KÄHLER-EINSTEIN BERGMAN METRICS ON PSEUDOCONVEX DOMAINS OF DIMENSION TWO

NIKHIL SAVALE & MING XIAO

Abstract. We prove that a two dimensional pseudoconvex domain of finite type with a Kähler-Einstein Bergman metric is biholomorphic to the unit ball. This answers an old question of Yau for such domains. The proof relies on asymptotics of derivatives of the Bergman kernel along critically tangent paths approaching the boundary, where the order of tangency equals the type of the boundary point being approached.

1. Introduction

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. There exist two natural canonical metrics defined in its interior. The first is the Bergman metric [3] defined using the Bergman kernel. The other is the complete Kähler-Einstein metric in D, whose existence was established by the work of Cheng-Yau and Mok-Yau [6, 24]. The importance of the metrics stems from their biholomorphic invariance property and intimate connections with the boundary geometry.

It is hence a natural question to ask when the two canonical metrics coincide; i.e. when the Bergman metric on the domain D is Kähler-Einstein. It was asked, in some form by Yau [29, pg. 679], whether this happens if and only if D is homogeneous. The reverse direction of Yau's question (i.e. if D is homogeneous, then the Bergman metric is Kähler-Einstein) follows from a simple observation using the Bergman invariant function (cf. Fu-Wong [13]). The challenging aspect of Yau's question is the forward direction which is still wide open in its full generality. It should be noted that homogeneous domains have been classified in [27] and the only smoothly bounded homogeneous domain is the ball, as a consequence of Wong [28] and Rosay [26].

A more tractable case of Yau's question is when D has strongly pseudoconvex smooth boundary. An explicit conjecture in this case was posed earlier by Cheng [5]: if the Bergman metric of a smoothly bounded strongly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the unit ball. Cheng's conjecture was confirmed by the combined work of Fu-Wong [13] and Nemirovski-Shafikov [25] in dimension two. In higher dimensions, it was proved more recently by Huang and the second author [19].

2000 *Mathematics Subject Classification.* 32F45, 32Q20, 81Q20 .

N. S. was partially supported by the DFG funded project CRC/TRR 191.

M. X. was partially supported by the NSF grants DMS-1800549 and DMS-2045104.

Since then there has been further work on Cheng's conjecture on Stein manifolds, and more generally on possibly singular Stein spaces, with strongly pseudoconvex boundary. See Huang-Li [18], Ebenfelt, Xu and the second author [8], as well as Ganguly-Sinha [14] for results along this line. Other variations of Cheng's conjecture can also be found in Li [21, 22] and references therein.

The proofs of Cheng's conjecture in [13, 19] fundamentally use Fefferman's asymptotic result [11] for the Bergman kernel, together with its connections to the CR invariant theory for the boundary geometry. In the broader context of pseudoconvex finite type domains, both tools are either absent or insufficiently understood. As a result, little progress was made towards understanding Yau's question in this context. To the best knowledge of the authors, the only known result was due to Fu-Wong [13]. They showed that, on a smoothly bounded, complete Reinhardt, pseudoconvex domain of finite type domain in $\mathbb{C}^2$, if the Bergman metric is Kähler-Einstein, then the domain is biholomorphic to the unit ball. Their proof utilized the non-tangential limit of the Bergman invariant function (see Fu [12]). Besides, their proof used the aid of a computer, again reflecting the intricacy of the problem in the more general finite type case.

Our main theorem below gives an affirmative answer to Yau's question for smoothly bounded pseudoconvex domains of finite type in dimension two.

Theorem 1. *Let $D \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. If the Bergman metric of D is Kähler-Einstein, then D is biholomorphic to the unit ball in $\mathbb{C}^2$.*

A key role is again played by the boundary asymptotics for the Bergman kernel. For two dimensional pseudoconvex domains of finite type, Hsiao and the first author [17] recently described the asymptotics of the Bergman kernel along transversal paths approaching the boundary. For our proof we shall need to extend this asymptotic result to tangential paths approaching a non-strongly pseudoconvex point on the boundary. The paths shall further be chosen to be *critically tangent*; their order of tangency with the boundary equals the type of the point on the boundary that is being approached (see Remark 5 below for a further discussion of this choice).

As a consequence of our main theorem, we also positively answer Yau's question for two dimensional bounded domains with real analytic boundary (such domains are always of finite type).

Corollary 2. *Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with real analytic boundary. If the Bergman metric of D is Kähler-Einstein, then D is biholomorphic to the unit ball in $\mathbb{C}^2$.*

The article is organized as follows. We begin with some preliminaries on the Bergman and Kähler-Einstein metrics in Section 2. In Section 3, we establish the asymptotics for the Bergman kernel and its derivatives along a critically tangent path. The leading term of the asymptotics is computed as well in terms of a model Bergman kernel on the complex plane. Then we carry out the requisite analysis of the model in Section 4. Finally we prove Theorem 1 in Section 5.

2. Preliminaries

In this section we begin with some requisite preliminaries on the Bergman and Kähler-Einstein metrics.

Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain. A boundary defining function is a smooth function $\rho \in C^\infty\left(\bar{D}\right)$ satisfying $D = \{\rho(z) < 0\} \subset \mathbb{C}^2$ and $d\rho|_{\partial D} \neq 0$. The CR and Levi-distributions on the boundary $X \coloneqq \partial D$ are defined via $T^{1,0}X = T^{1,0}\mathbb{C}^2 \cap T_{\mathbb{C}}X$ and $HX \coloneqq \mathrm{Re}\left[T^{1,0}X \oplus T^{0,1}X\right]$ respectively. The Levi form on the boundary is defined by

$$\begin{aligned} &\mathscr{L} \in \left(T^{1,0}X\right)^* \otimes \left(T^{0,1}X\right)^* \\ &\mathscr{L}\left(U, \bar{V}\right) \coloneqq \partial\bar{\partial}\rho\left(U, \bar{V}\right) = -\bar{\partial}\rho\left(\left[U, \bar{V}\right]\right) \end{aligned} \tag{2.1}$$

for $U, V \in T^{1,0}X$. The domain is called *strongly pseudoconvex* if the Levi form is positive definite; and *weakly pseudoconvex* (or simply *pseudoconvex*) if the Levi form is semi-definite.

We now recall the notion of finite type. There are two standard notions of finite type (D'Angelo and Kohn/Bloom-Graham) of a smooth real hypersurface M, and these happen to coincide in $\mathbb{C}^2$. (The reader is referred to [1] for more details). The domain is called of *finite type* (in the sense of Kohn/Bloom-Graham) if the Levi-distribution HX is bracket generating: $C^\infty(HX)$ generates TX under the Lie bracket. In particular the *type of a point* on the boundary $x \in X = \partial D$ is the smallest integer $r(x)$ such that $H_x X_{r(x)} = T_x X$, where the subspaces $HX_j \subset TX$, $j = 1, \ldots$ are inductively defined by

$$\begin{aligned} HX_1 &\coloneqq HX \\ HX_{j+1} &\coloneqq HX + \left[HX_j, HX\right], \quad \forall j \geq 1. \end{aligned} \tag{2.2}$$

In general, the function $x \mapsto r(x)$ is only upper semi-continuous. The finite type hypothesis is then equivalent to $r \coloneqq \max_{x \in X} r(x) < \infty$. Note that the type of a strongly pseudoconvex point x is $r(x) = 2$.

The Bergman projector of D is the orthogonal projector

$$K_D : L^2(D) \to L^2(D) \cap \mathcal{O}(D) \tag{2.3}$$

from square integrable functions onto the closed subspace of square-integrable holomorphic ones. Its Schwartz kernel, still denoted by $K_D(z, z') \in L^2(D \times D)$, is called the Bergman kernel of D. It is well-known to be smooth in the interior and positive along the diagonal. The Bergman metric is the Kähler metric in the interior defined by

$$g^D_{\alpha\bar{\beta}} \coloneqq \partial_\alpha \partial_{\bar{\beta}} \ln K_D(z, z).$$

Denote by $G = \det\left(g^D_{\alpha\bar{\beta}}\right)$ the determinant of the above metric. The Ricci tensor of g^D is by definition $R_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \ln G$. The Bergman metric is always Kähler, and is further said to be *Kähler-Einstein* if $R_{\alpha\bar{\beta}} = c g^D_{\alpha\bar{\beta}}$ for some constant c. Since D is a bounded domain, the sign of c must necessarily be negative (cf. [6, page 518]). The Bergman invariant function is defined by $B(z) := \frac{G(z)}{K_D(z,z)}$. It follows from the transformation formula of the Bergman kernel that the Bergman invariant function is invariant under biholomorphisms.

Next we briefly discuss the Kähler-Einstein metric. Recall the existence of a complete Kähler-Einstein metric on $D \subset \mathbb{C}^n$ is governed by the following Dirichlet problem:

$$\begin{aligned} J(u) := (-1)^n \det \begin{pmatrix} u & u_{\bar{\beta}} \\ u_\alpha & u_{\alpha\bar{\beta}} \end{pmatrix} &= 1 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D \end{aligned} \tag{2.4}$$

with $u > 0$ in D. Here u_α denotes $\partial_{z_\alpha} u$, and likewise for $u_{\bar{\beta}}$ and $u_{\alpha\bar{\beta}}$. The problem was first studied by Fefferman [11], and $J(\cdot)$ is often referred as Fefferman's complex Monge-Ampère operator. Cheng and Yau [6] proved the existence and uniqueness of an exact solution $u \in C^\infty(D)$ to (2.4), on a smoothly bounded strongly pseudoconvex domain D. The function u is called the Cheng–Yau solution; and $-\partial\partial \log u$ gives rise to a complete Kähler-Einstein metric on D. Mok-Yau [24] further showed a bounded domain admits a complete Kähler-Einstein metric if and only if it is a domain of holomorphy.

