KOSZUL COMPLEXES, BIRKHOFF NORMAL FORM AND THE MAGNETIC DIRAC OPERATOR

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Abstract. We consider the semi-classical Dirac operator coupled to a magnetic potential on a large class of manifolds including all metric contact manifolds. We prove a sharp local Weyl law and a bound on its eta invariant. In the absence of a Fourier integral parametrix, the method relies on the use of almost analytic continuations combined with the Birkhoff normal form and local index theory.

1. Introduction

Semi-classical analysis concerns the study of the spectrum of semi-classical (h-)pseudodifferential operators $A_h : C^\infty(X) \to C^\infty(X)$, $h \in (0,1]$, in the limit $h \to 0$ and is now the subject of several texts [10, 13, 17, 18, 21, 23, 28]. Standard examples of such operators include the Schroedinger operator $A_h = -h^2\Delta_X + V$ on a compact n-dimensional Riemannian manifold X with potential $V \in C^\infty(X)$. The clearest asymptotic result is given by the celebrated local Weyl law (cf. eg. [10] Ch. 10): assuming 0 is not a critical value of the symbol $\sigma(A) = a(x,\xi) \in C^\infty(T^*X)$, the number of eigenvalues $N(-ch, ch)$ of A_h in the interval $(-ch, ch)$ satisfies

$$N(-ch, ch) = O\left(h^{-n+1}\right) \tag{1.1}$$

as $h \to 0$, $\forall c > 0$. Similar results also exist in the case where 0 is a Morse-Bott critical level for the symbol (cf. [6]). In the critical case, the exponent in the local Weyl law may drop depending on the co-dimension of zero energy level $\Sigma_0^A := \{a(x,\xi) = 0\}$ and the signature of the normal Hessian. The local Weyl laws thus obtained are sharp and are proved using a parametrix construction for the evolution operator $e^{\frac{it}{h}A_h}$ as a Fourier integral operator.

In the context of non-scalar operators $A_h : C^\infty(X;E) \to C^\infty(X;E)$ acting on sections of a vector bundle E, fewer result are known. The simplest case is when the non-scalar symbol $a(x,\xi) \in C^\infty(T^*X;E)$ is smoothly diagonalizable near the zero energy level $\Sigma_0^A = \{\det(a(x,\xi)) = 0\}$. In this case similar Fourier integral methods apply (cf. [11, 21] or [12, 24] for an exposition in the microlocal/classical setting). For non-scalar operators another method is provided under the microhyperbolicity condition of Ivrii (cf. [18] Ch. 2,3 or [10] Ch. 12). In this paper, we study the particular

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case of the magnetic Dirac operator where neither diagonalizability nor the microhyperbolicity condition is satisfied.

More precisely, let $\left(X,g^{TX}\right)$ be an oriented Riemannian manifold of odd dimension $n=2m+1$ equipped with a spin structure. Let S be the corresponding spin bundle and let L be an auxiliary Hermitian line bundle. Fix a unitary connection A_0 on L and let $a\in\Omega^1\left(X;\mathbb{R}\right)$ be a one form. This gives a family of unitary connections on L via $\nabla^h=A_0+\frac{i}{h}a$ and a corresponding family of coupled magnetic Dirac operators

$$D_h\coloneqq hD_{A_0}+ic\left(a\right) \tag{1.2}$$

for $h\in(0,1]$.

In order to derive sharp spectral asymptotics, we shall make a couple of restrictive assumptions on the one form a and the metric g^{TX}. First, the one form a will be assumed to be a contact one form (i.e. one satisfying $a\wedge\left(da\right)^m>0$). This gives rise to the contact hyperplane $H=\ker\left(a\right)\subset TX$ as well as the Reeb vector field R defined via $i_Rda=0$, $i_Ra=1$.

To state the assumption on the metric, consider the contracted endomorphism $\mathfrak{J}:T_xX\to T_xX$ defined at each point $x\in X$ via

$$da\left(v_1,v_2\right)=g^{TX}\left(v_1,\mathfrak{J}v_2\right),\quad\forall v_1,v_2\in T_xX.$$

From the contact assumption, $\mathfrak{J}$ has a one dimensional kernel spanned by the Reeb vector field R. The endomorphism $\mathfrak{J}$ is clearly anti-symmetric with respect to the metric

$$g^{TX}\left(v_1,\mathfrak{J}v_2\right)=-g^{TX}\left(\mathfrak{J}v_1,v_2\right)$$

and hence its non-zero eigenvalues come in purely imaginary pairs $\pm i\mu$; $\mu>0$. The assumption on the metric $g^{TX}$is then as follows.

Definition 1.1. We say that the metric g^{TX} is *suitable* to the contact form a if there exist positive constants $0<\mu_1\leq\mu_2\leq\ldots\leq\mu_m$ (independent of $x\in X$) and a positive real function $\nu\left(x\right)>0$ such that

$$\mathrm{Spec}\left(\mathfrak{J}_x\right)=\left\{0,\pm i\mu_1\nu\left(x\right),\pm i\mu_2\nu\left(x\right),\ldots,\pm i\mu_m\nu\left(x\right)\right\} \tag{1.3}$$

$\forall x\in X$.

