbb{R};[0,1]\right)$ be any compactly supported supported function, such that $\theta=1$ near 0, and set

$$
\begin{aligned}
\mathcal{F}^{-1}\theta\left(x\right) &:= \check{\theta}\left(x\right)=\frac{1}{2\pi}\int e^{ix\xi}\theta\left(\xi\right)d\xi \\
\mathcal{F}_h^{-1}\theta\left(x\right) &:= \frac{1}{h}\check{\theta}\left(\frac{x}{h}\right)=\frac{1}{2\pi h}\int e^{\frac{i}{h}x\xi}\theta\left(\xi\right)d\xi
\end{aligned}
$$

to be its classical and semi-classical inverse Fourier transforms respectively. We shall then prove.

Theorem 1.1. *Let a be a non-resonant contact form and g^{TX} a strongly suitable metric. We then have a trace expansion*

$$
tr\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right]= \tag{1.9}
$$

$$
tr\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right]=h^{-m-1}\left(\sum_{j=0}^{N}f\left(\lambda\right)u_j\left(\lambda\right)h^{j/2}\right) \tag{1.10}
$$

$$
+\sum_{\gamma}e^{\frac{i}{h}T_\gamma}e^{i\frac{\pi}{2}\mathfrak{m}_\gamma}\sum_{j=0}^{N-2m-2}h^{j/2}\sum_{k=0}^{j}\lambda^k A_{\gamma,j,k}\theta\left(L_\gamma\right) \tag{1.11}
$$

$$
+O\left(h^{N/2-m-1}\right) \tag{1.12}
$$

for each $N\in\mathbb{N}$,$\lambda\in\mathbb{R}$. Here the second line on the right hand side above is a sum over the Reeb orbits of a. Furthermore; the terms appearing on the right hand side are as follows

(1) each u_j is a polynomial function in λ
(2) each $A_{\gamma,j,k}$ is a differential operator on $\mathbb{R}$ of order between k and j
(3) T_γ and L_γ denote the period and Riemannian length of the Reeb orbit respectively
(4) $\mathfrak{m}_\gamma$ denotes the Maslov index of a metaplectic lift of P_γ^+.

Finally, the leading contribution of each Reeb orbit γ is given by the multiplication operator

$$A_{\gamma,0,0}\theta=\frac{L_{\gamma}^{\#}}{2\pi}\frac{1}{\sqrt{\left|\det\left(1-P_{\gamma}^{+}\right)\right|}}\theta$$

with $L_{\gamma}^{\#}$ denoting the primitive length of the orbit.

An immediate consequence of the above trace formula is a little o estimate on the dimension of the kernel of D_h

$$k_h\coloneqq\dim\ker\left(D_h\right)=o\left(h^{-m}\right). \tag{1.13}$$

As another application, we shall prove a semiclassical limit formula for the (rescaled) eta invariant of the magnetic Dirac operator D_h. To state this, first let $R^{\perp}\subset TX$ denote the $2m$-dimensional orthogonal complement to the Reeb vector field. We may now define the endomorphisms $\left(\nabla^{TX}\mathfrak{J}\right)^{0}:R^{\perp}\rightarrow R^{\perp}$, $|\mathfrak{J}|:R^{\perp}\rightarrow R^{\perp}$, via

$$\begin{aligned}\left(\nabla^{TX}\mathfrak{J}\right)^{0}v&\coloneqq\left(\nabla_{v}^{TX}\mathfrak{J}\right)R,\quad\forall v\in R^{\perp},\\ |\mathfrak{J}|&\coloneqq\sqrt{-\mathfrak{J}^{2}}.\end{aligned} \tag{1.14}$$

We then have the following.

Theorem 1.2. *Let a be a non-resonant contact form and g^{TX} a strongly suitable metric. The rescaled eta invariant of the Dirac operator* (1.1) *satisfies*

$$\lim_{h\rightarrow0}h^{m}\eta\left(D_h\right)=-\frac{1}{2}\frac{1}{\left(2\pi\right)^{m+1}}\frac{1}{m!}\int_{X}\left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)^{0}\right]a\wedge\left(da\right)^{m}. \tag{1.15}$$

Before proceeding further we look at the limit formula formula above in the two special cases mentioned earlier.

(1) The dimension of the manifold $\dim X=3$ and $|da|=\mu_1$ has constant strength. In this case the limit (1.15) is given by the volume integral

$$\lim_{h\rightarrow0}h^{m}\eta\left(D_h\right)=-\frac{\mu_1}{8\pi^{2}}\int_{X}\left[i_{R}d^{*}da\right]dx.$$

(2) There is a smooth endomorphism $J:TX\rightarrow TX$, such that $\left(X^{2m+1},a,g^{TX},J\right)$ is a metric contact manifold (1.8). In this case the limit (1.15) is simply the volume

$$\lim_{h\rightarrow0}h^{m}\eta\left(D_h\right)=-\frac{m}{2}\frac{1}{\left(2\pi\right)^{m+1}}\mathrm{vol}\left(X\right).$$

A small time trace formula (1.9) was already proved in [26] assuming θ to be supported sufficiently close to the origin; much of this article attempts to extend the arguments therein to large supports. By the construction of appropriate trapping functions it is shown that the formula of [26] extends to large time when microlocalized away from the Reeb orbits. Near the Reeb orbits, the trace is studied via understanding the Birkhoff normal form of D_h near each orbit, using which it is reduced to the trace of a scalar effective Hamiltonian. The Birkhoff normal form procedure here combines the one in [26] with ones for scalar Hamiltonians [16, 17, 21, 30, 31] near periodic Hamiltonian orbits and hence requires the non-resonance assumption. The semiclassical asymptotics for the Dirac operator considered here were originally motivated by Taubes's proof of the three dimensional Weinstein conjecture [28] on the existence of Reeb orbits. The existence of

Reeb orbits, or the necessity of dynamical contributions (1.11), is still unresolved in higher dimensions.

The behavior of the eta invariant of Dirac operators has been studied under various operations (cf. [15] for a survey) and the formula (1.15) adds to a long list. A more precise relation between the eta invariant and the dynamics of geodesic flow has been studied on compact hyperbolic manifolds [23] and locally symmetric spaces of non-compact type [24]. The proof of such precise relations on general negatively curved manifolds is the subject of the hypo-elliptic Laplacian program of Bismut [4, 5].

Under the well known correspondence between semi-classical and microlocal analysis, the operator (1.1) corresponds to a hypo elliptic sub-Riemannian (sR) Dirac operator on the product $X \times S^1$. The Reeb orbits on X correspond to singular geodesics on the quasi-contact product suggesting a more general trace formula for sR Dirac operators. The eigenvalues of the symbol of the sR Dirac operator being the square root of the symbol of the sR Laplacian up to sign, similar trace formulas could be expected for the half-wave equation of the sR Laplacian. A systematic study of spectral asymptotics for sR Laplacians and related dynamics has been recently undertaken [9, 8].

The paper is organized as follows. In Section 2 we begin with the preliminaries of Dirac operators, Clifford representations and semi-classical analysis used in the paper. In Section 3 we breakup the trace (1.9) using a partition of unity adapted to the Reeb dynamics. By the construction of appropriate trapping functions it is shown here that the trace does not have non-local contributions when microlocalized away from the Reeb orbits. In Section 4 we generalize the Birkhoff normal form of [26] to one in a neighborhood of each Reeb orbit. This normal form is then used, via the construction of a similar trapping functions to reduce the trace asymptotics to $S^1 \times \mathbb{R}^{2m}$ in Section 5 leading to a proof of Theorem 1.1 in Section 6. In Section 7 we compute the second term in the local trace expansion of (1.10). This leads to the semi-classical limit formula for the eta invariant (1.15) in the final Section 8.

2. Preliminaries

2.1. **Spectral invariants of the Dirac operator.** Here we review the basic facts about Dirac operators used throughout the paper with [3] providing a standard reference. Consider a compact, oriented, Riemannian manifold $\left(X, g^{TX}\right)$ of odd dimension $n = 2m+1$. Let X be equipped with spin structure, i.e. a principal $\mathrm{Spin}\,(n)$ bundle $\mathrm{Spin}\,(TX) \to SO\,(TX)$ with an equivariant double covering of the principal $SO\,(n)$-bundle of orthonormal frames $SO\,(TX)$. The corresponding spin bundle $S = \mathrm{Spin}\,(TX) \times_{\mathrm{Spin}(n)} S_{2m}$ is associated to the unique irreducible representation of $\mathrm{Spin}\,(n)$. Let ∇^{TX} denote the Levi-Civita connection on TX. This lifts to the spin connection ∇^S on the spin bundle S. The Clifford multiplication endomorphism $c : T^*X \to S \otimes S^*$ may be defined (see 2.2) satisfying

$$c(a)^2 = -|a|^2, \quad \forall a \in T^*X.$$

Let L be a Hermitian line bundle on X. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(X;\mathbb{R})$ be a 1-form on X. This gives a family $\nabla^h = A_0 + \frac{i}{h}a$ of unitary connections on L. We denote by $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ the tensor product connection on $S \otimes L$. Each such connection defines a coupled Dirac operator

$$D_h \coloneqq hD_{A_0} + ic\,(a) = hc \circ \left(\nabla^{S\otimes L}\right) : C^\infty(X; S\otimes L) \to C^\infty(X; S\otimes L)$$

for $h \in (0,1]$. The operator D_h is elliptic and self-adjoint. It hence possesses a discrete spectrum of eigenvalues.

We define the eta function of D_h by the formula

$$\eta\left(D_h,s\right) := \sum_{\substack{\lambda\neq 0\\ \lambda\in\mathrm{Spec}(D_h)}} \mathrm{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{tr}\left(D_h e^{-tD_h^2}\right)dt, \tag{2.1}$$

$\forall s \in \mathbb{C}$. Here, and in the remainder of the paper, we use the convention that $\mathrm{Spec}(D_h)$ denotes a multiset with each eigenvalue of D_h being counted with its multiplicity. The above series converges for $\mathrm{Re}(s) > n$. It was shown in [1, 2] that the eta function possesses a meromorphic continuation to the entire complex s-plane and has no pole at zero. Its value at zero is defined to be the eta invariant of the Dirac operator

$$\eta_h := \eta\left(D_h, 0\right).$$

By including the zero eigenvalue in (2.1), with an appropriate convention, we may define a variant known as the reduced eta invariant by

$$\bar{\eta}_h := \frac{1}{2}\left\{k_h + \eta_h\right\}.$$

The eta invariant is unchanged under positive scaling

$$\eta\left(D_h,0\right) = \eta\left(cD_h,0\right); \quad \forall c>0. \tag{2.2}$$

Let $L_{t,h}$ denote the Schwartz kernel of the operator $D_h e^{-tD_h^2}$ on the product $X\times X$. Throughout the paper all Schwartz kernels will be defined with respect to the Riemannian volume density. Denote by $\mathrm{tr}\left(L_{t,h}\left(x,x\right)\right)$ the point-wise trace of $L_{t,h}$ along the diagonal. We may now analogously define the function

$$\eta\left(D_h,s,x\right) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{tr}\left(L_{t,h}\left(x,x\right)\right)dt, \tag{2.3}$$

$\forall s\in\mathbb{C}$, $x\in X$. In [6] theorem 2.6, it was shown that for $\mathrm{Re}(s) > -2$, the function $\eta\left(D_h,s,x\right)$ is holomorphic in s and smooth in x. From (2.3) it is clear that this is equivalent to

$$\mathrm{tr}\left(L_{t,h}\right) = O\left(t^{\frac{1}{2}}\right), \quad \text{as } t\to 0. \tag{2.4}$$

The eta invariant is then given by the convergent integral

$$\eta_h = \int_0^\infty \frac{1}{\sqrt{\pi t}}\mathrm{tr}\left(D_h e^{-tD_h^2}\right)dt. \tag{2.5}$$

2.2. **Clifford algebra and and its representations.** Here we review the construction of the spin representation of the Clifford algebra. The following being standard, is merely used to setup our conventions.

Consider a real vector space V of even dimension $2m$ with metric $\langle,\rangle$. Recall that its Clifford algebra $Cl\left(V\right)$ is defined as the quotient of the tensor algebra $T\left(V\right) := \oplus_{j=0}^{\infty}V^{\otimes j}$ by the ideal generated from the relations $v\otimes v + |v|^2 = 0$. Fix a compatible almost complex structure J and split $V\otimes\mathbb{C} = V^{1,0}\oplus V^{0,1}$ into the $\pm i$ eigenspaces of J. The complexification $V\otimes\mathbb{C}$ carries an induced $\mathbb{C}$-bilinear inner product $\langle,\rangle_{\mathbb{C}}$ as well as an induced Hermitian inner product $h^{\mathbb{C}}\left(,\right)$. Next, define $S_{2m} = \Lambda^* V^{1,0}$. Clearly S_{2m} is a complex vector space of dimension 2^m on which the unique irreducible (spin)-representation of the Clifford algebra $Cl(V) \otimes \mathbb{C}$ is defined by the rule

$$c_{2m}(v)\,\omega = \sqrt{2}\left(v^{1,0} \wedge \omega - \iota_{v^{0,1}}\omega\right), \quad v \in V, \omega \in S_{2m}.$$

The contraction above is taken with respect to $\langle , \rangle_{\mathbb{C}}$. It is clear that $c_{2m}(v) : \Lambda^{\text{even/odd}} \to \Lambda^{\text{odd/even}}$ switches the odd and even factors. For the Clifford algebra $Cl(W) \otimes \mathbb{C}$ of an odd dimensional vector space $W = V \oplus \mathbb{R}[e_0]$ there are exactly two irreducible representations. The first (spin)-representation $S_{2m+1} = \Lambda^* V^{1,0}$ is defined via

$$\begin{aligned} c_{2m+1}(v) &= c_{2m}(v), \quad v \in V \\ c_{2m+1}(e_0)\,\omega_{\text{even/odd}} &= \pm\frac{1}{i}\omega_{\text{even/odd}} \end{aligned} \tag{2.6}$$

while the other corresponds to the opposite sign convention in (2.6) above. We shall often use the shorthand's $c = c_{2m} = c_{2m+1}$ with the index $2m$, $2m+1$ implicitly understood.

Pick an orthonormal basis $e_1, e_2, \ldots, e_{2m}$ for V in which the almost complex structure is given by $Je_j = e_{j+m}$, $1 \leq j \leq m$. An $h^{\mathbb{C}}$-orthonormal basis for $V^{1,0}$ is now given by $w_j = \frac{1}{\sqrt{2}}(e_{j+m} + ie_j)$, $1 \leq j \leq m$. A basis for S_{2m} and $S^{\pm}_{2m+1}$ is given by

$$w_k = w_1^{k_1} \wedge \ldots \wedge w_m^{k_m}, \text{ with } k = (k_1, k_2, \ldots, k_m) \in \{0,1\}^m. \tag{2.7}$$

Ordering the above chosen bases lexicographically in k, we may define the Clifford matrices, of rank 2^m, via

$$\gamma_j^m = c(e_j), \quad 0 \leq j \leq 2m,$$

for each m . We note that the above is a slightly different convention from [26] adopted to simplify some formulas in Section 7. Again, we often write $\gamma_j^m = \gamma_j$ with the index m implicitly understood. Giving representations of the Clifford algebra, these matrices satisfy the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}. \tag{2.8}$$

We also set $\sigma_j = i\gamma_j$.

Next, one may further define the Clifford quantization map on the exterior algebra

$$\begin{aligned} c : \Lambda^* W \otimes \mathbb{C} &\to \operatorname{End}(S_{2m}) \\ c\left(e_0^{k_0} \wedge \ldots \wedge e_{2m}^{k_{2m}}\right) &= c(e_0)^{k_0} \ldots c(e_{2m})^{k_{2m}}. \end{aligned} \tag{2.9}$$

An easy computation yields

$$\gamma_0(\gamma_1\gamma_{m+1})\ldots(\gamma_m\gamma_{2m}) = \frac{1}{i^{m+1}}$$

and hence

$$\operatorname{tr}[\gamma_0 \ldots \gamma_{2m}] = \frac{1}{i^{m+1}}2^m.$$

Furthermore, if $e_0 \wedge \ldots \wedge e_{2m}$ is designated to give a positive orientation for W then for $\omega \in \Lambda^k W$ we have

$$c(*\omega) = i^{m+1}(-1)^{\frac{k(k+1)}{2}} c(\omega) \tag{2.10}$$

$$c(\omega)^* = (-1)^{\frac{k(k+1)}{2}} c(\omega) \tag{2.11}$$

under the Hodge star and $h^{\mathbb{C}}$-adjoint. The Clifford quantization map (2.9) is a linear surjection with kernel spanned by elements of the form $*\omega - i^{m+1}(-1)^{\frac{k(k+1)}{2}}\omega$. Thus, in particular one has linear isomorphisms

$$c : \Lambda^{\text{even/odd}} W \otimes \mathbb{C} \to \operatorname{End}(S_{2m}). \tag{2.12}$$

Next, given $(r_1, \ldots, r_m) \in \mathbb{R}^m \setminus 0$, we define

$$I_r := \{j | r_j \neq 0\} \subset \{1, 2, \ldots, m\} \tag{2.13}$$

$$Z_r := |I_r| \tag{2.14}$$

$$V_r := \bigoplus_{j \in I_r} \mathbb{C}[w_j] \subset V^{1,0} \tag{2.15}$$

$$\text{and} \quad w_r := \sum_{j=1}^{m} r_j w_j \in V_r. \tag{2.16}$$

Clearly, $\|w_r\| = |r|$. Denoting by $w_r^{\perp}$ the $h^{\mathbb{C}}$-orthogonal complement of $w_r \subset V_r$, one clearly has $V_r = \mathbb{C}[w_r] \oplus w_r^{\perp}$. We set

$$\begin{aligned} \mathtt{i}_r : \Lambda^* V_r &\to \Lambda^* V_r, \quad \text{via} \\ \mathtt{i}_r(\omega) &:= \frac{w_r}{|r|} \wedge \omega \\ \mathtt{i}_r\left(\frac{w_r}{|r|} \wedge \omega\right) &:= \omega \end{aligned} \tag{2.17}$$

for $\omega \in \Lambda^* w_r^{\perp}$. Clearly, $\mathtt{i}_r^2 = 1$ and $\mathtt{i}_r$ is a linear isomorphism between

$$\begin{aligned} \mathtt{i}_r : \Lambda^{\text{even}} V_r &\to \Lambda^{\text{odd}} V_r \\ \mathtt{i}_r : \Lambda^{\text{odd}} V_r &\to \Lambda^{\text{even}} V_r. \end{aligned}$$

Next, the endomorphism

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = (w_r \wedge + \iota_{\bar{w}_r}) : \Lambda^* V_r \to \Lambda^* V_r \tag{2.18}$$

has the form

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = \begin{bmatrix} & |r|\,\mathtt{i}_r \\ |r|\,\mathtt{i}_r & \end{bmatrix} \tag{2.19}$$

with respect to the decomposition $\Lambda^* V_r = \Lambda^{\text{odd}} V_r \oplus \Lambda^{\text{even}} V_r$. This finally allows us to write the eigenspaces of (2.18) as

$$V_r^{\pm} = (1 \pm \mathtt{i}_r)(\Lambda^{\text{even}} V_r) \tag{2.20}$$

with eigenvalue $\pm |r|$ respectively.

Finally we shall need an almost diagonalizability result for the restriction of Clifford multiplication to the sphere. Define $S(W) = \{v \in W | \, |v| = 1\}$ as well as the restriction

$$\begin{aligned} c : S(W) &\to \mathfrak{u}(S_{2m+1}) \\ c(v)^2 &= -\operatorname{Id}. \end{aligned} \tag{2.21}$$

The restriction of the spin bundle S_{2m+1} to the sphere $S(W)$ splits $S_{2m+1}|_{S(W)} = S_+(W) \oplus S_-(W)$ into the $\pm i$ eigenspaces of the c respectively. The summands

$S_+(W), S_-(W)$ maybe identified with the (non-trivial) bundle of positive and negative spinors on the sphere. The restriction c (2.21) is hence not globally diagonalizable over the sphere. We now identify $S(W) = \left\{\theta_0 e_0 + \dots \theta_{2m} e_{2m} \in W | \theta_0^2 + \dots + \theta_{2m}^2 = 1\right\}$ with the standard sphere in $S^{n-1} \subset \mathbb{R}^n$ using the chosen basis for W; with the induced basis (2.7) of S_{2m+1} giving identifications $\mathfrak{u}(S_{2m+1}) = \mathfrak{u}\left(\mathbb{C}^{2^m}\right)$, $U(S_{2m+1}) = U\left(\mathbb{C}^{2^m}\right)$. Thus

$$c(\theta) \coloneqq c(\theta_0 e_0 + \dots \theta_{2m} e_{2m}) = \sum_{j=0}^{2m} \theta_j \gamma_j \in C^\infty\left(S^{n-1}, \mathfrak{u}\left(\mathbb{C}^{2^m}\right)\right) \tag{2.22}$$

in this trivialization/coordinates. We now have.

Lemma 2.1. *For each $\rho \in \left(0, \frac{1}{8}\right)$, there exist smooth family of maps/functions $\mathtt{v}_t^\rho \in C^\infty\left(S^{n-1}; U\left(\mathbb{C}^{2^m}\right)\right)$; $a_{0,t}^\rho, a_{1,t}^\rho \in C^\infty\left([-1,1]_{\theta_0}\right)$, $t \in [0,1]$, such that*

(1) $\left|a_{j,t}^\rho\right| \leq \left(\frac{8}{\rho}\right)^{1/2}, \left|\partial_{\theta_0} a_{j,t}^\rho\right| \leq \left(\frac{8}{\rho}\right)^2$, $t \in [0,1]$, $j = 0, 1$.

(2) $\left\|\partial_t \mathtt{v}_t^\rho\right\| \leq \left(\frac{8}{\rho}\right)^2$, $\left\|\partial_{\theta_j} \mathtt{v}_t^\rho\right\| \leq \left(\frac{8}{\rho}\right)^4$, $t \in [0,1]$, $j = 0, \dots, 2m$.

(3)

$$a_{0,t}^\rho(\theta_0) = \begin{cases} \theta_0; & t \in \left[0, \frac{1}{2}\right] \\ 1; & t = 1,\ \theta_0 < 1 - \rho, \end{cases} \tag{2.23}$$

$$a_{1,t}^\rho(\theta_0) = \begin{cases} -1; & t \in \left[0, \frac{1}{2}\right] \\ 0; & t = 1,\ \theta_0 < 1 - \rho, \end{cases} \tag{2.24}$$

$$\mathtt{v}_t^\rho = \sigma_0; \quad t \in \left[0, \frac{1}{2}\right], \tag{2.25}$$

(4) we have the almost diagonalizability equation

$$\mathtt{v}_t^\rho(\theta)^* c(\theta) \mathtt{v}_t^\rho(\theta) = a_{0,t}^\rho(\theta_0)\gamma_0 + a_{1,t}^\rho(\theta_0)\left[\sum_{j=1}^{2m} \theta_j \gamma_j\right]. \tag{2.26}$$

Proof. The matrix

$$\mathtt{v} : S^{n-1} \setminus \{\theta_0 = 1\} \to U\left(\mathbb{C}^{2^m}\right) \tag{2.27}$$

$$\mathtt{v}(\theta) \coloneqq \sqrt{\frac{(1-\theta_0)}{2}}\sigma_0 - \frac{\theta_j}{\sqrt{2(1-\theta_0)}}\sigma_j \tag{2.28}$$

diagonalizes

$$\mathtt{v}^* c(\theta) \mathtt{v} = -\gamma_0 \tag{2.29}$$

away from the north-pole $\{\theta_0 = 1\}$. To get a map defined on the entire sphere, let $\chi_1^\rho \in C^\infty\left([-1,1]_{\theta_0}; \left[-1, 1 - \frac{\rho}{2}\right]\right)$ such that

$$\chi_1^\rho(\theta_0) = \begin{cases} \theta_0; & -1 \leq \theta_0 < 1 - \rho, \\ -1; & 1 - \frac{\rho}{2} \leq \theta_0 \leq 1, \end{cases} \tag{2.30}$$

with $\left|(\chi_1^\rho)'\right| \leq \frac{4}{\rho}$. Further let $\chi_0 \in C_c^\infty([-1,1]_t; [0,1])$ with $\chi_0 = 1$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $|\partial_t \chi_0| \leq 4$. Finally set $\chi_{1,t}^\rho = [1 - \chi_0(t)]^2 \chi_1^\rho - \left[1 - (1 - \chi_0(t))^2\right] \in C^\infty\left([-1,1]_{\theta_0}; \left[-1, 1 - \frac{\rho}{2}\right]\right)$

satisfying $\left|\left(\chi_{1,t}^{\rho}\right)'\right| \leq \frac{4}{\rho}$, $\left|\partial_t \chi_{1,t}^{\rho}\right| \leq 8$. Now $\chi_{2,t}^{\rho}\left(\theta_0\right) = \sqrt{\frac{1-\chi_{1,t}^{\rho}(\theta_0)^2}{1-\theta_0^2}} \in C^{\infty}\left([-1,1]_{\theta_0}\right)$ satisfies $\left|\chi_{2,t}^{\rho}\right| \leq \left(\frac{2}{\rho}\right)^{1/2}$, $\left|\left(\chi_{2,t}^{\rho}\right)'\right| \leq \left(\frac{4}{\rho}\right)^2$, $\left|\partial_t \chi_{2,t}^{\rho}\right| \leq \left(\frac{4}{\rho}\right)^2$. The family

$$\chi_t^{\rho} : S^{n-1} \to S^{n-1} \setminus \{\theta_0 = 1\}$$

$$\chi_t^{\rho}(\theta) \coloneqq \left(\chi_{1,t}^{\rho}(\theta_0), \chi_{2,t}^{\rho}(\theta_0)\theta_1, \ldots, \chi_{2,t}^{\rho}(\theta_0)\theta_{2m}\right) \tag{2.31}$$

now defines a family of maps on the entire sphere

$$\mathtt{v}_t^{\rho} : S^{n-1} \to U\left(\mathbb{C}^{2^m}\right)$$

$$\mathtt{v}_t^{\rho}(\theta) \coloneqq \mathtt{v}\left(\chi_t^{\rho}(\theta)\right). \tag{2.32}$$

The equation (2.26) now follows from (2.28), (2.29), (2.31) and (2.32) with

$$\begin{aligned} a_{0,t}^{\rho} &= -\theta_0 \chi_{1,t}^{\rho} - \left(1-\theta_0^2\right)\chi_{2,t}^{\rho} \\ a_{1,t}^{\rho} &= \chi_{1,t}^{\rho} - \theta_0 \chi_{2,t}^{\rho}. \end{aligned}$$

□

2.2.1. *Magnetic Dirac operator on* $\mathbb{R}^m$. Here we recall the spectrum of the magnetic Dirac operator

$$D_{\mathbb{R}^m} = \sum_{j=1}^{m} \left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} \left[\gamma_{2j}\left(h\partial_{x_j}\right) + i\gamma_{2j-1}x_j\right] \in \Psi_{\mathrm{cl}}^1\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right). \tag{2.33}$$

on $\mathbb{R}^m$ computed in [26]. Its square is computed in terms of the harmonic oscillator

$$D_{\mathbb{R}^m}^2 = \mathtt{H}_2 - ih\mathtt{R}_{2m+1}, \text{ with} \tag{2.34}$$

$$\mathtt{H}_2 = \frac{1}{2} \sum_{j=1}^{m} \mu_j \left[-\left(h\partial_{x_j}\right)^2 + x_j^2\right] \tag{2.35}$$

$$\mathtt{R}_{2m+1} = \frac{1}{2} \sum_{j=1}^{m} \mu_j \left[\gamma_{2j-1}\gamma_{2j}\right].$$

Define the lowering and raising operators $A_j = h\partial_{x_j} + x_j$, $A_j^* = -h\partial_{x_j} + x_j$ for $1 \leq j \leq m$, and the Hermite functions

$$\psi_{\tau,k}(x) \coloneqq \psi_{\tau}(x) \otimes w_k$$

$$\psi_{\tau}(x) \coloneqq \frac{1}{(\pi h)^{\frac{m}{4}} (2h)^{\frac{|\tau|}{2}} \sqrt{\tau!}} \left[\Pi_{j=1}^m \left(A_j^*\right)^{\tau_j}\right] e^{-\frac{|x|^2}{2h}}, \tag{2.36}$$

$$\text{for } \tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{N}_0^m.$$

We also set

$$E_{\tau} \coloneqq \bigoplus_{b \in \{0,1\}^{I_{\tau}}} \mathbb{C}\left[\prod_{j \in I_{\tau}} \left(\frac{c(w_j)A_j}{\sqrt{2\tau_j h}}\right)^{b_j} \psi_{\tau,0}\right]$$

with I_{τ}, V_{τ} as in (2.13), (2.15). One clearly has an isomorphism

$$\begin{aligned} \mathscr{I}_{\tau} : \Lambda^* V_{\tau} &\to E_{\tau} \\ \mathscr{I}_{\tau}\left(\bigwedge_{j \in I_{\tau}} w_j^{b_j}\right) &\coloneqq \prod_{j \in I_{\tau}} \left(\frac{c(w_j)A_j}{\sqrt{2\tau_j h}}\right)^{b_j} \psi_{\tau,0}. \end{aligned}$$

If $\mathtt{i}_\tau := \mathscr{I}_\tau \mathtt{i}_{r_\tau} \mathscr{I}_\tau^{-1} : E_\tau^{\mathrm{even/odd}} \to E_\tau^{\mathrm{odd/even}}$, the restriction of $D_{\mathbb{R}^m}$ to E_τ is of the form

$$D_{\mathbb{R}^m} = \begin{bmatrix} & |r_\tau|\,\mathtt{i}_\tau \\ |r_\tau|\,\mathtt{i}_\tau & \end{bmatrix}. \tag{2.37}$$

We may set

$$E_\tau^{\mathrm{even/odd}} := \mathscr{I}_\tau\left(\Lambda^{\mathrm{even/odd}} V_\tau\right)$$
$$E_\tau^{\pm} = \mathscr{I}_\tau\left(V_\tau^{\pm}\right) \tag{2.38}$$

and observe the Landau decomposition

$$L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right) = \mathbb{C}\left[\psi_{0,0}\right] \oplus \bigoplus_{\tau \in \mathbb{N}_0^m \setminus 0} \left(E_\tau^{\mathrm{even}} \oplus E_\tau^{\mathrm{odd}}\right). \tag{2.39}$$

The spectrum of (2.33) is given by Prop. 2.1 of [26].