We next make some observations on the Monge-Ampère operator for later applications. The left hand side of the first equation in (2.4) can further be invariantly written as $J(u) = u^{n+1} \det\left[\partial\bar{\partial}(-\ln u)\right]$. It may thus be computed in terms of any orthonormal frame $\{Z_\alpha\}_{\alpha=1}^n$ of $T^{1,0}\mathbb{C}^n$ as

$$J(u) = \det \begin{pmatrix} u & \bar{Z}_\beta u \\ Z_\alpha u & Z_\alpha \bar{Z}_\beta u - \left[Z_\alpha, \bar{Z}_\beta\right]^{0,1} u \end{pmatrix}. \tag{2.5}$$

This can be proved using the identity

$$\partial\bar{\partial} f\left(Z_\alpha, \bar{Z}_\beta\right) = Z_\alpha \bar{Z}_\beta (f) - \bar{\partial} f\left(\left[Z_\alpha, \bar{Z}_\beta\right]\right). \tag{2.6}$$

Here the normality of $\{Z_\alpha\}_{\alpha=1}^n$ means each of them has the same Euclidean norm as $\partial_{z_1}, \cdots, \partial_{z_n}$.

The following proposition gives an equivalent condition for the Bergman metric being Kähler-Einstein, which is easier to work with. The proof is similar to [13, Proposition 1.1] and [18, Proposition 3.3].

Proposition 3. *Let $D \subset \mathbb{C}^n, n \geq 2$, be a smoothly bounded pseudoconvex domain. Then its Bergman metric g^D is Kähler-Einstein if and only if the Bergman invariant function is constant $B(z) \equiv (n+1)^n \frac{\pi^n}{n!}$. This is also equivalent to the Bergman kernel K_D satisfying $J(K_D) = (-1)^n \frac{(n+1)^n \pi^n}{n!} K_D^{n+2}$.*

Proof. We start with the proof of the first assertion. Since the reverse direction is trivial, we only need to prove the forward part. Assume the Bergman metric of D is Kähler-Einstein.

Recall a smoothly bounded domain in $\mathbb{C}^n$ always has a strongly pseudoconvex boundary point. Therefore we can find a strongly pseudoconvex open connected piece M of ∂D. Fix $p \in M$. Next pick a small smoothly bounded strongly pseudoconvex domain $D' \subseteq D$ such that $D' \cap O = D \cap O$ and $\partial D' \cap O = \partial D \cap O =: M_0 \subseteq M$ for some small ball O in $\mathbb{C}^n$ centered at p.

Write $K_{D'}$ for the Bergman kernel of D'. Then by the localization of the Bergman kernel on pseudoconvex domains at a strongly pseudoconvex boundary point (cf. Theorem 4.2 in Engliš [10]), there is a smooth function Φ in a neighborhood of $D' \cup M_0$ such that

$$K_D = K_{D'} + \Phi \text{ on } D'. \tag{2.7}$$

Note that $K_{D'}$ obeys Fefferman asymptotic expansion on D' by [11]. Combining this with (2.7), we see for any defining function ρ of $D \cap O$ with $D \cap O = \{z \in O : \rho(z) < 0\}$, the Bergman kernel K_D also has the Fefferman type expansion in $D \cap O$:

$$K_D = \frac{\phi}{\rho^{n+1}} + \psi \log(-\rho) \quad \text{on } D \cap O. \tag{2.8}$$

Here ϕ and ψ are smooth in a neighborhood of $D' \cup M_0$ with ϕ nowhere zero on M_0.

Then by (2.8) and (the proof of) Theorem 1 of Klembeck [20], the Bergman metric of D is asymptotically of constant holomorphic sectional curvature $\frac{-2}{n+1}$ as $z \in D \to M_0$. Consequently, the Bergman metric of D is asymptotically of constant Ricci curvature -1 as $z \in D \to M_0$ (To prove the latter fact, alternatively one can apply a similar argument as page 510 of Cheng-Yau [6]). Therefore by the Kähler-Einstein assumption, we must have $R_{ij} = -g_{ij}$. This yields $\partial\bar{\partial} \log B \equiv 0$ in D. That is, $\log B$ is pluriharmonic in D.

Furthermore, by (2.7) and a similar argument as in the proof of Lemma 3.2 in [18], we have $B(z,z) \to \frac{(n+1)^n \pi^n}{n!}$ as $z \to M_0$. Now write $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ for the unit disk. Let $f : \bar{\Delta} \to O$ be an analytic disk attached to M_0. That is, f is holomorphic in Δ and continuous in $\bar{\Delta}$ with $f(\Delta) \subset O \cap D$ and $f(\partial\Delta) \subset M_0$. Then $\log B(f)$ is harmonic in Δ, continuous up to $\partial\Delta$, and takes constant value $\log \frac{(n+1)^n \pi^n}{n!}$ on $\partial\Delta$. This implies B takes the constant value $\frac{(n+1)^n \pi^n}{n!}$ on $f(\Delta)$. But since M_0 is strongly pseudoconvex, we can find a family $\mathcal{F}$ of analytic disks such that $\cup_{f \in \mathcal{F}} f(\Delta)$ fills up an open subset U of $O \cap D$(cf. [1]). Thus B is constant on U. Since B is real analytic and D is connected, we see $B \equiv \frac{(n+1)^n \pi^n}{n!}$.

Finally, a routine computation using the formula $J(u) = u^{n+1} \det\left(\partial\bar{\partial}(-\ln u)\right)$ yields that, $B(z) = c$ if and only if $J(K_D) = (-1)^n c K_D^{n+2}$. Then the second assertion of the proposition follows immediately. □

3. The Bergman kernel and its derivatives

To prove Theorem 1, we shall fundamentally use the asymptotics of the Bergman kernel on pseudoconvex domains of finite type. In this section, we first briefly recall some classical and recent known work, and then prove new results for asymptotics of the Bergman kernel.

In Section 2, we already made use of Fefferman's Bergman kernel asymptotics in the strongly pseudoconvex case. Let D be a strongly pseudoconvex domain with a defining function $\rho \in C^{\infty}\left(\bar{D}\right)$. Fefferman [11] showed that the Bergman kernel of the domain D has an asymptotic expansion

$$K_{D}\left(z,z\right)=a\left(z\right)\rho^{-n-1}+b\left(z\right)\ln\left(-\rho\right) \tag{3.1}$$

for some functions $a\left(z\right), b\left(z\right) \in C^{\infty}\left(\bar{D}\right)$.

Recently, the asymptotics in (3.1) were extended to pseudoconvex domains of finite type in $\mathbb{C}^{2}$ by Hsiao and the first author [17, Theorem 2]. They established the full asymptotic expansion of the Bergman kernel described along transversal paths approaching the boundary. This is not suitable for our proof of Theorem 1. We shall need the asymptotic expansion of the Bergman kernel, and its derivatives, along certain critically tangent paths (see Section 1 and Remark 5) approaching the boundary. Besides, we also need information of the leading coefficient in the asymptotics.

To state our result, now let $D\subset\mathbb{C}^{2}$ be a pseudoconvex domain of finite type. Fix $x^{*}\in X=\partial D$ on the boundary of the domain of type $r=r\left(x^{*}\right)$. Let $U_{1},U_{2}\coloneqq JU_{1}\in C^{\infty}\left(HX\right)$ be two local orthonormal sections of the Levi distribution and $U_{3}\in C^{\infty}\left(TX\right)$, $U_{3}\perp HX$ to be a unit normal to the Levi distribution. One then extends U_{1} to a local unit length vector field in the interior of D. Set $U_{2}=JU_{1}$ to be an extension of U_{2} to the interior of D. Choose an extension of U_{3} of unit length and that is orthogonal to U_{1},U_{2}. Set $U_{0}=-JU_{3}$ (so that $U_{3}=JU_{0}$). It is easy to see that U_{0} is of unit length and normal to the boundary $U_{0}\perp TX$ near $x^{*}\in X$. Replacing U_{3} by $-U_{3}$ if needed, we assume U_{0} is outward-pointing to D. This also gives a local boundary defining function ρ via $U_{0}\left(\rho\right)=1,\ \rho|_{X}=0$. Note that the flow of the normal vector field U_{0} also gives a locally defined projection $\pi:D\rightarrow X=\partial D$ onto the boundary. The pairs of vector fields define CR vector fields $Z=\frac{1}{2}\left(U_{1}-iU_{2}\right), W=\frac{1}{2}\left(U_{0}-iU_{3}\right)\in T^{1,0}\mathbb{C}^{2}$.

Next, from a system of coordinates $x=\left(x_{1},x_{2},x_{3}\right)$ on the boundary centered at $x^{*}\in X$, we assign weights to local functions and vector fields. To define these, first the weight of a monomial x^{α}, $\alpha\in\mathbb{N}_{0}^{3}$, is $w.\alpha\coloneqq\alpha_{1}+\alpha_{2}+r\alpha_{3}$, with $w(x)=w\left(x_{1},x_{2},x_{3}\right)\coloneqq\left(1,1,r\right)$. The weight $w\left(f\right)$ of a function $f\in C^{\infty}\left(X\right)$ is then the minimum weight of the monomials appearing in its Taylor series at $x^{*}=0$. Finally, the weight $w\left(U\right)$ of a smooth vector field $U=\sum_{j=1}^{3}f_{j}\partial_{x_{j}}$ is $w\left(U\right)\coloneqq\min\left\{w\left(f_{1}\right)-1,w\left(f_{2}\right)-1,w\left(f_{3}\right)-r\right\}$.