Before proceeding further, we give two examples of suitable metrics.

(1) The dimension of the manifold $\dim X=3$. In this case any metric g^{TX} is suitable as $\mathrm{Spec}\left(\mathfrak{J}_x\right)=\left\{0,\pm i\left|da\right|\right\}$ has only two non-zero eigenvalues.
(2) There is a smooth endomorphism $J:TX\to TX$, such that $\left(X^{2m+1},a,g^{TX},J\right)$ is a metric contact manifold. That is, we have

$$\begin{aligned} J^2v_1 &= -v_1+a\left(v_1\right)R,\\ g^{TX}\left(v_1,Jv_2\right) &= da\left(v_1,v_2\right),\quad\forall v_1,v_2\in T_xX. \end{aligned} \tag{1.4}$$

In this case the nonzero eigenvalues of $\mathfrak{J}_x=J_x$ are $\pm i$ (each with multiplicity m). For any given contact form a there exists an infinite dimensional space of $\left(g^{TX},J\right)$ satisfying (1.4). This case in particular includes all strictly pseudo-convex CR manifolds.

In addition to the local Weyl law we shall also be interested in the asymptotics of the eta invariant $\eta_h=\eta\left(D_h\right)$ of the Dirac operator, formally its signature (see 2.1 for a definition) . The main result is now stated as follows.

Theorem 1.2. *Under the contact and suitability assumptions on a,g^{TX}, the local Weyl counting function and eta invariant of D_h satisfy the sharp asymptotics*

$$N\left(-ch,ch\right) = O\left(h^{-m}\right) \tag{1.5}$$

$$\eta_h = O\left(h^{-m}\right) \tag{1.6}$$

as $h\to 0$.

We note that the exponents above are significantly lower than (1.1). This is again partly attributed to the high co-dimension of Σ_0^D.

The proof of the asymptotic result Theorem 1.2 above will be based on a functional trace expansion. To state the trace expansion involved, set $\nu_0:=\mu_1\left[\min_{x\in X}\nu\left(x\right)\right]$ and choose $f\in C_c^\infty\left(-\sqrt{2\nu_0},\sqrt{2\nu_0}\right)$. Pick real numbers $0<T'<T$ and let $\theta\in C_c^\infty\left(\left(-T,T\right);\left[0,1\right]\right)$ such that $\theta\left(x\right)=1$ on $\left(-T',T'\right)$. Let

$$\begin{aligned}\mathcal{F}^{-1}\theta\left(x\right) &:= \check{\theta}\left(x\right)=\frac{1}{2\pi}\int e^{ix\xi}\theta\left(\xi\right)d\xi\\ \mathcal{F}_h^{-1}\theta\left(x\right) &:= \frac{1}{h}\check{\theta}\left(\frac{x}{h}\right)=\frac{1}{2\pi h}\int e^{\frac{i}{h}x\xi}\theta\left(\xi\right)d\xi\end{aligned}$$

be its classical and semi-classical inverse Fourier transforms respectively. We shall then prove.

Theorem 1.3. *Let a,g^{TX} be a contact form and suitable metric respectively. There exist smooth functions $u_j\in C^\infty\left(\mathbb{R}\right)$ such that there is a trace expansion*

$$tr\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right]= \tag{1.7}$$

$$tr\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right]=h^{-m-1}\left(\sum_{j=0}^{N-1}f\left(\lambda\right)u_j\left(\lambda\right)h^{j/2}+O\left(h^{N/2}\right)\right) \tag{1.8}$$

for T sufficiently small and for each $N\in\mathbb{N}$,$\lambda\in\mathbb{R}$.

Again, the trace (1.7) should be compared with the wave trace expansions for scalar and microhyperbolic operators ([10] ch. 10, 12) although a different scale of size $\sqrt{h}$ is being used. In the absence of a Fourier integral parametrix or microhyperbolicity our strategy is to combine the use of almost analytic continuations with local index theory expansions. We first show that the trace is $O(h^\infty)$ in the region $\text{spt}(\theta) \subset \{T > |x| \geq h^\epsilon\}$, $\epsilon \in \left(0, \frac{1}{2}\right)$ (see Lemma 3.1). Here the the lack of microhyperbolicity for the symbol poses a difficulty in the use of almost analytic continuations (cf. [10] ch. 12, see also [9]). We however show that this can be overcome with a closer understanding of the total symbol of D via its Birkhoff normal form. It is in deriving the Birkhoff normal form then that Koszul complexes are used and the assumptions on a, g^{TX} required. The local index theory method (cf. [4, 20]) finally provides the expansion in the region $\text{spt}(\theta) \subset \{|x| < h^\epsilon\}$ (see Lemma 3.2).