Proposition 2.2. *An orthogonal decomposition of $L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right)$ consisting of eigenspaces of the magnetic Dirac operator $D_{\mathbb{R}^m}$ (2.33) is given by*

$$L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right) = \mathbb{C}\left[\psi_{0,0}\right] \oplus \bigoplus_{\tau \in \mathbb{N}_0^m \setminus 0} \left(E_\tau^{+} \oplus E_\tau^{-}\right).$$

Here $E_\tau^{\pm}$, as in (2.38), have dimension $2^{Z_\tau - 1}$ and correspond to the eigenvalues $\pm\sqrt{\mu.\tau h}$ respectively.

2.3. **The Semi-classical calculus.** Finally, here we review the semi-classical pseudodifferential calculus used throughout the paper with [18, 32] being the detailed references. Much of this being reviewed in [26], we only highlight some new aspects. Let $\mathfrak{gl}(l)$ denote the space of all $l \times l$ complex matrices. For $A = (a_{ij}) \in \mathfrak{gl}(l)$ we denote $|A| = \max_{ij} |a_{ij}|$. Denote by $\mathcal{S}\left(\mathbb{R}^n; \mathbb{C}^l\right)$ the space of Schwartz maps $f : \mathbb{R}^n \to \mathbb{C}^l$. We define the symbol space $S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ as the space of maps $a : (0,1]_h \to C^\infty\left(\mathbb{R}^{2n}_{x,\xi}; \mathfrak{gl}(l)\right)$ such that each of the semi-norms

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} \langle \xi \rangle^{-m+|\beta|} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\right|$$

is finite $\forall \alpha, \beta \in \mathbb{N}_0^n$. Such a symbol is said to lie in the more refined class $a \in S^m_{\mathrm{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ if there exists an h-independent sequence a_k, $k = 0, 1, \ldots$ of symbols such that $a - \left(\sum_{k=0}^N h^k a_k\right) \in h^{N+1} S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$, $\forall N$. The symbol classes $S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$, $S^m_{\mathrm{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ as above can be Weyl quantized to define one-parameter families of operators $a^W \in \Psi^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right), \Psi^m_{\mathrm{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ with Schwartz kernels given by

$$a^W := \frac{1}{(2\pi h)^n} \int e^{i(x-y).\xi/h} a\left(\frac{x+y}{2}, \xi; h\right) d\xi$$

This class of operators is closed under the standard operations of composition and formal-adjoint. Furthermore the class is invariant under changes of coordinates and basis for $\mathbb{C}^l$. This allows one to define invariant classes of operators $\Psi^m(X;E), \Psi^m_{\mathrm{cl}}(X;E)$ on $C^\infty(X;E)$ associated to any complex, Hermitian vector bundle $\left(E, h^E\right)$ on a smooth compact manifold X.

For $A \in \Psi^m_{\mathrm{cl}}(X;E)$, its principal symbol is well-defined as an element in $\sigma(A) \in S^m(X; \mathrm{End}(E)) \subset C^\infty(X; \mathrm{End}(E))$. One has that $\sigma(A) = 0$ if and only if $A \in h\Psi^m_{\mathrm{cl}}(X;E)$. We remark that $\sigma(A)$ is the restriction of standard symbol in [32] to the refined class $\Psi_{\mathrm{cl}}^{m}(X;E)$ and is locally given by the first coefficient a_0 in the expansion in h of its Weyl symbol. The principal symbol satisfies the basic relations $\sigma(AB)=\sigma(A)\sigma(B)$, $\sigma(A^*)=\sigma(A)^*$ with the formal adjoints being defined with respect to the same Hermitian metric h^E. The principal symbol map has an inverse given by the quantization map $\mathrm{Op}: S^m(X;\mathrm{End}(E))\rightarrow\Psi_{\mathrm{cl}}^{m}(X;E)$ satisfying $\sigma(\mathrm{Op}(a))=a\in S^m(X;\mathrm{End}(E))$. We remark that this quantization map is non-canonical and depends on the choice of an open cover, with local trivializations for E, and a subordinate partition of unity. We often use the alternate notation $\mathrm{Op}(a)=a^W$. For a scalar function $b\in S^0(X)$, it is clear from the multiplicative property of the symbol that $\left[a^W,b^W\right]\in h\Psi_{\mathrm{cl}}^{m-1}(X;E)$ and we define $H_b(a):=\frac{i}{h}\sigma\left(\left[a^W,b^W\right]\right)\in S^{m-1}(X;\mathrm{End}(E))$. We note that $H_b(a)$ depends on the quantization scheme, in particular the local trivializations used in defining Op. However one has $H_b(a)=\{a,b\}$ is given by the Poisson bracket when both sides are computed in the same defining trivialization.

The wavefront set of an operator $A\in\Psi_{\mathrm{cl}}^{m}(X;E)$ can be defined invariantly as a subset $WF(A)\subset\overline{T^*X}$ of the fibrewise radial compactification of its cotangent bundle. If the local Weyl symbol of A is given by a then $(x_0,\xi_0)\notin WF(A)$ if and only if there exists an open neighborhood $(x_0,\xi_0;0)\in U\subset\overline{T^*X}\times(0,1]_h$ such that $a\in h^\infty\langle\xi\rangle^{-\infty}C^k\left(U;\mathbb{C}^l\right)$ for all k. The wavefront set satisfies the basic properties $WF(A+B)\subset WF(A)\cap WF(B)$, $WF(AB)\subset WF(A)\cap WF(B)$ and $WF(A^*)=WF(A)$. The wavefront set $WF(A)=\emptyset$ is empty if and only if $A\in h^\infty\Psi^{-\infty}(X;E)$. We say that two operators $A=B$ microlocally on $U\subset\overline{T^*X}$ if $WF(A-B)\cap U=\emptyset$.

An operator $A\in\Psi_{\mathrm{cl}}^{m}(X;E)$ is said to be elliptic if $\langle\xi\rangle^m\sigma(A)^{-1}$ exists and is uniformly bounded on T^*X. If $A\in\Psi_{\mathrm{cl}}^{m}(X;E)$, $m>0$, is formally self-adjoint such that $A+i$ is elliptic then it is essentially self-adjoint (with domain $C_c^\infty(X;E)$) as an unbounded operator on $L^2(X;E)$. Its resolvent $(A-z)^{-1}\in\Psi_{\mathrm{cl}}^{-m}(X;E)$, $z\in\mathbb{C}$, $\mathrm{Im}z\neq0$, now exists and is pseudo-differential by an application of Beals's lemma. Given a Schwartz function $f\in\mathcal{S}(\mathbb{R})$, the Helffer-Sjöstrand formula now expresses the function $f(A)$ of such an operator in terms of its resolvent and an almost analytic continuation $\tilde{f}$ via

$$f(A)=\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}(z)(A-z)^{-1}dzd\bar{z}.$$

We then also have $WF(f(A))\subset\Sigma_{\mathrm{spt}(f)}^{A}:=\bigcup_{\lambda\in\mathrm{spt}(f)}\Sigma_\lambda^A$ where

$$\Sigma_\lambda^A=\left\{(x,\xi)\in T^*X|\det(\sigma(A)(x,\xi)-\lambda I)=0\right\}. \tag{2.40}$$

is classical λ-energy level of A.

2.3.1. *The class $\Psi_\delta^m(X;E)$.* We shall need also more exotic class of scalar symbols $S_\delta^m\left(\mathbb{R}^{2n};\mathbb{C}\right)$ defined for each $0<\delta<\frac{1}{2}$. A function $a:(0,1]_h\rightarrow C^\infty\left(\mathbb{R}_{x,\xi}^{2n};\mathbb{C}\right)$ is said to be in this class if and only if

$$\|a\|_{\alpha,\beta}:=\sup_{x,\xi,h}\langle\xi\rangle^{-m+|\beta|}h^{(|\alpha|+|\beta|)\delta}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi;h)\right| \tag{2.41}$$

is finite $\forall\alpha,\beta\in\mathbb{N}_0^n$. This class of operators is closed under the standard operations of composition, adjoint and changes of coordinates allowing the definition of the exotic pseudo-differential algebra $\Psi_\delta^m(X)$ on a compact manifold. The class

$S_{\delta}^{m}(X)$ is a family of functions $a:(0,1]_{h}\rightarrow C^{\infty}(T^{*}X;\mathbb{C})$ satisfying the estimates (2.41) in every coordinate chart and induced trivialization. Such a family can be quantized to $a^{W}\in\Psi_{\delta}^{m}(X)$ satisfying $a^{W}b^{W}=(ab)^{W}+h^{1-2\delta}\Psi_{\delta}^{m+m'-1}(X)$, $\frac{i}{h^{1-2\delta}}\sigma\left(\left[a^{W},b^{W}\right]\right)=[\{a,b\}]$ for another $b\in S_{\delta}^{m'}(X)$. The operators in $\Psi_{\delta}^{0}(X)$ are uniformly bounded on $L^{2}(X)$. Finally, the wavefront an operator $A\in\Psi_{\delta}^{m}(X;E)$ is similarly defined and satisfies the same basic properties as before.

3. Dynamical partitions

The trace formula of Theorem 1.1 was proved in [26] assuming θ to be supported in a sufficiently small interval near 0. In this case only the local contribution to the trace (1.10) appears. It now thus suffices to consider θ supported away from 0 and prove the following.

Lemma 3.1. *For $\theta\in C_{c}^{\infty}((T_{0},\infty);[0,1])$, $T_{0}>0$, one has*

(3.1)

$$tr\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_{h}^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right]=$$

$$tr\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right]=\sum_{\gamma}e^{\frac{i}{h}L_{\gamma}}e^{i\frac{\pi}{2}\mathfrak{m}_{\gamma}}\sum_{j=0}^{N-2m-2}h^{j/2}\sum_{k=0}^{j}\lambda^{k}A_{\gamma,j,k}\theta\left(T_{\gamma}\right)$$

$$+O\left(h^{N/2-m-1}\right)\tag{3.2}$$

for all $\lambda\in\mathbb{R}$, with the right hand side above being the same as the dynamical contribution (1.11) *in* (1.12).

To prove Lemma 3.1 we shall split the trace via a microlocal partition of unity adapted to the Reeb dynamics. To this end we first need a description of the contact form in a neighborhood of each Reeb orbit.

3.1. **Normal structure for the contact form.**

Let $\gamma\subset X$ be a primitive closed Reeb orbit with period T_{γ}. For a point $p\in\gamma$, the linearized Poincare return map $P_{\gamma}^{+}:H_{p}\rightarrow H_{p}$ restricted to the contact hyperplane then has the decomposition (1.2) as before. For each corresponding eigenvalue in this decomposition, define the following model quadratic functions on $\mathbb{R}^{2m}$

$$\begin{aligned}
&\text{Elliptic case:} & Q_{j}^{e} &= \frac{1}{2}\left(x_{j}^{2}+x_{j+m}^{2}\right), & 1\leq j\leq N_{e}\\
&\text{Hyperbolic case:} & Q_{j}^{h} &= x_{N_{e}+j}x_{N_{e}+j+m}, & 1\leq j\leq N_{h}\\
&\text{Loxodromic case:} & Q_{j}^{l,\mathrm{Re}} &= x_{m-2j+2}x_{2m-2j+1}-x_{m-2j+1}x_{2m-2j+2}, & 1\leq j\leq N_{l}\\
& & Q_{j}^{l,\mathrm{Im}} &= x_{m-2j+1}x_{2m-2j+1}+x_{m-2j+2}x_{2m-2j+2}, & 1\leq j\leq N_{l}
\end{aligned}\tag{3.3}$$

Also let $Q^{h,-}=\frac{\pi}{2}\sum_{j=1}^{N_{h}^{-}}\left(x_{N_{e}+j}^{2}+x_{N_{e}+j+m}^{2}\right)$ be the quadratic whose Hamiltonian flow rotates negative hyperbolic blocks by π.

In the theorem below we let $\gamma^{0}\coloneqq S^{1}\times\{0\}\subset S^{1}\times\mathbb{R}^{2m}$. We shall use θ or x_{0} interchangeably to denote the circular S^{1} variable. We also let $\chi^{-}\in C_{c}^{\infty}\left(0,\frac{1}{2}\right)_{\theta}$, $\chi^{+}\in C_{c}^{\infty}\left(\frac{1}{2},1\right)_{\theta}$ be non-negative functions with total integral 1. We now have the following normal structure for the contact form a near a nonresonant γ.

Proposition 3.2. *There exists a diffeomorphism $\kappa : \Omega_\gamma^0 \to \Omega_\gamma$ between some neighborhood of $\gamma^0 \subset \Omega_\gamma^0$ and some neighborhood of the Reeb orbit $\gamma \subset \Omega_\gamma$ such that*

$$\kappa^* a = \underbrace{\left(T_\gamma + \chi^- Q^{h,-} + \chi^+ \varphi^+\right)}_{=:\varphi} d\theta + \frac{1}{2}\sum_{j=1}^{m}\left(x_j dx_{j+m} - x_{j+m} dx_j\right) \tag{3.4}$$

modulo $O\left(Q^\infty\right)$. Here $\varphi^+ = \varphi^+\left(Q\right)$ in (3.4) *is a function on $\mathbb{R}^{2m}$ of the quadratics* (3.3) *with linear term*

$$\varphi^+ = \sum_{j=1}^{N_e} \beta_j Q_j^e + \sum_{j=1}^{N_h} \alpha_j Q_j^h + \sum_{j=1}^{N_l}\left(\alpha_j^0 Q_j^{l,Re} + \beta_j^0 Q_j^{l,Im}\right) + O\left(Q^2\right). \tag{3.5}$$

Proof. Choose Darboux coordinates $(x, y; z)$ centered at p in which $a = dz + \frac{1}{2}\sum_{j=1}^{m}\left(x_j dx_{j+m} - x_{j+m} dx_j\right)$. Then $\Sigma = \{z = 0\} \subset X$ defines a local Poincare section transverse to the Reeb vector field ∂_z in these coordinates. The Reeb flow gives rise to a symplectic return map and a return time function

$$\begin{aligned} P_\Sigma : (\Sigma, da) &\to (\Sigma, da) \\ T_\Sigma : \Sigma &\to \mathbb{R} \end{aligned} \tag{3.6}$$

which satisfy the relation

$$P_\Sigma^* a - a = dT_\Sigma \tag{3.7}$$

(cf. [14] Prop. 2.1). The linearization of P_Σ at 0 being P_γ^+, has the same spectrum Spec $\left(P_\gamma^+\right)$. Under the nonresonance assumption, such a symplectic map is a composition of the Hamiltonian diffeomorphisms

$$P_\Sigma = e^{H_{\varphi^+}} \circ e^{H_{Q^{h,-}}}, \tag{3.8}$$

modulo $O\left(Q^\infty\right)$, for a function φ^+ of the form (3.5) (cf. [20, 27]). We now compute $\left(e^{H_{Q^{h,-}}}\right)^* a = a$ and

$$\begin{aligned} \frac{d}{dt}\left(e^{tH_{\varphi^+}}\right)^* a\Big|_{t=0} &= i_{H_{\varphi^+}} da + d i_{H_{\varphi^+}} a \\ &= d\varphi^+ - d\left[\frac{1}{2}\sum_{j=1}^{m}\left(x_j \varphi^+_{x_j} + x_{j+m}\varphi^+_{x_{j+m}}\right)\right]. \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9) we now have

$$T_\Sigma = T_\gamma + \underbrace{\varphi^+ - \frac{1}{2}\sum_{j=1}^{m}\left(x_j \varphi^+_{x_j} + x_{j+m}\varphi^+_{x_{j+m}}\right)}_{T_\Sigma^+ :=}. \tag{3.10}$$

Next, let us compute the return map and return time, associated to the Poincare section $\Sigma_0 = \{\theta = 0\}$, for the model contact form (3.4) on $S^1 \times \mathbb{R}^{2m}$. Its Reeb vector field R_0 is easily computed

$$R_0 = \begin{cases} \frac{1}{T_\gamma}\left(\partial_\theta + \chi^- H_{Q^{h,-}}\right), & \theta \in \left(0, \frac{1}{2}\right) \\ \frac{1}{T_\gamma + \chi^+ T_\Sigma^+}\left(\partial_\theta + \chi^+ H_{\varphi^+}\right), & \theta \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{3.11}$$

To compute the return map and time, first note that each of the quadratics (3.3) Poisson commutes with φ^{+} of the form (3.5). Hence each of these quadratics is constant along the Hamilton flow of $H_{\varphi^{+}}$. An easy calculation upon differentiating (3.5) yields that the quantity $\frac{1}{2}\sum_{j=1}^{m}\left(x_j\varphi^{+}_{x_j}+x_{j+m}\varphi^{+}_{x_{j+m}}\right)$ maybe expressed in terms of the same quadratic functions and is thus also constant along the Hamilton flow of $H_{\varphi^{+}}$. Thus T^{+}_{Σ} (3.10) is constant along the Hamilton flow of $H_{\varphi^{+}}$. The return map and time of (3.11) are now easily computed to be $e^{H_{\varphi^{+}}}\circ e^{H_{Q^{h,-}}}$ and T_{Σ} respectively.

Finally, with the return map and time of the Poincare section Σ being the same as in the model case, a Moser style argument maybe applied to complete the proof. $\square$

In the proof above we have modified arguments from [16] Thm. 2.7 from the elliptic case. A general non-degenerate case appears for geodesic flows in [31]. We shall call a chart $\kappa:\mathit{\Omega}^0_{\gamma}\rightarrow\mathit{\Omega}_{\gamma}$ given by the Proposition above a Darboux-Reeb chart near γ.

Next fix a constant $\delta\in\left(0,\frac{1}{2}\right)$. Define a dilation on each Darboux-Reeb chart

$$\begin{gathered}\varrho_{\delta}:\mathit{\Omega}^0_{\gamma}\rightarrow\mathit{\Omega}^0_{\gamma}\\ \varrho_{\delta}\left(x_0;x_1,\ldots,x_{2m}\right)=\left(x_0;h^{\delta}x_1,\ldots,h^{\delta}x_{2m}\right)\end{gathered}$$

and also denote by $\varrho_{\delta}:\mathit{\Omega}_{\gamma}\rightarrow\mathit{\Omega}_{\gamma}$ the corresponding dilation of $\mathit{\Omega}_{\gamma}$. For each subset S of $\mathit{\Omega}^0_{\gamma}$ (or $\mathit{\Omega}_{\gamma}$) we denote by $S^{\delta}\coloneqq\varrho_{\delta}(S)$ its (h-dependent) image under the dilation. We also denote by $\tilde{S}\subset T^*X$ the inverse image under the projection $\pi:T^*X\rightarrow X$. Letting $\Gamma\coloneqq\{\gamma_v\}_{v=1}^{M}$ be the set of all primitive Reeb orbits, we set $\mathit{\Omega}\coloneqq\cup_{v=1}^{M}\mathit{\Omega}_{\gamma_v}$. Below let $\Gamma\subset\Omega\subset\mathit{\Omega}$ be any subcover of the system of Darboux-Reeb charts and denote $C_{\varepsilon,T}\coloneqq B_{\mathbb{R}^{2m}}(\varepsilon)\times(-T,T)_{x_0}\subset\mathbb{R}^n_x$ the cylinder of radius ε and height T in Euclidean space. We now have the following elementary lemma.

Lemma 3.3. *For each $\delta\in\left(0,\frac{1}{2}\right)$, $T>0$ there exists an $\varepsilon>0$ of the following significance: each point $x\in X\setminus\Omega^{\delta}$ has a Darboux chart $\varphi_x:N_x\xrightarrow{\sim}C_{\varepsilon h^{\delta},T}\subset\mathbb{R}^n$, $N_x\subset X\setminus\Gamma$, centered at x satisfying*

$$\left(\varphi_x^{-1}\right)^*a=dx_0+\sum_{j=1}^{m}\left(x_jdx_{j+m}-x_{j+m}dx_j\right). \tag{3.12}$$

Proof. The Reeb trajectory $\gamma_x\coloneqq e^{tR}(x)\,,-T<t<T$, $x\in X\setminus\Omega^{\delta}$, being non-self-intersecting the existence of a chart of height T is similar to the Darboux theorem. It only remains to show that one may choose a chart into a cylinder of radius εh^{δ} for ε uniform in h. By compactness, a radius of an h-independent size $\varepsilon=O(1)$ works for points in the h-independent set $x\in X\setminus\Omega_0$, for $\Omega_0\subset\Omega$. For points $x\in\bar{\Omega}_0\setminus\Omega^{\delta}$, non-resonance implies that the linearizations $\left(P^{+}_{\gamma}\right)^k-\left(P^{+}_{\gamma}\right)^l$, $k,l\in\mathbb{Z}$, of the Poincare return maps $P^k_{\Sigma}-P^l_{\Sigma}$ (3.6) at 0 are invertible. Here the Poincare sections are again given by $\{x_0=0\}$ in terms of the Darboux-Reeb coordinates on Ω_0. One may hence shrink Ω_0 to arrange $\left\|P^k_{\Sigma}(x)-P^l_{\Sigma}(x)\right\|\geq C\left\|(x_1,\ldots x_{2m})\right\|$, $\forall x\in\bar{\Omega}_0$, $|k|\leq N_T,|l|\leq N_T$, where $N_T\coloneqq\max_{\gamma\in\Gamma}\frac{T}{T_{\gamma}}$. From here one finds a uniform ε such that $\forall x\in\left(\bar{\Omega}_0\setminus\Omega^{\delta}_{\gamma}\right)\cap\{x_0=0\}$ the first N iterates under P_{Σ} of the ball $B_{\mathbb{R}^{2m}}\left(\varepsilon_xh^{\delta}\right)$ are disjoint. The Reeb flow-outs $e^{tR}\left[B_{\mathbb{R}^{2m}}\left(\varepsilon_xh^{\delta}\right)\right],-T<t<T$, of the balls being non-self-intersecting, a chart satisfying (3.12) comes from a Moser style argument. $\square$

For each Darboux chart $\varphi_x : N_x \xrightarrow{\sim} C_{\varepsilon h^\delta, T} \subset \mathbb{R}^n$ as above we set $N_x^0 := \varphi_x^{-1}\left(C_{\frac{\varepsilon h^\delta}{8}, \frac{T}{8}}\right)$. The chart is called trivialized if it comes equipped with an orthonormal trivialization of the spin bundle. Below for each h-independent constant c we denote by a shorthand the h-dependent constant $c_\delta := ch^\delta$.

We now come to the construction of dynamical partitions. Below, the energy levels Σ_I^D above are as in (2.40). Let $T > 0$, $\tau > 0$, $\delta \in \left(0, \frac{1}{2}\right)$ and $\Gamma \subset \Omega \subset \mathit{\Omega}$ be a subcover of the system of Darboux-Reeb charts as before. A (Ω, τ, δ)-microlocal partition of unity is defined to be a collection of zeroth-order pseudo-differential operators $\mathcal{P} = \left\{A_u \in \Psi_\delta^0(X) \,| 0 \leq u \leq N_h\right\} \cup \left\{B_v \in \Psi_\delta^0(X) \,| 1 \leq v \leq M\right\}$ satisfying

$$\begin{aligned}
\sum_{u=0}^{N_h} A_u + \sum_{v=1}^{M} B_v \quad &= \quad 1 \\
N_h \quad &= \quad O\left(h^{-\delta}\right) \\
WF(A_0) \subset \quad U_0 \subset \quad &\overline{T^*X} \setminus \Sigma^D_{\left[-\frac{\tau_\delta}{64}, \frac{\tau_\delta}{64}\right]} \\
WF(A_u) \Subset \quad U_u \subset \quad &\Sigma^D_{[-\tau_\delta, \tau_\delta]} \cap \tilde{N}^0_{x_u}, \; 1 \leq u \leq N \\
WF(B_v) \Subset \quad V_v \subset \quad &\Sigma^D_{[-\tau_\delta, \tau_\delta]} \cap \tilde{\Omega}^\delta_{\gamma_v}, \; 1 \leq v \leq M
\end{aligned} \tag{3.13}$$

for some open cover $\{U_u\}_{u=0}^N \cup \{V_v\}_{v=1}^M$ of T^*X and for some collection of trivialized Darboux charts $N := \{N_{x_u}\}_{u=1}^N \subset X \setminus \Gamma$. For such a partition $\mathcal{P}$ define the pairs of indices

$$\begin{aligned}
I_{\mathcal{P}} &= \left\{(u, u') \,| u \leq u', \, WF(A_u) \cap WF(A_{u'}) \neq \emptyset\right\} \\
J_{\mathcal{P}} &= \left\{(u, v) \,| WF(A_u) \cap WF(B_v) \neq \emptyset\right\}.
\end{aligned} \tag{3.14}$$

An augmentation $(\mathcal{P}; \mathcal{V}, \mathcal{W})$ of this partition consists of an additional collection of open sets $\mathcal{V} = \left\{V^1_{uu'}\right\}_{(u,u') \in I_{\mathcal{P}}} \cup \left\{V^2_{uv}\right\}_{(u,v) \in J_{\mathcal{P}}}$, $\mathcal{W} = \left\{W^1_{uu'}\right\}_{(u,u') \in I_{\mathcal{P}}} \cup \left\{W^2_{uv}\right\}_{(u,v) \in J_{\mathcal{P}}}$ satisfying

$$\begin{aligned}
WF(A_u) \cap WF(A_{u'}) \quad &\subset \quad W^1_{uu'} \\
&\quad\;\; \cap \\
WF(A_u) \cup WF(A_{u'}) \quad &\subset \quad V^1_{uu'} \Subset \Sigma^D_{[-2\tau_\delta, 2\tau_\delta]} \cap \tilde{N}_{x_u}, \\
WF(A_u) \cap WF(B_v) \quad &\subset \quad W^2_{uv} \\
&\quad\;\; \cap \\
WF(A_u) \cup WF(B_v) \quad &\subset \quad V^2_{uv} \Subset \Sigma^D_{[-2\tau_\delta, 2\tau_\delta]} \cap \tilde{N}_{x_u}.
\end{aligned} \tag{3.15}$$

Next with $d = \sigma(D)$, for each pair of indices in (3.14) we set

$$T_{uu'} := \frac{1}{\inf_{(g, \mathrm{v}) \in \mathcal{G}_{uu'} \times S^0_\delta(X; U(S))} |H_{g,\mathrm{v}} d|}, \tag{3.16}$$

$$S_{uv} := \frac{1}{\inf_{(g, \mathrm{v}) \in \mathcal{H}_{uv} \times S^0_\delta(X; U(S))} |H_{g,\mathrm{v}} d|}, \quad \text{with} \tag{3.17}$$

$$\mathcal{G}_{uu'} := \left\{g \in S^0_\delta(X; [0,1]) \mid g|_{W^1_{uu'}} = 1, \; g|_{\left(V^1_{uu'}\right)^c} = 0\right\} \tag{3.18}$$

$$\mathcal{H}_{uv} := \left\{g \in S^0_\delta(X; [0,1]) \mid g|_{W^2_{uv}} = 1, \; g|_{(V^2_{uv})^c} = 0\right\} \tag{3.19}$$

and $|H_{g,\mathtt{v}}d| \coloneqq \sup \|\{\mathtt{v}^* d\mathtt{v}, g\}\|$ with the bracket being computed in terms of the chosen and induced trivialization/coordinates on $N_{x_u}, \tilde{N}_{x_u}$. A function in $\mathcal{G}_{uu'}$ or $\mathcal{H}_{uv}$ shall be referred to as a trapping/microlocal weight function.