In [7, Prop. 3.2] (see also [1]) it was shown that a coordinate system $x = (x_1, x_2, x_3)$ on the boundary centered at x^* may be chosen so that

$$Z|_X = \frac{1}{2}\left[\underbrace{\partial_{x_1} + (\partial_{x_2} p)\,\partial_{x_3} - i\left(\partial_{x_2} - (\partial_{x_1} p)\,\partial_{x_3}\right)}_{=:Z_0} + R\right], \tag{3.2}$$

where $p(x_1, x_2)$ is a homogeneous, subharmonic (and non-harmonic) real polynomial of degree and weight r. We note that r must be even. Besides, p has no purely holomorphic or anti-holomorphic terms in $z_1 = x_1 + ix_2$ in its Taylor expansion at 0. Moreover, $R = \sum_{j=1}^{3} r_j(x)\,\partial_{x_j}$ is a real vector field of weight $w(R) \geq 0$.

The coordinate system (x_1, x_2, x_3) on the boudary is next extended to the interior of the domain by being constant in the normal direction $U_0(x_j) = 0$, $j = 1, 2, 3$. Then $x' := (\rho, x_1, x_2, x_3)$ serve as coordinates on the interior of the domain near x^* in which $U_0 = \partial_\rho$. We also extend the notion of weights to the new coordinate system. The weight of a monomial $\rho^{\alpha_0} x^{\alpha}$ is defined as $w'(\rho^{\alpha_0} x^{\alpha}) = w'.\alpha' := r\alpha_0 + \alpha_1 + \alpha_2 + r\alpha_3$, with $w'(x') = w'(\rho, x_1, x_2, x_3) := (r; 1, 1, r)$ now denoting the augmented weight vector. We again define the weight $w(f)$ of a smooth function $f \in C^\infty(D)$ near x^* as the minimum weight of the monomials appearing in its Taylor series at x^* in these coordinates. Finally, the weight $w(U)$ of a smooth vector field $U = f_0\partial_\rho + \sum_{j=1}^{3} f_j \partial_{x_j}$ is $w(U) := \min\{w(f_0) - r, w(f_1) - 1, w(f_2) - 1, w(f_3) - r\}$. Note that one has $w(U) \geq -r$, and $w(U) > -r$ if $f_0(0) = f_3(0) = 0$.

Below $O(k)$ denotes a vector field of weight k or higher. By a rescaling of the x_3 coordinate, and at the cost of scaling the polynomial $p(x_1, x_2)$, we may also arrange $U_3|_{x^*=0} = \pm\partial_{x_3}$. By the fact that $[Z, \bar{Z}] = [-\Delta p(z_1)\,\frac{i}{2}\partial_{x_3}] + O(-1)$ and the pseudoconvexity condition (2.1), one can show that it must be ∂_{x_3}. But the sign is irrelevant to our proof, and thus we will not elaborate it here. Therefore we have

$$U_3 = \partial_{x_3} + O(-r+1). \tag{3.3}$$

Next let $V \in C^\infty(HX)$ denote another locally defined section of the Levi distribution. This defines a local *tangential path* approaching x^* via

$$z(\epsilon) := \Big(\underbrace{e^{\epsilon V} x^*}_{=\pi(z(\epsilon))}, \underbrace{-\epsilon^r}_{=\rho(z(\epsilon))}\Big) \in D, \quad \epsilon > 0. \tag{3.4}$$

Note the above path is indeed tangential to the boundary; its tangent vector at x^* is in the Levi-distribution $\frac{dz}{d\epsilon}\big|_{\epsilon=0} = V_{x^*} \in H_{x^*}X$. The order of tangency the path makes with the boundary is the type of the point $r(x^*)$. Writing $V = \sum_{j=1}^{3} g_j \partial_{x_j}$, we associate the section V with a point

$$z_{1,V} := (x_{1,V}, x_{2,V}) = (g_1(0), g_2(0)) \in \mathbb{R}^2. \tag{3.5}$$

In the computation of the leading asymptotics of the Bergman kernel K_D (see (3.7) in Theorem 4), one will further see the appearance of the *model*

Bergman kernel B_p corresponding to the subharmonic polynomial p in (3.2). For the readers' convenience, we briefly recall the notion of model Bergman kernel. For that, we consider the L^2 orthogonal projector from $L^2(\mathbb{C}_{z_1})$ to H^2_p. Here

$$H^2_p \coloneqq \left\{ f \in L^2(\mathbb{C}_{z_1}) \mid \bar{\partial}_p f = 0 \right\}; \quad \text{and} \quad \bar{\partial}_p \coloneqq \partial_{\bar{z}_1} + \partial_{\bar{z}_1} p.$$

Then B_p is defined to be the Schwartz kernel of this projector. More discussion and analysis of the model Bergman kernel follows in Section 4.

We now state the necessary asymptotics result for the Bergman kernel and its derivatives. Below $\partial^{\alpha'} = \left(\frac{1}{2}U_0\right)^{\alpha_0} Z^{\alpha_1} \bar{Z}^{\alpha_2} \left(\frac{1}{2}U_3\right)^{\alpha_3}$ denotes a mixed derivative along the respective vector fields for $\alpha' = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^4$.

Theorem 4. *Let $D \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. For any point $x^* \in X = \partial D$ on the boundary, of type $r = r(x^*)$, the Bergman kernel and its derivatives satisfy the following asymptotics for each $N \in \mathbb{N}$:*

(3.6)

$$\begin{aligned} &\partial^{\alpha'} K_D(z, z) \\ &= \sum_{j=0}^{N} \frac{1}{(-2\rho)^{2 + \frac{2 + w'.\alpha'}{r} - \frac{1}{r}j}} a_j + \sum_{j=0}^{N} a'_j (-\rho)^j \log(-\rho) + O\left((-\rho)^{\frac{1}{r}(N+1) - 2 - \frac{2 + w'.\alpha'}{r}} \right), \end{aligned}$$

for some set of numbers a_j, a'_j as $z \to x^*$ tangentially to the boundary along the path (3.4).

Furthermore, the leading term can be computed in terms of the model Bergman kernel of the subharmonic polynomial p as

$$a_0 = \delta_{0\alpha_3} . \left[\partial_{z_1}^{\alpha_1} \partial_{\bar{z}_1}^{\alpha_2} \underbrace{\left(\frac{1}{\pi} \int_0^\infty e^{-s} s^{1 + \frac{2}{r} + \alpha_0} B_p\left(s^{\frac{1}{r}} z_1\right) ds \right)}_{=: \tilde{B}_{p,\alpha_0}(z_1)} \right]_{z_1 = z_{1,V}} . \tag{3.7}$$

Proof. The proof is similar to [17, Thm. 2]. We shall only point out the necessary modifications.

In [17, Sec. 4] the following space of symbols $\hat{S}^m_{\frac{1}{r}}\left(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{R}_t\right)$, $m \in \mathbb{R}$, in the variables $(\rho, x, \rho', y; t) \in \mathbb{C}^2_z \times \mathbb{C}^2_w \times \mathbb{R}_t$ was defined. This is the space of smooth functions satisfying the symbolic estimates

(3.8)

$$\left| \partial_\rho^{\alpha_0} \partial_{\rho'}^{\beta_0} \partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(\rho, x, \rho', y, t) \right| \leq C_{N, \alpha\beta\gamma} \langle t \rangle^{m - \gamma + \frac{w'.(\alpha' + \beta')}{r}} \frac{\left(1 + \left|t^{\frac{1}{r}} \hat{x}\right| + \left|t^{\frac{1}{r}} \hat{y}\right|\right)^{N(\alpha', \beta', \gamma)}}{\left(1 + \left|t^{\frac{1}{r}} \hat{x} - t^{\frac{1}{r}} \hat{y}\right|\right)^{-N}},$$

for each $(x, y, \rho, \rho', t, N) \in \mathbb{R}^6_{x,y} \times \mathbb{R}^2_{\rho,\rho'} \times \mathbb{R}_t \times \mathbb{N}$ and $(\alpha', \beta', \gamma) \in \mathbb{N}_0^4 \times \mathbb{N}_0^4 \times \mathbb{N}_0$ with $\alpha' = (\alpha_0, \alpha)$, $\beta' = (\beta_0, \beta)$. Here $N(\alpha', \beta', \gamma) \in \mathbb{N}$ depends only on the given indices, $\langle t\rangle := \sqrt{1+t^2}$ denotes the Japanese bracket while the notation $\hat{x} = (x_1, x_2)$ denotes the first two coordinates of the tuple $x = (x_1, x_2, x_3)$. Below $\hat{S}\left(\mathbb{R}^2_{\hat{x}} \times \mathbb{R}^2_{\hat{y}}\right)$ further denotes the space of restrictions of functions in $\hat{S}^m_{\frac{1}{r}}$ to $x_3, y_3, \rho, \rho' = 0$ and $t = 1$.