There is a large recent literature for semi-classical problems in the presence of magnetic fields (see [15] for a survey). In particular the extensive book of Ivrii [17] specifically considers the case of the magnetic Dirac operator in ch. 17. The Birkhoff normal form here (5.13) generalizes proposition 17.2.1 therein. Our use of normal forms should also be compared to its use in scalar cases from [8, 14, 22].

The asymptotic problem of the eta invariant (1.6) was earlier considered by the author in [25] where a non-sharp estimate was proved, under no assumptions on a, g^{TX}, via the use of the heat trace. This asymptotic problem was first considered and applied in [26] in the proof of the three-dimensional Weinstein conjecture using Seiberg-Witten theory. The three-dimensional case has been further explored in [27].

The paper is organized as follows. In Section 2 begin with preliminary notions used throughout the paper including basic facts about Clifford representations, Dirac operators and the semi-classical calculus. In 2.2.1 we we compute the spectrum of a model magnetic Dirac operator on $\mathbb{R}^m$ using Clifford representations and the harmonic oscillator. In Section 3 we perform certain reductions towards proving Theorem 1.3 including a time scale breakup of the trace into Lemma 3.1 and Lemma 3.2. These reductions are then used in Section 4 to further reduce Lemma 3.1 to the case of a Euclidean magnetic Dirac operator on $\mathbb{R}^n$. In Section 5 we obtain the Birkhoff normal form for the Euclidean magnetic Dirac operator on $\mathbb{R}^n$ from Section 4. It is here in 5.1 that Koszul complexes are employed for the normal form. In Section 6 we show how the normal form is used in proving Lemma 3.1 via the use of almost analytic continuations. In Section 7 we prove Lemma 3.2 using the methods of local index theory. In Section 8 we show how to prove the spectral estimates of Theorem 1.2 via the trace expansion Theorem 1.3. Finally, in Section A we prove some spectral estimates useful in Section 4 and Section 5.

2. Preliminaries

2.1. **Spectral invariants of the Dirac operator.** Here we review the basic facts about Dirac operators used throughout the paper with [3] providing a standard reference. Consider a compact, oriented, Riemannian manifold $\left(X, g^{TX}\right)$ of odd dimension $n = 2m + 1$. Let X be equipped with spin structure, i.e. a principal $\text{Spin}(n)$ bundle $\text{Spin}(TX) \to SO(TX)$ with an equivariant double covering of the principal $SO(n)$-bundle of orthonormal frames $SO(TX)$. The corresponding spin bundle $S = \mathrm{Spin}(TX) \times_{\mathrm{Spin}(n)} S_{2m}$ is associated to the unique irreducible representation of $\mathrm{Spin}(n)$. Let ∇^{TX} denote the Levi-Civita connection on TX. This lifts to the spin connection ∇^S on the spin bundle S. The Clifford multiplication endomorphism $c : T^*X \to S \otimes S^*$ may be defined (see 2.2) satisfying

$$c(a)^2 = -|a|^2, \quad \forall a \in T^*Y.$$

Let L be a Hermitian line bundle on Y. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(Y;\mathbb{R})$ be a 1-form on Y. This gives a family $\nabla^h = A_0 + \frac{i}{h}a$ of unitary connections on L. We denote by $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ the tensor product connection on $S \otimes L$. Each such connection defines a coupled Dirac operator

$$D_h \coloneqq hD_{A_0} + ic(a) = hc \circ \left(\nabla^{S\otimes L}\right) : C^\infty(Y; S\otimes L) \to C^\infty(Y; S \otimes L)$$

for $h \in (0,1]$. Each Dirac operator D_h is elliptic and self-adjoint. It hence possesses a discrete spectrum of eigenvalues.

We define the eta function of D_h by the formula

(2.1)
$$\eta(D_h, s) \coloneqq \sum_{\substack{\lambda\neq 0\\ \lambda\in\mathrm{Spec}(D_h)}} \mathrm{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}\left(D_h e^{-tD_h^2}\right) dt.$$

Here, and in the remainder of the paper, we use the convention that $\mathrm{Spec}(D_h)$ denotes a multiset with each eigenvalue of D_h being counted with its multiplicity. The above series converges for $\mathrm{Re}(s) > n$. It was shown in [1, 2] that the eta function possesses a meromorphic continuation to the entire complex s-plane and has no pole at zero. Its value at zero is defined to be the eta invariant of the Dirac operator

$$\eta_h \coloneqq \eta(D_h, 0).$$

By including the zero eigenvalue in (2.1), with an appropriate convention, we may define a variant known as the reduced eta invariant by

$$\bar\eta_h \coloneqq \frac{1}{2}\left\{k_h + \eta_h\right\}.$$

The eta invariant is unchanged under positive scaling

$$\eta(D_h, 0) = \eta(cD_h, 0); \quad \forall c > 0. \tag{2.2}$$

Let $L_{t,h}$ denote the Schwartz kernel of the operator $D_h e^{-tD_h^2}$ on the product $X \times X$. Throughout the paper all Schwartz kernels will be defined with respect to the Riemannian volume density. Denote by $\mathrm{tr}(L_{t,h}(x,x))$ the point-wise trace of $L_{t,h}$ along the diagonal. We may now analogously define the function

$$\eta(D_h, s, x) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}(L_{t,h}(x,x))\, dt. \tag{2.3}$$