Finally, the *extension/trapping time* of an augmented (Ω, τ, δ)-partition $(\mathcal{P}; \mathcal{V}, \mathcal{W})$ is set to be

$$T_{(\mathcal{P};\mathcal{V},\mathcal{W})} \coloneqq \min \left\{ \min \left\{T_{uu'}\right\}_{(u,u') \in I_{\mathcal{P}}}, \min \left\{S_{uv}\right\}_{(u,v) \in J_{\mathcal{P}}} \right\}. \tag{3.20}$$

Proposition 3.4. *Let $T > 0$, $\delta \in \left(0, \frac{1}{2}\right)$ and $\Gamma \subset \Omega \subset \varOmega$ be a subcover. Then for each τ sufficiently small one has an augmented (Ω, τ, δ)-partition of unity $(\mathcal{P}; \mathcal{V}, \mathcal{W})$ with*

$$T_{(\mathcal{P};\mathcal{V},\mathcal{W})} > T. \tag{3.21}$$

Proof. By Lemma 3.3 there exists $\varepsilon > 0$ such that each $x \in X \setminus \Omega^\delta$ has a Darboux chart $\varphi_x : N_x \xrightarrow{\sim} C_{\varepsilon h^\delta, T} \subset \mathbb{R}^n$ centered at x of radius $\varepsilon_\delta = \varepsilon h^\delta$ and height T. Next with $(x'', \xi'') = (x_{m+1}, \dots, x_{2m}; \xi_{m+1}, \dots, \xi_{2m})$ being a subset of the coordinates on $\mathbb{R}^{2n}_{x,\xi}$ set $C''_{\varepsilon_\delta, T} \coloneqq \left\{x''^2 + \xi''^2 < \varepsilon_\delta^2,\ -T < x_0 < T\right\} \subset \mathbb{R}^{2n}_{x,\xi}$. Also for each $\tau > 0$, set

$$\begin{aligned} U_{\varepsilon_\delta, \tau_\delta, T} \coloneqq \Bigg\{ & \xi_0^2 + 2\sum_{j=1}^m \mu_j \left(x_j^2 + \xi_j^2\right) < \tau_\delta^2, \\ & x''^2 + \xi''^2 < \left(\frac{\varepsilon_\delta}{2}\right)^2,\ -\frac{T}{2} < x_0 < \frac{T}{2} \Bigg\} \subset C''_{\varepsilon_\delta, T}. \end{aligned} \tag{3.22}$$

Then by the preliminary Birkhoff normal form procedure of [26] Sec. 5 (eqns 5.1, 5.5, 5.6, 5.7, 5.8) there exists $0 < \tau \ll 1$ sufficiently small of the following significance: there is a neighborhood $M_u \subset \tilde{N}_{x_u}$ of $\tilde{N}^0_{x_u} \cap \Sigma_0^D$, a Hamiltonian symplectomorphism

$$\begin{aligned} \kappa_u \coloneqq e^{H_{f_1}} \circ e^{H_{f_0}} &: U_{\varepsilon_\delta, \tau_\delta, T} \to M_u \\ \kappa_u \left(x_0, 0, x''; 0, 0, \xi''\right) &= \left(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}; -1, \frac{x''}{\sqrt{2}}, \frac{\xi''}{\sqrt{2}}\right) \end{aligned}$$

(see [26] pgs. 1812-1813 for f_0, f_1) in terms of the chosen coordinates on each, a self-adjoint endomorphism $c_A \in C^\infty\left(U_{\varepsilon_\delta, \tau_\delta, T}; i\mathfrak{u}\left(2^m\right)\right)$ and functions $\left\{r_j \in C^\infty\left(U_{\varepsilon_\delta, \tau_\delta, T}\right)\right\}_{j=0}^{2m}$ vanishing to second order along Σ_0^D such that

$$\begin{aligned} e^{ic_A}\left[\left(e^{H_{f_1}} \circ e^{H_{f_0}}\right)^* d\right] e^{-ic_A} &= H_1 + \sigma_j r_j, \quad \text{with} \\ H_1 &\coloneqq \xi_0 \sigma_0 + \sum_{j=1}^m \left(2\mu_j\right)^{\frac{1}{2}} \left(x_j \sigma_{2j-1} + \xi_j \sigma_{2j}\right). \end{aligned} \tag{3.23}$$

Taylor expand $r_0 = \sum r_{00}\left(x_0, x''; \xi_0, \xi''\right) \xi_0^2 + r_j^1 x_j + r_j^2 \xi_j$, with r_j^1, r_j^2 vanishing to first order along Σ_0^D. A further conjugation of the above (3.23) by $e^{\left[r_j^1 (2\mu_j)^{-\frac{1}{2}} \sigma_{2j-1} + r_j^2 (2\mu_j)^{-\frac{1}{2}} \sigma_{2j}\right]\sigma_0}$ sets $r_j^1 = r_j^2 = 0$ while a symplectic change of variables in x_0 sets $r_{00} = 0$. Now set

$$\begin{aligned} \left(\tilde{\theta}_0, \tilde{\theta}_1, \dots, \tilde{\theta}_{2m}\right) &\coloneqq \left(\xi_0, \left(2\mu_1\right)^{\frac{1}{2}} x_1, \left(2\mu_1\right)^{\frac{1}{2}} \xi_1, \dots, \left(2\mu_m\right)^{\frac{1}{2}} x_m, \left(2\mu_m\right)^{\frac{1}{2}} \xi_m\right) \\ &\quad + \left(0, r_1, \dots, r_{2m}\right) \\ \tilde{\theta}' &= \left(\tilde{\theta}_1, \dots, \tilde{\theta}_{2m}\right) \end{aligned} \tag{3.24}$$

and note from (3.23) that the eigenvalues of the symbol d are $\pm\left|\tilde{\theta}\right|$. We clearly have

$$\begin{aligned}\kappa_u^{-1}\left(M_u\right)\cap\Sigma_0^D &= U_{\varepsilon_\delta,\tau_\delta,T}\cap\left\{\tilde{\theta}=0\right\}\\ &= U_{\varepsilon_\delta,\tau_\delta,T}\cap\left\{\xi_0=x_1=\xi_1=\ldots=x_m=\xi_m=0\right\}\end{aligned}\tag{3.25}$$

and we may set

$$\theta_j=\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}\in C^\infty\left(U_{\varepsilon_\delta,\tau_\delta,T}\setminus\Sigma_0^D;S^{n-1}\right).\tag{3.26}$$

If we denote by o_N the set of functions that vanish to order N along Σ_0^D, we have

$$\begin{aligned}\left\{\tilde{\theta}_0,x_0\right\}-1&=o_1\\ \left\{\tilde{\theta}_j,x_0\right\}&=o_1,j\geq 1,\\ \left\{\tilde{\theta}_j,x''\right\}&=o_1,j\geq 0,\\ \left\{\tilde{\theta}_j,\xi''\right\}&=o_1,j\geq 0,\\ \left\{\tilde{\theta}_0,\tilde{\theta}_j\right\}&=o_2,j\geq 0,\\ \left\{\tilde{\theta}_j,\tilde{\theta}_k\right\}\quad\text{or}\quad\left\{\tilde{\theta}_j,\tilde{\theta}_k\right\}-1&=o_1,k>j\geq 0,\\ r_j&=o_2,j\geq 0.\end{aligned}\tag{3.27}$$

By (3.22), (3.25) $U_{\varepsilon_\delta,\tau_\delta,T}$ denotes a collar neighborhood of radius τ_δ of Σ_0^D. Hence by shrinking τ if necessary, we may assume

$$\begin{aligned}\left|\left\{\tilde{\theta}_j,x_0\right\}\right|&\leq 2, && j\geq 0,\\ \left|\left\{\left|\tilde{\theta}\right|,x_0\right\}\right|&\leq 2, && \left|\tilde{\theta}\right|\neq 0,\\ \left|\left\{\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|},x_0\right\}\right|&\leq\frac{4}{\left|\tilde{\theta}\right|}, && \left|\tilde{\theta}\right|\neq 0,\ j\geq 0,\\ \left|\frac{1}{\varepsilon_\delta}\left\{\tilde{\theta}_j,\left|(x'',\xi'')\right|\right\}\right|&\leq\frac{1}{T}, && j\geq 0,\\ \left|\frac{1}{\varepsilon_\delta}\left\{\left|\tilde{\theta}\right|,\left|(x'',\xi'')\right|\right\}\right|&\leq\frac{1}{T}, && \left|\tilde{\theta}\right|\neq 0,\\ \left|\frac{1}{\varepsilon_\delta}\left\{\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|},\left|(x'',\xi'')\right|\right\}\right|&\leq\frac{1}{T\left|\tilde{\theta}\right|}, && \left|\tilde{\theta}\right|\neq 0,\ j\geq 0,\\ \left|\left\{\tilde{\theta}_0,\tilde{\theta}_j\right\}\right|&\leq\frac{\left|\tilde{\theta}\right|}{T}, && j\geq 0,\\ \left|\left\{\tilde{\theta}_0,\left|\tilde{\theta}\right|\right\}\right|&\leq\frac{\left|\tilde{\theta}\right|}{T}, && \left|\tilde{\theta}\right|\neq 0,\end{aligned}$$

$$\left|\left\{\tilde{\theta}_0, \frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}\right\}\right| \leq \frac{1}{T}, \qquad \left|\tilde{\theta}\right| \neq 0,\, j \geq 0,$$

$$\frac{1}{4}\left[\xi_0^2 + 2\sum_{j=1}^{m}\mu_j\left(x_j^2+\xi_j^2\right)\right] \leq \sum_{j=0}^{2m}\tilde{\theta}_j^2 \leq 4\left[\xi_0^2 + 2\sum_{j=1}^{m}\mu_j\left(x_j^2+\xi_j^2\right)\right] \tag{3.28}$$

on $U_{\varepsilon_\delta,\tau_\delta,T}$ and set

$$\tilde{U}_{\varepsilon_\delta,\tau_\delta,T} := \left\{\sum_{j=0}^{2m}\tilde{\theta}_j^2 < \left(\frac{\tau_\delta}{8}\right)^2,\, x''^2+\xi''^2 < \left(\frac{\varepsilon_\delta}{8}\right)^2,\, -\frac{T}{8} < x_0 < \frac{T}{8}\right\} \subset U_{\varepsilon_\delta,\tau_\delta,T}.$$

It is clear from the above construction that a finite set $\left\{\kappa_{p_u}\left(\tilde{U}_{\frac{2\varepsilon_\delta}{3},\frac{2\tau_\delta}{3},\frac{2T}{3}}\right)\right\}_{u=1}^{N_h}$, $N_h = O\left(h^{-\delta}\right)$, covers $\Sigma^D_{\left[-\frac{\tau_\delta}{16},\frac{\tau_\delta}{16}\right]} \setminus \tilde{\Omega}^\delta_{\gamma_v}$. Next define

$$\begin{aligned} U_0 &= \overline{T^*X} \setminus \Sigma^D_{\left[-\frac{\tau_\delta}{32},\frac{\tau_\delta}{32}\right]} \\ U_u &= \kappa_{p_u}\left(\tilde{U}_{\frac{2\varepsilon_\delta}{3},\frac{2\tau_\delta}{3},\frac{2T}{3}}\right) \\ V_v &= \Sigma^D_{\left[-\frac{\tau_\delta}{8},\frac{\tau_\delta}{8}\right]} \cap \tilde{\Omega}^\delta_{\gamma_v}. \end{aligned}$$

Choose $\mathcal{P} = \left\{A_u \in \Psi^0_\delta\right\}_{0\leq u\leq N} \cup \left\{B_v \in \Psi^0_\delta(X)\right\}_{1\leq v\leq M}$ to be any microlocal partition of unity subordinate to this cover. We then augment this partition by

$$\begin{aligned} W^1_{uu'} &= \kappa_{p_u}\left(\tilde{U}_{\varepsilon_\delta,\tau_\delta,T}\right) \subset \tilde{N}_{x_u} \\ W^2_{uv} &= \kappa_{p_u}\left(\tilde{U}_{\varepsilon_\delta,\tau_\delta,T}\right) \subset \tilde{N}_{x_u} \\ V^1_{uu'} &= \kappa_{p_u}\left(\tilde{U}_{4\varepsilon_\delta,4\tau_\delta,4T}\right) \subset \tilde{N}_{x_u} \\ V^2_{uv} &= \kappa_{p_u}\left(\tilde{U}_{4\varepsilon_\delta,4\tau_\delta,4T}\right) \subset \tilde{N}_{x_u} \end{aligned}$$

where $(u,u') \in I_{\mathcal{P}}$ and $(u,v) \in J_{\mathcal{P}}$ lie in the corresponding index sets. Clearly the above satisfy (3.13), (3.15).

It remains to verify (3.21). To this end, let $\chi \in C_c^\infty([-4,4];[0,1])$, be a cutoff such that $\chi = 1$ on $[-2,2]$ and $|\chi'| \leq 1$. For $\rho \in \left(0,\frac{1}{8}\right)$ fixed, define a function $\varphi_\rho \in C^\infty\left([-1,1]_{\theta_0};[0,1]\right)$ such that $\varphi_\rho(\theta_0) = \begin{cases} 1; & \text{for } \theta_0 \in [1-\rho,1] \\ 0; & \text{for } \theta_0 \in [-1,1-2\rho] \end{cases}$ and $\left|\varphi'_\rho\right| \leq \frac{2}{\rho}$.
Set

$$\begin{aligned} \beta\left(\tilde{\theta}\right) &:= \sqrt{\left|\tilde{\theta}\right|^2 - \varphi_\rho(\theta_0)\left|\tilde{\theta}'\right|^2} \\ &= \sqrt{\left|\tilde{\theta}_0\right|^2 + (1-\varphi_\rho)\left|\tilde{\theta}'\right|^2}. \end{aligned}$$

For $\theta_0 \in [-1,1-2\rho]$, $\varphi_\rho = 0$ and $\beta\left(\tilde{\theta}\right) = \left|\tilde{\theta}\right|$. While for $\theta_0 \in [1-2\rho,1]$, we have $\left|\tilde{\theta}\right| \geq \sqrt{\left|\tilde{\theta}\right|^2 - \varphi_\rho(\theta_0)\left|\tilde{\theta}'\right|^2} = \beta\left(\tilde{\theta}\right) = \sqrt{\left|\tilde{\theta}_0\right|^2 + (1-\varphi_\rho)\left|\tilde{\theta}'\right|^2} \geq \left|\tilde{\theta}_0\right| = \theta_0\left|\tilde{\theta}\right| \geq$

$\frac{1}{2}\left|\tilde{\theta}\right|$ for $\rho\in\left(0,\frac{1}{8}\right)$ as chosen. Thus $\left|\tilde{\theta}\right|\geq\beta\left(\tilde{\theta}\right)\geq\frac{1}{2}\left|\tilde{\theta}\right|$in both cases and we may for each $1\leq u\leq N$, define the microlocal weight function

$$g_{u}:=\left(\kappa_{p_{u}}^{-1}\right)^{*}\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_{\delta}}\right)\chi\left(\frac{16x_{0}}{T}\right)\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)\in C_{c}^{\infty}\left(\kappa_{p_{u}}\left(\tilde{U}_{\varepsilon_{\delta},\tau_{\delta},T}\right)\right)$$

in terms of the relevant coordinates on $\tilde{U}_{\varepsilon_{\delta},\tau_{\delta},T}$.

Next, with $\mathtt{v}_{t}^{\rho}\in C^{\infty}\left(S^{n-1};U\left(\mathbb{C}^{2^{m}}\right)\right)$ as in Lemma 2.1, we choose for each $1\leq u\leq N$ a symbol $\tilde{\mathtt{v}}_{u}\in S_{\delta}^{0}\left(X;U\left(S\right)\right)$ satisfying

$$\tilde{\mathtt{v}}_{u}:=\begin{cases}\mathtt{v}_{8\left|\tilde{\theta}\right|/\tau_{\delta}}^{\rho}\left(\theta\right); & \left|\tilde{\theta}\right|<\frac{\tau_{\delta}}{8}\\ \mathtt{v}_{1}^{\rho}\left(\theta\right); & \left|\tilde{\theta}\right|\geq\frac{\tau_{\delta}}{8},\end{cases}$$

on $\kappa_{p_{u}}\left(\tilde{U}_{\varepsilon_{\delta},\tau_{\delta},T}\right)$, with $\tilde{\theta},\theta$ given by (3.24), (3.26). Since the conjugate of the symbol d of the Dirac operator is $e^{ic_{A}}de^{-ic_{A}}=\sigma_{j}\tilde{\theta}_{j}=i\left|\tilde{\theta}\right|c\left(\theta\right)$ by (3.23) on $\kappa_{p_{u}}\left(\tilde{U}_{\varepsilon,\tau,T}\right)$, we may compute from Lemma 2.1

$$\left(\tilde{\mathtt{v}}_{u}\right)^{*}e^{ic_{A}}de^{-ic_{A}}\tilde{\mathtt{v}}_{u}=\begin{cases}\tilde{\theta}_{0}\sigma_{0}-\left[\sum_{j=1}^{2m}\tilde{\theta}_{j}\sigma_{j}\right]; & \left|\tilde{\theta}\right|\leq\frac{\tau}{16}\\ \left|\tilde{\theta}\right|\mathtt{v}_{1}^{\rho}\left(\theta\right)^{*}c\left(\theta\right)\mathtt{v}_{1}^{\rho}\left(\theta\right); & \left|\tilde{\theta}\right|\geq\frac{\tau}{8}\end{cases}\tag{3.29}$$

on $\kappa_{p_{u}}\left(\tilde{U}_{\varepsilon,\tau,T}\right)$. Furthermore; Lemma 2.1 also gives

$$\left|\tilde{\theta}\right|\mathtt{v}_{1}^{\rho}\left(\theta\right)^{*}c\left(\theta\right)\mathtt{v}_{1}^{\rho}\left(\theta\right)=\begin{cases}\left|\tilde{\theta}\right|\sigma_{0}; & \theta_{0}\leq1-\rho,\\ \left|\tilde{\theta}\right|\left[a_{0,1}^{\rho}\left(\theta_{0}\right)\sigma_{0}+a_{1,1}^{\rho}\left(\theta_{0}\right)\sum_{j=1}^{2m}\theta_{j}\sigma_{j}\right]; & \theta_{0}>1-\rho.\end{cases}$$

Choose $\mathtt{v}_{u}\in S_{\delta}^{0}\left(X;U\left(S\right)\right)$ to be a symbol satisfying

$$\mathtt{v}_{u}=e^{-ic_{A}}\tilde{\mathtt{v}}_{u}\tag{3.30}$$

on $\kappa_{p_{u}}\left(\tilde{U}_{\varepsilon,\tau,T}\right)$.

We now compute for $\left|\tilde{\theta}\right|>\frac{\tau_{\delta}}{8}$, $\theta_{0}\leq1-2\rho$;

$$\begin{aligned}\left|H_{g_{u},\mathtt{v}_{u}}\left(d\right)\right| & =\left|\left\{\mathtt{v}_{u}^{*}d\mathtt{v}_{u},g_{u}\right\}\right|\\
 & =\left|\left\{\left|\tilde{\theta}\right|,g_{u}\right\}\sigma_{0}\right|\\
 & =\left|\frac{16}{T}\left\{\left|\tilde{\theta}\right|,x_{0}\right\}\frac{\chi'\left(\frac{16x_{0}}{T}\right)}{\chi\left(\frac{16x_{0}}{T}\right)}g_{u}\right.\\
 & \quad\left.+\frac{16}{\varepsilon_{\delta}}\left\{\left|\tilde{\theta}\right|,\left|\left(x'',\xi''\right)\right|\right\}\frac{\chi'\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}{\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}g_{u}\right|\\
 & \leq\frac{64}{T}\end{aligned}\tag{3.31}$$

using (3.28).

While for $\left|\tilde{\theta}\right| > \frac{\tau_\delta}{8}$, $1-2\rho \le \theta_0 \le 1-\rho$;

$$
\begin{aligned}
\left|H_{g_u,\mathtt{v}_u}(d)\right| &= \left|\{\mathtt{v}_u^* d\mathtt{v}_u, g_u\}\right| \\
&= \left|\left\{\left|\tilde{\theta}\right|, g_u\right\}\sigma_0\right| \\
&= \left|\frac{16}{T}\left\{\left|\tilde{\theta}\right|, x_0\right\}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u + \right. \\
&\quad + \frac{16}{\varepsilon_\delta}\left\{\left|\tilde{\theta}\right|, \left|(x'',\xi'')\right|\right\}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u \\
&\quad + \frac{8\varphi_\rho}{\beta\tau_\delta}\left\{\left|\tilde{\theta}\right|, \left|\tilde{\theta}'\right|^2\right\}\frac{\chi'\left(\frac{16\beta(\tilde{\theta})}{\tau_\delta}\right)}{\chi\left(\frac{16\beta(\tilde{\theta})}{\tau_\delta}\right)}g_u \\
&\quad \left. + \frac{8\varphi_\rho'}{\beta\tau_\delta}\left\{\left|\tilde{\theta}\right|, \frac{\tilde{\theta}_0}{\left|\tilde{\theta}\right|}\right\}\left|\tilde{\theta}'\right|^2\frac{\chi'\left(\frac{16\beta(\tilde{\theta})}{\tau_\delta}\right)}{\chi\left(\frac{16\beta(\tilde{\theta})}{\tau_\delta}\right)}g_u\right| \\
&\le \frac{8^4}{\rho T}
\end{aligned}
\tag{3.32}
$$

using (3.28).

Now for $\left|\tilde{\theta}\right| > \frac{\tau_\delta}{8}$, $\theta_0 > 1-\rho$; we compute

$$
\begin{aligned}
\left|H_{g_u,\mathtt{v}_u}(d)\right| &= \left|\{\mathtt{v}_u^* d\mathtt{v}_u, g_u\}\right| \\
&= \left|\left\{\left|\tilde{\theta}\right|\left[a_{0,1}^\rho(\theta_0)\sigma_0 + a_{1,1}^\rho(\theta_0)\sum_{j=1}^{2m}\theta_j\sigma_j\right], g_u\right\}\right| \\
&\le \left|\left\{\left|\tilde{\theta}\right|, g_u\right\}\right| + \left|\left|\tilde{\theta}\right|\left\{a_{0,1}^\rho(\theta_0)\sigma_0 + a_{1,1}^\rho(\theta_0)\sum_{j=1}^{2m}\theta_j\sigma_j, g_u\right\}\right|
\end{aligned}
$$

with

$$
\begin{aligned}
\left|\left\{\left|\tilde{\theta}\right|,g_{u}\right\}\right| = & \left|\frac{16}{T}\left\{\left|\tilde{\theta}\right|,x_{0}\right\}\frac{\chi'\left(\frac{16x_{0}}{T}\right)}{\chi\left(\frac{16x_{0}}{T}\right)}g_{u}\right.\\
 & +\frac{16}{\varepsilon_{\delta}}\left\{\left|\tilde{\theta}\right|,\left|\left(x'',\xi''\right)\right|\right\}\frac{\chi'\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}{\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}g_{u}\\
 & \left.+\frac{8}{\beta\tau_{\delta}}\left\{\left|\tilde{\theta}\right|,\tilde{\theta}_{0}^{2}\right\}\frac{\chi'\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_{\delta}}\right)}{\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_{\delta}}\right)}g_{u}\right|\\
 & \leq\frac{8^{4}}{T} \qquad (3.33)
\end{aligned}
$$

and

$$
\begin{aligned}
 & \left|\left|\tilde{\theta}\right|\left\{a_{0,1}^{\rho}\left(\theta_{0}\right)\sigma_{0}+a_{1,1}^{\rho}\left(\theta_{0}\right)\sum_{j=1}^{2m}\theta_{j}\sigma_{j},g_{u}\right\}\right|\\
= & \left|\frac{16\left|\tilde{\theta}\right|}{T}\left(a_{0,1}^{\rho}\right)'\sigma_{0}\left\{\frac{\tilde{\theta}_{0}}{\left|\tilde{\theta}\right|},x_{0}\right\}\frac{\chi'\left(\frac{16x_{0}}{T}\right)}{\chi\left(\frac{16x_{0}}{T}\right)}g_{u}\right.\\
 & +\frac{16\left|\tilde{\theta}\right|}{T}\left(a_{1,1}^{\rho}\right)'\left(\sum_{j=1}^{2m}\theta_{j}\sigma_{j}\right)\left\{\frac{\tilde{\theta}_{0}}{\left|\tilde{\theta}\right|},x_{0}\right\}\frac{\chi'\left(\frac{16x_{0}}{T}\right)}{\chi\left(\frac{16x_{0}}{T}\right)}g_{u}\\
 & +\frac{16\left|\tilde{\theta}\right|}{T}a_{1,1}^{\rho}\left(\sum_{j=1}^{2m}\left\{\frac{\tilde{\theta}_{j}}{\left|\tilde{\theta}\right|},x_{0}\right\}\sigma_{j}\right)\frac{\chi'\left(\frac{16x_{0}}{T}\right)}{\chi\left(\frac{16x_{0}}{T}\right)}g_{u}\\
 & +16\left|\tilde{\theta}\right|\left(a_{0,1}^{\rho}\right)'\sigma_{0}\frac{1}{\varepsilon_{\delta}}\left\{\frac{\tilde{\theta}_{0}}{\left|\tilde{\theta}\right|},\left|\left(x'',\xi''\right)\right|\right\}\frac{\chi'\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}{\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}g_{u}\\
 & +16\left|\tilde{\theta}\right|\left(a_{1,1}^{\rho}\right)'\left(\sum_{j=1}^{2m}\theta_{j}\sigma_{j}\right)\frac{1}{\varepsilon_{\delta}}\left\{\frac{\tilde{\theta}_{0}}{\left|\tilde{\theta}\right|},\left|\left(x'',\xi''\right)\right|\right\}\frac{\chi'\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}{\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}g_{u}\\
 & +16\left|\tilde{\theta}\right|a_{1,1}^{\rho}\left(\sum_{j=1}^{2m}\frac{1}{\varepsilon_{\delta}}\left\{\frac{\tilde{\theta}_{j}}{\left|\tilde{\theta}\right|},\left|\left(x'',\xi''\right)\right|\right\}\sigma_{j}\right)\frac{\chi'\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}{\chi\left(\frac{16\left|\left(x'',\xi''\right)\right|}{\varepsilon_{\delta}}\right)}g_{u}\\
 & +\left|\tilde{\theta}\right|\frac{8}{\beta\tau_{\delta}}\left(a_{0,1}^{\rho}\right)'\sigma_{0}\left\{\frac{\tilde{\theta}_{0}}{\left|\tilde{\theta}\right|},\tilde{\theta}_{0}^{2}\right\}\frac{\chi'\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_{\delta}}\right)}{\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_{\delta}}\right)}g_{u}
\end{aligned}
$$

$$+\left|\tilde{\theta}\right|\frac{8}{\beta\tau_\delta}\left(a^\rho_{0,1}\right)'\left(\sum_{j=1}^{2m}\theta_j\sigma_j\right)\left\{\frac{\tilde{\theta}_0}{\left|\tilde{\theta}\right|},\tilde{\theta}_0^2\right\}\frac{\chi'\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_\delta}\right)}{\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_\delta}\right)}g_u$$

$$+\left|\tilde{\theta}\right|\frac{8}{\beta\tau_\delta}a^\rho_{1,1}\left(\sum_{j=1}^{2m}\left\{\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|},\tilde{\theta}_0^2\right\}\sigma_j\right)\frac{\chi'\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_\delta}\right)}{\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_\delta}\right)}g_u\Bigg|$$

$$\leq\left(\frac{8}{\rho}\right)^2\frac{8^4}{T}. \tag{3.34}$$

using Lemma 2.1 and (3.28).