Next a generalization of this space is defined via

$$\hat{S}^{m,k}_{\frac{1}{r}} := \bigoplus_{p+q+p'+q'\leq k} (tx_3)^p (t\rho)^q (ty_3)^{p'} \left(t\rho'\right)^{q'} \hat{S}^m_{\frac{1}{r}}, \tag{3.9}$$

for each $(m,k) \in \mathbb{R} \times \mathbb{N}_0$. Finally, the subspace of classical symbols $\hat{S}^m_{\frac{1}{r},\mathrm{cl}} \subset \hat{S}^m_{\frac{1}{r}}$ comprises of those symbols for which there exist $a_{jpp'qq'}(\hat{x}, \hat{y}) \in \hat{S}\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, $j, p, p', q, q' \in \mathbb{N}_0$, such that the following belongs to $\hat{S}^{m-(N+1)\frac{1}{r}, N+1}_{\frac{1}{r}}$ for each $N \in \mathbb{N}_0$:

$$a(x,y,t) - \sum_{j=0}^{N} \sum_{p+q+p'+q'\leq j} t^{m-\frac{1}{r}j} (tx_3)^p (t\rho)^q (ty_3)^{p'} \left(t\rho'\right)^{q'} a_{jpp'qq'}\left(t^{\frac{1}{r}}\hat{x}, t^{\frac{1}{r}}\hat{y}\right). \tag{3.10}$$

The space $\hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ is now defined similarly to (3.9). The principal symbol of such an element $a \in \hat{S}^m_{\frac{1}{r},\mathrm{cl}}$ is defined to be the function

$$\sigma_L(a) := a_{00000} \in \hat{S}\left(\mathbb{R}^2 \times \mathbb{R}^2\right).$$

Now, following the proof of [15, Prop. 7.6], there exists a smooth phase function $\Phi(z,w)$ defined locally on a neighbourhood $U \times U$ of (x^*, x^*) in $\bar{D} \times \bar{D} \subset \mathbb{C}^2_z \times \mathbb{C}^2_w$ such that

$$\begin{aligned} &\Phi(z,w) - x_3 + y_3 \\ &= -i\rho\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))} - i\rho'\sqrt{-\sigma_{\triangle_X}(y,(0,0,1))} + O(|\rho|^2) + O(\left|\rho'\right|^2), \end{aligned} \tag{3.11}$$

where $q_0(z, d_z\Phi)$ and $q_0(w, -\overline{d}_w\Phi)$ vanish to infinite order on $\{\rho = 0\}$ and on $\{\rho' = 0\}$, respectively. Here $\triangle_X$ denotes the real Laplace operator on the boundary $X = \partial D$ of the domain, while $q_0 = \sigma(\Box_f)$ denotes the principal symbol of the complex Laplace-Beltrami operator $\Box_f = \bar{\partial}^*_f\bar{\partial} + \bar{\partial}\bar{\partial}^*_f$ on the domain. The proofs of [17, Lemma 17] and [17, Lemma 20] can be repeated to obtain the following description for the Bergman kernel: for some $a(z,w,t) \in \hat{S}^{1+\frac{2}{r}}_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{R}_t\right)$ one has

$$K_D(z,w) = \frac{1}{\pi}\int_0^\infty e^{i\Phi(z,w)t} a(z,w,t)\,dt \quad \left(\mathrm{mod}\ C^\infty\left((U \times U) \cap \left(\overline{D} \times \overline{D}\right)\right)\right) \tag{3.12}$$

with $\sigma_L(a) = B_p$ being the model Bergman kernel defined prior to the statement of this theorem.

We need to differentiate the last description (3.12). For that, we adopt the notion of weights we defined before Theorem 4. By construction, the chosen vector fields $\left(U_0, Z, \overline{Z}, U_3\right)$ have weights $(-r, -1, -1, -r)$ respectively. Furthermore, the leading parts in their weight expansions are given by

$$\left(U_0, Z, \overline{Z}, U_3\right) = \left(\partial_\rho, Z_0 + O(0), \bar{Z}_0 + O(0), \partial_{x_3} + O(-r+1)\right), \tag{3.13}$$

Here $Z_0 \coloneqq \frac{1}{2}[\partial_{x_1} + (\partial_{x_2} p)\,\partial_{x_3} - i\,(\partial_{x_2} - (\partial_{x_1} p)\,\partial_{x_3})]$ is now understood as a locally defined vector field in the interior of the domain. Next we observe from definitions of the symbol spaces (3.8), (3.9) that a vector field U of weight $w(U)$ maps

$$U : \hat{S}^{m}_{\frac{1}{r},\mathrm{cl}} \to \hat{S}^{m-\frac{1}{r}w(U)}_{\frac{1}{r},\mathrm{cl}}. \tag{3.14}$$

The equations (3.11), (3.13), (3.14) now allow us to differentiate (3.12) to obtain: for some $a_\alpha(z; w, t) \in \hat{S}^{1+\frac{2+w'.\alpha}{r},\alpha_0+\alpha_3}_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{R}_t\right)$ one has

$$\partial^\alpha K_D(z,z) = \frac{1}{\pi}\int_0^\infty e^{i\Phi(z,z)t} a_\alpha(z,z,t)\,dt \quad \left(\mathrm{mod}\ C^\infty\left((U\times U)\cap\left(\overline{D}\times\overline{D}\right)\right)\right)$$
$$\text{with} \quad a_\alpha = \left(Z_0^{\alpha_1}\bar{Z}_0^{\alpha_2}B_p\right)t^{1+\frac{2+w'.\alpha}{r}} + \hat{S}^{1+\frac{1+w'.\alpha}{r},\alpha_0+\alpha_3}_{\frac{1}{r},\mathrm{cl}}. \tag{3.15}$$

Recall the vector field $V = \sum_{j=1}^{3} g_j \partial_{x_j} \in C^\infty(HX)$ lies in the Levi distribution. By (3.2), its ∂_{x_3}-component function has weight $w(g_3) \geq r-1$. Thus along the flow of V, and consequently along the path $z(\epsilon)$ in (3.4), the coordinate functions satisfy

$$(x_1, x_2, x_3, \rho) = \left(\epsilon g_1(0) + O\left(\epsilon^2\right), \epsilon g_2(0) + O\left(\epsilon^2\right), O\left(\epsilon^r\right), -\epsilon^r\right). \tag{3.16}$$

The last two equations (3.15) and (3.16) now combine to give the theorem. □

Remark 5. (Critical tangency) The path $z(\epsilon)$ in (3.4) is particularly chosen to be critically tangent to the boundary. Namely its order of tangency with the boundary is the type $r(x^*)$ of the boundary point $x^* \in \partial D$ that is being approached. This order of tangency is critical in the sense that it is the maximum for which the expansion in (3.6) can be proved. As for a higher order of tangency (i.e., ρ having vanishing order higher than r at $\epsilon = 0$), the terms in the symbolic expansion of $a_\alpha \in \hat{S}^{1+\frac{2+w'.\alpha}{r},\alpha_0+\alpha_3}_{\frac{1}{r},\mathrm{cl}}$ in (3.15) become increasing in order and not asymptotically summable. This means in particular, the double summation in (3.10) would be asymptotically non-summable along the path. A critically tangent path is necessary in our proof below since for such a path the leading coefficient (3.7) picks up information of the model Bergman kernel at the arbitrary tangent vector V. For a path tangent at a lesser order, the leading coefficient only depends on the value of the model kernel B_p at the origin.

4. Analysis of the model kernel

In Section 3, we introduced the model Bergman kernel B_p, corresponding to a subharmonic, homogeneous polynomial $p(x_1, x_2)$. As we see from Theorem 4, it plays an important role in the asymptotics of the Bergman kernel K_D of D. To prepare for the proof of Theorem 1, we need to further analyze this model Bergman kernel B_p. For convenience, we will also write $p(x_1, x_2)$ as $p(z_1)$, where $z_1 = x_1 + ix_2$.

4.1. Expansion of the model kernel and first few coefficients.

First we will work out the expansion of the model Bergman kernel B_p, and compute the values of the first few coefficients in the expansion. As usual, for a smooth function f on $\mathbb{C}_{z_1}$, we write $f_{z_1} = \partial_{z_1} f = \frac{\partial f}{\partial z_1}$, and likewise for $f_{\bar{z}_1}$ and $f_{z_1\bar{z}_1}$.

Proposition 6. *For any $z_1 \in \mathbb{R}^2$, with $\Delta p(z_1) \neq 0$, the model Bergman kernel on diagonal satisfies the asymptotics*

$$\left[\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2} B_p\right]\left(t^{\frac{1}{r}} z_1\right) = \frac{t^{1-\frac{2+|\alpha|}{r}}}{2\pi}\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}\left[\sum_{j=0}^{N} b_j t^{-j} + O\left(t^{-N-1}\right)\right] \tag{4.1}$$

for each $N \in \mathbb{N}$ as $t \to \infty$. Moreover, the first four terms in the asymptotics are given by

$$b_0 = 4q; \quad b_1 = q^{-2}Q; \quad b_2 = \frac{1}{6}\partial_{z_1}\partial_{\bar{z}_1}\left[q^{-3}Q\right];$$

$$b_3 = \frac{q}{48}\left\{[q^{-1}\partial_{z_1}\partial_{\bar{z}_1}]^2 q^{-3}Q - q^{-4}Q\left[\partial_{z_1}\partial_{\bar{z}_1}\right]q^{-3}Q - q^{-1}[\partial_{\bar{z}_1}\left(q^{-3}Q\right)][\partial_{z_1}\left(q^{-3}Q\right)]\right\}; \tag{4.2}$$

where $q := \frac{1}{4}\Delta p = p_{z_1\bar{z}_1}$ and $Q := qq_{z_1\bar{z}_1} - q_{z_1}q_{\bar{z}_1}$ are defined in terms of the polynomial p.