In [5] theorem 2.6, it was shown that for $\mathrm{Re}(s) > -2$, the function $\eta\left(D_h, s, x\right)$ is holomorphic in s and smooth in x. From (2.3) it is clear that this is equivalent to

$$\mathrm{tr}\left(L_{t,h}\right) = O\left(t^{\frac{1}{2}}\right), \quad \text{as } t \to 0. \tag{2.4}$$

The eta invariant is then given by the convergent integral

$$\eta_h = \int_0^\infty \frac{1}{\sqrt{\pi t}} \mathrm{tr}\left(D_h e^{-tD_h^2}\right) dt. \tag{2.5}$$

2.2. **Clifford algebra and and its representations.** Here we review the construction of the spin representation of the Clifford algebra. The following being standard, is merely used to setup our conventions and subsequently compute the spectrum of the model magnetic Dirac operator on $\mathbb{R}^m$ in 2.2.1.

Consider a real vector space V of even dimension $2m$ with metric $\langle , \rangle$. Recall that its Clifford algebra $Cl\left(V\right)$ is defined as the quotient of the tensor algebra $T\left(V\right) := \oplus_{j=0}^{\infty} V^{\otimes j}$ by the ideal generated from the relations $v \otimes v + |v|^2 = 0$. Fix a compatible almost complex structure J and split $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into the $\pm i$ eigenspaces of J. The complexification $V \otimes \mathbb{C}$ carries an induced $\mathbb{C}$-bilinear inner product $\langle , \rangle_{\mathbb{C}}$ as well as an induced Hermitian inner product $h^{\mathbb{C}}\left(, \right)$. Next, define $S_{2m} = \Lambda^* V^{1,0}$. Clearly S_{2m} is a complex vector space of dimension 2^m on which the unique irreducible (spin)-representation of the Clifford algebra $Cl\left(V\right) \otimes \mathbb{C}$ is defined by the rule

$$c_{2m}\left(v\right)\omega = \sqrt{2}\left(v^{1,0} \wedge \omega - \iota_{v^{0,1}}\omega\right), \quad v \in V, \omega \in S_{2m}.$$

The contraction above is taken with respect to $\langle , \rangle_{\mathbb{C}}$. It is clear that $c_{2m}\left(v\right) : \Lambda^{\text{even/odd}} \to \Lambda^{\text{odd/even}}$ switches the odd and even factors. For the Clifford algebra $Cl\left(W\right) \otimes \mathbb{C}$ of an odd dimensional vector space $W = V \oplus \mathbb{R}\left[e_0\right]$ there are exactly two irreducible representations. These two (spin)-representations $S_{2m+1}^{+} = S_{2m+1}^{-} = \Lambda^* V^{1,0}$ are defined via

$$\begin{aligned} c_{2m+1}^{\pm}\left(v\right) &= c_{2m}\left(v\right), \quad v \in V \\ c_{2m+1}^{+}\left(e_0\right)\omega_{\text{even/odd}} &= -c_{2m+1}^{-}\left(e_0\right)\omega_{\text{even/odd}} = \pm i \omega_{\text{even/odd}}. \end{aligned} \tag{2.6}$$

Throughout the rest of the paper, we stick with the positive convention and use the shorthands $c = c_{2m}$, $c = c_{2m+1}^{+}$ when the index $2m$, $2m+1$ implicitly understood.

Pick an orthonormal basis $e_1, e_2, \ldots, e_{2m}$ for V in which the almost complex structure is given by $Je_{2j-1} = e_{2j}$, $1 \leq j \leq m$. An $h^{\mathbb{C}}$-orthonormal basis for $V^{1,0}$ is now given by $w_j = \frac{1}{\sqrt{2}}\left(e_{2j} + ie_{2j-1}\right)$, $1 \leq j \leq m$. A basis for S_{2m} and $S_{2m+1}^{\pm}$ is given by $w_k = w_1^{k_1} \wedge \ldots \wedge w_m^{k_m}$ with $k = \left(k_1, k_2, \ldots, k_m\right) \in \{0,1\}^m$. Ordering the above chosen bases lexicographically in k, we may define the Clifford matrices, of rank 2^m, via

$$\gamma_j^m = c\left(e_j\right), \quad 0 \leq j \leq 2m,$$

for each m. Again, we often write $\gamma_j^m = \gamma_j$ with the index m implicitly understood. Giving representations of the Clifford algebra, these matrices satisfy the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}. \tag{2.7}$$