Now for $\frac{\tau_\delta}{16}\leq\left|\tilde{\theta}\right|\leq\frac{\tau_\delta}{8}$, $\chi\left(\frac{16\beta\left(\tilde{\theta}\right)}{\tau_\delta}\right)=1$ and we may compute

$$\begin{aligned}
\left|H_{g_u,\mathtt{v}_u}(d)\right| &= \left|\left\{\mathtt{v}_u^* d\mathtt{v}_u, g_u\right\}\right| \\
&= \left|\left\{\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}{}^*\left[\sigma_j\tilde{\theta}_j\right]\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}, g_u\right\}\right| \\
&= \Bigg|\frac{16}{T}\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}{}^*\left[\sigma_j\left\{\tilde{\theta}_j,x_0\right\}\right]\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u \\
&+\frac{128}{\tau_\delta T}\left[\partial_t\mathtt{v}^\rho_t|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\left|\tilde{\theta}\right|,x_0\right\}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u \\
&+\frac{128}{\tau_\delta T}\left[\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\partial_t\mathtt{v}^\rho_t|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\left|\tilde{\theta}\right|,x_0\right\}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u \\
&+\frac{16}{T}\left[\partial_{\theta_k}\mathtt{v}^\rho_t|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\frac{\tilde{\theta}_k}{\left|\tilde{\theta}\right|},x_0\right\}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u \\
&+\frac{16}{T}\left[\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\partial_{\theta_k}\mathtt{v}^\rho_t|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\frac{\tilde{\theta}_k}{\left|\tilde{\theta}\right|},x_0\right\}\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u \\
&+\frac{16}{\varepsilon_\delta}\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}{}^*\left[\sigma_j\left\{\tilde{\theta}_j,\left|(x'',\xi'')\right|\right\}\right]\mathtt{v}^{\rho}_{8\left|\tilde{\theta}\right|/\tau_\delta}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u
\end{aligned}$$

$$
\begin{aligned}
&+\frac{128}{\tau_\delta\varepsilon_\delta}\left[\partial_t \mathtt{v}_t^\rho|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\mathtt{v}^\rho_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\left|\tilde{\theta}\right|,\left|(x'',\xi'')\right|\right\}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\\
&+\frac{128}{\tau_\delta\varepsilon_\delta}\left[\mathtt{v}^\rho_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\partial_t \mathtt{v}_t^\rho|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\left|\tilde{\theta}\right|,\left|(x'',\xi'')\right|\right\}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\\
&+\frac{16}{\varepsilon_\delta}\left[\partial_{\theta_k} \mathtt{v}_t^\rho|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\mathtt{v}^\rho_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\frac{\tilde{\theta}_k}{\left|\tilde{\theta}\right|},\left|(x'',\xi'')\right|\right\}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\\
&+\frac{16}{\varepsilon_\delta}\left[\mathtt{v}^\rho_{8\left|\tilde{\theta}\right|/\tau_\delta}\right]^*\left[\sigma_j\tilde{\theta}_j\right]\left[\partial_{\theta_k} \mathtt{v}_t^\rho|_{t=8\left|\tilde{\theta}\right|/\tau_\delta}\right]\left\{\frac{\tilde{\theta}_k}{\left|\tilde{\theta}\right|},\left|(x'',\xi'')\right|\right\}\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\Bigg|
\end{aligned}
$$

$$
\leq\left(\frac{8}{\rho}\right)^4\frac{8^4}{T} \tag{3.35}
$$

using Lemma 2.1 and (3.28).

Finally for $\left|\tilde{\theta}\right|\leq\frac{\tau_\delta}{16}$ again $\chi\left(\frac{16\beta(\tilde{\theta})}{\tau_\delta}\right)=1$ and we may use (3.29) to compute

$$
\begin{aligned}
\left|H_{g_u,\mathtt{v}_u}(d)\right| &= \left|\left\{\mathtt{v}_u^* d\mathtt{v}_u, g_u\right\}\right|\\
&=\left|\left\{\tilde{\theta}_0\sigma_0-\left[\sum_{j=1}^{2m}\tilde{\theta}_j\sigma_j\right],g_u\right\}\right|\\
&=\Bigg|\frac{16}{T}\left\{\tilde{\theta}_0,x_0\right\}\sigma_0\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u\\
&\quad-\frac{16}{T}\left(\sum_{j=1}^{2m}\left\{\tilde{\theta}_j,x_0\right\}\sigma_j\right)\frac{\chi'\left(\frac{16x_0}{T}\right)}{\chi\left(\frac{16x_0}{T}\right)}g_u\\
&\quad+\frac{16}{\varepsilon_\delta}\left\{\tilde{\theta}_0,\left|(x'',\xi'')\right|\right\}\sigma_0\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\\
&\quad-\frac{16}{\varepsilon_\delta}\left(\sum_{j=1}^{2m}\left\{\tilde{\theta}_j,\left|(x'',\xi'')\right|\right\}\sigma_j\right)\frac{\chi'\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}{\chi\left(\frac{16\left|(x'',\xi'')\right|}{\varepsilon_\delta}\right)}g_u\Bigg|\\
&\leq\frac{8^2}{T}
\end{aligned}\tag{3.36}
$$

using (3.28). Since $\rho\in\left(0,\frac{1}{8}\right)$ is fixed and T arbitrary, the proposition follows from (3.31)-(3.36). $\square$

Next, given an augmented (Ω,τ,δ)-partition of unity $(\mathcal{P};\mathcal{V},\mathcal{W})$ the trace (3.1) from the Helffer-Sjöstrand formula is clearly the sum of traces of the following four kinds

$$\begin{aligned}
\mathcal{T}^{\theta}_{A_u,A_v}(D) &:= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\operatorname{tr}\left[A_u\left(\frac{1}{\sqrt{h}}D-z\right)^{-1}A_v\right]dzd\bar{z}\\
\mathcal{T}^{\theta}_{A_u,B_v}(D) &:= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\operatorname{tr}\left[A_u\left(\frac{1}{\sqrt{h}}D-z\right)^{-1}B_v\right]dzd\bar{z}\\
\mathcal{T}^{\theta}_{B_v,A_u}(D) &:= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\operatorname{tr}\left[B_v\left(\frac{1}{\sqrt{h}}D-z\right)^{-1}A_u\right]dzd\bar{z}\\
\mathcal{T}^{\theta}_{B_u,B_v}(D) &:= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\operatorname{tr}\left[B_u\left(\frac{1}{\sqrt{h}}D-z\right)^{-1}A_v\right]dzd\bar{z}.
\end{aligned}\tag{3.37}$$

Next we state a modification of [26] Lemma 3.3. Below $V^1_{uu'}$, $W^1_{uu'}$, $T_{uu'}$ are as in (3.15), (3.16).

Lemma 3.5. *Let $D' \in \Psi^1_{cl}(X;S)$ be essentially self-adjoint such that $D = D'$ microlocally on $V^1_{uu'}$. Then for $\theta \in C^\infty_c((T_0,T_{uu'});[0,1])$ one has*

$$\mathcal{T}^{\theta}_{A_u,A_v}(D) = \mathcal{T}^{\theta}_{A_u,A_v}(D') \quad \textit{mod } h^\infty.$$

Proof. The lemma is essentially the same as [26] Lemma 3.3 with a couple of changes. First our cutoffs lie in the more exotic class $\Psi^0_\delta(X)$. However these have the same basic composition and wavefront properties needed in the proof of [26]. Next our definition of trapping time (3.16) is more general than that in [26] eq. 3.5 since an additional conjugation by a unitary symbol $\mathtt{v} \in S^0_\delta(X;U(E))$ is allowed in the definition (3.16) here. This is however easily overcome; let $\theta \in C^\infty_c\left(\left(T'_0,T^0_{uu'}\right);[0,1]\right)$ be such that $T_0 < T'_0$, $T^0_{uu'} < T_{uu'}$. There hence exists $(g,\mathtt{v}) \in \mathcal{G}_{uu'} \times S^0(X;U(S))$ with $|H_{g,\mathtt{v}}d| < \frac{1}{S_{uu'}}$. We choose $\mathtt{V} \in \Psi^0_\delta(X;S)$ unitary with $\sigma(\mathtt{V}) = [\mathtt{v}]$ and note $H_{g,\mathtt{v}}d = H_g(\mathtt{V}^* d\mathtt{V})$ in terms a quantization defined using the chosen coordinates/trivialization on N_{x_u}. Now, the proof of [26] Lemma 3.3 carries through with the conjugates $\mathtt{V}^*D\mathtt{V}$, $\mathtt{V}^*D'\mathtt{V}$, $\mathtt{V}^*A_u\mathtt{V}$ and $\mathtt{V}^*A_v\mathtt{V}$. ☐

We also note that similar lemmas as above hold for the traces $\mathcal{T}^{\theta}_{A_u,B_v}(D)$ and $\mathcal{T}^{\theta}_{B_v,A_u}(D)$ in (3.37). Next we show that the first three traces in (3.37) are $O(h^\infty)$ when $\operatorname{spt}(\theta)$ is contained within the extension time.

Lemma 3.6. *Let $(\mathcal{P};\mathcal{V},\mathcal{W})$ be an augmented (Ω,τ,δ)-partition of unity. Then for each $\theta \in C^\infty_c((T_0,T);[-1,1])$ with $T < T_{(\mathcal{P};\mathcal{V},\mathcal{W})}$ one has*

$$\mathcal{T}^{\theta}_{A_u,A_v}(D),\ \mathcal{T}^{\theta}_{A_u,B_v}(D),\ \mathcal{T}^{\theta}_{B_v,A_u}(D) = O(h^\infty).$$

Proof. The proof is the same as [26] Lemma 3.1 (cf. eq. 3.2). One only has to quantify the smallness of $\operatorname{spt}(\theta)$ assumed therein. The proof in [26] carries through in so far as $\operatorname{spt}(\theta)$ is contained in each of $\{(T_0,T_{uu'})\}_{(u,u')\in I_{\mathcal{P}}}$, $\{(T_0,S_{uv})\}_{(u,v)\in J_{\mathcal{P}}}$ as required by Lemma 3.5. This is guaranteed for $T < T_{(\mathcal{P};\mathcal{V},\mathcal{W})}$ by (3.20). ☐

Given θ,Ω there exists by 3.4 an (Ω,τ,δ)-partition of unity with an extension time large enough to guarantee the hypothesis of Lemma 3.6. Splitting the trace in such fashion, it then suffices to consider the asymptotics of the fourth trace $\mathcal{T}^{\theta}_{B_u,B_v}(D)$ in (3.37). Since B_u and B_v have disjoint micro-supports for $u \neq v$; it suffices to consider $\mathcal{T}^{\theta}_{B_v,B_v}(D)$. Since these are localized near the Reeb orbits, they shall first require an understanding of the Birkhoff normal form for D near each orbit done in the next section. We shall return to $\mathcal{T}^{\theta}_{B_v,B_v}(D)$ in Section 5.

4. Birkhoff normal form near a Reeb orbit

In this section we derive a Birkhoff normal form for the Dirac operator in a neighborhood of each Reeb orbit. First, consider a Darboux-Reeb chart near γ and choose an orthonormal frame $\left\{e_j = w_j^k \partial_{x_k}\right\}, 0 \leq j \leq 2m$ for the tangent bundle on Ω_γ. Here we use the convention that $x_0 = \theta$ is the circular variable on $\Omega^0_\gamma \subset S^1 \times \mathbb{R}^{2m}$ and shall use these interchangeably. We hence have

$$w_j^k g_{kl} w_r^l = \delta_{jr}, \tag{4.1}$$

where g_{kl} is the metric in these coordinates and the Einstein summation convention is being used. Let Γ^l_{jk} be the Christoffel symbols for the Levi-Civita connection in the orthonormal frame e_i satisfying $\nabla_{e_j} e_k = \Gamma^l_{jk} e_l$. This orthonormal frame induces an orthonormal frame u_j, $1 \leq j \leq 2^m$, for the spin bundle S. We further choose a local orthonormal section $\mathtt{1}(x)$ for the Hermitian line bundle L and define via $\nabla^{A_0}_{e_j} \mathtt{1} = \Upsilon_j(x) \mathtt{1}$, $0 \leq j \leq 2m$ the Christoffel symbols of the unitary connection A_0 on L. In terms of the induced frame $u_j \otimes \mathtt{1}$, $1 \leq j \leq 2^m$, for $S \otimes L$ the Dirac operator (1.1) has the form (cf. [3] Section 3.3)

$$\begin{aligned} D &= \gamma^j w_j^k P_k + h\left(\frac{1}{4}\Gamma^l_{jk}\gamma^j\gamma^k\gamma_l + \Upsilon_j \gamma^j\right), \quad \text{where} && (4.2)\\ P_k &= h\partial_{x_k} + i a_k, && (4.3)\end{aligned}$$

and the one form a is given by (3.4).

The expression in (4.2) is formally self-adjoint with respect to the Riemannian density $e^0 \wedge \ldots \wedge e^{2m} = \sqrt{g}dx := \sqrt{g}dx^0 \wedge \ldots \wedge dx^{2m}$ with $g = \det(g_{ij})$. To get an operator self-adjoint with respect to the Euclidean density dx one expresses the Dirac operator in the framing $g^{\frac{1}{4}} u_j \otimes \mathtt{1}, 1 \leq j \leq 2^m$. In this new frame the expression (4.2) for the Dirac operator needs to be conjugated by $g^{\frac{1}{4}}$ and hence the term $h\gamma^j w_j^k g^{-\frac{1}{4}}\left(\partial_{x_k} g^{\frac{1}{4}}\right)$ added. Hence, the Dirac operator in the new frame has the form

$$D = \left[\sigma^j w_j^k (\xi_k + a_k)\right]^W + hE \in \Psi^1_{\mathrm{cl}}\left(\Omega^0_\gamma; \mathbb{C}^{2^m}\right),$$

with $\sigma^j = i\gamma^j$, for some self-adjoint endomorphism $E(x) \in C^\infty\left(\Omega^0_\gamma; i\mathfrak{u}\left(\mathbb{C}^{2^m}\right)\right)$.

The one form a is given in terms of these Darboux-Reeb coordinates by the same formula (3.4)

$$a = \varphi d\theta + \frac{1}{2}\sum_{j=1}^m (x_j dx_{j+m} - x_{j+m} dx_j) + a^\infty_\gamma$$

with a^∞_γ denoting a form on Ω_γ vanishing to infinite order along γ. Picking a cutoff $\chi_\gamma \in C^\infty_c(\Omega_\gamma)$ that equals 1 on Ω_γ we may extend the one form to all of $S^1 \times \mathbb{R}^{2m}$ via

$$a = \underbrace{\varphi d\theta + \frac{1}{2}\sum_{j=1}^m (x_j dx_{j+m} - x_{j+m} dx_j)}_{=:a^0} + \chi_\gamma a^\infty_\gamma$$

The functions w_j^k are extended such that

$$\left(w_j^k \partial_{x_k} \otimes dx^j\right)\big|_{(K_s^0)^c} = \partial_{x_0} \otimes dx^0 + \sum_{j=1}^{m} \mu_j^{\frac{1}{2}} \left(\partial_{x_j} \otimes dx^j + \partial_{x_{j+m}} \otimes dx^{j+m}\right)$$

(and hence $g|_{(K_s^0)^c} = dx_0^2 + \sum_{j=1}^m \mu_j \left(dx_j^2 + dx_{j+m}^2\right))$ outside a compact neighborhood $\Omega_\gamma^0 \Subset K_s^0$. The endomorphism $E(x) \in C_c^\infty\left(\mathbb{R}^n; i\mathfrak{u}\left(\mathbb{C}^{2^m}\right)\right)$ is extended to an arbitrary self-adjoint endomorphism of compact support. This now gives the operator

$$D_0 = \left[\sigma^j w_j^k \left(\xi_k + a_k^0\right)\right]^W + \chi_\gamma \sigma^j a_{\gamma,j}^\infty + hE \in \Psi_{\mathrm{cl}}^1\left(S^1 \times \mathbb{R}^{2m}; \mathbb{C}^{2^m}\right) \tag{4.4}$$

as a well defined formally self adjoint operators on $S^1 \times \mathbb{R}^{2m}$. Furthermore, the symbols of $D_0 + i$ being elliptic in the class $S^0(g)$ for the order functions $g = \sqrt{1 + \sum_{k=0}^{2m} \left(\xi_k + a_k\right)^2}$ it is essentially self adjoint (see [12] Ch. 8).

4.1. Birkhoff normal form for the Dirac operator.

Next, we derive a Birkhoff normal form for the Dirac operator (4.4) on $S^1 \times \mathbb{R}^{2m}$. First consider the function

$$f_0 := \sum_{j=1}^{m} \left(x_j x_{j+m} + \xi_j \xi_{j+m}\right) \in C^\infty\left(\mathbb{R}^{2m}\right).$$

If H_{f_0} and $e^{tH_{f_0}}$ denote the Hamilton vector field and time t flow of f_0 respectively then it is easy to compute

$$e^{\frac{\pi}{4} H_{f_0}}\left(x_j, \xi_j; x_{j+m}\xi_{j+m}\right) = \left(\frac{x_j + \xi_{j+m}}{\sqrt{2}}, \frac{-x_{j+m} + \xi_j}{\sqrt{2}}; \frac{x_{j+m} + \xi_j}{\sqrt{2}}, \frac{-x_j + \xi_{j+m}}{\sqrt{2}}\right).$$

We abbreviate $(x', \xi') = (x_1, \ldots, x_m; \xi_1, \ldots, \xi_m)$,
$(x'', \xi'') = (x_{m+1}, \ldots, x_{2m}; \xi_{m+1}, \ldots, \xi_{2m})$ and $(x, \xi) = (x_0, x', x''; \xi_0, \xi', \xi'')$.

Using Egorov's theorem, the operator (4.4) is conjugated to

$$e^{\frac{i\pi}{4h} f_0^W} D_0 e^{-\frac{i\pi}{4h} f_0^W} = d_0^W, \quad \text{with} \tag{4.5}$$

$$d_0 = \sigma^j w_{j,f_0}^0 \left(\xi_0 + \varphi_{f_0}\right) + \sqrt{2}\left(\sigma^j w_{j,f_0}^k \xi_k + \sigma^j w_{j,f_0}^{k+m} x_k\right) + \sigma^j r_j^\infty + O(h) \tag{4.6}$$

(4.7)

$$\text{where } w_{j,f_0}^k = \left(e^{-\frac{\pi}{4} H_{f_0}}\right)^* w_j^k$$

$$\varphi_{f_0} = \left(e^{-\frac{\pi}{4} H_{f_0}}\right)^* \varphi$$

$$r_j^\infty = \left(e^{-\frac{\pi}{4} H_{f_0}}\right)^* \chi_\gamma a_{\gamma,j}^\infty \tag{4.8}$$

Using the formulas (3.3), (3.5) we may also calculate

$$
\begin{aligned}
\varphi_{f_0} &= T_\gamma + \chi^- \tilde{Q}^{h,-} + \chi^+ \varphi^+ \left(\tilde{Q}\right) \quad \text{with,} \\
\tilde{Q}_j^e &= \frac{1}{4}\left[\left(x_j - \xi_{j+m}\right)^2 + \left(x_{j+m} - \xi_j\right)^2\right] \\
\tilde{Q}_j^h &= \frac{1}{2}\left(x_{N_e+j} - \xi_{N_e+j+m}\right)\left(x_{N_e+j+m} - \xi_{N_e+j}\right) \\
\tilde{Q}_j^{l,\mathrm{Re}} &= \frac{1}{2}\left(x_{m-2j+2} - \xi_{2m-2j+2}\right)\left(x_{2m-2j+1} - \xi_{m-2j+1}\right) \\
&\quad -\frac{1}{2}\left(x_{m-2j+1} - \xi_{2m-2j+1}\right)\left(x_{2m-2j+2} - \xi_{m-2j+2}\right) \\
\tilde{Q}_j^{l,\mathrm{Im}} &= \frac{1}{2}\left(x_{m-2j+1} - \xi_{2m-2j+1}\right)\left(x_{2m-2j+1} - \xi_{m-2j+1}\right) \\
&\quad +\frac{1}{2}\left(x_{m-2j+2} - \xi_{2m-2j+2}\right)\left(x_{2m-2j+2} - \xi_{m-2j+2}\right) \quad \text{and} \\
\tilde{Q}^{h,-} &= \frac{\pi}{4}\sum_{j=1}^{N_h^-}\left[\left(x_{N_e+j} - \xi_{N_e+j+m}\right)^2 + \left(x_{N_e+j+m} - \xi_{N_e+j}\right)^2\right]
\end{aligned}
$$

Next, set

$$
\begin{aligned}
\bar{\varphi}_{f_0} = \bar{\varphi} &= T_\gamma + \chi^- \bar{Q}^{h,-} + \chi^+ \varphi^+ \left(\bar{Q}\right) \quad \text{with,} && (4.9) \\
\bar{Q}_j^e &= \frac{1}{4}\left[\xi_{j+m}^2 + x_{j+m}^2\right] \\
\bar{Q}_j^h &= -\frac{1}{2} x_{N_e+j+m}\xi_{N_e+j+m} \\
\bar{Q}_j^{l,\mathrm{Re}} &= \frac{1}{2}\left(x_{2m-2j+2}\xi_{2m-2j+1} - x_{2m-2j+1}\xi_{2m-2j+2}\right) \\
\bar{Q}_j^{l,\mathrm{Im}} &= -\frac{1}{2}\left(x_{2m-2j+1}\xi_{2m-2j+1} + x_{2m-2j+2}\xi_{2m-2j+2}\right) \quad \text{and} \\
\bar{Q}^{h,-} &= \frac{\pi}{4}\sum_{j=1}^{N_h^-}\left[\xi_{N_e+j+m}^2 + x_{N_e+j+m}^2\right] && (4.10)
\end{aligned}
$$

Below denote by $o_N', o_N'' \subset S_{\mathrm{cl}}^1\left(T^*S^1 \times \mathbb{R}^{4m}; \mathbb{C}^l\right)$ the subspace of self-adjoint symbols $a : (0,1]_h \to C^\infty\left(T^*S^1 \times \mathbb{R}^{4m}; i\mathfrak{u}\left(2^m\right)\right)$ such that each of the coefficients a_k, $k = 0, 1, 2, \ldots$ in its symbolic expansion vanishes to order N in $(\xi_0 + \bar{\varphi}, x', \xi')$ and (x'', ξ'') respectively. We also denote by o_N', o_N'' the space of Weyl quantizations of the respective symbols. One clearly has $\varphi_{f_0} = \bar{\varphi} + o_1' o_1''$. A Taylor expansion of d_0 (4.6) now gives $r_j^0 \in o_2'$, $r_j^1 \in o_1' o_1''$, $r_j^\infty \in o_\infty''$, $0 \le j \le 2m$, such that

$$
d_0 = \sqrt{2}\sigma^j\left(\bar{w}_j^0\left(\xi_0 + \bar{\varphi}\right) + \bar{w}_j^k \xi_k + \bar{w}_j^{k+m} x_k\right) + \sigma^j\left(r_j^0 + r_j^1 + r_j^\infty\right) + O(h)
$$

and where $\bar{w}_j^k(x_0) = w_j^k(x_0, 0, 0)$.

On squaring using (4.1) we obtain

$$\left(d_0^W\right)^2 = Q_0^W + o_2' o_1'' + o_\infty'' + O(h), \quad \text{with}$$

(4.11)

$$Q_0 = \begin{bmatrix}\xi_0+\bar{\varphi} & \xi' & x'\end{bmatrix}\begin{bmatrix}\bar{g}^{00}(x_0) & \bar{g}^{k0}(x_0) & \bar{g}^{(k+m)0}(x_0)\\ \bar{g}^{0l}(x_0) & \bar{g}^{kl}(x_0) & \bar{g}^{k(l+m)}(x_0)\\ \bar{g}^{0(l+m)}(x_0) & \bar{g}^{(k+m)l}(x_0) & \bar{g}^{(k+m)(l+m)}(x_0)\end{bmatrix}\begin{bmatrix}\xi_0+\bar{\varphi}\\ \xi'\\ x'\end{bmatrix}.$$

Here $\bar{g}^{kl}(x_0) = 2g^{kl}(x_0,0,0)$ and g^{kl} the components of the inverse metric in Reeb Darboux coordinates along the orbit and

$$\bar{g}^{00}(x_0) = \frac{1}{T_\gamma^2\,|R|^2}.$$

Next we consider another function f_1 of the form

$$f_1 = \frac{1}{2}\begin{bmatrix}x' & \xi'\end{bmatrix}\begin{bmatrix}\alpha_{m\times m}(x_0) & \gamma_{m\times m}(x_0)\\ \gamma^t_{m\times m}(x_0) & \beta_{m\times m}(x_0)\end{bmatrix}\begin{bmatrix}x'\\ \xi'\end{bmatrix}$$

where α,β and γ are matrix valued functions of the given orders with α,β symmetric. An easy computation now shows

$$\begin{aligned}\left(e^{H_{f_1}}\right)^*\begin{bmatrix}\xi_0+\bar{\varphi}\\ x'\\ \xi'\end{bmatrix} &= e^{\Lambda}\begin{bmatrix}\xi_0+\bar{\varphi}\\ x'\\ \xi'\end{bmatrix} + o_2' \quad \text{with}\\ \Lambda(x_0) &= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -I_{m\times m}\\ 0 & I_{m\times m} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & \alpha_{m\times m}(x_0) & \gamma_{m\times m}(x_0)\\ 0 & \gamma^t_{m\times m}(x_0) & \beta_{m\times m}(x_0)\end{bmatrix}.\end{aligned}$$

From the suitability assumption (1.7), we have that there exists a smooth matrix valued functions α,β and γ such that

$$\begin{aligned}\left(e^{H_{f_1}}\right)^* Q_0 &= \begin{bmatrix}\xi_0+\bar{\varphi} & \xi' & x'\end{bmatrix} e^{\Lambda^t}\begin{bmatrix}\bar{g}^{00}(x_0) & \bar{g}^{k0}(x_0) & \bar{g}^{(k+m)0}(x_0)\\ \bar{g}^{0l}(x_0) & \bar{g}^{kl}(x_0) & \bar{g}^{k(l+m)}(x_0)\\ \bar{g}^{0(l+m)}(x_0) & \bar{g}^{(k+m)l}(x_0) & \bar{g}^{(k+m)(l+m)}(x_0)\end{bmatrix} e^{\Lambda}\begin{bmatrix}\xi_0+\bar{\varphi}\\ \xi'\\ x'\end{bmatrix}\\ &= Q_1 \coloneqq \bar{g}^{00}(x_0)(\xi_0+\bar{\varphi})^2 + \left[\sum_{j=1}^m \mu_j\left(x_j^2+\xi_j^2\right)\right]\end{aligned}$$

(4.12)

$$+\,2\sum_{j=1}^m(\xi_0+\bar{\varphi})\left[h_j^0(x_0)\,\xi_j + h_j^1(x_0)\,x_j\right] + o_3'$$

and where

$$\begin{bmatrix}\bar{g}^{00}(x_0)\\ h_j^0(x_0)\\ h_j^1(x_0)\end{bmatrix} = e^{\Lambda^t}\begin{bmatrix}\bar{g}^{00}(x_0)\\ \bar{g}^{0l}(x_0)\\ \bar{g}^{0(l+m)}(x_0)\end{bmatrix}.$$

Next, if

$$f_2 = (\xi_0+\bar{\varphi})\begin{bmatrix}\frac{1}{\mu}\xi' & \frac{1}{\mu}x'\end{bmatrix}\begin{bmatrix}0 & -I_{m\times m}\\ I_{m\times m} & 0\end{bmatrix}\begin{bmatrix}h_j^0(x_0)\\ h_j^1(x_0)\end{bmatrix}$$

we may compute

$$\left(e^{H_{f_2}}\right)^* Q_1 = Q_2 := \bar{g}^{00}\left(x_0\right)\left(\xi_0+\bar{\varphi}\right)^2 + \left[\sum_{j=1}^m \mu_j\left(x_j^2+\xi_j^2\right)\right] + o_3'. \tag{4.13}$$

Finally, letting L_γ denote the length of the Reeb orbit note

$$L_\gamma = \exp\left\{-\frac{1}{2}\frac{\int_0^1 dx_0\left(g^{00}\right)^{-1/2}\left(\ln g^{00}\right)}{\int_0^1 dx_0\left(g^{00}\right)^{-1/2}}\right\}$$

and set

$$a\left(x_0\right) := \left(g^{00}\right)^{1/2}\int_0^\theta d\theta'\left(g^{00}\right)^{-1/2}\ln\left[T_\gamma L_\gamma\left(g^{00}\right)^{1/2}\right]$$

to compute

$$\left(e^{H_{a\xi}}\right)^* Q_2 = \frac{1}{L_\gamma^2}\left(\xi_0+\bar{\varphi}\right)^2 + \left[\sum_{j=1}^m \mu_j\left(x_j^2+\xi_j^2\right)\right] + o_3'. \tag{4.14}$$

Letting

$$H_2 = \frac{1}{2}\sum_{j=1}^m \mu_j\left(x_j^2+\xi_j^2\right),$$

using (4.11), (4.12), (4.13) and (4.14) Egorov's theorem now gives

(4.15)

$$d_{00}^W := e^{\frac{i}{h}a\xi^W} e^{\frac{i}{h}f_2^W} e^{\frac{i}{h}f_1^W} d_0^W e^{-\frac{i}{h}f_1^W} e^{-\frac{i}{h}f_2^W} e^{-\frac{i}{h}a\xi^W} = \left(\sum_{j=0}^{2m}\sigma_j b_j\right)^W + ho_0 \quad \text{with}$$

$$\sum_{j=0}^{2m} b_j^2 = \left(\frac{1}{L_\gamma^2}\left(\xi_0+\bar{\varphi}\right)^2 + 2H_2\right)^W + o_2' o_1'' + o_\infty''.$$

Another Taylor expansion in the variables $(\xi_0+\bar{\varphi}, x', \xi'; x'', \xi'')$ gives $A = (a_{jk}(x_0)) \in C^\infty\left(S^1; \mathfrak{so}(n)\right)$ and $r_{j,0} \in o_1' o_1''$, $r_{j,1} \in o_2'$, $r_{j,\infty} \in o_\infty''$, $j = 0, \ldots, 2m$, such that

$$e^{-A}\begin{bmatrix} b_0 \\ \vdots \\ b_{2m}\end{bmatrix} = \begin{bmatrix} \frac{1}{L_\gamma}\left(\xi_0+\bar{\varphi}\right) \\ \left(2\mu_1\right)^{\frac{1}{2}} x_1 \\ \left(2\mu_1\right)^{\frac{1}{2}}\xi_1 \\ \vdots \\ \left(2\mu_m\right)^{\frac{1}{2}} x_m \\ \left(2\mu_m\right)^{\frac{1}{2}}\xi_m\end{bmatrix} + \begin{bmatrix} r_{0,0} \\ \vdots \\ r_{2m,0}\end{bmatrix} + \begin{bmatrix} r_{0,1} \\ \vdots \\ r_{2m,1}\end{bmatrix} + \begin{bmatrix} r_{0,\infty} \\ \vdots \\ r_{2m,\infty}\end{bmatrix}.$$

We may now set $c_A = \frac{1}{i}a_{jk}\sigma^j\sigma^k \in C^\infty\left(S^1; i\mathfrak{u}\left(2^m\right)\right)$ and compute

$$e^{ic_A^W} d_{00}^W e^{-ic_A^W} = d_1^W, \quad \text{where} \tag{4.16}$$

$$d_1 = H_1 + \sigma^j\left(r_{j,0}+r_{j,1}+r_{j,\infty}\right) + O(h), \quad \text{and} \tag{4.17}$$

$$H_1 := \frac{1}{L_\gamma}\left(\xi_0+\bar{\varphi}\right)\sigma_0 + \sum_{j=1}^m \left(2\mu_j\right)^{\frac{1}{2}}\left(x_j\sigma_{2j-1}+\xi_j\sigma_{2j}\right). \tag{4.18}$$

Finally, if we further Taylor expand $r_{0,0}+r_{0,1}=l_{00}\left(x_0,\frac{1}{L}\left(\xi_0+\bar{\varphi}\right),x'',\xi''\right)+x_1l_{01}+\xi_1l_{02}+\ldots+\xi_ml_{0(2m)}$, then a further conjugation of d_1^W by $e^{ic_2^W}$; $c_2=\frac{1}{i}l_{0k}\sigma^0\sigma^k$, it is possible to make $r_{0,0}+r_{0,1}$ independent of (x',ξ') in (4.17).