Proof. The proof uses some rescaling arguments. Following [23, Sec. 4.1], we introduce the rescaling operator $\delta_{t^{-\frac{1}{r}}} : \mathbb{C} \to \mathbb{C}$ given by $\delta_{t^{-\frac{1}{r}}}(z_1) := t^{-\frac{1}{r}} z_1, t > 0$. Recall when introducing B_p, we defined $\bar{\partial}_p := \partial_{\bar{z}_1} + \partial_{\bar{z}_1} p$. The corresponding Kodaira Laplacian on functions $\Box_p = \bar{\partial}_p^*\bar{\partial}_p$ then gets rescaled to the operator

$$\left(\delta_{t^{-\frac{1}{r}}}\right)_* \Box_p = t^{-\frac{2}{r}}\Box_t$$

where $\Box_t := \bar{\partial}_t^*\bar{\partial}_t$, and $\bar{\partial}_t := \partial_{\bar{z}_1} + t\left(\partial_{\bar{z}_1} p\right)$.

We pause to introduce two more Bergman type kernel functions that are defined similarly as B_p. Firstly denote by the shorthand $\mathcal{B}_t := B_{tp}$, $t > 0$, the model Bergman kernel associated to the rescaled homogeneous polynomial tp. Next define the weighted space $L^2_{tp}(\mathbb{C}_{z_1}) := \left\{f | e^{-tp} f \in L^2(\mathbb{C}_{z_1})\right\}$, and denote by $\mathcal{O}(\mathbb{C}_{z_1})$ the space of entire functions on $\mathbb{C}_{z_1}$. The L^2 orthogonal projector $\mathcal{B}_t^p$ from $L^2_{tp}(\mathbb{C}_{z_1})$ to $L^2_{tp}(\mathbb{C}_{z_1}) \cap \mathcal{O}(\mathbb{C}_{z_1})$ is then seen to be related to the kernel B_t by the relation

$$\mathcal{B}_t\left(z_1, z_1'\right) = e^{-tp(z_1)-tp\left(z_1'\right)}\mathcal{B}_t^p\left(z_1, z_1'\right). \tag{4.3}$$

This follows on simply noting that multiplication by e^{-tp} is a isomorphism from $L^2_{tp}(\mathbb{C}_{z_1})$ to $L^2(\mathbb{C}_{z_1})$ and $L^2_{tp}(\mathbb{C}_{z_1}) \cap \mathcal{O}(\mathbb{C}_{z_1})$ to H^2_{tp} respectively.

Moreover, $\mathcal{B}_t$ can be equivalently understood as the Bergman projector for the trivial holomorphic line bundle on $\mathbb{C}$ with Hermitian metric $h_t = e^{-tp}$. The curvature of this metric is $t\underbrace{\left(2\partial_{z_1}\partial_{\bar{z}_1}p\right)}_{=\frac{1}{2}\Delta p}dz_1 \wedge d\bar{z}_1$. Its eigenvalue is Δp.

In [17, Thm. 14], the Bergman kernel of B_t was related to the model via

$$B_p\left(t^{\frac{1}{r}}z_1, t^{\frac{1}{r}}z_1'\right) = t^{-\frac{2}{r}}\mathcal{B}_t\left(z_1, z_1'\right). \tag{4.4}$$

Furthermore, in its proof the following spectral gap property for $\Box_t$ was observed

$$\mathrm{Spec}\left(\Box_t\right) \subset \{0\} \cup \left[c_1 t^{2/r} - c_2, \infty\right)$$

for some $c_1, c_2 > 0$.

At a point $z_1 \in \mathbb{C}$, where $\Delta p(z_1) \neq 0$, the asymptotics of $\mathcal{B}_t(z_1, z_1)$ as $t \to \infty$ are thus the standard asymptotics for the Bergman kernel on tensor powers of a positive line bundle (cf. [16, Thm. 1.6]). There is an asymptotic expansion

$$\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}\mathcal{B}_t(z_1) = \frac{t}{2\pi}\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}\left[\sum_{j=0}^{N} b_j t^{-j} + O\left(t^{-N-1}\right)\right] \tag{4.5}$$

for each $N \in \mathbb{N}$ as $t \to \infty$. The last two equations (4.4) and (4.5) combine to prove (4.1).

It remains to compute the first four coefficients in (4.5). For that we will make use of (4.3), by which it suffices to find the corresponding coefficients in the expansion of $\mathcal{B}_t^p$. The computations for the latter can be found in [9, (6.2) and Theorem 9]. In order to see the specialization of the formulas therein to the special case here, we note the Kähler metric $g = \partial\bar{\partial}p$ with potential p has component $g_{1\bar{1}} = q = \partial_{z_1}\partial_{\bar{z}_1}p$. The only non-zero Christoffel symbols are $\overline{\Gamma_{11}^1} = \Gamma_{\bar{1}\bar{1}}^{\bar{1}} = q^{-1}\partial_{\bar{z}_1}q$. Furthermore, the only non-zero components of the Riemannian, Ricci and scalar curvatures respectively are given by the following. Here we follow the convention of curvatures in [9, pp. 6], which may differ from that of some other papers by a negative sign.

$$R_{1\bar{1}1\bar{1}} = \partial_{z_1}\partial_{\bar{z}_1}q - q^{-1}\left(\partial_{z_1}q\right)\left(\partial_{\bar{z}_1}q\right) = q^{-1}Q; \quad \mathrm{Ric}_{1\bar{1}} = q^{-2}Q; \quad R = q^{-3}Q.$$

The corresponding Laplace operator L_1 of [9, (2.10)] in our special context is given by $L_1 = q^{-1}\partial_{z_1}\partial_{\bar{z}_1}$. We now bring these specializations into [9, (6.2) and Theorem 9] to obtain the values of the coefficients b_0, b_1, b_2 and b_3. For instance, we note that the tensors appearing the computation of b_3 are those arising from $\sigma_8 = \sigma_9 = \sigma_{10} = q^{-4}Q\left[\partial_{z_1}\partial_{\bar{z}_1}\right]q^{-3}Q$, $\sigma_{12} = \sigma_{13} =$

$q^{-1}\left[\partial_{\bar{z}_1}\left(q^{-3}Q\right)\right]\left[\partial_{z_1}\left(q^{-3}Q\right)\right]$ and $\sigma_{14}=\left[q^{-1}\partial_{z_1}\partial_{\bar{z}_1}\right]^2 q^{-3}Q$ in the notation of [9, Theorem 9]. □

Remark 7. Although we computed the values of $b_0,\cdots,b_3$ in Proposition 6, we will only use b_3 in the proof of Theorem 1.

4.2. **Models with vanishing expansion coefficients.** Having shown that the model kernel $B_p\left(t^{\frac{1}{r}}z_1\right)$ admits an asymptotic expansion at $t\to\infty$, we ask when the terms of the asymptotic expansion are eventually zero, or in other words, $b_j=0$ for j sufficiently large. This is relevant to our theorem below. We prove the following somewhat surprising result which shows the vanishing of the third coefficient is already restrictive. As above, let $p\left(x_1,x_2\right)$ be a subharmonic and non-harmonic homogeneous polynomial of degree r.

Theorem 8. *Suppose the third term b_3 vanishes in the asymptotic expansion* (4.1) *of the model kernel B_p corresponding to p. Then there exists some real number $c_0>0$ such that $q=c_0\left(z_1\bar{z}_1\right)^{\frac{r}{2}-1}$. Here as before, $q:=\frac{1}{4}\Delta p$.*

To prove the theorem, we carry out some Hermitian analysis. For that, we start with a few definitions and lemmas. In the remainder of this subsection, we will write z instead of z_1 for simplicity.

Definition 9. Let $f\in\mathbb{C}[z,\zeta]$ be a polynomial of two variables. Fix $a\in\mathbb{C}$. Let $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. We say f is *divisible* by $(z+a\zeta)^k$ with coefficient λ, denoted by $f\sim D_a(k,\lambda)$, if $f(z,\zeta)=(z+a\zeta)^k\hat{f}(z,\zeta)$ for some $\hat{f}\in\mathbb{C}[z,\zeta]$ with $\hat{f}(-a,1)=\lambda$.

It is clear that if $f\sim D_a(k,\lambda)$ with $k\geq 1$, then we have $f\sim D_a(k-1,0)$. In the following, we say $f\in\mathbb{C}[z,\zeta]$ is *Hermitian* if $f(z,\bar{z})$ is real-valued for every $z\in\mathbb{C}$.

Lemma 10. *Let $f\in\mathbb{C}[z,\zeta]$ be a nonconstant Hermitian homogeneous polynomial of two variables. Then there exist $a\in\mathbb{C},k\geq 1$ and a nonzero $\lambda\in\mathbb{C}$ such that $f\sim D_a(k,\lambda)$. Moreover, if $f\neq cz^m\zeta^m$ for every real number $c\neq 0$ and integer $m\geq 1$, then we can further choose $a\neq 0$.*

Proof. Write d for the degree of f. Since f is homogeneous, we have

$$f(z,\zeta)=\zeta^d f\left(\frac{z}{\zeta},1\right). \tag{4.6}$$

By assumption, $f(\eta,1)\in\mathbb{C}[\eta]$ is nonconstant, for otherwise $f(z,\zeta)$ is not Hermitian. Using the fundamental theorem of algebra, we may write

$$f(\eta,1)=c\eta^m\prod_{j=1}^{l}(\eta-a_j)^{k_j}. \tag{4.7}$$

Here $c\in\mathbb{C}$ is nonzero, and $m,l\geq 0$ and $k_j\geq 1$ satisfy $m+\sum_{j=1}^{l}k_j\leq d$. Moreover, a_j's are distinct nonzero complex numbers. When $l=0$, the above equation is understood as $f(\eta,1)=c\eta^m$. By (4.6) and (4.7), we have

$$f(z,\zeta)=cz^m\zeta^n\prod_{j=1}^{l}(z-a_j\zeta)^{k_j},\quad \text{where}\quad n=d-m-\sum_{j=1}^{l}k_j. \tag{4.8}$$

We first consider the case where $l=0$. In this case, $f(z,\zeta)=cz^m\zeta^n$. Since f is nonconstant and Hermitian, we must have $c\in\mathbb{R}, c\neq 0$, and $n=m\geq 1$. The conclusion of the lemma follows if we choose $a=0, k=m\geq 1, \lambda=c\neq 0$.