Next, one may further define the Clifford quantization map on the exterior algebra

$$\begin{aligned} c : \Lambda^* W \otimes \mathbb{C} &\rightarrow \mathrm{End}\,(S_{2m}) \\ c\left(e_0^{k_0} \wedge \ldots \wedge e_{2m}^{k_{2m}}\right) &= c\,(e_0)^k \ldots c\,(e_{2m})^{k_{2m}}\,. \end{aligned} \tag{2.8}$$

An easy computation yields

$$c\,(e_0 \wedge \ldots \wedge e_{2m}) = i^{m+1}.$$

Furthermore, if $e_0 \wedge \ldots \wedge e_{2m}$ is designated to give a positive orientation for W then for $\omega \in \Lambda^k W$ we have

$$c\,(*\omega) = i^{m+1}\,(-1)^{\frac{k(k+1)}{2}}\,c\,(\omega) \tag{2.9}$$

$$c\,(\omega)^* = (-1)^{\frac{k(k+1)}{2}}\,c\,(\omega) \tag{2.10}$$

under the Hodge star and $h^{\mathbb{C}}$-adjoint. The Clifford quantization map (2.8) is a linear surjection with kernel spanned by elements of the form $*\omega - i^{m+1}\,(-1)^{\frac{k(k+1)}{2}}\,\omega$. Thus, in particular one has linear isomorphisms

$$c : \Lambda^{\mathrm{even/odd}} W \otimes \mathbb{C} \rightarrow \mathrm{End}\,(S_{2m})\,. \tag{2.11}$$

Next, given $(r_1, \ldots, r_m) \in \mathbb{R}^m \setminus 0$, we define

$$I_r := \{j | r_j \neq 0\} \subset \{1, 2, \ldots, m\} \tag{2.12}$$

$$Z_r := |I_r| \tag{2.13}$$

$$V_r := \bigoplus_{j \in I_r} \mathbb{C}\,[w_j] \subset V^{1,0} \tag{2.14}$$

$$\text{and} \quad w_r := \sum_{j=1}^m r_j w_j \in V_r. \tag{2.15}$$

Clearly, $\|w_r\| = |r|$. Denoting by $w_r^\perp$ the $h^{\mathbb{C}}$-orthogonal complement of $w_r \subset V_r$, one clearly has $V_r = \mathbb{C}\,[w_r] \oplus w_r^\perp$. Hence

$$\begin{aligned} \Lambda^{\mathrm{even}} V_r &= \left(\Lambda^{\mathrm{even}} w_r^\perp\right) \oplus \frac{w_r}{|r|} \wedge \left(\Lambda^{\mathrm{odd}} w_r^\perp\right) \\ \Lambda^{\mathrm{odd}} V_r &= \left(\Lambda^{\mathrm{odd}} w_r^\perp\right) \oplus \frac{w_r}{|r|} \wedge \left(\Lambda^{\mathrm{even}} w_r^\perp\right). \end{aligned} \tag{2.16}$$

Next, we define

$$\begin{aligned} \mathtt{i}_r : \Lambda^* V_r &\rightarrow \Lambda^* V_r, \quad \text{via} \\ \mathtt{i}_r(\omega) &:= \frac{w_r}{|r|} \wedge \omega \\ \mathtt{i}_r\left(\frac{w_r}{|r|} \wedge \omega\right) &:= \omega \end{aligned} \tag{2.17}$$

for $\omega \in \Lambda^* w_r^{\perp}$. Clearly, $\mathtt{i}_r^2 = 1$ with the decomposition (2.16) implying that $\mathtt{i}_r$ is a linear isomorphism between

$$\begin{aligned} \mathtt{i}_r : \Lambda^{\text{even}} V_r &\rightarrow \Lambda^{\text{odd}} V_r \\ \mathtt{i}_r : \Lambda^{\text{odd}} V_r &\rightarrow \Lambda^{\text{even}} V_r. \end{aligned}$$

Next, the endomorphism

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = (w_r \wedge + \iota_{\bar{w}_r}) : \Lambda^* V_r \rightarrow \Lambda^* V_r \tag{2.18}$$

has the form

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = \begin{bmatrix} & |r|\, \mathtt{i}_r \\ |r|\, \mathtt{i}_r & \end{bmatrix} \tag{2.19}$$

with respect to the decomposition $\Lambda^* V_r = \Lambda^{\text{odd}} V_r \oplus \Lambda^{\text{even}} V_r$. This finally allows us to write the eigenspaces of (2.18) as

$$V_r^{\pm} = (1 \pm \mathtt{i}_r)\left(\Lambda^{\text{even}} V_r\right) \tag{2.20}$$

with eigenvalue $\pm |r|$ respectively.