4.1.1. *Weyl product and Koszul complexes.* We now derive a formal Birkhoff normal form for the symbol d_1 in (4.17). Since much of what follows here proceeds in a similar fashion to [26] Section 5, we refer there for necessary modifications to avoid repetition of arguments. First denote by $R=C^\infty\left(S^1_{x_0}\right)$ the ring of real valued functions on the circle. Further define

$$S\coloneqq R\left[\!\left[\xi_0+\bar{\varphi},x',\xi';x'',\xi'';h\right]\!\right]$$

the ring of formal power series in the further given $4m+2$ variables with coefficients in R. The ring $S\otimes\mathbb{C}$ is now equipped with the Weyl product

$$a*b\coloneqq\left[e^{\frac{ih}{2}\left(\partial_{r_1}\partial_{s_2}-\partial_{r_2}\partial_{s_1}\right)}\left(a\left(s_1,r_1;h\right)b\left(s_2,r_2;h\right)\right)\right]_{x=s_1=s_2,\xi=r_1=r_2},$$

(again using the convention $\theta=x_0$) corresponding to the composition formula for pseudo-differential operators, with

$$[a,b]\quad\coloneqq\quad a*b-b*a$$

being the corresponding Weyl bracket. It is an easy exercise to show that for $a,b\in S$ real valued, the commutator $i\left[a,b\right]\in S$ is real valued.

Next, we define a filtration on S. Each monomial $h^k\left(\xi_0+T_\gamma\right)^a\left(x'\right)^{\alpha'}\left(\xi'\right)^{\beta'}\left(x''\right)^{\alpha''}\left(\xi''\right)^{\beta''}$ in S is given the weight $2k+a+|\alpha'|+|\beta'|+|\alpha''|+|\beta''|$. The ring S is equipped with a decreasing filtration

$$\begin{aligned}S=O_0\quad&\supset\quad O_1\supset\ldots\supset O_N\supset\ldots,\\ \bigcap_N O_N\quad&=\quad\{0\},\end{aligned}$$

where O_N consists of those power series with monomials of weight N or more. Similar filtrations

$$\begin{aligned}S&=O'_0\supset O'_1\supset\ldots\supset O'_N\supset\ldots\\ S&=O''_0\supset O''_1\supset\ldots\supset O''_N\supset\ldots\end{aligned}$$

maybe defined with O'_N, O''_N consisting of power series in those monomials with $2k+a+|\alpha'|+|\beta'|\geq N$ and $2k+|\alpha''|+|\beta''|\geq N$ respectively. It is an exercise to show that

$$\begin{aligned}O_N*O_M\quad&\subset\quad O_{N+M}\\ \left[O_N,O_M\right]\quad&\subset\quad ihO_{N+M-2}\end{aligned}$$

and similar inclusions holding for its primed versions. The associated grading is given by

$$S=\bigoplus_{N=0}^{\infty}S_N$$

where S_N consists of those power series with monomials of weight exactly N . We also define the quotient ring $D_N\coloneqq S/O_{N+1}$ whose elements may be identified with the set of homogeneous polynomials with monomials of weight at most N. The ring D_N is also similarly graded and filtered. In similar vein, we may also define the ring

$$S(m) = S \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$$

of $R \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$ valued formal power series in $(\xi_0 + \bar{\varphi}, x', \xi'; h)$. The ring $S(m)$ is equipped with an induced product $*$ and decreasing filtration

$$\begin{aligned} O_0(m) &\supset O_1(m) \supset \ldots \supset O_N(m) \supset \ldots, \\ \bigcap_N O_N(m) &= \{0\}, \end{aligned}$$

where $O_N(m) = O_N \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$. It is again a straightforward exercise to show that for $a, b \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ self-adjoint, the commutator $i[a,b] \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ is self-adjoint.

4.1.2. *Koszul complexes.* Let us now again consider the $2m$ and $2m+1$ dimensional real inner product spaces $V = \mathbb{R}[e_1, \ldots, e_{2m}]$ and $W = \mathbb{R}[e_0] \oplus V$ from 2.2. Considering the chain groups $D_N \otimes \Lambda^k V$, $k = 0, 1, \ldots, n$, one may define four differentials

$$\begin{aligned} w_x^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(x_j e_{2j-1} \wedge + \xi_j e_{2j} \wedge \right) \\ i_x^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(x_j i_{e_{2j-1}} + \xi_j i_{e_{2j}} \right) \\ w_\partial^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} e_{2j-1} \wedge + \partial_{\xi_j} e_{2j} \wedge \right) \\ i_\partial^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} i_{e_{2j-1}} + \partial_{\xi_j} i_{e_{2j}} \right). \end{aligned}$$

Similarly, we may consider the chain groups $D_N \otimes \Lambda^k W$, $k = 0, 1, \ldots, n$, one may define four differentials

$$\begin{aligned} w_x &= \frac{1}{L_\gamma} (\xi_0 + \bar{\varphi}) e_0 \wedge + 2^{\frac{1}{2}} w_x^0 \\ i_x &= \frac{1}{L_\gamma} (\xi_0 + \bar{\varphi}) i_{e_0} + 2^{\frac{1}{2}} i_x^0 \\ w_\partial &= \partial_{\xi_0} e_0 \wedge + 2^{\frac{1}{2}} w_\partial^0 \\ i_\partial &= \partial_{\xi_0} i_{e_0} + 2^{\frac{1}{2}} i_\partial^0. \end{aligned}$$

Next, we define twisted Koszul differentials on $D_N \otimes \Lambda^k V$ via

$$\begin{aligned} \tilde{w}_\partial^0 &= \frac{i}{h} \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\mathrm{ad}_{x_j} e_{2j-1} \wedge + \mathrm{ad}_{\xi_j} e_{2j} \wedge \right) = \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} e_{2j} \wedge - \partial_{\xi_j} e_{2j-1} \wedge \right) \\ \tilde{i}_\partial^0 &= \frac{i}{h} \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\mathrm{ad}_{x_j} i_{e_{2j-1}} + \mathrm{ad}_{\xi_j} i_{e_{2j}} \right) = \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} i_{e_{2j}} - \partial_{\xi_j} i_{e_{2j-1}} \right). \end{aligned}$$

We note that the above are symplectic adjoints to their untwisted counterparts with respect to the symplectic pairing $\sum_{j=1}^m e_{2j-1} \wedge e_{2j}$ on V.

Similar twisted Koszul differentials on $D_N \otimes \Lambda^k W$ are defined via

$$\begin{aligned} \tilde{w}_\partial &= \frac{1}{L_\gamma}\left(\mathrm{ad}_{\xi_0+\bar{\varphi}}\right) e_0\wedge + 2^{\frac{1}{2}}\tilde{w}^0_\partial \\ \tilde{i}_\partial &= \frac{1}{L_\gamma}\left(\mathrm{ad}_{\xi_0+\bar{\varphi}}\right) i_{e_0} + 2^{\frac{1}{2}}\tilde{i}^0_\partial . \end{aligned}$$

We note that in what follows works with any leading terms replacing $e_0\wedge$ and i_{e_0} above that would serve as differentials.

We now compute the twisted combinatorial Laplacian to be

$$\begin{aligned} \tilde{\Delta}^0 &= \tilde{w}^0_\partial i^0_x + i^0_x \tilde{w}^0_\partial \\ &= -\left(w^0_x \tilde{i}^0_\partial + \tilde{i}^0_\partial w^0_x\right) \\ &= \sum_{j=1}^m \mu_j \left[\xi_j \partial_{x_j} - x_j \partial_{\xi_j} + e_{2j} i_{e_{2j-1}} - e_{2j-1} i_{e_{2j}}\right]. \end{aligned}$$

One may similarly define $\tilde{\Delta}$. Next, we define the space of twisted $\tilde{\Delta}^0$-harmonic, $\bar{\varphi}$-commuting, x_0- independent elements

$$\begin{aligned} \mathcal{H}^k_N &= \left\{\omega \in D_N \otimes \Lambda^k W \middle|\, \tilde{\Delta}^0\omega = 0,\, \partial_{x_0}\omega = 0,\, \mathrm{ad}_{\bar{\varphi}}\omega = 0\right\} \\ \mathcal{H}^k &= \left\{\omega \in S \otimes \Lambda^k W \middle|\, \tilde{\Delta}^0\omega = 0,\, \partial_{x_0}\omega = 0,\, \mathrm{ad}_{\bar{\varphi}}\omega = 0\right\}. \end{aligned}$$

The following version of the Hodge decomposition theorem follows in a similar fashion to [26] Lemma 5.1. We only note that the ξ_0-independence in the definition of $\mathcal{H}^k_N$ from [26] is here replaced by the condition $\mathrm{ad}_{\xi_0+\bar{\varphi}}\omega = 0$, which on account of non-resonance is equivalent to $\mathrm{ad}_{\xi_0}\omega = \partial_{x_0}\omega = 0$, $\mathrm{ad}_{\bar{\varphi}}\omega = 0$.

Lemma 4.1. *The k-th chain group is spanned by the three subspaces*

$$D_N \otimes \Lambda^k W = \mathbb{R}\left[Im\left(i_x \tilde{w}_\partial\right), Im\left(\tilde{w}_\partial i_x\right), \mathcal{H}^k_N\right].$$

4.1.3. *Formal Birkhoff normal form.* As in [26] section 5.2 the Koszul complexes now allow us to complete the Birkhoff normal form procedure for the symbol d_1 in (4.17). Define the Clifford quantization of an element in $a \in S \otimes \Lambda^k W$, using (2.9) as an element in

$$c_0(a) := i^{\frac{k(k+1)}{2}} c(a) \in S(m).$$

This gives an isomorphism

$$c_0 : S \otimes \Lambda^{\mathrm{odd/even}} W \to S \otimes i\mathfrak{u}_{\mathbb{C}}\left(2^m\right) \tag{4.19}$$

of real elements of the even or odd exterior algebra with self-adjoint elements in $S(m)$. In a fashion similar to [26] we may now prove the following formal Birkhoff normal form for the symbol d_1. Below the symbol $H_1$is as in (4.18).

Proposition 4.2. *There exist $f \in O_1' \cap O_3$, $a \in O_2 \otimes \Lambda^{even} W$ and $\omega \in \mathcal{H}^{odd} \cap O_1' \cap O_2$ such that*

$$e^{ic_0(a)} e^{\frac{i}{h}f} d_1 e^{-\frac{i}{h}f} e^{-ic_0(a)} = H_1 + c_0(\omega). \tag{4.20}$$

5. Reduction to $S^1 \times \mathbb{R}^{2m}$

We now return to the study of the traces $\mathcal{T}^{\theta}_{B_v,B_v}(D)$ of the fourth kind in (3.37). The asymptotics of these traces can be reduced to $S^1 \times \mathbb{R}^{2m}$. This however first requires a modification lemma as Lemma 3.5 and the definition and construction of another trapping time/function.

Let $\Gamma \subset \Omega \subset \mathit{\Omega}$ be any subcover $\delta \in \left(0, \frac{1}{2}\right)$ and $\tau > 0$ as before. We define an trapping time in a similar fashion to (3.16)

$$T_v := \frac{1}{\inf_{(g,\mathrm{v}) \in \mathcal{G}_v \times S^0(X;U(S))} |H_{g,\mathrm{v}} d|}$$
$$\mathcal{G}_v := \left\{ g \in S^0_\delta\left(X;[0,1]\right) \mid g|_{\Sigma^D_{[-\tau,\tau]} \cap \tilde{\Omega}_{\gamma_v}} = 1,\ g|_{\left(\Sigma^D_{[-8\tau,8\tau]} \cap \tilde{\Omega}_{\gamma_v}\right)^c} = 0 \right\}$$

and set

$$T^{\tau}_{(\Omega,\mathit{\Omega})} := \min_{1 \le v \le M} T_v.$$

We now have an analog of 3.4.

Proposition 5.1. *Let $\mathit{\Omega}$ be a collection of Darboux-Reeb charts and $T > 0$. Then for each τ sufficiently small there exists an open sub-cover $\Gamma \subset \Omega \subset \mathit{\Omega}$ such that*

$$T^{\tau}_{(\Omega,\mathit{\Omega})} > T. \tag{5.1}$$

Proof. The proof is similar to 3.4 with a some modifications that we precise. Let $0 < \varepsilon \ll 1$, be sufficiently small such that for each Reeb orbit γ_v the set $A_\varepsilon := S^1_{x_0} \times B_{\mathbb{R}^{2m}}(\varepsilon) \subset \mathit{\Omega}^0_{\gamma_v}$ is contained inside the Darboux-Reeb chart 3.2. Next for $(x', \xi') = (x_1, \ldots, x_m; \xi_1, \ldots, \xi_m)$, $(x'', \xi'') = (x_{m+1}, \ldots, x_{2m}; \xi_{m+1}, \ldots, \xi_{2m})$ set $\tilde{C}_\varepsilon := \left\{ x''^2 + \xi''^2 < \varepsilon^2 \right\} \subset T^* S^1_{x_0} \times \mathbb{R}^{4m}_{x',x'',\xi',\xi''}$. Also set

$$U_{\varepsilon,\tau} := \left\{ \frac{1}{L^2_\gamma}\left(\xi_0 + \bar{\varphi}\right)^2 + 2\sum_{j=1}^{m} \mu_j \left(x_j^2 + \xi_j^2\right) < \tau^2,\ x''^2 + \xi''^2 < \varepsilon^2 \right\} \subset \tilde{C}_\varepsilon$$

with $\bar{\varphi} = \bar{\varphi}(x'', \xi'')$ as in (4.9). Also denote by o'_N, o''_N functions which vanish to order N in $(\xi_0 + \bar{\varphi}, x', \xi')$ and (x'', ξ'') respectively. Then as in 4.1 (eqns (4.5), (4.15), (4.16), (4.17) and (4.18)) there exists $0 < \tau \ll 1$ sufficiently small of the following significance: for each $1 \le v \le M$ there exists a neighborhood $M_v \subset \tilde{A}_\varepsilon$ of $\tilde{A}_{\frac{\varepsilon}{8}} \cap \Sigma^D_0$, a Hamiltonian symplectomorphism

$$\kappa_v := e^{H_{f_1}} \circ e^{H_{f_0}} : U_{\varepsilon,\tau} \to M_v$$
$$\kappa_v\left(x_0, 0, x''; -\bar{\varphi}, 0, \xi''\right) = \left(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}; -\bar{\varphi}, \frac{x''}{\sqrt{2}}, \frac{\xi''}{\sqrt{2}} \right)$$

a self-adjoint endomorphism $c_A \in C^\infty\left(U_{\varepsilon,\tau}; i\mathfrak{u}\left(2^m\right)\right)$, functions $r_{j,0} \in o'_1 o''_1$,$r_{j,1} \in o'_2$, $r_{j,\infty} \in o''_\infty$, $j = 0, \ldots, 2m$, such that

$$e^{ic_A}\left(\left(e^{H_{f_1}} \circ e^{H_{f_0}}\right)^* d\right) e^{-ic_A} = H_1 + \sigma^j r_{j,0} + \sigma^j r_{j,1} + \sigma^j r_{j,\infty}, \tag{5.2}$$

with H_1 as in (4.18).

Also note that the terms $r_{0,0}+r_{0,1}$ maybe assumed to be $(x';\xi')$ independent as observed after (4.18). Now set

(5.3)
$$\left(\tilde{\theta}_0,\tilde{\theta}_1,\ldots,\tilde{\theta}_{2m}\right)=\left(\frac{1}{L_\gamma}\left(\xi_0+\bar{\varphi}\right),(2\mu_1)^{\frac{1}{2}}x_1,(2\mu_1)^{\frac{1}{2}}\xi_1,\ldots,(2\mu_m)^{\frac{1}{2}}x_m,(2\mu_m)^{\frac{1}{2}}\xi_m\right)$$
$$+\left(r_{0,0},r_{1,0},\ldots,r_{2m,0}\right)+\left(r_{0,1},r_{1,1},\ldots,r_{2m,1}\right)+\left(r_{0,\infty},r_{1,\infty},\ldots,r_{2m,\infty}\right)$$
$$\tilde{\theta}'=\left(\tilde{\theta}_1,\ldots,\tilde{\theta}_{2m}\right)$$

and note from (3.23) that the eigenvalues of the symbol d are $\pm\left|\tilde{\theta}\right|$. We clearly have $U_{\varepsilon,\tau}\cap\Sigma_0^D=\left\{\tilde{\theta}=0\right\}\cap\Sigma_0^D$and we may set

$$\theta_j=\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}\in C^\infty\left(U_{\varepsilon,\tau}\setminus\Sigma_0^D;S^{n-1}\right). \tag{5.4}$$

We now compute

$$\left\{\tilde{\theta}_0,x_0\right\}-\frac{1}{L_\gamma}=o_1'+o_1''+\mathsf{o}_\infty''$$
$$\left\{\tilde{\theta}_j,x_0\right\}=o_1'+o_1''+\mathsf{o}_\infty'',\quad j\geq 1,$$
$$\left\{\tilde{\theta}_j,x''\right\}=o_1'+\mathsf{o}_\infty'',\quad j\geq 1,$$
$$\left\{\tilde{\theta}_j,\xi''\right\}=o_1'+\mathsf{o}_\infty'',\quad j\geq 1,$$
$$\left\{\tilde{\theta}_0,\tilde{\theta}_j\right\}=o_2'+o_1'o_1''+\mathsf{o}_\infty'',\quad j\geq 0,$$
$$\left\{\tilde{\theta}_j,\tilde{\theta}_k\right\}\quad\text{or}\ \left\{\tilde{\theta}_j,\tilde{\theta}_k\right\}-1=o_1'+o_1''+\mathsf{o}_\infty''\quad k>j\geq 0, \tag{5.5}$$

similar to (3.27). Note that the bracket $\left\{\tilde{\theta}_0,\tilde{\theta}_j\right\}$ is still $o_2+\mathsf{o}_\infty$ due to the $(x';\xi')$-independence of r_0 in $\tilde{\theta}_0$. In this case however, unlike (3.27) the brackets $\left\{\tilde{\theta}_0,x''\right\},\left\{\tilde{\theta}_0,\xi''\right\}$ may not be $o_1'+\mathsf{o}_\infty''$ due to the presence of the $\bar{\varphi}\left(x'',\xi''\right)$ term in $\tilde{\theta}_0$. However the quadratics

$$\begin{aligned}\hat{Q}_j^e&=\left(\xi_{j+m}^2+x_{j+m}^2\right)^{\frac{\varepsilon}{T}}\\ \hat{Q}_j^h&=\left(x_{N_e+j+m}^2+\xi_{N_e+j+m}^2\right)^{\frac{\varepsilon}{T}}\\ \hat{Q}_j^{l,\mathrm{Re}}&=\left(x_{2m-2j+1}^2+x_{2m-2j+2}^2\right)^{\frac{\varepsilon}{T}}\\ \hat{Q}_j^{l,\mathrm{Im}}&=\left(\xi_{2m-2j+1}^2+\xi_{2m-2j+2}^2\right)^{\frac{\varepsilon}{T}}\end{aligned} \tag{5.6}$$

are seen to satisfy

$$\left\{\tilde{\theta}_0,\hat{Q}\right\}-\frac{1}{L}\left\{\bar{\varphi},\hat{Q}\right\}=o_1'+\mathsf{o}_\infty'' \tag{5.7}$$

$$\left|\left\{\bar{\varphi},\hat{Q}\right\}\right|\leq\frac{\varepsilon}{T}\underbrace{\left(m\sup_{(x'',\xi'')\leq\varepsilon}\left|\partial_{\bar{Q}}\bar{\varphi}\right|\right)}_{=:c_0},\quad\hat{Q}\neq 0, \tag{5.8}$$

where $\bar{\varphi}$ is considered as a function of the quadratics $\bar{Q}$ as in (4.9). Hence for ε, τ sufficiently small, the bracket relations (5.5), (5.7) and (5.8) again imply

$$
\begin{aligned}
\left|\left\{\tilde{\theta}_j, x_0\right\}\right| &\leq 2, & j \geq 0,\\
\left|\left\{\left|\tilde{\theta}\right|, x_0\right\}\right| &\leq 2, & \left|\tilde{\theta}\right| \neq 0,\\
\left|\left\{\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}, x_0\right\}\right| &\leq \frac{4}{\left|\tilde{\theta}\right|}, & \left|\tilde{\theta}\right| \neq 0,\ j \geq 0,\\
\left|\frac{1}{\varepsilon}\left\{\tilde{\theta}_j, \hat{Q}\right\}\right| &\leq \frac{2c_0}{LT}, & j \geq 0,\\
\left|\frac{1}{\varepsilon}\left\{\left|\tilde{\theta}\right|, \hat{Q}\right\}\right| &\leq \frac{2c_0}{LT}, & \left|\tilde{\theta}\right| \neq 0,\\
\left|\frac{1}{\varepsilon}\left\{\frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}, \hat{Q}\right\}\right| &\leq \frac{2c_0}{LT\left|\tilde{\theta}\right|}, & \left|\tilde{\theta}\right| \neq 0,\ j \geq 0,\\
\left|\left\{\tilde{\theta}_0, \tilde{\theta}_j\right\}\right| &\leq \frac{\left|\tilde{\theta}\right|}{T}, & j \geq 0,\\
\left|\left\{\tilde{\theta}_0, \left|\tilde{\theta}\right|\right\}\right| &\leq \frac{\left|\tilde{\theta}\right|}{T}, & \left|\tilde{\theta}\right| \neq 0,\\
\left|\left\{\tilde{\theta}_0, \frac{\tilde{\theta}_j}{\left|\tilde{\theta}\right|}\right\}\right| &\leq \frac{1}{T}, & \left|\tilde{\theta}\right| \neq 0,\ j \geq 0,
\end{aligned}
$$

(5.9)

$$\frac{1}{4}\left[\left(\xi_0 + \bar{\varphi}\right)^2 + 2\sum_{j=1}^{m}\mu_j\left(x_j^2 + \xi_j^2\right)\right] \leq \sum_{j=0}^{2m}\tilde{\theta}_j^2 \leq 4\left[\left(\xi_0 + \bar{\varphi}\right)^2 + 2\sum_{j=1}^{m}\mu_j\left(x_j^2 + \xi_j^2\right)\right]$$

on $U_{\varepsilon,\tau}$. Again define

$$\tilde{U}_{\varepsilon,\tau} := \left\{\sum_{j=0}^{2m}\tilde{\theta}_j^2 < \tau^2,\ x''^2 + \xi''^2 < \varepsilon^2\right\} \subset U_{\varepsilon,\tau}.$$

We now set

$$\Omega_{\gamma_v} := \left\{\hat{Q}_j < \left(\frac{\varepsilon}{16m}\right)^2\right\} \tag{5.10}$$

To verify (5.1) again let $\chi \in C_c^{\infty}\left([-4,4];[0,1]\right)$, be a cutoff such that $\chi = 1$ on $[-2,2]$ and $|\chi'| \leq 1$. Also for $\rho \in \left(0, \frac{1}{8}\right)$ fixed, define a function $\varphi_\rho \in C^{\infty}\left([-1,1]_{\theta_0};[0,1]\right)$ such that $\varphi_\rho\left(\theta_0\right) = \begin{cases} 1; & \text{for } \theta_0 \in [1-\rho, 1] \\ 0; & \text{for } \theta_0 \in [-1, 1-2\rho] \end{cases}$ and $\left|\varphi_\rho'\right| \leq \frac{2}{\rho}$.

The trapping function in this case is now modified to

$$\begin{aligned} g_v \coloneqq & \chi\left(\frac{\beta\left(\tilde{\theta}\right)}{\tau}\right)\left[\prod_j \chi\left(\frac{\hat{Q}_j}{\left(\varepsilon/16m\right)^2}\right)\right] \in C_c^\infty\left(\Sigma^D_{[-8\tau,8\tau]}\cap\tilde{\Omega}_{\gamma_v}\right) \quad \text{where} \\ \beta\left(\tilde{\theta}\right) \coloneqq & \sqrt{\left|\tilde{\theta}\right|^2-\varphi_\rho\left(\theta_0\right)\left|\tilde{\theta}'\right|^2} \\ = & \sqrt{\left|\tilde{\theta}_0\right|^2+\left(1-\varphi_\rho\right)\left|\tilde{\theta}'\right|^2} \quad \text{satisfying} \\ \frac{\left|\tilde{\theta}\right|}{2} \leq & \beta\left(\tilde{\theta}\right)\leq\left|\tilde{\theta}\right| \end{aligned}$$

as before in terms of the relevant coordinates on $\tilde{U}_{\varepsilon,\tau}$. With $\mathtt{v}_u$ now defined in a similar fashion to (3.30), one may again estimate $\left|H_{g_u,\mathtt{v}_u}(d)\right| = O\left(\frac{1}{T}\right)$ as in (3.31)-(3.36) using (5.7) and (5.9) to complete the proof. □

Next; we have a lemma reducing the trace asymptotics to $S^1\times\mathbb{R}^{2m}$. First choose T sufficiently large such that spt $(\theta)\subset[-T,T]$. Then choose τ sufficiently small and an open sub-cover $\Gamma\subset\Omega\subset\mathit{\Omega}$ with $T^\tau_{(\Omega,\mathit{\Omega})}>T$. Finally and as observed before, by choosing τ even smaller if necessary, one may also find an (Ω,τ,δ) partition to arrange $T_{(\mathcal{P};\mathcal{V},\mathcal{W})}>T$; reducing us to study of the asymptotics of $\mathcal{T}^\theta_{B_v,B_v}(D)$. We now show that (5.1) allows a further reduction to $S^1\times\mathbb{R}^{2m}$. Below, the operator D_0 is as in (4.4).