We next assume $l\geq 1$. Then by (4.8), the conclusion of the lemma follows if we choose $a=-a_1\neq 0, k=k_1\geq 1, \lambda=ca_1^m\prod_{j=2}^{l}(a_1-a_j)^{k_j}\neq 0$. This proves the first part of Lemma 10.

Note if f is not a multiple of $z^m\zeta^m$ for any integer m, then it can only be the latter case, and this establishes the second part of Lemma 10. □

We next extend the above definition to rational functions.

Definition 11. Let $g\in\mathbb{C}(z,\zeta)$ be a rational function. Write $g=\frac{f_1}{f_2}$, where $f_1,f_2\in\mathbb{C}[z,\zeta]$ and $f_2\neq 0$. If $f_i\sim D_a(k_i,\lambda_i), 1\leq i\leq 2$, with $k_1,k_2\geq 0$ and $\lambda_2\neq 0$, then we say $g\sim D_a(k_1-k_2,\frac{\lambda_1}{\lambda_2})$. Note that k_1-k_2 could be negative.

Note if $g\in\mathbb{C}(z,\zeta)$ and $g\sim D_a(k,\lambda)$, then we have $g\sim D_a(k-1,0)$. We next make a few more observations.

Lemma 12. *If $g\in\mathbb{C}(z,\zeta)$ and $g\sim D_a(k,\lambda)$ for some $a\in\mathbb{C}$, then the following hold:*

(1) $\partial_z g\sim D_a(k-1,k\lambda)$ and $\partial_\zeta g\sim D_a(k-1,ak\lambda)$;

(2) $\partial_z\partial_\zeta g\sim D_a(k-2,ak(k-1)\lambda)$.

Proof. Write $g=\frac{f_1}{f_2}$ with $f_1,f_2\in\mathbb{C}[z,\zeta], f_2\neq 0$. Write $f_i=(z+a\zeta)^{k_i}h_i$ for $1\leq i\leq 2$, where $h_1,h_2\in\mathbb{C}[z,\zeta], k_1,k_2\geq 0, k_1-k_2=k$ and $h_2(-a,1)\neq 0$, $\frac{h_1(-a,1)}{h_2(-a,1)}=\lambda$. A routine computation yields

$$\begin{aligned}\partial_z g&=\frac{f_2\partial_z f_1-f_1\partial_z f_2}{f_2^2}\\&=\frac{(k_1-k_2)(z+a\zeta)^{k_1+k_2-1}h_1h_2+(z+a\zeta)^{k_1+k_2}(h_2\partial_z h_1-h_1\partial_z h_2)}{(z+a\zeta)^{2k_2}h_2^2}.\end{aligned}$$

Then it is clear that $\partial_z g\sim D_a(k-1,k\lambda)$. Similarly one can show $\partial_\zeta g\sim D_a(k-1,ak\lambda)$. This finishes the proof of part (1). The conclusion in part (2) follows immediately from part (1). □

The statements in the next lemma follow from direct computations. We omit the proof.

Lemma 13. *Let $g_1,g_2\in\mathbb{C}(z,\zeta)$ and $a\in\mathbb{C}$. Assume $g_i\sim D_a(k_i,\lambda_i)$ for $1\leq i\leq 2$ where $k_i\in\mathbb{Z}$ and $\lambda_i\in\mathbb{C}$, then the following hold:*

(1) $g_1g_2\sim D_a(k_1+k_2,\lambda_1\lambda_2)$;

(2) $cg_1\sim D_a(k_1,c\lambda_1)$ for any complex number c;

(3) $g_1 + g_2 \sim D_a(k_1, \lambda_1 + \lambda_2)$ *if* $k_1 = k_2$*; and* $g_1 + g_2 \sim D_a(k_1, \lambda_1)$ *if* $k_1 < k_2$*;*

(4) In addition assume $\lambda_2 \neq 0$*. Then* $\frac{g_1}{g_2} \sim D_a\left(k_1 - k_2, \frac{\lambda_1}{\lambda_2}\right)$.

We are now ready to prove Theorem 8.

Proof of Theorem 8. Recall $q = \partial_z \partial_{\bar{z}} p$ and $Q = q(\partial_z \partial_{\bar{z}} p) - (\partial_z q)(\partial_{\bar{z}} q)$ are real polynomials in $\mathbb{C}[z, \bar{z}]$. Note we can assume q is nonconstant, for otherwise the conclusion is trivial. We will identify $p(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]$ with its complexification $p(z, \zeta) \in \mathbb{C}[z, \zeta]$ (where we replace $\bar{z}$ by a new variable ζ). Moreover, since $p(z, \bar{z})$ is real-valued, $p(z, \zeta)$ is Hermitian. Likewise for $q(z, \bar{z})$ and $Q(z, \bar{z})$. To establish Theorem 8, it suffices to show that $q(z, \zeta) = c_0 z^m \zeta^m$ for some constant c_0 and integer $m \geq 1$. Seeking a contraction, suppose the conclusion fails. Then by Lemma 10, we can find some complex numbers $a \neq 0, \lambda \neq 0$, and some integer $k \geq 1$ such that $q \sim D_a(k, \lambda)$. That is, we can write $q(z, \zeta) = (z + a\zeta)^k h$, where $h \in \mathbb{C}[z, \zeta]$ and $h(-a, 1) = \lambda$. A direct computation yields the following holds for some $\hat{h} \in \mathbb{C}[z, \zeta]$.

$$Q(z, \zeta) = -ak(z + a\zeta)^{2k-2} h^2 + (z + a\zeta)^{2k-1} \hat{h}.$$

Thus we have $Q \sim D_a\left(2k - 2, -ak\lambda^2\right)$. By assumption $b_3 \equiv 0$. We multiply it by $\frac{48}{q}$ and use the standard complexification to get

(4.9)
$$\left[q^{-1} \partial_z \partial_\zeta\right]^2 q^{-3} Q - q^{-4} Q \left[\partial_z \partial_\zeta\right] q^{-3} Q - q^{-1} \left[\partial_\zeta \left(q^{-3} Q\right)\right] \left[\partial_z \left(q^{-3} Q\right)\right] = 0.$$

On the other hand, by Lemma 13, $q^3 \sim D_a\left(3k, \lambda^3\right)$ and $q^{-3} Q \sim D_a\left(-k - 2, -\frac{ak}{\lambda}\right)$. Then by Lemma 12,

$$\partial_z \left(q^{-3} Q\right) \sim D_a \left(-k - 3, \frac{ak(k+2)}{\lambda}\right); \quad \partial_\zeta \left(q^{-3} Q\right) \sim D_a \left(-k - 3, \frac{a^2 k(k+2)}{\lambda}\right).$$

Using the above and Lemma 13, we can compute the last term on the left hand side of (4.9):

$$-q^{-1} \left[\partial_\zeta \left(q^{-3} Q\right)\right] \left[\partial_z \left(q^{-3} Q\right)\right] \sim D_a \left(-3k - 6, -\frac{a^3 k^2 (k+2)^2}{\lambda^3}\right).$$

Similarly, we compute the first two terms on the left hand side of (4.9):

$$\left[q^{-1} \partial_z \partial_\zeta\right]^2 q^{-3} Q \sim D_a \left(-3k - 6, -\frac{a^3 k(k+2)(k+3)(2k+4)(2k+5)}{\lambda^3}\right);$$

$$-q^{-4} Q \left[\partial_z \partial_\zeta\right] q^{-3} Q \sim D_a \left(-3k - 6, -\frac{a^3 k^2 (k+2)(k+3)}{\lambda^3}\right).$$

Consequently, the left hand side of (4.9) equals to $D_a(-3k - 6, T)$, where

$$T = -\frac{a^3 k(k+2)}{\lambda^3} \left[k(k+2) + (k+3)(2k+4)(2k+5) + k(k+3)\right] \neq 0.$$

This means the left hand side of (4.9) is nonzero, a contradiction. The proof is completed. $\square$

Remark 14. It would be interesting to compare our work with that of Bedford and Pinchuk [2].

4.3. **The case** $p = \frac{c}{2}\left(z_1\bar{z}_1\right)^{\frac{r}{2}}$**.** We next consider the particular case when $p = \frac{c}{2}\left(z_1\bar{z}_1\right)^{\frac{r}{2}}$ for $c > 0$ (recall r must be even). Here it becomes possible to compute the Bergman kernel B_p explicitly.