2.2.1. *Magnetic Dirac operator on* $\mathbb{R}^m$. We now define the magnetic Dirac operator on $\mathbb{R}^m$ via

$$D_{\mathbb{R}^m} = \sum_{j=1}^{m} \left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} \left[\gamma_{2j}\left(h\partial_{x_j}\right) + i\gamma_{2j-1} x_j\right] \in \Psi^1_{\text{cl}}\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right). \tag{2.21}$$

Its square is computed in terms of the harmonic oscillator

$$D^2_{\mathbb{R}^m} = \mathtt{H}_2 - ih\mathtt{R}_{2m+1}, \text{ with} \tag{2.22}$$

$$\begin{aligned} \mathtt{H}_2 &= \tfrac{1}{2} \sum_{j=1}^{m} \mu_j \left[-\left(h\partial_{x_j}\right)^2 + x_j^2\right] \\ \mathtt{R}_{2m+1} &= \tfrac{1}{2} \sum_{j=1}^{m} \mu_j \left[\gamma_{2j-1}\gamma_{2j}\right]. \end{aligned} \tag{2.23}$$

It is an easy exercise to show that

$$\mathtt{R}_{2m+1} w_k = \frac{i}{2}\left[\sum_{j=1}^{m} (-1)^{k_j - 1} \mu_j\right] w_k. \tag{2.24}$$

Next, define the lowering and raising operators $A_j = h\partial_{x_j} + x_j$, $A_j^* = -h\partial_{x_j} + x_j$ for $1 \leq j \leq m$, and the Hermite functions

$$\begin{aligned} \psi_{\tau,k}(x) &:= \psi_\tau(x) \otimes w_k \\ \psi_\tau(x) &:= \frac{1}{(\pi h)^{\frac{m}{4}} (2h)^{\frac{|\tau|}{2}} \sqrt{\tau!}} \left[\Pi_{j=1}^m \left(A_j^*\right)^{\tau_j}\right] e^{-\frac{|x|^2}{2h}}, \\ &\qquad \text{for } \tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{N}_0^m. \end{aligned} \tag{2.25}$$

It is well known that $\psi_{\tau,k}(x)$ form an orthonormal basis for $L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right)$. Furthermore we have the standard relations

$$\begin{aligned} \left[A_j, A_j^*\right] &= 2h \\ \mathtt{H}_2 &= \frac{1}{2}\sum_{j=1}^m \mu_j \left(A_j A_j^* - 1\right). \end{aligned} \tag{2.26}$$

It is clear from (2.22), (2.24) and (2.26) that each $\psi_{\tau,k}(x)$ is an eigenvector of $D^2_{\mathbb{R}^m}$ with eigenvalue

$$\lambda_{\tau,k} = h\sum_{j=1}^m \left(2\tau_j + 1 + (-1)^{k_j - 1}\right) \frac{\mu_j}{2}.$$

Hence, clearly the kernel of $D_{\mathbb{R}^m}$ is one dimensional and spanned by $\psi_{0,0} = e^{-\frac{|x|^2}{2h}}$. We now find a decomposition of $L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right)$ into eigenspaces of $D_{\mathbb{R}^m}$. First, if we define

$$\overline{\partial} = \frac{1}{2}\sum_{j=1}^m \left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} c(w_j) A_j, \tag{2.27}$$

then one quickly computes

$$\overline{\partial}^* = -\frac{1}{2}\sum_{j=1}^m \left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} c(\overline{w}_j) A_j^* \tag{2.28}$$

and

$$D_{\mathbb{R}^m} = \sqrt{2}\left(\overline{\partial} + \overline{\partial}^*\right). \tag{2.29}$$

For each $\tau \in \mathbb{N}_0^m \setminus 0$, we define I_τ, V_τ as in (2.12), (2.14) and set

$$E_\tau := \bigoplus_{b\in\{0,1\}^{I_\tau}} \mathbb{C}\left[\prod_{j\in I_\tau} \left(\frac{c(w_j) A_j}{\sqrt{2\tau_j h}}\right)^{b_j} \psi_{\tau,0}\right].$$

It is clear that we have an orthogonal decomposition

$$L^2\left(\mathbb{R}^m; \mathbb{C}^{2^m}\right) = \mathbb{C}\left[\psi_{0,0}\right] \oplus \bigoplus_{\tau\in\mathbb{N}_0^m\setminus 0} E_\tau.$$

Furthermore, we have the isomorphism

$$\begin{aligned}\mathscr{I}_\tau : \Lambda^* V_\tau &\rightarrow E_\tau \\ \mathscr{I}_\tau\left(\bigwedge_{j\in I_\tau} w_j^{b_j}\right) &:= \prod_{j\in I_\tau}\left(\frac{c\left(w_j\right)A_j}{\sqrt{2\tau_j h}}\right)^{b_j}\psi_{\tau,0}.\end{aligned}$$