Proposition 5.2. *For each $1\leq v\leq M$, one has*

$$\mathcal{T}^\theta_{B_v,B_v}(D) = \underbrace{tr\left[B^0_v f\left(\frac{D_0}{\sqrt{h}}\right)\check{\theta}\left(\frac{\lambda\sqrt{h}-D_0}{h}\right)B^0_v\right]}_{\coloneqq\mathcal{T}^\theta_{B^0_v,B^0_v}(D_0)} \quad \text{mod } h^\infty$$

for cutoffs $B^0_v\in\Psi^0_\delta\left(S^1\times\mathbb{R}^{2m}\right)$, with $WF\left(B^0_v\right)\Subset\Sigma^{D_0}_{[-\tau_\delta,\tau_\delta]}\cap\tilde{\Omega}^\delta_{\gamma_v}$.

Proof. The proof is again similar to [26] Prop. 4.1, provided the smallness of spt (θ) is quantified. First one has an analog of Lemma 3.5: for $D'\in\Psi^1_{\mathrm{cl}}(X;S)$ essentially self-adjoint, with $D=D'$ microlocally on $\Sigma^D_{[-8\tau,8\tau]}\cap\tilde{\mathit{\Omega}}_{\gamma_v}$, and $\theta\in C_c^\infty\left(\left(T'h^\epsilon,T_v\right);[0,1]\right)$ one has

$$\mathcal{T}^\theta_{B_v,B_v}(D)=\mathcal{T}^\theta_{B_v,B_v}(D') \quad \text{mod } h^\infty$$

(since B_v has microsupport in $\Sigma^{D_1}_{[-\tau_\delta,\tau_\delta]}\cap\tilde{\Omega}^\delta_{\gamma_v}$ and hence on $\Sigma^{D_1}_{[-\tau,\tau]}\cap\tilde{\Omega}_{\gamma_v}$). Now as $D=D_0$ on $\mathit{\Omega}_{\gamma_v}$ by construction (4.4) and hence microlocally on $\Sigma^D_{[-8\tau,8\tau]}\cap\tilde{\mathit{\Omega}}_{\gamma_v}$; the proof in [26] is seen to carry through provided spt(θ) is contained in each of $\left\{\left(T'h^\epsilon,T_v\right)\right\}$, $1\leq v\leq M$. But this is guaranteed by our choice of an appropriate subcover $\Gamma\subset\Omega\subset\mathit{\Omega}$ satisfying (5.1)and spt $(\theta)\subset[-T,T]$. □

Next, we show how the Birkhoff normal form maybe used to perform a further reduction on the trace. First note that we may similarly use (2.9) to define a self-adjoint Clifford-Weyl quantization map

$$c_0^W\coloneqq\mathrm{Op}\otimes c_0 : S^0_{\mathrm{cl}}\left(T^*S^1\times\mathbb{R}^{4m};\mathbb{C}\right)\otimes\Lambda^{\mathrm{odd/even}}W\to\Psi^0_{\mathrm{cl}}\left(S^1\times\mathbb{R}^{2m};\mathbb{C}^{2^m}\right)$$

which maps real valued symbols $S_{\mathrm{cl}}^{0}\left(T^{*}S^{1}\times\mathbb{R}^{4m};\mathbb{R}\right)\otimes\Lambda^{\mathrm{odd/even}}W$ to self-adjoint operators in $\Psi_{\mathrm{cl}}^{0}\left(S^{1}\times\mathbb{R}^{2m};\mathbb{C}^{2^{m}}\right)$. Similarly we define a space of real-valued, twisted $\tilde{\Delta}^{0}$-harmonic, $\bar{\varphi}$-commuting, x_{0}- independent symbols

$$\mathcal{H}^{k}S_{\mathrm{cl}}^{0}\coloneqq\left\{ \omega\in S_{\mathrm{cl}}^{0}\left(T^{*}S^{1}\times\mathbb{R}^{4m};\mathbb{R}\right)\otimes\Lambda^{k}W|\,\tilde{\Delta}^{0}\omega=0,\,H_{\bar{\varphi}}\omega=0,\,\partial_{x_{0}}\omega=0\right\} .$$

Next, an application of Borel's lemma by virtue of (4.5), (4.16) and (4.20) gives the existence of

$$\begin{aligned}\bar{a} & \sim\sum_{j=0}^{\infty}h^{j}\bar{a}_{j}\in S_{\mathrm{cl}}^{0}\left(T^{*}S^{1}\times\mathbb{R}^{4m};\mathbb{R}\right)\otimes\Lambda^{\mathrm{odd}}W\\
\bar{r} & \sim\sum_{j=0}^{\infty}h^{j}\bar{r}_{j}\in S_{\mathrm{cl}}^{0}\left(T^{*}S^{1}\times\mathbb{R}^{4m};\mathbb{R}\right)\otimes\Lambda^{\mathrm{odd}}W\\
\bar{f} & \sim\sum_{j=0}^{\infty}h^{j}\bar{f}_{j}\in S_{\mathrm{cl}}^{0}\left(T^{*}S^{1}\times\mathbb{R}^{4m};\mathbb{R}\right)\\
\bar{\omega} & \sim\sum_{j=0}^{\infty}h^{j}\bar{\omega}_{j}\in\mathcal{H}^{\mathrm{odd}}S_{\mathrm{cl}}^{0}\end{aligned}$$

such that

$$e^{ic_{0}^{W}(\bar{a})}e^{\frac{i}{h}\bar{f}^{W}}d_{0}^{W}e^{-\frac{i}{h}\bar{f}^{W}}e^{-ic_{0}^{W}(\bar{a})}=\underbrace{H_{1}^{W}+c_{0}^{W}\left(\bar{\omega}\right)}_{\coloneqq\bar{D}}+c_{0}^{W}\left(\bar{r}\right) \tag{5.11}$$

on $S^{1}\times\mathbb{R}^{2m}$. Here $\left\{ \bar{r}_{j}\right\} _{j\in\mathbb{N}_{0}},\bar{f}_{0}$, $\bar{\omega}_{0}$ vanish to infinite, second and second order respectively along

$$\varGamma=\left\{ \xi_{0}+\bar{\varphi}=x'=\xi'=x''=\xi''=0\right\} .$$

Moreover $\bar{f}_{0}$, $\bar{\omega}_{0}$ vanish to first order along

$$\varGamma'=\left\{ \xi_{0}+\bar{\varphi}=x'=\xi'\right\} .$$

We may hence choose $\bar{\omega}_{0}$ having an expansion

$$\bar{\omega}_{0}=\left(\xi_{0}+\bar{\varphi}\right)\omega_{00}+\sum_{j=1}^{m}\left(\omega_{0j}z_{j}+\bar{\omega}_{0j}\bar{z}_{j}\right) \tag{5.12}$$

in terms of the complex coordinates $z_{j}=x'_{j}+iy'_{j}$ with

$$\left\Vert \bar{\omega}_{0j}\right\Vert _{C^{0}}\leq\varepsilon$$

arbitrarily small.

Next we show that one may pass from the trace asymptotics of D_{0} to $\bar{D}$(4.4). Below we set $\bar{B}_{v}=e^{ic_{0}^{W}(\bar{a})}e^{\frac{i}{h}\bar{f}^{W}}B_{v}^{0}e^{-\frac{i}{h}\bar{f}^{W}}e^{-ic_{0}^{W}(\bar{a})}$. Note that $\bar{B}_{v}=1$ on an h^{δ} size neighborhood of $\varGamma$ by construction.

Proposition 5.3. *For each $1\leq v\leq M$, we have*

$$\mathcal{T}_{B_{v}^{0},B_{v}^{0}}^{\theta}\left(D_{0}\right)=\mathcal{T}_{\bar{B}_{v},\bar{B}_{v}}^{\theta}\left(\bar{D}\right)\quad mod\ h^{\infty}.$$

Proof. By choosing an appropriately small Ω in terms of Reeb Darboux coordinates as in (5.10), we may find a cutoff of the form $A=\chi\left(\frac{\bar{D}^2+\left(x''^2+\xi''^2\right)^W}{h^{2\delta}}\right)$, $\chi\in C_c^\infty(\mathbb{R})$, that is microlocally 1 on $WF\left(\bar{B}_v\right)$. We then have by the Helffer-Sjöstrand formula (5.13)

$$\mathcal{T}^\theta_{B_v^0,B_v^0}(D_0)-\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\bar{D}\right)=\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}(z)\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\operatorname{tr}\left[\bar{B}_v\varDelta_z A\bar{B}_v\right]dzd\bar{z}\mod h^\infty,$$

with

$$\varDelta_z=\left(\frac{1}{\sqrt{h}}\left(\bar{D}+c_0^W(\bar{r})\right)-z\right)^{-1}c_0^W(\bar{r})\left(\frac{1}{\sqrt{h}}\left(\bar{D}\right)-z\right)^{-1}.$$

Since $\bar{r}$ vanishes to infinite order along $\varGamma$, symbolic calculus gives

$$c_0^W(\bar{r})=R_N\left[\bar{D}^N+\left(\bar{\varphi}^W\right)^N\right]\quad\forall N,$$

for some $R_N\in\Psi^0_{\mathrm{cl}}\left(S^1\times\mathbb{R}^{2m};\mathbb{C}^{2^m}\right)$. From which the commutation $\left[\bar{D},\bar{\varphi}^W\right]=0$ gives

$$\varDelta_z=\left(\frac{1}{\sqrt{h}}\left(\bar{D}+c_0^W(\bar{r})\right)-z\right)^{-1}S_N\left(\frac{1}{\sqrt{h}}\left(\bar{D}\right)-z\right)^{-1}\left[\bar{D}^2+\left(x''^2+\xi''^2\right)^W\right]^N\quad\forall N,$$

for some $S_N\in\Psi^0_{\mathrm{cl}}\left(S^1\times\mathbb{R}^{2m};\mathbb{C}^{2^m}\right)$. Now

$$\varDelta_z A=\left(\frac{1}{\sqrt{h}}\left(\bar{D}+c_0^W(\bar{r})\right)-z\right)^{-1}S_N\left(\frac{1}{\sqrt{h}}\left(\bar{D}\right)-z\right)^{-1}h^{2\delta N}\chi_N\left(\frac{\bar{D}^2+\left(x''^2+\xi''^2\right)^W}{h^{2\delta}}\right)\quad\forall N,$$

for $\chi_N(x)=x^N\chi(x)\in C_c^\infty(\mathbb{R})$. Plugging this last equation into (5.13) gives the result. □

6. Trace Asymptotics

In this section we finish the proof of Lemma 3.1 and hence Theorem 1.1. By the reductions 5.2 and 5.3 of the last section it suffices to consider the trace $\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\bar{D}\right)$.

Proof of Lemma 3.1. We begin with the orthogonal Landau decomposition (2.39)

(6.1)

$$L^2\left(S^1\times\mathbb{R}^{2m};\mathbb{C}^{2^m}\right)=L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)\otimes\underbrace{\left(\mathbb{C}\left[\psi_{0,0}\right]\oplus\bigoplus_{\varLambda\in\mu.\left(\mathbb{N}_0^m\setminus 0\right)}\left[E^{\mathrm{even}}_\varLambda\oplus E^{\mathrm{odd}}_\varLambda\right]\right)}_{=L^2\left(\mathbb{R}^m_{x'};\mathbb{C}^{2^m}\right)}\text{ where}$$

$$E^{\mathrm{even}}_\varLambda:=\bigoplus_{\substack{\tau\in\mathbb{N}_0^m\setminus 0\\ \varLambda=\mu.\tau}}E^{\mathrm{even}}_\tau$$

$$E^{\mathrm{odd}}_\varLambda:=\bigoplus_{\substack{\tau\in\mathbb{N}_0^m\setminus 0\\ \varLambda=\mu.\tau}}E^{\mathrm{odd}}_\tau$$

according to the eigenspaces of the squared magnetic Dirac operator $D^2_{\mathbb{R}^m}$ (2.33) on $\mathbb{R}^m$. It is clear from (4.18) that

$$H_1^W=\frac{1}{L_\gamma}\left(\xi_0+\bar{\varphi}\right)^W\sigma_0+D_{\mathbb{R}^m}$$

in terms of the above decomposition. Furthermore one has the commutation relations

$$\begin{aligned}\left[\sigma_0, D_{\mathbb{R}^m}^2\right] &= 0\\ \left[c_0^W\left(\bar{\omega}\right), D_{\mathbb{R}^m}^2\right] &= ihc_0^W\left(\tilde{\Delta}^0\bar{\omega}\right)=0\end{aligned}$$

since $\bar{\omega}$ in (5.11) is $\tilde{\Delta}^0$-harmonic. Hence $\bar{D}$ preserves the decomposition (6.1) and we may consider the restriction of its traces to the eigenspaces of $D_{\mathbb{R}^m}^2$. Namely, let $E_0 \coloneqq \mathbb{C}\left[\psi_{0,0}\right],\ E_\Lambda \coloneqq E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}},\ E_{>0}\coloneqq \bigoplus_{\Lambda\in\mu.\left(\mathbb{N}_0^m\setminus 0\right)} E_\Lambda$ and $\mathtt{P}_0,\ \mathtt{P}_\Lambda,\ \mathtt{P}_{>0}\coloneqq \bigoplus_{\Lambda\in\mu.\left(\mathbb{N}_0^m\setminus 0\right)}\mathtt{P}_\Lambda$ denote the summands and the corresponding projections of (6.1). It is then clear that $\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\bar{D}\right)=\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_0\bar{D}\mathtt{P}_0\right)+\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_{>0}\bar{D}\mathtt{P}_{>0}\right)$.

Set

$$\begin{aligned}\bar{D}_0\coloneqq \mathtt{P}_0\bar{D}\mathtt{P}_0 &: L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)\to L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)\\ \bar{D}_\Lambda\coloneqq \mathtt{P}_\Lambda\bar{D}\mathtt{P}_\Lambda &: L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''};E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}\right)\to L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''};E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}\right),\ \Lambda>0.\end{aligned}$$

The restrictions of the $c_0^W\left(\bar{\omega}\right)$ term in $\bar{D}$ are

$$\begin{aligned}\varOmega_0\coloneqq \mathtt{P}_0 c_0^W\left(\bar{\omega}\right)\mathtt{P}_0 &: L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)\to L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)\\ \varOmega_\Lambda\coloneqq \mathtt{P}_\Lambda c_0^W\left(\bar{\omega}\right)\mathtt{P}_\Lambda &: L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''};E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}\right)\to L^2\left(S^1_{x_0}\times\mathbb{R}^m_{x''};E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}\right),\ \Lambda>0.\end{aligned}$$

The operator $\varOmega_0=\alpha_0^W\in\Psi^0_{\text{cl}}\left(S^1_{x_0}\times\mathbb{R}^m_{x''}\right)$ is pseudo-differential operator with symbol vanishing to second order along $\varGamma''=\left\{\xi_0+\bar{\varphi}=x''=\xi''=0\right\}$. Also, quantizing the expansion (5.12) gives

$$c_0^W\left(\bar{\omega}\right)=\left(\xi_0+\bar{\varphi}\right)^W\underbrace{c_0^W\left(\omega_{00}\right)}_{=O_{L^2\to L^2}(\varepsilon)}+\sum_{j=1}^m\left[\underbrace{c_0^W\left(\omega_{0j}\right)}_{=O_{L^2\to L^2}(\varepsilon)}A_j+A_j^*\underbrace{c_0^W\left(\bar{\omega}_{0j}\right)}_{=O_{L^2\to L^2}(\varepsilon)}\right]+O\left(h\right)$$

Knowing the action of the lowering and raising operators A_j, A_j^* on each eigenstate (2.36) of $D^2_{\mathbb{R}^m}$ then gives the estimate

$$\varOmega_\Lambda=\left(\xi_0+\bar{\varphi}\right)^W O_{L^2\to L^2}\left(\varepsilon\right)+O_{L^2\to L^2}\left(\varepsilon\sqrt{\Lambda h}\right)+O_{L^2\to L^2}\left(h\right) \tag{6.2}$$

with all constants above being uniform in $\Lambda>0$.

Next, we consider $\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_{>0}\bar{D}\mathtt{P}_{>0}\right)$ by computing the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}-z\right)$, $z\in\mathbb{C}$, to each E_Λ, $\Lambda>0$, eigenspace in (6.1). Using (2.37) this has the form

$$\begin{aligned}\bar{D}_\Lambda\left(z\right) &\coloneqq \mathtt{P}_\Lambda\left(\frac{1}{\sqrt{h}}\bar{D}-z\right)\mathtt{P}_\Lambda\\ &= \frac{1}{\sqrt{h}}\begin{bmatrix}-\left(\xi_0+\bar{\varphi}\right)-z\sqrt{h} & \left(\sqrt{2\Lambda h}\right)^W\\ \left(\sqrt{2\Lambda h}\right)^W & \xi_0+\bar{\varphi}-z\sqrt{h}\end{bmatrix}+\frac{1}{\sqrt{h}}\varOmega_\Lambda\end{aligned}$$

with respect to the $\mathbb{Z}_2$- grading $E_\Lambda=E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}$. Here we leave the identification $\mathtt{i}_\tau$ in (2.37) between the odd and even parts as being understood. Let $\varepsilon_0>0$ be such that $f \in C_c^\infty\left(-\sqrt{2\mu_1}+\varepsilon_0, \sqrt{2\mu_1}-\varepsilon_0\right)$. Set $R_\Lambda(z) = [r_\Lambda(z)]^W$

$$r_\Lambda(z) := \frac{\sqrt{h}\begin{bmatrix} -(\xi_0+\bar{\varphi}) - z\sqrt{h} & \left(\sqrt{2\Lambda h}\right) \\ \left(\sqrt{2\Lambda h}\right) & \xi_0+\bar{\varphi}-z\sqrt{h}\end{bmatrix}}{z^2 h - (\xi_0+\bar{\varphi})^2 - 2\Lambda h}$$

which is well defined for $|\mathrm{Re} z| \leq \sqrt{2\mu_1}-\varepsilon_0 < \inf_{\mathbb{R}^n}\sqrt{2\Lambda}$, and h sufficiently small. We now compute

$$\begin{aligned} \left\|R_\Lambda(z)\bar{D}_\Lambda(z) - I\right\| &\leq C\varepsilon + O(h) \\ \left\|\bar{D}_\Lambda(z)R_\Lambda(z) - I\right\| &\leq C\varepsilon + O(h) \end{aligned}$$

using (6.2) with the constants above being uniform in Λ. Choosing ε sufficiently small in (6.2) shows that the inverse $\bar{D}_\Lambda(z)^{-1}$ exists and is $O(R_\Lambda(z)) = O\left(h^{-\frac{1}{2}}\right)$ uniformly. Thus the resolvent $\left(\mathtt{P}_{>0}\bar{D}\mathtt{P}_{>0}-z\right)^{-1}$ extends holomorphically to the strip $|\mathrm{Re} z| \leq \sqrt{2\mu_1}-\varepsilon_0$ and picking an almost analytic continuation for f in the Helffer-Sjöstrand formula (3.37) supported in this strip gives $\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_{>0}\bar{D}\mathtt{P}_{>0}\right) = 0$.

We now consider $\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_0\bar{D}\mathtt{P}_0\right)$. The cutoffs maybe taken to be of the form $\bar{B}_v = \chi\left(\frac{\left(x''^2+\xi''^2\right)^W}{h^{2\delta}}\right)\chi\left(\frac{\mathtt{H}_2+\left((\xi_0+\bar{\varphi})^W\right)^2}{h^{2\delta}}\right)$, with $\mathtt{H}_2$ being the harmonic oscillator (2.35), to compute

$$\mathcal{T}^\theta_{\bar{B}_v,\bar{B}_v}\left(\mathtt{P}_0\bar{D}\mathtt{P}_0\right) = \frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}(z)\,\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\mathrm{tr}\,\left[\bar{B}_v^0\left(\frac{1}{\sqrt{h}}\bar{D}_0 - z\right)^{-1}\bar{B}_v^0\right]dzd\bar{z} \tag{6.3}$$

where $\bar{B}_v^0 = \chi\left(\frac{\left(x''^2+\xi''^2\right)^W}{h^{2\delta}}\right)\chi\left(\frac{\left((\xi_0+\bar{\varphi})^W\right)^2}{h^{2\delta}}\right)$ and

$$\bar{D}_0 = -\frac{1}{L_\gamma}(\xi_0+\bar{\varphi})^W + \alpha_0^W$$

being the effective Hamiltonian. The above being a scalar operator, (6.3) now reduces to the usual trace formula microlocalized near the Hamiltonian trajectory $\Gamma'' = \{\xi_0+\bar{\varphi} = x'' = \xi'' = 0\}$ of $\frac{1}{L_\gamma}(\xi_0+\bar{\varphi})$. The formula (3.1) now follows on identifying the period, symplectic action and return map of this trajectory to be L_γ, T_γ and P_γ^+ respectively (cf. [10, 11] Ch 7. for an identification of the Maslov index in terms of the metaplectic group). $\square$

7. Local trace expansion: computation of the second coefficient

In this section we study the trace expansion of a function of the operator $\frac{D}{\sqrt{h}}$. We first recall the following which appears as Proposition 7.1 of [26].

Proposition 7.1. *There exist tempered distributions $u_j \in \mathcal{S}'(\mathbb{R}_s)$, $j = 0,1,2,\ldots$, such that one has a trace expansion*

$$tr\,\phi\left(\frac{D}{\sqrt{h}}\right) = h^{-n/2}\left(\sum_{j=0}^N u_j(\phi)\,h^{j/2}\right) + h^{(N+1-n)/2}O\left(\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^N\hat{\phi}^{(k)}\right\|_{L^1}\right) \tag{7.1}$$

for each $N \in \mathbb{N}$, $\phi \in \mathcal{S}(\mathbb{R}_s)$.

The coefficient u_0 in (7.1) was computed in Proposition 7.4 of [26]. Our main task here is the computation of the next coefficient u_1. The calculation here is similar to that of the second coefficient of the symplectic Bergman kernel (cf. [22] Ch. 8) using the local index theory method.

To this end we first briefly recall the construction of the distributions u_j. Fixing the point $p \in X$ there is an orthonormal basis $e_{0,p} = \frac{R}{|R|}, \{e_{j,p}, e_{j+m,p}\}_{j=1}^m \in R^\perp$, of the tangent space at p consisting of eigenvectors of $\mathfrak{J}_p$ with eigenvalues $0, \pm i\mu_j$, $j = 1, \ldots, m$, such that

$$da\,(p) = \sum_{j=1}^m \mu_j e_{j,p}^* \wedge e_{j+m,p}^*. \tag{7.2}$$

Using the parallel transport from this basis fix a geodesic coordinate system $(x_0, \ldots, x_{2m})$ on an open neighborhood of $p \in \Omega$. Let $e_j = w_j^k \partial_{x_k}$, $0 \le j \le 2m$, be the local orthonormal frame of TX obtained by parallel transport of $e_{j,p} = \partial_{x_j}\big|_p$, $0 \le j \le 2m$, along geodesics. Hence we again have $w_j^k g_{kl} w_r^l = \delta_{jr}$; $w_j^k\big|_p = \delta_j^k$ with g_{kl} being the components of the metric in these coordinates. Choose an orthonormal basis $\{s_{j,p}\}_{j=1}^{2^m}$ for S_p in which Clifford multiplication

$$c\,(e_j)\big|_p = \gamma_j \tag{7.3}$$

is standard. Choose an orthonormal basis $\mathtt{l}_p$ for L_p. Parallel transport the bases $\{s_{j,p}\}_{j=1}^{2^m}$, $\mathtt{l}_p$ along geodesics using the spin connection ∇^S and unitary family of connections $\nabla^h = A_0 + \frac{i}{h} a$ to obtain trivializations $\{s_j\}_{j=1}^{2^m}$, $\mathtt{l}$ of S, L on Ω. Since Clifford multiplication is parallel, the relation (7.3) now holds on Ω. The connection $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ can be expressed in this frame and these coordinates as

$$\nabla^{S\otimes L} = d + A_j^h dx^j + \Gamma_j dx^j, \tag{7.4}$$

where each A_j^h is a Christoffel symbol of ∇^h and each Γ_j is a Christoffel symbol of the spin connection ∇^S. Since the section $\mathtt{l}$ is obtained via parallel transport along geodesics, the connection coefficient A_j^h maybe written in terms of the curvature $F_{jk}^h dx^j \wedge dx^k$ of ∇^h via

$$A_j^h(x) = \int_0^1 d\rho \left(\rho x^k F_{jk}^h(\rho x)\right). \tag{7.5}$$

The dependence of the curvature coefficients F_{jk}^h on the parameter h is seen to be linear in $\frac{1}{h}$ via

$$F_{jk}^h = F_{jk}^0 + \frac{i}{h}(da)_{jk} \tag{7.6}$$

despite the fact that they are expressed in the h dependent frame $\mathtt{l}$. This is because a gauge transformation from an h independent frame into $\mathtt{l}$ changes the curvature coefficient by conjugation. Since L is a line bundle this is conjugation by a function and hence does not change the coefficient. Furthermore, the coefficients in the Taylor expansion of (7.6) at 0 maybe expressed in terms of the covariant derivatives $\left(\nabla^{A_0}\right)^l F_{jk}^0$, $\left(\nabla^{A_0}\right)^l (da)_{jk}$ evaluated at p. Next, using the Taylor expansion

$$(da)_{jk} = (da)_{jk}(0) + x^l a_{jkl}, \tag{7.7}$$

we see that the connection $\nabla^{S\otimes L}$ has the form

$$\nabla^{S\otimes L}=d+\left[\frac{i}{h}\left(\frac{x^k}{2}\left(da\right)_{jk}(0)+x^kx^lA_{jkl}\right)+x^kA^0_{jk}+\Gamma_j\right]dx^j \tag{7.8}$$

where

$$\begin{aligned}A^0_{jk} &= \int_0^1 d\rho\left(\rho F^0_{jk}\left(\rho x\right)\right)\\ A_{jkl} &= \int_0^1 d\rho\left(\rho a_{jkl}\left(\rho x\right)\right)\end{aligned}$$

and Γ_j are all independent of h. Finally from (7.3) and (7.8) may write down the expression for the Dirac operator (1.1) also given as $D=hc\circ\left(\nabla^{S\otimes L}\right)$ in terms of the chosen frame and coordinates to be

(7.9)

$$D=\gamma^rw^j_r\left[h\partial_{x_j}+i\frac{x^k}{2}\left(da\right)_{jk}(0)+ix^kx^lA_{jkl}+h\left(x^kA^0_{jk}+\Gamma_j\right)\right]$$

(7.10)

$$\begin{aligned}&=\gamma^r\left[w^j_rh\partial_{x_j}+iw^j_r\frac{x^k}{2}\left(da\right)_{jk}(0)+\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]+\\ &\gamma^r\left[iw^j_rx^kx^lA_{jkl}+hw^j_r\left(x^kA^0_{jk}+\Gamma_j\right)-\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]\in\Psi^1_{\mathrm{cl}}\left(\Omega^0_s;\mathbb{C}^{2^m}\right)\end{aligned}$$

In the second expression above both square brackets are self-adjoint with respect to the Riemannian density $e^1\wedge\ldots\wedge e^n=\sqrt{g}dx:=\sqrt{g}dx^1\wedge\ldots\wedge dx^n$ with $g=\det\left(g_{ij}\right)$. Again one may obtain an expression self-adjoint with respect to the Euclidean density dx in the framing $g^{\frac{1}{4}}u_j\otimes\mathtt{1},1\leq j\leq 2^m$, with the result being an addition of the term $h\gamma^jw^k_jg^{-\frac{1}{4}}\left(\partial_{x_k}g^{\frac{1}{4}}\right)$.