Theorem 15. *The model Bergman kernel corresponding to the homogeneous subhamonic polynomial* $p = \frac{c}{2}\left(z_1\bar{z}_1\right)^{\frac{r}{2}}$ *is given by*

(4.10)
$$B_p\left(z_1, z_1'\right) = \frac{re^{-\left[p(z_1)+p\left(z_1'\right)\right]}c^{\frac{2}{r}}}{2\pi}G\left(c^{\frac{2}{r}}z_1\overline{z_1'}\right), \quad \textit{where}$$
(4.11)
$$G\left(x\right) \coloneqq \sum_{\alpha=0}^{\frac{r}{2}-1}\frac{x^{\alpha}}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)} + x^{\frac{r}{2}-1}e^{x^{\frac{r}{2}}}\left[\sum_{\alpha=0}^{\frac{r}{2}-1}\frac{\Gamma\left(\frac{2(\alpha+1)}{r}\right)-\Gamma\left(\frac{2(\alpha+1)}{r}, x^{\frac{r}{2}}\right)}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)}\right]$$

is given in terms of the incomplete gamma function $\Gamma\left(a, u\right) \coloneqq \int_u^{\infty} t^{a-1}e^{-t}dt$*,* $u > 0$*.*

Proof. From the formulas $\Box_p = \bar{\partial}_p^*\bar{\partial}_p$ and $\bar{\partial}_p \coloneqq \partial_{\bar{z}_1} + \partial_{\bar{z}_1}p = \partial_{\bar{z}_1} + \frac{cr}{4}z_1^{\frac{r}{2}}\bar{z}_1^{\frac{r}{2}-1}$, an orthonormal basis for $\ker\left(\Box_p\right)$ is easily found to be

$$s_{\alpha} \coloneqq \left(\frac{1}{2\pi}\frac{r}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)}c^{\frac{2(\alpha+1)}{r}}\right)^{1/2} z_1^{\alpha}e^{-p}, \quad \alpha \in \mathbb{N}_0.$$

Since $B_p = \sum s_{\alpha}\overline{s_{\alpha}}$, we have

$$B_p\left(z_1, z_1'\right) = \frac{re^{-\left[p(z_1)+p\left(z_1'\right)\right]}}{2\pi}\sum_{\alpha\in\mathbb{N}_0}\frac{1}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)}c^{\frac{2(\alpha+1)}{r}}\left(z_1\overline{z_1'}\right)^{\alpha}. \tag{4.12}$$

To compute the above in a closed form, consider the series

$$F\left(y\right) \coloneqq \sum_{\alpha=0}^{\infty}\frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)} = \sum_{\alpha=0}^{s-1}\frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)} + \underbrace{\sum_{\alpha=s}^{\infty}\frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)}}_{F_0(y)\coloneqq},$$

for $s = \frac{r}{2}$. Differentiating the second term in the series and using $\Gamma\left(z+1\right) = z\Gamma\left(z\right)$ yields $F_0'\left(y\right) = F_0\left(y\right) + \sum_{\alpha=0}^{s-1}\frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)}$ for $y > 0$. This ODE can be solved (uniquely) with the boundary condition $F_0\left(0\right) = 0$ to give

$$F_0\left(y\right) = e^{y}\left[\sum_{\alpha=0}^{s-1}\frac{\Gamma\left(\frac{\alpha+1}{s}\right)-\Gamma\left(\frac{\alpha+1}{s}, y\right)}{\Gamma\left(\frac{\alpha+1}{s}\right)}\right] \tag{4.13}$$

in terms of the incomplete gamma function. Thus in particular we have computed $F(y) := y^{\frac{1}{s}-1} G\left(y^{\frac{1}{s}}\right)$, where G is as defined in (4.11). Finally we note from (4.12) that

$$B_p(z, z') = \frac{r e^{-\left[p(z_1)+p\left(z_1'\right)\right]} c^{\frac{2}{r}}}{2\pi} x^{s-1} F(x^s),$$

for $x = c^{\frac{2}{r}} z_1 \overline{z_1'}$, completing the proof. □

5. Proof of the main theorem

In this section we finally prove Theorem 1.

Proof of Theorem 1. It suffices to show that D is strongly pseudoconvex, or the type $r = 2$ along the boundary, as thereafter one can apply Fu-Wong [13] and Nemirovski-Shafikov [25]. To this end, suppose $x^* \in \partial D$ is a point on the boundary of type $r = r(x^*) \geq 2$. By (2.5) and Proposition 3, under the assumption of Theorem 1, the Bergman kernel $K = K_D$ of the domain satisfies the following Monge-Ampère equation inside D.

(5.1)
$$\begin{aligned} J(K) = \det &\begin{pmatrix} K & \bar{Z}K & \bar{W}K \\ ZK & \left(Z\bar{Z} - \left[Z, \bar{Z}\right]^{0,1}\right)K & \left(Z\bar{W} - \left[Z, \bar{W}\right]^{0,1}\right)K \\ WK & \left(W\bar{Z} - \left[W, \bar{Z}\right]^{0,1}\right)K & \left(W\bar{W} - \left[W, \bar{W}\right]^{0,1}\right)K \end{pmatrix} \\ &= \frac{9\pi^2}{2} K^4. \end{aligned}$$

Here we have used the orthonormal frame of $T^{1,0}\mathbb{C}^2$ given by $Z = \frac{1}{2}(U_1 - iU_2)$, $W = \frac{1}{2}(U_0 - iU_3)$ defined prior to Theorem 4. Using (3.2) and (3.13), we compute the $(0,1)$ components of the commutators above:

$$\begin{aligned} \left[Z, \bar{Z}\right]^{0,1} &= \left[-\Delta p(z_1) \frac{i}{2} \partial_{x_3}\right]^{0,1} + O(-1) \\ &= \frac{\Delta p(z_1)}{2} \left(W - \bar{W}\right)^{0,1} + O(-1) \\ &= -\frac{\Delta p(z_1)}{2} \bar{W} + O(-1); \\ \left[Z, \bar{W}\right]^{0,1} &= O(-r); \quad \left[W, \bar{Z}\right]^{0,1} = O(-r); \quad \left[W, \bar{W}\right]^{0,1} = O(-2r+1). \end{aligned}$$

This allows us to compute the most singular term in the asymptotics of both sides of (5.1) as $z \to x^*$ along the tangential path $z(\epsilon)$ in (3.4). By Theorem 4, one obtains along $z(\epsilon)$,

$$J(K) = \left[(-2\rho)^{-2-\frac{2}{r}}\right]^4 \left[\det \begin{pmatrix} \tilde{B}_{p,0} & \partial_{\bar{z}_1}\tilde{B}_{p,0} & \tilde{B}_{p,1} \\ \partial_{z_1}\tilde{B}_{p,0} & \partial_{z_1}\partial_{\bar{z}_1}\tilde{B}_{p,0} + \left[\frac{\Delta p}{2}\right]\tilde{B}_{p,1} & \partial_{z_1}\tilde{B}_{p,1} \\ \tilde{B}_{p,1} & \partial_{\bar{z}_1}\tilde{B}_{p,1} & \tilde{B}_{p,2} \end{pmatrix} (z_{1,V}) + o_\epsilon(1)\right]$$

$$K^4 = \left[(-2\rho)^{-2-\frac{2}{r}}\right]^4 \left[\tilde{B}_{p,0}\left(z_{1,V}\right)^4 + o_\epsilon(1)\right].$$

Here we say a function ϕ is $o_\epsilon(1)$ if $\phi(\epsilon)$ goes to 0 as $\epsilon \to 0^+$. (Recall $\rho = -\epsilon^r$ along the path). Thus comparing the leading coefficients in the asymptotics gives the following equation

(5.2)

$$\det\begin{pmatrix} \tilde{B}_{p,0} & \partial_{\bar{z}_1}\tilde{B}_{p,0} & \tilde{B}_{p,1} \\ \partial_{z_1}\tilde{B}_{p,0} & \partial_{z_1}\partial_{\bar{z}_1}\tilde{B}_{p,0} + \left[\frac{\Delta p}{2}\right]\tilde{B}_{p,1} & \partial_{z_1}\tilde{B}_{p,1} \\ \tilde{B}_{p,1} & \partial_{\bar{z}_1}\tilde{B}_{p,1} & \tilde{B}_{p,2}\end{pmatrix}(z_1) = \frac{9\pi^2}{2}\tilde{B}_{p,0}(z_1)^4,$$

at each $z_1 \in \mathbb{R}^2$, for the model Bergman kernel. Here $\tilde{B}_{p,\alpha_0}$ is as defined in (3.7).

Finally, one chooses z_1 such that $\Delta p(z_1) \neq 0$ and substitutes $z_1 \mapsto t^{\frac{1}{r}}z_1$ in the last equation (5.2) above for the model. The terms involved in the above equation are then of the following form from the definition (3.7).

$$\begin{aligned}\tilde{B}_{p,\alpha_0}\left(t^{\frac{1}{r}}z_1\right) &= \frac{1}{\pi}\int_0^\infty e^{-s}s^{1+\frac{2}{r}+\alpha_0}B_p\left(s^{\frac{1}{r}}t^{\frac{1}{r}}z_1\right)ds \\ &= \frac{t^{-2-\frac{2}{r}-\alpha_0}}{\pi}\int_0^\infty e^{-\frac{\tau}{t}}\tau^{1+\frac{2}{r}+\alpha_0}B_p\left(\tau^{\frac{1}{r}}z_1\right)d\tau\end{aligned}$$

Next we use Proposition 6 to obtain an asymptotic expansion for the above and its derivatives. Namely, the kernel $\tau^{1+\frac{2}{r}+\alpha_0}B_p\left(\tau^{\frac{1}{r}}z_1\right) \in S^{2+\alpha_0}_{\tau,\mathrm{cl}}$ is a classical symbol by 6 and thus standard asymptotics for its Laplace transform (cf. [4, eqn. 1.6]) give

(5.3)

$$\begin{aligned}&\left[\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}\tilde{B}_{p,\alpha_0}\right]\left(t^{\frac{1}{r}}z_1\right) \\ &= t^{1-\frac{2+\alpha_1+\alpha_2}{r}}\left[\sum_{j=0}^{N+2+\alpha_0}c_jt^{-j} + \sum_{j=0}^{N}d_jt^{-(3+\alpha_0+j)}\ln t + O\left(t^{-(3+\alpha_0+N)}\right)\right],\end{aligned}$$

$\forall N \in \mathbb{N}$, as $t \to \infty$. The logarithmic terms above arise from integrating terms of order τ^j, $j<0$, in the classical expansion of the given symbol. In particular the leading logarithmic term is $d_0 = \frac{1}{2\pi^2}\partial_{z_1}^{\alpha_1}\partial_{\bar{z}_1}^{\alpha_2}b_{3+\alpha_0}$.