Each E_τ hence has dimension 2^{Z_τ} and is closed under $c\left(w_j\right)A_j,\ c\left(\overline{w}_j\right)A_j^*$ for $1\leq j\leq m$. We again have

$$\begin{aligned}E_\tau &= E_\tau^{\text{even}}\oplus E_\tau^{\text{odd}}, \quad \text{where} \\ E_\tau^{\text{even/odd}} &:= \mathscr{I}_\tau\left(\Lambda^{\text{even/odd}}V_\tau\right),\end{aligned} \tag{2.30}$$

thus giving the Landau decomposition

$$L^2\left(\mathbb{R}^m;\mathbb{C}^{2^m}\right)=\mathbb{C}\left[\psi_{0,0}\right]\oplus\bigoplus_{\tau\in\mathbb{N}_0^m\setminus 0}\left(E_\tau^{\text{even}}\oplus E_\tau^{\text{odd}}\right). \tag{2.31}$$

The Dirac operator $D_{\mathbb{R}^m}$ by virtue of (2.27), (2.28), (2.29) preserves and acts on E_τ via

$$c\left(\frac{w_{r_\tau}+\bar{w}_{r_\tau}}{\sqrt{2}}\right)=\left(w_{r_\tau}\wedge+\iota_{\bar{w}_{r_\tau}}\right),$$

under the isomorphism $\mathscr{I}_\tau$, where $r_\tau:=\left(\sqrt{\tau_1\mu_1 h},\ldots,\sqrt{\tau_m\mu_m h}\right)$ and w_{r_τ} is as in (2.15). Hence, if we define $\mathtt{i}_\tau:=\mathscr{I}_\tau\mathtt{i}_{r_\tau}\mathscr{I}_\tau^{-1}:E_\tau^{\text{even/odd}}\rightarrow E_\tau^{\text{odd/even}}$ we have that the restriction of $D_{\mathbb{R}^m}$ to E_τ is of the form

$$D_{\mathbb{R}^m} = \begin{bmatrix} & \left|r_\tau\right|\mathtt{i}_\tau \\ \left|r_\tau\right|\mathtt{i}_\tau & \end{bmatrix} \tag{2.32}$$

via (2.19). Also note that since $E_\tau^{\text{even/odd}}\subset\mathscr{I}_\tau\left(C^\infty\left(\mathbb{R}^m\right)\otimes\Lambda^{\text{even/odd}}V^{1,0}\right)$ respectively, one has

$$c\left(e_0\right)E_\tau^{\text{even/odd}} = \pm iE_\tau^{\text{even/odd}} \tag{2.33}$$

using (2.6). The eigenspaces for $D_{\mathbb{R}^m}$ are now given by

$$E_\tau^{\pm} = \mathscr{I}_\tau\left(V_\tau^{\pm}\right), \tag{2.34}$$

via (2.20) with eigenvalue $\pm\left|r_\tau\right|=\pm\sqrt{\mu.\tau h}$ respectively. We now summarize.

Proposition 2.1. *An orthogonal decomposition of $L^2\left(\mathbb{R}^m;\mathbb{C}^{2^m}\right)$ consisting of eigenspaces of the magnetic Dirac operator $D_{\mathbb{R}^m}$ (2.21) is given by*

$$L^2\left(\mathbb{R}^m;\mathbb{C}^{2^m}\right)=\mathbb{C}\left[\psi_{0,0}\right]\oplus\bigoplus_{\tau\in\mathbb{N}_0^m\setminus 0}\left(E_\tau^{+}\oplus E_\tau^{-}\right).$$

Here $E_\tau^{\pm}$, as in (2.34), have dimension $2^{Z_\tau-1}$ and correspond to the eigenvalues $\pm\sqrt{\mu.\tau h}$ respectively.

2.3. **The Semi-classical calculus.** Finally, here we review the semi-classical pseudodifferential calculus used throughout the paper with [13, 28] being the detailed references. Let $\mathfrak{gl}(l)$ denote the space of all $l \times l$ complex matrices. For $A = (a_{ij}) \in \mathfrak{gl}(l)$ we denote $|A| = \max_{ij} |a_{ij}|$. Denote by $\mathcal{S}\left(\mathbb{R}^n; \mathbb{C}^l\right)$ the space of Schwartz maps $f : \mathbb{R}^n \to \mathbb{C}^l$. We define the symbol space $S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ as the space of maps $a : (0,1]_h \to C^\infty\left(\mathbb{R}^{2n}_{x,\xi}; \mathfrak{gl}(l)\right)$ such that each of the semi-norms

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} \langle\xi\rangle^{-m+|\beta|} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\right|$$

is finite $\forall \alpha, \beta \in \mathbb{N}_0^n$. Such a symbol is said to lie in the more refined class $a \in S^m_{\text{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ if there exists an h-independent sequence a_k, $k = 0, 1, \ldots$ of symbols such that

$$a - \left(\sum_{k=0}^N h^k a_k\right) \in h^{N+1} S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right), \ \forall N. \tag{2.35}$$