Let i_g be the injectivity radius of g^{TX}. Define the cutoff $\chi\in C^\infty_c(-1,1)$ such that $\chi=1$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$. We now modify the functions w^k_j, outside the ball $B_{i_g/2}(p)$, such that $w^k_j=\delta^k_j$ (and hence $g_{jk}=\delta_{jk}$) are standard outside the ball $B_{i_g}(p)$ of radius i_g centered at p. This again gives

$$\begin{aligned}\mathbb{D}=\gamma^r&\left[w^j_rh\partial_{x_j}+iw^j_r\frac{x^k}{2}\left(da\right)_{jk}(0)+\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]+\\ &\chi\left(\left|x\right|/i_g\right)\gamma^r\left[iw^j_rx^kx^lA_{jkl}+hw^j_r\left(x^kA^0_{jk}+\Gamma_j\right)-\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]\\ &\qquad\in\Psi^1_{\mathrm{cl}}\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)\end{aligned} \tag{7.11}$$

as a well defined operator on $\mathbb{R}^n$ formally self adjoint with respect to $\sqrt{g}dx$. Again $\mathbb{D}+i$ being elliptic in the class $S^0(m)$ for the order function

$$m=\sqrt{1+g^{jl}\left(\xi_j+\frac{x^k}{2}\left(da\right)_{jk}(0)\right)\left(\xi_l+\frac{x^r}{2}\left(da\right)_{lr}(0)\right)},$$

the operator $\mathbb{D}$ is essentially self adjoint. Also as observed in [26] Section 7

$$\mathrm{tr}\ \phi\left(\frac{D}{\sqrt{h}}\right)(p,\cdot)=\mathrm{tr}\ \phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(0,\cdot) \tag{7.12}$$

mod h^∞.

We now introduce the rescaling operator $\mathscr{R}:C^{\infty}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)\rightarrow C^{\infty}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)$; $\left(\mathscr{R}s\right)(x)\coloneqq s\left(\frac{x}{\sqrt{h}}\right)$. Conjugation by $\mathscr{R}$ amounts to the rescaling of coordinates $x\rightarrow x\sqrt{h}$. A Taylor expansion in (7.11) now gives the existence of classical (h-independent) self-adjoint, first-order differential operators $\mathtt{D}_{j}=a_{j}^{k}\left(x\right)\partial_{x_{k}}+b_{j}\left(x\right)$, $j=0,1\ldots$, with polynomial coefficients (of degree at most $j+1$) as well as h-dependent self-adjoint, first-order differential operators $\mathrm{E}_{N+1}=\sum_{|\alpha|=N+1}x^{\alpha}\left[c_{\alpha}^{k}\left(x;h\right)\partial_{x_{k}}+d_{\alpha}\left(x;h\right)\right]$, $N\in\mathbb{N}$, with uniformly C^{∞} bounded coefficients $c_{j,\alpha}^{k}$, $d_{j,\alpha}$ such that

$$\mathscr{R}\mathbb{D}\mathscr{R}^{-1} = \sqrt{h}\mathtt{D}\quad\text{with} \tag{7.13}$$

$$\mathtt{D} = \left(\sum_{j=0}^{N}h^{j/2}\mathtt{D}_{j}\right)+h^{(N+1)/2}\mathrm{E}_{N+1},\ \forall N. \tag{7.14}$$

The coefficients of the polynomials $a_{j}^{k}\left(x\right),\,b_{j}\left(x\right)$ again involve the covariant derivatives of the curvatures $F^{TX},F^{A_{0}}$ and da evaluated at p. It is now clear from (7.13) that

$$\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)\left(x,x'\right)=h^{-n/2}\phi\left(\mathtt{D}\right)\left(\frac{x}{\sqrt{h}},\frac{x'}{\sqrt{h}}\right). \tag{7.15}$$

Next, let $I_{j}=\left\{k=\left(k_{0},k_{1},\ldots\right)|k_{\alpha}\in\mathbb{N},\ \sum k_{\alpha}=j\right\}$ denote the set of partitions of the integer j and set

$$\mathtt{C}_{j}^{z}=\sum_{k\in I_{j}}\left(z-\mathtt{D}_{0}\right)^{-1}\left[\Pi_{\alpha}\mathtt{D}_{k_{\alpha}}\left(z-\mathtt{D}_{0}\right)^{-1}\right]. \tag{7.16}$$

The coefficient u_{j} in the expansion (7.1) is now the total integral over X of a smooth family of distributions $u_{j,p}\in C^{\infty}\left(X;\mathcal{S}'\left(\mathbb{R}_{s}\right)\right)$ parametrized by X

$$\begin{aligned}
u_{j} &= \int_{X}u_{j,p},\qquad\text{where}\\
u_{j,p} &= \mathrm{tr}\,U_{j,p}\ \text{ and}\\
U_{j,p}\left(\phi\right) &= -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{\phi}\left(z\right)\mathtt{C}_{j}^{z}\left(0,0\right)dzd\bar{z}\in\mathrm{End}S_{p}^{TX}.
\end{aligned}$$

It was further shown in [26] that each $u_{j,p}$ is point-wise given by a linear combination of the following elementary distributions

$$v_{a}\left(s\right)\coloneqq s^{a},\ a\in\mathbb{N}_{0} \tag{7.17}$$

$$v_{a,b,c,\varLambda}\left(s\right)\coloneqq\partial_{s}^{a}\left[\left|s\right|s^{b}\left(s^{2}-2\varLambda\right)^{c-\frac{1}{2}}H\left(s^{2}-2\varLambda\right)\right], \tag{7.18}$$
$$\left(a,b,c;\varLambda\right)\in\mathbb{N}_{0}\times\mathbb{Z}\times\mathbb{N}_{0}\times\mu.\left(\mathbb{N}_{0}^{m}\setminus0\right).$$

To now state the computation of u_{1}; first define $\mathscr{P}_{j}^{\pm}:T_{p}X\rightarrow\ker\left(\pm i\mu_{j}-\mathfrak{J}\right)$, $1\leq j\leq m$, the projections onto the eigenspaces of $\mathfrak{J}$ with eigenvalue $\pm i\mu_{j}$ respectively in (1.7). Also set $\frac{d_{j}}{2}=d_{j}^{+}=d_{j}^{-}=\dim\ker\left(\pm i\mu_{j}-\mathfrak{J}\right)$ and $\mathscr{P}_{j}\coloneqq\mathscr{P}_{j}^{+}+\mathscr{P}_{j}^{-}$. Next, define the endomorphism

$$\begin{aligned}
\left(\nabla^{TX}\mathfrak{J}\right)^{0}:T_{p}X &\rightarrow T_{p}X\\
\left(\nabla^{TX}\mathfrak{J}\right)^{0}v &\coloneqq \left(\nabla_{v}^{TX}\mathfrak{J}\right)R,\quad v\in T_{p}X,
\end{aligned}$$

agreeing with (1.14) on $R^{\perp}$, and set $\left(\nabla^{TX}\mathfrak{J}\right)_{j}\coloneqq\mathscr{P}_{j}\left(\nabla^{TX}\mathfrak{J}\right)^{0}\mathscr{P}_{j}$, $1\leq j\leq m$.

We then have the following.

Proposition 7.2. *The second coefficient u_1 of (7.1) is given by*

$$u_{1,p}(s) = c_{1;1}v_1 + \sum_{\Lambda\in\mu.(\mathbb{N}_0^m\setminus 0)} c_{1;1,-2,0,\Lambda}(p)\, v_{1,-2,0,\Lambda}(s)$$

$$+ \sum_{\Lambda\in\mu.(\mathbb{N}_0^m\setminus 0)} c_{1;0,-3,0,\Lambda}(p)\, v_{0,-3,0,\Lambda}(s), \quad \textit{where} \tag{7.19}$$

$$c_{1;1} = -\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^{m+1}}\left[tr\, \mathfrak{J}^{-2}\left(\nabla^{TX}\mathfrak{J}\right)^0\right] \quad \textit{and} \tag{7.20}$$

(7.21)

$$c_{1;1,-2,0,\Lambda}(p) = c_{1;0,-3,0,\Lambda}(p) = \begin{cases} -\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^{m+1}}\tau\left[\frac{1}{d_j} tr\ \left(\nabla^{TX}\mathfrak{J}\right)_j\right]; & \textit{if } \Lambda = \mu_j\tau \textit{ for some } j,\\ 0; & \textit{otherwise.}\end{cases}$$

Proof. We begin by noting the first two terms in (7.14)

$$\mathtt{D}_0 = \gamma^j\left[\partial_{x_j} + i\frac{x^k}{2}(da)_{jk}(0)\right] \tag{7.22}$$

$$= \gamma^0\partial_{x_0} + \underbrace{\gamma^j\left[\partial_{x_j} + \frac{i\mu_j(p)}{2}x_{j+m}\right] + \gamma^{j+m}\left[\partial_{x_{j+m}} - \frac{i\mu_j(p)}{2}x_j\right]}_{:=\mathtt{D}_{00}} \tag{7.23}$$

$$\mathtt{D}_1 = \frac{i}{3}\gamma^j x^k x^l \underbrace{\left(\nabla_{e_l} da\right)_{jk}(0)}_{=:\mathtt{A}_{jkl}} \tag{7.24}$$

$$= \frac{i}{3}\gamma^j x^k x^l \underbrace{g^{TX}\left(e_j, \left(\nabla_{e_l}\mathfrak{J}\right)e_k\right)}_{=:\mathtt{A}_{jkl}} \tag{7.25}$$

using (7.2), (7.7). For future reference we also note that

$$\mathtt{D}_0^2 = -\partial_{x_0}^2 + \underbrace{\sum_{j=1}^m\left[-\partial_{x_j}^2 - \partial_{x_{j+m}}^2 + i\mu_j\left(x_{j+m}\partial_{x_j} - x_j\partial_{x_{j+m}}\right) + \frac{1}{4}\left(x_j^2 + x_{j+m}^2\right)\right] - i\mathtt{F}_m}_{=:\mathtt{D}_{00}^2}$$

$$\mathtt{F}_m = \mu_j\left[\sum_{j=1}^m \gamma^j\gamma^{j+m}\right]$$

gives the complex harmonic oscillator.

As in the computation for u_0 in [26], we compute u_1 by computing the expansions of the heat traces tr $e^{-t\mathtt{D}^2}$, tr $\mathtt{D}e^{-t\mathtt{D}^2}$. First note that following (7.13), (7.14) we may compute

$$\mathtt{D}^2 = \mathtt{D}_0^2 + \sqrt{h}\left\{\mathtt{D}_0, \mathtt{D}_1\right\} + O(h).$$

An application of Duhamel's principle then yields

$$e^{-t\mathtt{D}^2} = e^{-t\mathtt{D}_0^2} - \sqrt{h}\left(\underbrace{\int_0^t e^{-(t-s)\mathtt{D}_0^2}\left\{\mathtt{D}_0,\mathtt{D}_1\right\}e^{-s\mathtt{D}_0^2}ds}_{=:\mathtt{U}_{10}}\right) + O(h). \tag{7.26}$$

We compute

$$\{\mathtt{D}_0,\mathtt{D}_1\} = \frac{i}{3}\mathtt{A}_{jkl}\left\{-2x^kx^l\partial_{x_j}+\gamma^k\gamma^jx^l+\gamma^l\gamma^jx^k-2\left(ia_j\right)x^kx^l\right\}. \tag{7.27}$$

Next set $\mu_{j+m}=\mu_j$, $1\leq j\leq m$, and note Mehler's formula

$$\begin{aligned} e^{-t\mathtt{D}_0^2}\left(x,y\right) &= e^{t\partial_{x_0}^2}e^{-t\mathtt{D}_{00}^2} \\ &= \frac{e^{-\frac{(x_0-y_0)^2}{4t}}}{\sqrt{4\pi t}}\left(\prod_{j=1}^m\frac{\mu_j}{4\pi\sinh\mu_jt}\right)m_t\left(x',y'\right)e^{it\mathtt{F}_m}, \end{aligned} \tag{7.28}$$

$$\begin{aligned} m_t\left(x',y'\right) &= \exp\left\{-\frac{\mu_j}{4\tanh\mu_jt}\left(\left(x_j-y_j\right)^2+\left(x_{j+m}-y_{j+m}\right)^2\right)\right. \\ &\qquad\left.+\frac{\mu_j}{2}\tanh\left(\frac{\mu_jt}{2}\right)\left(x_jy_j+x_{j+m}y_{j+m}\right)\right\} \\ &= \exp\left\{-\frac{\mu_j}{4\tanh\mu_jt}\left({x_j}^2+x_{j+m}^2+y_j^2+y_{j+m}^2\right)+\frac{\mu_j}{2\sinh\mu_jt}\left(x_jy_j+x_{j+m}y_{j+m}\right)\right\}, \end{aligned} \tag{7.29}$$

where $(x';y')=(x_1,\ldots,x_{2m};y_1,\ldots,y_{2m})$. We may now substitute (7.27) and(7.28) into (7.26). This gives a formula for $\mathtt{U}_{10}\left(0,0\right)$ as an integral over s and x. Furthermore one observes that the x -integral is an odd integral which must evaluate to 0. Hence we have

$$u_1\left(e^{-ts^2}\right)=-\mathrm{tr}\ \mathtt{U}_{10}\left(0,0\right)=0. \tag{7.30}$$

We now compute the second term in $\mathrm{tr}\ \mathtt{D}e^{-t\mathtt{D}^2}$. First differentiate (7.26) using (7.14) to obtain

$$\begin{aligned} \mathtt{D}e^{-t\mathtt{D}^2} &= \mathtt{D}_0e^{-t\mathtt{D}_0^2} \\ &\quad -\sqrt{h}\left(\mathtt{D}_0\int_0^te^{-(t-s)\mathtt{D}_0^2}\left\{\mathtt{D}_0,\mathtt{D}_1\right\}e^{-s\mathtt{D}_0^2}ds-\mathtt{D}_1e^{-t\mathtt{D}_0^2}\right)+O\left(h\right). \end{aligned}$$

The $O\left(\sqrt{h}\right)$ term above maybe rewritten symmetrically

$$\begin{aligned} &\mathtt{U}_{11} \\ &:= \mathtt{D}_0\int_0^te^{-(t-s)\mathtt{D}_0^2}\left\{\mathtt{D}_0,\mathtt{D}_1\right\}e^{-s\mathtt{D}_0^2}ds-\mathtt{D}_1e^{-t\mathtt{D}_0^2} \\ &= \underbrace{\int_0^t\left(\mathtt{D}_0e^{-(t-s)\mathtt{D}_0^2}\right)\mathtt{D}_1\left(\mathtt{D}_0e^{-s\mathtt{D}_0^2}\right)ds}_{=:\mathtt{K}_1} \\ &\quad+\underbrace{\frac{1}{2}\int_0^te^{-(t-s)\mathtt{D}_0^2}\left\{\mathtt{D}_0^2,\mathtt{D}_1\right\}e^{-s\mathtt{D}_0^2}ds}_{=:\mathtt{K}_2} \\ &\quad-\frac{1}{2}\left(\mathtt{D}_1e^{-t\mathtt{D}_0^2}+e^{-t\mathtt{D}_0^2}\mathtt{D}_1\right) \end{aligned} \tag{7.31, 7.32}$$

using an integration by parts argument. It is clear from (7.24) that

$$\mathtt{D}_1 e^{-t\mathtt{D}_0^2}(0,0) = 0$$

with the same being true of its adjoint

$$e^{-t\mathtt{D}_0^2}\mathtt{D}_1(0,0) = 0.$$

Similarly the adjointness property for

$$\int_0^t e^{-(t-s)\mathtt{D}_0^2}\left(\mathtt{D}_0^2\mathtt{D}_1\right)e^{-s\mathtt{D}_0^2}ds \qquad \text{and}$$
$$\int_0^t e^{-(t-s)\mathtt{D}_0^2}\left(\mathtt{D}_1\mathtt{D}_0^2\right)e^{-s\mathtt{D}_0^2}ds$$

gives

$$\mathtt{K}_2(0,0) = \left[\int_0^t e^{-(t-s)\mathtt{D}_0^2}\left(\mathtt{D}_1\mathtt{D}_0^2\right)e^{-s\mathtt{D}_0^2}ds\right](0,0).$$

We now compute

$$\begin{aligned}
&\mathtt{K}_1\\
=\;& \int_0^t ds\left(\mathtt{D}_0 e^{-(t-s)\mathtt{D}_0^2}\right)\mathtt{D}_1\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)\\
=\;& \int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left[\gamma^\mu\left(\partial_{x_\mu}+ia_\mu\right)\right]\left(\frac{i}{3}\gamma^j x^k x^l \mathtt{A}_{jkl}\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)\\
=\;& \int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3}\gamma^k\gamma^j x^l \mathtt{A}_{jkl}\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)\\
&+\int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3}\gamma^l\gamma^j x^k \mathtt{A}_{jkl}\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)\\
&-\int_0^t ds\, 2e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3} x^k x^l \mathtt{A}_{jkl}\right)\left(\partial_{x_j}+ia_j\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)\\
&-\underbrace{\int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3}\gamma^j x^k x^l \mathtt{A}_{jkl}\right)\left[\gamma^\mu\left(\partial_{x_\mu}+ia_\mu\right)\right]\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)}_{=\mathtt{K}_2}.
\end{aligned}$$

Hence we now simplify (7.32) to

$$\begin{aligned}
&\mathtt{U}_{11}\\
=\;& \underbrace{\int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3}\gamma^k\gamma^j x^l \mathtt{A}_{jkl}\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)}_{=:\mathtt{L}_1}\\
&+\underbrace{\int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3}\gamma^l\gamma^j x^k \mathtt{A}_{jkl}\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)}_{=:\mathtt{L}_2}\\
&-\underbrace{2\int_0^t ds\, e^{-(t-s)\mathtt{D}_0^2}\left(\frac{i}{3} x^k x^l \mathtt{A}_{jkl}\right)\left(\partial_{x_j}+ia_j\right)\left(\mathtt{D}_0 e^{-s\mathtt{D}_0^2}\right)}_{=:\mathtt{L}_3}
\end{aligned} \tag{7.33}$$

We now evaluate traces of each of the kernels $\mathtt{L}_1, \mathtt{L}_2$ and $\mathtt{L}_3$.

First compute

$$\mathtt{D}_0 e^{-t\mathtt{D}_0^2}(x,0) = -\left[\frac{\gamma^0 x_0}{2t} + \sum_{\mu=1}^{m} \frac{\mu_\mu}{2\tanh\mu_\mu t}\left(\gamma^\mu x_\mu + \gamma^{\mu+m} x_{\mu+m}\right)\right]\frac{e^{-\frac{x_0^2}{4t}}}{\sqrt{4\pi t}} m_t(x',0)\, e^{it\mathtt{F}_m}$$

$$+\left[\sum_{\mu=1}^{m} \frac{i\mu_\mu}{2}\left(\gamma^\mu x_{\mu+m} - \gamma^{\mu+m} x_\mu\right)\right]\frac{e^{-\frac{x_0^2}{4t}}}{\sqrt{4\pi t}} m_t(x',0)\, e^{it\mathtt{F}_m}. \tag{7.34}$$

and set

$$\begin{aligned}
\tilde{m}_t(x,y) &:= \frac{e^{-\frac{(x_0-y_0)^2}{4t}}}{\sqrt{4\pi t}} m_t(x',y') \\
E(x';s,t) &:= m_{t-s}(0,x')\, m_s(x',0) \\
\tilde{E}(x;s,t) &:= \tilde{m}_{t-s}(0,x)\, \tilde{m}_s(x,0) \\
\frac{1}{\rho_\mu(t)} &:= \begin{cases} \frac{1}{2t}; & \mu = 0, \\ \frac{\mu_\mu}{2\tanh\mu_\mu t}; & 1 \le \mu \le 2m. \end{cases}
\end{aligned}$$

Plugging (7.28) and (7.34) into (7.33) gives

$$\operatorname{tr} \mathtt{L}_1(0,0) = \mathtt{A}_{jkl}\left\{ \underbrace{-\int_0^t ds \int dx E(x;s,t)\frac{x_\mu x_l}{\rho_\mu(s)} \operatorname{tr}\left[\frac{i}{3}\gamma^k\gamma^j\gamma^\mu e^{it\mathtt{F}_m}\right]}_{=:\mathtt{I}_{10}^{jkl}} \right.$$

$$\left. + \underbrace{\int_0^t ds \int dx E(x;s,t)(ia_\mu x_l) \operatorname{tr}\left[\frac{i}{3}\gamma^k\gamma^j\gamma^\mu e^{it\mathtt{F}_m}\right]}_{=:\mathtt{I}_{11}^{jkl}} \right\} \tag{7.35}$$

$$\operatorname{tr} \mathtt{L}_2(0,0) = \mathtt{A}_{jkl}\left\{ \underbrace{-\int_0^t ds \int dx E(x;s,t)\frac{x_\mu x_k}{\rho_\mu(s)} \operatorname{tr}\left[\frac{i}{3}\gamma^l\gamma^j\gamma^\mu e^{it\mathtt{F}_m}\right]}_{=:\mathtt{I}_{20}^{jkl}} \right.$$

$$\left. + \underbrace{\int_0^t ds \int dx E(x;s,t)(ia_\mu x_k) \operatorname{tr}\left[\frac{i}{3}\gamma^l\gamma^j\gamma^\mu e^{it\mathtt{F}_m}\right]}_{=:\mathtt{I}_{21}^{jkl}} \right\}. \tag{7.36}$$

Since the function $E(x;s,t)$ is an even function in x, we must have $\mu = l$ for the x integral in $\mathtt{L}_{10}^{jkl}$ to be non-zero. Similarly, we must have $\mu, l > 0$ with $|\mu - l| = m$ for the x integral in $\mathtt{I}_{11}^{jkl}$ to be non-zero. We now note that for indices $p < q < r$;

$$\begin{aligned}
\operatorname{tr}\left[i\gamma^p e^{it\mathtt{F}_m}\right] &= \begin{cases} 2^m\left(\Pi_{j=1}^m \sinh\mu_j t\right); & p = 0 \\ 0 & \text{otherwise.} \end{cases} \\
\operatorname{tr}\left[i\gamma^p\gamma^q\gamma^r e^{it\mathtt{F}_m}\right] &= \begin{cases} -i2^m\frac{\left(\Pi_{j=1}^m \sinh\mu_j t\right)}{\tanh(\mu_q t)}; & p = 0 \text{ and } r - q = m \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This now implies that the coefficient of $\mathtt{A}_{jkl}$ in $\mathtt{L}_1$ is zero unless exactly one of the indices(i,j,k) is zero; and the other two are either equal or differ by m. A similar analysis also give that the coefficient of $\mathtt{A}_{jkl}$ in $\mathtt{L}_2$, $\mathtt{L}_3$ is zero unless exactly one of (i,j,k) is zero. Furthermore

$$\begin{aligned}\operatorname{tr} \mathtt{L}_3(0,0) &= 2\mathtt{A}_{jkl}\left\{\underbrace{\int_0^t ds\int dx E(x;s,t)\,x_k x_l\left(\frac{\delta^{0j}}{2s}-\frac{x_0x_j}{2s\rho_j(s)}\right)\operatorname{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]}_{=:\mathtt{l}_{30}^{jkl}}\right.\\ &\left.\quad+\underbrace{\int_0^t ds\int dx E(x;s,t)\left(\frac{ia_jx_0x_kx_l}{2s}\right)\operatorname{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]}_{=:\mathtt{l}_{31}^{jkl}}\right\}.\end{aligned} \tag{7.37}$$

For future reference we define $\mathtt{l}_1^{jkl}=\mathtt{l}_{10}^{jkl}+\mathtt{l}_{11}^{jkl}$, $\mathtt{l}_2^{jkl}=\mathtt{l}_{20}^{jkl}+\mathtt{l}_{21}^{jkl}$, $\mathtt{l}_3^{jkl}=\mathtt{l}_{30}^{jkl}+\mathtt{l}_{31}^{jkl}$ and $\mathtt{u}_{11}^{jkl}\coloneqq\mathtt{l}_1^{jkl}+\mathtt{l}_2^{jkl}+\mathtt{l}_3^{jkl}$.

We may now make the three cases.

Case (i) $j=0$

Again as observed before we must have either $k=l$ or $|k-l|=m$. If $1\le k=l\le m$, we compute

$$\begin{aligned}&\mathtt{l}_1^{0kk}+\mathtt{l}_2^{0kk}\\ &\coloneqq \mathtt{l}_{10}^{0kk}+\mathtt{l}_{11}^{0kk}+\mathtt{l}_{20}^{0kk}+\mathtt{l}_{21}^{0kk}\\ &=-\int_0^t ds\int dx\tilde{E}(x;s,t)\frac{\mu_k x_k^2}{\tanh(\mu_k s)}\operatorname{tr}\left[\frac{i}{3}\gamma^k\gamma^0\gamma^k e^{it\mathtt{F}_m}\right]\\ &-\int_0^t ds\int dx\tilde{E}(x;s,t)\left(i\mu_k x_k^2\right)\operatorname{tr}\left[\frac{i}{3}\gamma^k\gamma^0\gamma^{k+m}e^{it\mathtt{F}_m}\right]\\ &=\frac{1}{3}2^m\left(\Pi_{j=1}^m\sinh\mu_j t\right)\int_0^t ds\left[\frac{\mu_k}{\tanh(\mu_k t)}-\frac{\mu_k}{\tanh(\mu_k s)}\right]\left(\int dx\, x_k^2\tilde{E}(x;s,t)\right)\end{aligned} \tag{7.38}$$

$$=\frac{1}{3}\frac{1}{\sqrt{4\pi t}}2^m\left(\Pi_{j=1}^m\sinh\mu_j t\right)\int_0^t ds\left[\frac{\mu_k}{\tanh(\mu_k t)}-\frac{\mu_k}{\tanh(\mu_k s)}\right]\left(\int dx'\, x_k^2 E(x';s,t)\right) \tag{7.39}$$

$$=\frac{1}{3}\frac{2\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{\sqrt{4\pi t}}\int_0^t ds\left[\frac{\mu_k}{\tanh(\mu_k t)}-\frac{\mu_k}{\tanh(\mu_k s)}\right]\left[\frac{\sinh\mu_k s\,\sinh\mu_k(t-s)}{\mu_k\sinh\mu_k t}\right] \tag{7.40}$$

$$=\frac{1}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{\mu_k}\frac{1}{\sqrt{4\pi t}}\frac{1}{\sinh\mu_k t}\left[\frac{(\mu_k t)\cosh(\mu_k t)-\sinh(\mu_k t)}{\tanh(\mu_k t)}-(\mu_k t)\sinh(\mu_k t)\right]$$

and

$$\begin{aligned}
&\mathtt{I}_3^{0kk}\\
&= \frac{1}{t}\frac{1}{\sqrt{4\pi t}}\int E\left(x';s,t\right)x_k^2\mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]\\
&= \frac{1}{3}\frac{1}{t}\frac{1}{\sqrt{4\pi t}}2^m\left(\Pi_{j=1}^m\sinh\mu_j t\right)\int ds\int dx' E\left(x';s,t\right)x_k^2 \qquad (7.41)\\
&= \frac{1}{3}\frac{2\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{t}\frac{1}{\sqrt{4\pi t}}\int_0^t ds\left[\frac{\sinh\mu_k s\,\sinh\mu_k\left(t-s\right)}{\mu_k\sinh\mu_k t}\right] \qquad (7.42)\\
&= \frac{1}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{\mu_k}\frac{1}{\sqrt{4\pi t}}\frac{1}{\left(\mu_k t\right)\sinh\mu_k t}\left[\left(\mu_k t\right)\cosh\left(\mu_k t\right)-\sinh\left(\mu_k t\right)\right]. \qquad (7.43)
\end{aligned}$$

Here we have used one of the integrals

$$\begin{aligned}
\int dx'\,E\left(x';s,t\right) &= \prod_{j=1}^m\frac{\mu_j}{4\pi\sinh\mu_j t}\\
\int dx'\,x_k^2E\left(x';s,t\right) &= 2\left(\prod_{j=1}^m\frac{\mu_j}{4\pi\sinh\mu_j t}\right)\left[\frac{\sinh\mu_k s\,\sinh\mu_k\left(t-s\right)}{\mu_k\sinh\mu_k t}\right]\\
\int dx'\,x_k^2x_l^2E\left(x';s,t\right) &= 4\left(\prod_{j=1}^m\frac{\mu_j}{4\pi\sinh\mu_j t}\right)\left[\frac{\sinh\mu_k s\,\sinh\mu_k\left(t-s\right)}{\mu_k\sinh\mu_k t}\right]\left[\frac{\sinh\mu_l s\,\sinh\mu_l\left(t-s\right)}{\mu_l\sinh\mu_l t}\right]\\
\int dx'\,x_k^4E\left(x';s,t\right) &= 12\left(\prod_{j=1}^m\frac{\mu_j}{4\pi\sinh\mu_j t}\right)\left[\frac{\sinh\mu_k s\,\sinh\mu_k\left(t-s\right)}{\mu_k\sinh\mu_k t}\right]^2
\end{aligned}$$

in (7.38), (7.41) and one of

$$\begin{aligned}
\int_0^t ds\,\sinh\mu_k s\,\sinh\mu_k\left(t-s\right) &= \frac{1}{2\mu_k}\left[\left(\mu_k t\right)\cosh\left(\mu_k t\right)-\sinh\left(\mu_k t\right)\right]\\
\int_0^t ds\,\cosh\mu_k s\,\sinh\mu_k\left(t-s\right) &= \frac{1}{2\mu_k}\left(\mu_k t\right)\sinh\left(\mu_k t\right)\\
\int_0^t ds\,s\,\sinh\mu_k s\,\sinh\mu_k\left(t-s\right) &= \frac{1}{\left(2\mu_k\right)^2}\left(\mu_k t\right)\left[\left(\mu_k t\right)\cosh\left(\mu_k t\right)-\sinh\left(\mu_k t\right)\right]\\
\int_0^t ds\,s\,\sinh\mu_k s\,\cosh\mu_k\left(t-s\right) &= \frac{1}{\left(2\mu_k\right)^2}\left[\left(\mu_k t\right)\cosh\left(\mu_k t\right)-\sinh\left(\mu_k t\right)+\left(\mu_k t\right)^2\sinh\left(\mu_k t\right)\right]
\end{aligned}$$

in (7.39), (7.42). The sum of (7.40) and (7.43) now gives

$$\begin{aligned}
\mathtt{u}_{11}^{0kk}\left(0,0\right) &:= \mathtt{I}_1^{0kk}+\mathtt{I}_2^{0kk}+\mathtt{I}_3^{0kk}\\
&= \frac{1}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{\mu_k}\frac{1}{\sqrt{4\pi t}}\left[\frac{\mu_k t}{\left(\sinh\mu_k t\right)^2}-\frac{1}{\mu_k t}\right]. \qquad (7.44)
\end{aligned}$$

A similar computation yields the same answer for $\mathtt{u}_{11}^{0kk}\left(0,0\right)$ if $k>m$.