The above allows us to compute the asymptotics of both sides of the equation (5.2) as $t \to \infty$. In particular the right hand side of (5.2) is seen to contain the logarithmic term

$$\frac{9\pi^2}{2}b_3^4\left(\frac{1}{2\pi^2}t^{-2-\frac{2}{r}}\ln t\right)^4$$

in its asymptotic expansion. Such a term involving the fourth power of a logarithm is missing from the left hand side of (5.2). This particularly gives $b_3 = 0$.

Using Theorem 8, it now follows that $q(z,\bar{z}) = c_0(z_1\bar{z}_1)^{\frac{r}{2}-1}$ for some $c_0 > 0$. Since p has no purely holomorphic or anti-holomorphic terms in z_1, this gives $p = \frac{c}{2}(z_1\bar{z}_1)^{\frac{r}{2}}$ for some $c > 0$.

However, the model kernel B_p for this potential $p = \frac{c}{2}(z_1\bar{z}_1)^{\frac{r}{2}}$ was computed in Theorem 15. Suppose $r > 2$. By Theorem 15 and definition of $\tilde{B}_{p,\alpha_0}$ in (3.7),

$$\tilde{B}_{p,\alpha_0}(0) = \frac{1}{\pi}\Gamma\left(2+\frac{2}{r}+\alpha_0\right)B_p(0) = \frac{1}{2\pi^2}\Gamma\left(2+\frac{2}{r}+\alpha_0\right)\frac{r}{\Gamma\left(\frac{2}{r}\right)}c^{\frac{2}{r}};$$

$$\left[\partial_{z_1}\tilde{B}_{p,\alpha_0}\right](0) = \left[\partial_{\bar{z}_1}\tilde{B}_{p,\alpha_0}\right](0) = 0;$$

$$\begin{aligned}\left[\partial_{z_1}\partial_{\bar{z}_1}\tilde{B}_{p,\alpha_0}\right](0) &= \frac{1}{\pi}\Gamma\left(2+\frac{4}{r}+\alpha_0\right)\left[\partial_{z_1}\partial_{\bar{z}_1}B_p\right](0)\\ &= \frac{1}{2\pi^2}\Gamma\left(2+\frac{4}{r}+\alpha_0\right)\frac{r}{\Gamma\left(\frac{4}{r}\right)}c^{\frac{4}{r}}.\end{aligned}$$

Plugging the above into (5.2) with $z_1 = 0$, and noting $\Delta p(0) = 0$ as $r > 2$, we obtain

$$\begin{aligned}&\left(\frac{r}{2\pi^2}\right)^3\frac{\Gamma\left(2+\frac{4}{r}\right)}{\Gamma\left(\frac{4}{r}\right)}c^{\frac{8}{r}}\left[\frac{\Gamma\left(2+\frac{2}{r}\right)}{\Gamma\left(\frac{2}{r}\right)}\frac{\Gamma\left(4+\frac{2}{r}\right)}{\Gamma\left(\frac{2}{r}\right)} - \frac{\Gamma\left(3+\frac{2}{r}\right)}{\Gamma\left(\frac{2}{r}\right)}\frac{\Gamma\left(3+\frac{2}{r}\right)}{\Gamma\left(\frac{2}{r}\right)}\right]\\ &= \frac{9\pi^2}{2}\left[\frac{r}{2\pi^2}\frac{\Gamma\left(2+\frac{2}{r}\right)}{\Gamma\left(\frac{2}{r}\right)}c^{\frac{2}{r}}\right]^4.\end{aligned}$$

Using $\Gamma(z+1) = z\Gamma(z)$, the above simplifies to the equation

$$\left(1+\frac{4}{r}\right)\left(2+\frac{2}{r}\right) = \frac{9}{4}\left(1+\frac{2}{r}\right)^2.$$

Solving this quadratic equation yields $r = 2$, a plain contradiction. This finishes the proof. $\square$

Remark 16. Note in our proof above, we compared the $\left(t^{-2-\frac{2}{r}}\ln t\right)^4$ term on both sides of (5.2). For that, we only used the information of b_3, where b_3 arises in the coefficient of the first $\ln t$ term in the asymptotics for the model Bergman kernel (see (5.3)). The authors also compared the non-logarithmic terms on two sides of (5.2): the $\left(t^{-2-\frac{2}{r}}\right)^4$ and $\left(t^{-2-\frac{2}{r}}\right)^4 t^{-1}$ terms, whose calculations then involve b_0 and b_1. Nevertheless, we only got tautologies and thus derived no contradiction. It is interesting to compare this with the proofs of Cheng's conjecture. In dimension 2, Fu-Wong [13] used information of the logarithmic term in the Fefferman expansion of the Bergman kernel (3.1); while in higher dimension, Huang and the second author [19] utilized information of the non-logarithmic term (principal singular term) in the expansion (3.1).

References

[1] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real submanifolds in complex space and their mappings*, vol. 47 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1999.

[2] E. Bedford and S. I. Pinchuk, *Domains in* $\mathbf{C}^2$ *with noncompact groups of holomorphic automorphisms*, Mat. Sb. (N.S.), 135(177) (1988), pp. 147–157, 271.

[3] S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, No. V, American Mathematical Society, Providence, R.I., revised ed., 1970.

[4] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, (1976), pp. 123–164. Astérisque, No. 34–35.

[5] S.-Y. Cheng, *Open Problems, Conference on Nonlinear Problems in Geometry held in Katata, Sep. 3-8, 1979*, Tohoku University, Department of Math., Sendai, 1979.

[6] S. Y. Cheng and S. T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math., 33 (1980), pp. 507–544.

[7] M. Christ, *Embedding compact three-dimensional CR manifolds of finite type in* $\mathbf{C}^n$, Ann. of Math. (2), 129 (1989), pp. 195–213.

[8] P. Ebenfelt, M. Xiao, and H. Xu, *On the classification of normal stein spaces and finite ball quotients with Bergman-Einstein metrics*, Int. Math. Res. Not. IMRN, (2022), pp. 15240–15270.

[9] M. Engliš, *The asymptotics of a Laplace integral on a Kähler manifold*, J. Reine Angew. Math., 528 (2000), pp. 1–39.

[10] ———, *Pseudolocal estimates for* $\overline{\partial}$ *on general pseudoconvex domains*, Indiana Univ. Math. J., 50 (2001), pp. 1593–1607.

[11] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math., 26 (1974), pp. 1–65.

[12] S. Fu, *Geometry of Reinhardt domains of finite type in* $\mathbb{C}^2$, J. Geom. Anal., 6 (1996), pp. 407–431.

[13] S. Fu and B. Wong, *On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics*, Mathematical Research Letters, 4 (1997), pp. 697–703.

[14] S. Ganguly and S. Sinha, *Bergman-Einstein metric on two-dimensional Stein spaces*, arXiv:2210.12323v1, (2022).

[15] C.-Y. Hsiao, *Projections in several complex variables*, Mém. Soc. Math. Fr. (N.S.), (2010), p. 131.

[16] C.-Y. Hsiao and G. Marinescu, *Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles*, Comm. Anal. Geom., 22 (2014), pp. 1–108.

[17] C.-Y. Hsiao and N. Savale, *Bergman–Szegő kernel asymptotics in weakly pseudoconvex finite type cases*, J. Reine Angew. Math., (2022), pp. 173–223.

[18] X. Huang and X. Li, *Bergman-Einstein metric on a Stein space with a strongly pseudoconvex boundary*, arXiv: 2008.03645v1, (2020).

[19] X. Huang and M. Xiao, *Bergman-Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries*, J. Reine Angew. Math., 770 (2021), pp. 183–203.

[20] P. F. Klembeck, *Kähler metrics of negative curvature, the Bergmann metric near the boundary, and the Kobayashi metric on smooth bounded strictly pseudoconvex sets*, Indiana Univ. Math. J., 27 (1978).

[21] S.-Y. Li, *Characterization for balls by potential function of Kähler-Einstein metrics for domains in* $\mathbb{C}^n$, Comm. Anal. Geom., 13 (2005), pp. 461–478.

[22] ———, *On characterization for a class of pseudo-convex domains with positive constant pseudo-scalar curvature on their boundaries*, Comm. Anal. Geom., 17 (2009), pp. 17–35.

[23] G. Marinescu and N. Savale, *Bochner Laplacian and Bergman kernel expansion of semipositive line bundles on a Riemann surface*, Math. Ann., 389 (2024), pp. 4083–4124.

[24] N. Mok and S.-T. Yau, *Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*, in The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), vol. 39 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1983, pp. 41–59.

[25] S. Y. Nemirovski and R. G. Shafikov, *Conjectures of Cheng and Ramadanov*, Uspekhi Mat. Nauk, 61 (2006), pp. 193–194.

[26] J.-P. Rosay, *Sur une caractérisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes*, Annales de l'institut Fourier, 29 (1979), pp. 91–97.

[27] E. B. Vinberg, S. G. Gindikin, and I. I. Pjateckiĭ-Šapiro, *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moskov. Mat. Obšč., 12 (1963), pp. 359–388.

[28] B. Wong, *Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group.*, Invent. Math., 41 (1977), pp. 253–258.

[29] S. T. Yau, *Problem section*, in Seminar on Differential Geometry, vol. 102 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, N.J., 1982, pp. 669–706.

Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany

Email address: nsavale@math.uni-koeln.de

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA

Email address: m3xiao@ucsd.edu