We now consider the possibility$l = k + m$ and compute

$$
\begin{aligned}
&\mathtt{l}_1^{0k(k+m)} + \mathtt{l}_2^{0k(k+m)} \\
&= \mathtt{l}_{10}^{0kk} + \mathtt{l}_{11}^{0kk} + \mathtt{l}_{20}^{0kk} + \mathtt{l}_{21}^{0kk} \\
&= -\int_0^t ds \int dx E\left(x; s, t\right) \frac{x_{k+m}^2}{\rho_{k+m}\left(s\right)} \mathrm{tr}\left[\frac{i}{3}\gamma^k \gamma^0 \gamma^{k+m} e^{it\mathtt{F}_m}\right] \\
&+ \int_0^t ds \int dx E\left(x; s, t\right) \left(i\mu_k x_{k+m}^2\right) \mathrm{tr}\left[\frac{i}{3}\gamma^k \gamma^j \gamma^k e^{it\mathtt{F}_m}\right] \\
&- \int_0^t ds \int dx E\left(x; s, t\right) \frac{x_k^2}{\rho_k\left(s\right)} \mathrm{tr}\left[\frac{i}{3}\gamma^{k+m} \gamma^j \gamma^k e^{it\mathtt{F}_m}\right] \\
&+ \int_0^t ds \int dx E\left(x; s, t\right) \left(-i\mu_k x_k^2\right) \mathrm{tr}\left[\frac{i}{3}\gamma^{k+m} \gamma^j \gamma^{k+m} e^{it\mathtt{F}_m}\right] \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathtt{l}_3^{0k(k+m)} \\
=\;& \frac{1}{t}\frac{1}{\sqrt{4\pi t}} \int E\left(x'; s, t\right) x_k x_{k+m} \mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right] \\
=\;& 0.
\end{aligned}
$$

Hence

$$
\mathtt{u}_{11}^{0k(k+m)}\left(0,0\right) = 0. \tag{7.45}
$$

A similar computation in the case $k = l + m$ shows $\mathtt{u}_{11}^{0(k+m)k}\left(0,0\right) = 0$.

Case (ii) $k = 0$

Again as observed before we must have either $j = l$ or $|j - l| = m$. If $1 \leq j = l \leq m$, we compute

$$
\begin{aligned}
&\mathtt{I}_1^{j0j}+\mathtt{I}_2^{j0j}\\
&\coloneqq \mathtt{I}_{10}^{j0j}+\mathtt{I}_{11}^{j0j}+\mathtt{I}_{20}^{j0j}+\mathtt{I}_{21}^{j0j}\\
&=\int_0^t ds\int dx\tilde{E}\left(x;s,t\right)\left(\frac{\mu_j}{2\tanh\left(\mu_j s\right)}\right)x_j^2\mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]\\
&-\int_0^t ds\int dx\tilde{E}\left(x;s,t\right)\left(i\frac{\mu_j}{2}x_j^2\right)\mathrm{tr}\left[\frac{i}{3}\gamma^0\gamma^j\gamma^{j+m}e^{it\mathtt{F}_m}\right]\\
&+\int_0^t ds\int dx\tilde{E}\left(x;s,t\right)\frac{x_0^2}{2s}\mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]\\
&=\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\int_0^t ds\left(\frac{\mu_j}{\tanh\left(\mu_j s\right)}\right)\left[\frac{\sinh\mu_j s\,\sinh\mu_j\left(t-s\right)}{\mu_j\sinh\mu_j t}\right]\\
&-\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\int_0^t ds\frac{\mu_j}{\tanh\left(\mu_j t\right)}\left[\frac{\sinh\mu_j s\,\sinh\mu_j\left(t-s\right)}{\mu_j\sinh\mu_j t}\right]\\
&+\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\int_0^t ds\left(\frac{t-s}{t}\right)\\
&=\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{t}{2}\\
&-\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{2\mu_j}\frac{1}{\tanh\left(\mu_j t\right)}\frac{1}{\sinh\mu_j t}\left[\left(\mu_j t\right)\cosh\mu_j t-\sinh\mu_j t\right]\\
&+\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{t}{2}
\end{aligned}
\tag{7.46}
$$

and

$$
\begin{aligned}
&\mathtt{I}_3^{j0j}\\
&=-\int_0^t ds\int dx\tilde{E}\left(x;s,t\right)\frac{1}{2s}\frac{\mu_j}{\tanh\mu_j s}x_0^2x_j^2\mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right]\\
&=-\frac{1}{3}\frac{2}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\int_0^t ds\left(\frac{t-s}{t}\right)\left[\frac{\cosh\mu_j s\,\sinh\mu_j\left(t-s\right)}{\sinh\mu_j t}\right]\\
&=-\frac{1}{3}\frac{1}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{\left(\mu_j t\right)\sinh\mu_j t}\frac{1}{2\mu_j}\left[\left(\mu_j t\right)\cosh\mu_j t-\sinh\mu_j t+\mu_j^2t^2\sinh\mu_j t\right]
\end{aligned}
\tag{7.47}
$$

The sum of (7.46) and (7.47) now gives

$$
\mathtt{u}_{11}^{j0j}\left(0,0\right)=\frac{1}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{\left(2\pi\right)^m}\frac{1}{2\mu_j}\frac{1}{\sqrt{4\pi t}}\left[\frac{1}{\mu_j t}-\frac{\mu_j t}{\left(\sinh\mu_j t\right)^2}\right]
\tag{7.48}
$$

If $l = j + m$, we compute

$$\begin{aligned}
&\mathtt{1}_{1}^{j0(j+m)} + \mathtt{1}_{2}^{j0(j+m)} \\
&:= \mathtt{1}_{10}^{j0(j+m)} + \mathtt{1}_{11}^{j0(j+m)} + \mathtt{1}_{20}^{j0(j+m)} + \mathtt{1}_{21}^{j0(j+m)} \\
&= -\int_0^t ds \int dx \tilde{E}(x;s,t) \left(\frac{\mu_j}{2\tanh(\mu_j s)}\right) x_{j+m}^2 \mathrm{tr}\left[\frac{i}{3}\gamma^0\gamma^j\gamma^{j+m}e^{it\mathtt{F}_m}\right] \\
&- \int_0^t ds \int dx \tilde{E}(x;s,t) \left(i\frac{\mu_j}{2}x_{j+m}^2\right) \mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right] \\
&- \int_0^t ds \int dx \tilde{E}(x;s,t) \frac{x_0^2}{2s} \mathrm{tr}\left[\frac{i}{3}\gamma^{j+m}\gamma^j\gamma^0 e^{it\mathtt{F}_m}\right] \\
&= \frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\int_0^t ds \frac{\mu_j}{\tanh(\mu_j s)}\frac{1}{\tanh(\mu_j t)}\left[\frac{\sinh\mu_j s\,\sinh\mu_j(t-s)}{\mu_j\sinh\mu_j t}\right] \\
&- \frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\int_0^t ds \left[\frac{\sinh\mu_j s\,\sinh\mu_j(t-s)}{\mu_j\sinh\mu_j t}\right] \\
&- \frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\int_0^t ds \left(\frac{t-s}{t}\right)\frac{1}{\tanh(\mu_j t)} \\
&= -\frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j}\frac{\mu_j t}{\tanh(\mu_j t)} \\
&- \frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j^2\sinh\mu_j t}\left[(\mu_j t)\cosh\mu_j t - \sinh\mu_j t\right] \\
&\frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\frac{t}{2\tanh(\mu_j t)} \\
&= -\frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j}\left[(\mu_j t)\cosh\mu_j t - \sinh\mu_j t\right] \qquad (7.49)
\end{aligned}$$

and

$$\begin{aligned}
&\mathtt{1}_{3}^{j0(j+m)} \\
&= \mathtt{1}_{31}^{j0(j+m)} \\
&= 2\int_0^t ds \int dx E(x;s,t)\left(\frac{i\mu_j x_0^2 x_{j+m}^2}{4s}\right)\mathrm{tr}\left[\frac{i}{3}\gamma^0 e^{it\mathtt{F}_m}\right] \\
&= \frac{1}{3}\frac{2i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\int_0^t ds\left(\frac{t-s}{t}\right)\left[\frac{\sinh\mu_j s\,\sinh\mu_j(t-s)}{\mu_j\sinh\mu_j t}\right] \\
&= \frac{1}{3}\frac{i}{\sqrt{4\pi t}}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j^2\sinh\mu_j t}\left[(\mu_j t)\cosh\mu_j t - \sinh\mu_j t\right] \qquad (7.50)
\end{aligned}$$

The sum of (7.49) and (7.50) gives

$$\mathtt{u}_{11}^{j0(j+m)}(0,0) = 0. \qquad (7.51)$$

A similar computation also yields $\mathtt{u}_{11}^{(j+m)0j}(0,0) = 0$.

Case (iii) $l = 0$

Again as observed before we must have either $j = k$ or $|j - k| = m$. The tensor

$\mathtt{A}_{jkl}$ in (7.24) being anti-symmetric in j,k; we have $\mathtt{u}_{11}^{jj0}(0,0)=0$. On the other hand, the expressions for $\mathtt{l}_1^{jkl}+\mathtt{l}_2^{jkl}$ and $\mathtt{l}_3^{jkl}$ are symmetric in k,l. Hence we find

$$\mathtt{u}_{11}^{j(j+m)0}(0,0) = \mathtt{u}_{11}^{j0(j+m)}(0,0)=0 \tag{7.52}$$

$$\mathtt{u}_{11}^{(j+m)j0}(0,0) = \mathtt{u}_{11}^{(j+m)0j}(0,0)=0 \tag{7.53}$$

as in the previous case.

To sum up, from (7.32), (7.33), (7.35), (7.36), (7.37), (7.44), (7.45), (7.48), (7.51), (7.52) and (7.53) we have finally have

$$\begin{aligned} u_1\left(se^{-ts^2}\right) &= -\mathrm{tr}\ \mathtt{U}_{11}(0,0)\\ &= -\mathtt{A}_{jkl}\mathtt{u}_{11}^{jkl}\\ &= -\frac{\mathtt{A}_{0kk}}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{\mu_k}\frac{1}{\sqrt{4\pi t}}\left[\frac{\mu_k t}{(\sinh\mu_k t)^2}-\frac{1}{\mu_k t}\right]\\ &\quad -\frac{\mathtt{A}_{j0j}}{3}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j}\frac{1}{\sqrt{4\pi t}}\left[\frac{1}{\mu_j t}-\frac{\mu_j t}{(\sinh\mu_j t)^2}\right]\\ &= -\mathtt{A}_{j0j}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j}\frac{1}{\sqrt{4\pi t}}\left[\frac{1}{\mu_j t}-\frac{\mu_j t}{(\sinh\mu_j t)^2}\right]. \end{aligned} \tag{7.54}$$

A simple computation using Laplace transforms now shows

$$-\mathtt{A}_{j0j}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j^2 t}\frac{1}{\sqrt{4\pi t}} = -\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^{m+1}}\cdot\underbrace{\frac{\mathtt{A}_{j0j}}{\mu_j^2}}_{=\mathrm{tr}\ \mathfrak{J}^{-2}(\nabla^{TX}\mathfrak{J})^0}.v_1\left(se^{-ts^2}\right) \tag{7.55}$$

$$\begin{aligned} \mathtt{A}_{j0j}\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\frac{1}{2\mu_j}\frac{1}{\sqrt{4\pi t}}\frac{\mu_j t}{(\sinh\mu_j t)^2} &= \frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^m}\mathtt{A}_{j0j}\frac{\sqrt{t}}{\sqrt{\pi}}\left[\sum_{\tau=1}^{\infty}\tau e^{-2\tau\mu_j t}\right]\\ &= -\frac{\left(\Pi_{j=1}^m\mu_j\right)}{(2\pi)^{m+1}}\underbrace{\mathtt{A}_{j0j}}_{=\frac{1}{d_j}\mathrm{tr}\ (\nabla^{TX}\mathfrak{J})_j}\left[\sum_{\tau=1}^{\infty}\tau\left(v_{1,-2,0,\Lambda}+v_{0,-3,0,\Lambda}\right)\left(se^{-ts^2}\right)\right] \end{aligned} \tag{7.56}$$

where $2\Lambda=2\tau\mu_j=2\tau\mu_j$ in the last equation (7.56) above.

Thus, (7.30), (7.54), (7.55) and (7.56) show that the two sides of (7.19) evaluate equally on test functions e^{-ts^2}, se^{-ts^2}. Differentiating k times and setting $t=1$; they evaluate equally on test functions $s^{2k}e^{-s^2}$, $s^{2k+1}e^{-s^2}$ for each k. The density of this set of functions in Schwartz space $\mathcal{S}(\mathbb{R})$ now gives the result. □

We end with a corollary of the above computation useful in the next section.

Corollary 7.3. *The improper integral converges*

$$\int_0^\infty u_1\left(se^{-ts^2}\right)\frac{dt}{\sqrt{\pi t}} = -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\frac{1}{m!}\int_X\left[tr\ \frac{1}{|\mathfrak{J}|}\left(\nabla^{TX}\mathfrak{J}\right)^0\right]a\wedge(da)^m.$$

Proof. This is a calculation from (7.54)

$$\begin{aligned}
\int_0^\infty u_1\left(se^{-ts^2}\right)\frac{dt}{\sqrt{\pi t}} &= -\frac{1}{2}\int_X dx \mathtt{A}_{j0j}\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^{m+1}}\int_0^\infty \frac{1}{\mu_j t}\left[\frac{1}{\mu_j t}-\frac{\mu_j t}{(\sinh \mu_j t)^2}\right]dt \\
&= -\frac{1}{2}\int_X dx\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^{m+1}}\frac{\mathtt{A}_{j0j}}{\mu_j}\int_0^\infty \frac{1}{u}\left[\frac{1}{u}-\frac{u}{(\sinh u)^2}\right]du \\
&= -\frac{1}{2}\int_X dx\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^{m+1}}\frac{\mathtt{A}_{j0j}}{\mu_j}\left[-\frac{1}{u}+\frac{2}{e^{2u}-1}\right]_0^\infty \\
&= \frac{1}{2}\int_X dx\frac{\left(\Pi_{j=1}^m \mu_j\right)}{(2\pi)^{m+1}}\frac{\mathtt{A}_{j0j}}{\mu_j}\left[\lim_{u\to 0}\frac{1+2u-e^{2u}}{u\left(e^{2u}-1\right)}\right] \\
&= -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\int_X \underbrace{\frac{\mathtt{A}_{j0j}}{\mu_j}}_{=\mathrm{tr}\ \frac{1}{|R||\mathfrak{J}|}(\nabla^{TX}\mathfrak{J})^0}\underbrace{\left(\Pi_{j=1}^m \mu_j\right)dx}_{=\frac{1}{m!}|R|a\wedge(da)^m}.
\end{aligned}$$

□

8. Semiclassical limit of the eta invariant

In this section we prove the semiclassical limit formula for the eta invariant of Theorem 1.2. First, from [26] Cor. 7.3, the distributions $u_j \in \mathcal{S}'(\mathbb{R})$ of (7.1) are smooth near 0. Hence

$$u_j^{\pm}(x) := 1_{[0,\infty)}(\pm x)\, u_j(x) \in \mathcal{S}'(\mathbb{R})$$

are well defined tempered distributions and we similarly define $f^{\pm}$ for any $f \in \mathcal{S}(\mathbb{R})$. We now have two term asymptotics for irregular functional traces similar to 7.1.

Lemma 8.1. *For any $f \in \mathcal{S}(\mathbb{R})$,*

$$\text{(8.1)} \qquad tr\ f^{\pm}\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}}u_0^{\pm}(f) + h^{-m}u_1^{\pm}(f) + o\left(h^{-m}\right).$$

Proof. We begin by proving an improved local Weyl law. To this end, choose $\theta \in C_c^\infty(\mathbb{R};[0,1])$ such that $\theta(x) = 1$ near 0 and $\check{\theta}(\xi) \geq \frac{1}{4}$ for $|\xi| \leq 1$ in (1.9). For each $\epsilon > 0$, set $\theta_\epsilon(x) = \theta(\epsilon x)$ and let $N(a,b)$ denote the number of eigenvalues of D_h in the interval (a,b). Choosing $f(x) \geq 0$ with $f(0) = 1$, the trace expansion (1.9) with $\lambda = 0$ now gives

$$\frac{1}{\epsilon h}N(-\epsilon h, \epsilon h)\left(\frac{1}{4}+O\left(\sqrt{\epsilon h}\right)\right) \leq \mathrm{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{\epsilon h}\check{\theta}\left(\frac{-D}{\epsilon h}\right)\right] = h^{-m-1}\left[u_0(0) + O_\epsilon(h)\right].$$

Hence for $\epsilon > 0$ fixed and $h \ll 1$ depending on ϵ, we have an improved local Weyl law

$$\text{(8.2)} \qquad N(-\epsilon h, \epsilon h) = O\left(\epsilon h^{-m}\right).$$

From here (1.13) follows.

Now, to prove (8.1) first observe that by virtue of 7.1 we may assume $f \in C_c^\infty\left(-\sqrt{2\mu_1}, \sqrt{2\mu_1}\right)$. Next define the spectral measure $\mathfrak{M}_f(\lambda') := \sum_{\lambda \in \mathrm{Spec}\left(\frac{D}{\sqrt{h}}\right)} f(\lambda)\,\delta(\lambda - \lambda')$. It is clear that the expansion (1.9) to its first two terms may be written as

$$\mathfrak{M}_f * \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda) = h^{-m-\frac{1}{2}}\left(f(\lambda)\,u_0(\lambda) + h^{1/2}f(\lambda)\,u_1(\lambda) + O(h)\right)$$

where $\theta_{\frac{1}{2}}(x) = \theta\left(\frac{x}{\sqrt{h}}\right)$. Both sides above involving Schwartz functions in λ, the remainder maybe replaced by $O\left(\frac{h}{\langle\lambda\rangle^2}\right)$. We may then integrate to obtain

$$\int_{-\infty}^{0} d\lambda \int d\lambda' \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda-\lambda')\,\mathfrak{M}_f(\lambda') = h^{-m-\frac{1}{2}}\left(\int_{-\infty}^{0} d\lambda f(\lambda)\,u_0(\lambda) + h^{1/2}\int_{-\infty}^{0} d\lambda f(\lambda)\,u_1(\lambda) + O(h)\right). \tag{8.3}$$

Now note

$$\int_{-\infty}^{0} d\lambda \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda-\lambda') = 1_{(-\infty,0]}(\lambda') + \phi\left(\frac{\lambda'}{\sqrt{h}}\right) \tag{8.4}$$

where $\phi(x) \coloneqq \int_{-\infty}^{0} dt\check{\theta}(t-x) - 1_{(-\infty,0]}(x)$ is a function that is rapidly decaying with all derivatives, odd and smooth on $\mathbb{R}_x \setminus 0$. Next let $\chi \in C_c^\infty(\mathbb{R};[0,1])$ be an even function equal to 1 near 0 and set $\phi_R(x) = \chi\left(\frac{x}{R}\right)\phi(x)$ for each $R > 0$. We now compute

$$\begin{aligned}
&\int d\lambda' \left[\phi\left(\frac{\lambda'}{\sqrt{h}}\right) - \phi_R\left(\frac{\lambda'}{\sqrt{h}}\right)\right]\mathfrak{M}_f(\lambda') \\
=&\int d\lambda' \left[1-\chi\left(\frac{\lambda'}{R\sqrt{h}}\right)\right]\phi\left(\frac{\lambda'}{\sqrt{h}}\right)\mathfrak{M}_f(\lambda') \\
=&O\left(h^{-m}\sum_{k\geq R}\langle k\rangle^{-\infty}\right) \\
=&O\left(\frac{h^{-m}}{R}\right)
\end{aligned} \tag{8.5}$$

from the local Weyl law (8.2).

Next for $\epsilon > 0$, we observe

$$\begin{aligned}
&\left|\phi_R(x) - \phi_R * \check{\theta}_\epsilon(x)\right| \\
=&\left|\int dy\,[\phi_R(x) - \phi_R(x-\epsilon y)]\,\check{\theta}(y)\right| \\
\leq&O_N(1)\left[\left\langle\frac{x}{\epsilon}\right\rangle^{-N} + \epsilon\langle x\rangle^{-N}\right] \quad \forall N \in \mathbb{N}.
\end{aligned} \tag{8.6}$$

Now consider a pairing corresponding to the first term above with $\mathfrak{M}_f(\lambda')$

$$\begin{aligned}
&\int d\lambda' \left\langle\frac{\lambda'}{\epsilon\sqrt{h}}\right\rangle^{-N}\mathfrak{M}_f(\lambda') \\
=&\int d\lambda' 1_{[-R',R']}\left(\frac{\lambda'}{\epsilon\sqrt{h}}\right)\left\langle\frac{\lambda'}{\epsilon\sqrt{h}}\right\rangle^{-N}\mathfrak{M}_f(\lambda') \\
&+\int d\lambda' \left(1-1_{[-R',R']}\right)\left(\frac{\lambda'}{\epsilon\sqrt{h}}\right)\left\langle\frac{\lambda'}{\epsilon\sqrt{h}}\right\rangle^{-N}\mathfrak{M}_f(\lambda').
\end{aligned} \tag{8.7}$$

The support of $1_{[-R',R']}\left(\frac{\lambda'}{\epsilon\sqrt{h}}\right)$ can be covered by $O(R')$ intervals of size $\epsilon\sqrt{h}$, which combined with the local Weyl law gives that the first term above is $O(R'\epsilon h^{-m})$. The second term on the other hand, observing $\left(1-1_{[-R',R']}\right)\left(\frac{\lambda'}{\epsilon\sqrt{h}}\right)\left\langle\frac{\lambda'}{\epsilon\sqrt{h}}\right\rangle^{-N} =$

$O\left(\frac{1}{R'}\left\langle\frac{\lambda'}{\sqrt{h}}\right\rangle^{-N+1}\right)$, is $O\left(\frac{1}{R'}h^{-m}\right)$. On choosing $R' = \frac{1}{\sqrt{\epsilon}}$, this gives (8.7) is $O\left(\sqrt{\epsilon}h^{-m}\right)$. A similar estimate

$$\int d\lambda' \epsilon \left\langle\frac{\lambda'}{\sqrt{h}}\right\rangle^{-N} \mathfrak{M}_f(\lambda') = O\left(\epsilon h^{-m}\right) \tag{8.8}$$

combined with (8.6) gives

$$\int d\lambda' \left[\phi_R\left(\frac{\lambda'}{\sqrt{h}}\right) - \phi_R * \check{\theta}_\epsilon\left(\frac{\lambda'}{\sqrt{h}}\right)\right]\mathfrak{M}_f(\lambda') = O_R\left(\sqrt{\epsilon}h^{-m}\right). \tag{8.9}$$

The second term above has an expansion on integrating (1.9) against ϕ_R

$$\begin{aligned}\int d\lambda' \phi_R * \check{\theta}_\epsilon\left(\frac{\lambda'}{\sqrt{h}}\right)\mathfrak{M}_f(\lambda') &= h^{-m}\left[\int d\lambda \phi_R(\lambda) f(0) u_0(0) + O_{R,\epsilon}(h)\right]\\ &= O_{R,\epsilon}\left(h^{-m+1}\right).\end{aligned} \tag{8.10}$$

Finally putting together (8.3), (8.4), (8.5), (8.9) and (8.10) gives

$$\begin{aligned}\operatorname{tr} f^-\left(\frac{D}{\sqrt{h}}\right) &= \int d\lambda' 1_{(-\infty,0]}(\lambda')\mathfrak{M}_f(\lambda')\\ &= h^{-m-\frac{1}{2}}\left(\int_{-\infty}^0 d\lambda f(\lambda) u_0(\lambda) + h^{1/2}\int_{-\infty}^0 d\lambda f(\lambda) u_1(\lambda)\right)\\ &\quad + O\left(\frac{h^{-m}}{R}\right) + O_R\left(\sqrt{\epsilon}h^{-m}\right) + O_{R,\epsilon}\left(h^{-m+1}\right)\end{aligned}$$

from which (8.1) follows on choosing each of $\frac{1}{R}$, ϵ, h sufficiently small depending on the preceding parameters. $\square$

We now come to the proof of Theorem 1.2.

Proof of Theorem 1.2. We begin by using the invariance of η under positive scaling to write

$$\begin{aligned}\eta_h = \eta\left(\frac{D}{\sqrt{h}}\right) &= \int_0^\infty dt \frac{1}{\sqrt{\pi t}} \operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]\\ &= \int_0^\varepsilon dt \frac{1}{\sqrt{\pi t}} \operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right] + \int_\varepsilon^\infty dt \frac{1}{\sqrt{\pi t}} \operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right].\end{aligned} \tag{8.11}$$

The equation 4.5 pg. 859 of [25] with $r = \frac{1}{h}$ translates to the estimate

$$\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right] = O\left(h^{-m}e^{ct}\right) \tag{8.12}$$

giving that the first integral of (8.11) is $O\left(\sqrt{\varepsilon}h^{-m}\right)$. The second integral is evaluated to be $\operatorname{tr} E_\varepsilon\left(\frac{D}{\sqrt{h}}\right) = \operatorname{tr} \frac{1}{\varepsilon}E\left(\frac{\varepsilon D}{\sqrt{h}}\right)$ where

$$E(x) = \operatorname{sign}(x)\operatorname{erfc}(|x|) = \operatorname{sign}(x)\cdot\frac{2}{\sqrt{\pi}}\int_{|x|}^\infty e^{-s^2}ds$$

with the convention $\operatorname{sign}(0) = 0$. The functions E, E_ε are rapidly decaying with all derivatives, odd and smooth on $\mathbb{R}_x \setminus 0$. Hence (8.1) gives

$$\operatorname{tr} E_\varepsilon\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}}\left[u_0(E_\varepsilon)\right] + h^{-m}\left[u_1(E_\varepsilon)\right] + o\left(h^{-m}\right)$$

where the evaluations above again make sense on account of the smoothness of u_0, u_1 near 0. As observed from [26] Prop. 7.4, the coefficient u_0 is an even function of λ. Since E_ε is odd, the first evaluation above is 0. The second is evaluated from definition to

$$\begin{aligned} u_1\left(E_\varepsilon\right) &= \int_\varepsilon^\infty u_1\left(se^{-ts^2}\right)\frac{dt}{\sqrt{\pi t}} \\ &= -\frac{1}{2}\frac{1}{\left(2\pi\right)^{m+1}}\frac{1}{m!}\int_X\left[\mathrm{tr}\ \frac{1}{\left|\mathfrak{J}\right|}\left(\nabla^{TX}\mathfrak{J}\right)^0\right]a\wedge\left(da\right)^m+O\left(\varepsilon\right) \end{aligned}$$

following the Corollary 7.3. Choosing ε sufficiently small and putting everything together

$$\eta_h = h^{-m}\left(-\frac{1}{2}\frac{1}{\left(2\pi\right)^{m+1}}\frac{1}{m!}\int_X\left[\mathrm{tr}\ \frac{1}{\left|\mathfrak{J}\right|}\left(\nabla^{TX}\mathfrak{J}\right)^0\right]a\wedge\left(da\right)^m\right)+o\left(h^{-m}\right)$$

as required. □

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