Symbols as above can be Weyl quantized to define one-parameter families of operators $a^W : \mathcal{S}\left(\mathbb{R}^n; \mathbb{C}^l\right) \to \mathcal{S}\left(\mathbb{R}^n; \mathbb{C}^l\right)$ with Schwartz kernels given by

$$a^W := \frac{1}{(2\pi h)^n} \int e^{i(x-y).\xi/h} a\left(\frac{x+y}{2}, \xi; h\right) d\xi$$

We denote by $\Psi^m_{\text{cl}}\left(\mathbb{R}^n; \mathbb{C}^l\right)$ the class of operators thus obtained by quantizing $S^m_{\text{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$. This class of operators is closed under the standard operations of composition and formal-adjoint. Indeed, the Weyl symbols of the composition and adjoint satisfy

$$\begin{aligned} a^W \circ b^W &= (a * b)^W \\ &:= \left[e^{\frac{ih}{2}\left(\partial_{r_1}\partial_{s_2} - \partial_{r_2}\partial_{s_1}\right)} \left(a\left(s_1, r_1; h\right) b\left(s_2, r_2; h\right)\right)\right]^W_{x=s_1=s_2, \xi=r_1=r_2} \\ \left(a^W\right)^* &= (a^*)^W . \end{aligned} \tag{2.36}$$

Furthermore the class is invariant under changes of coordinates and basis for $\mathbb{C}^l$. This allows one to define an invariant class of operators $\Psi^m_{\text{cl}}(X; E)$ on $C^\infty(X; E)$ associated to any complex vector bundle on a smooth compact manifold X. These define uniformly in h bounded operators between the Sobolev spaces $H^s(X; E) \to H^{s-m}(X; E)$ with the h-dependent norm on each Sobolev space defined via

$$\|u\|_{H^s(X)} := \left\|\left(1 + h^2 \nabla^{E*} \nabla^E\right)^{s/2} u\right\|_{L^2}, \quad s \in \mathbb{R},$$

with respect to any metric g^{TX}, h^E on X, E and unitary connection ∇^E.

For $A \in \Psi^m_{\text{cl}}(X; E)$, its principal symbol is well-defined as an element in $\sigma(A) \in S^m(X; \text{End}(E)) \subset C^\infty(X; \text{End}(E))$. One has that $\sigma(A) = 0$

if and only if $A\in h\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$. We remark that $\sigma\left(A\right)$ is the restriction of standard symbol in [28] to the refined class $\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$ and is locally given by the first coefficient a_{0} in the expansion of its Weyl symbol. The principal symbol satisfies the basic relations $\sigma\left(AB\right)=\sigma\left(A\right)\sigma\left(B\right)$, $\sigma\left(A^{*}\right)=\sigma\left(A\right)^{*}$ with the formal adjoints being defined with respect to the same Hermitian metric h^{E}. The principal symbol map has an inverse given by the quantization map $\mathrm{Op}:S^{m}\left(X;\mathrm{End}\left(E\right)\right)\rightarrow\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$ satisfying $\sigma\left(\mathrm{Op}\left(a\right)\right)=a\in S^{m}\left(X;\mathrm{End}\left(E\right)\right)$. We often use the alternate notation $\mathrm{Op}\left(a\right)=a^{W}$. For a scalar function $b\in S^{m}\left(X\right)$, it is clear from the multiplicative property of the symbol that $\left[a^{W},b^{W}\right]\in h\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$ and we define $H_{b}\left(a\right):=\frac{i}{h}\sigma\left(\left[a^{W},b^{W}\right]\right)\in S^{m}\left(X;\mathrm{End}\left(E\right)\right)$. If a is self adjoint and b real, then it is easy to see that $H_{b}\left(a\right)$ is self-adjoint. We then define $\left|H_{b}\left(a\right)\right|=\max_{\lambda\in\mathrm{Spec}\;H_{b}\left(a\right)}\left|\lambda\right|$.

The wavefront set of an operator $A\in\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$ can be defined invariantly as a subset $WF\left(A\right)\subset\overline{T^{*}X}$ of the fibrewise radial compatification of its cotangent bundle. If the local Weyl symbol of A is given by a then $\left(x_{0},\xi_{0}\right)\notin WF\left(A\right)$ if and only if there exists an open neighborhood $\left(x_{0},\xi_{0};0\right)\in U\subset\overline{T^{*}X}\times\left(0,1\right]_{h}$ such that $a\in h^{\infty}\left\langle \xi\right\rangle ^{-\infty}C^{k}\left(U;\mathbb{C}^{l}\right)$ for all k. The wavefront set satisfies the basic properties $WF\left(A+B\right)\subset WF\left(A\right)\cap WF\left(B\right)$, $WF\left(AB\right)\subset WF\left(A\right)\cap WF\left(B\right)$ and $WF\left(A^{*}\right)=WF\left(A\right)$. The wavefront set $WF\left(A\right)=\emptyset$ is empty if and only if $A\in h^{\infty}\Psi^{-\infty}\left(X;E\right)$. We say that two operators $A=B$ microlocally on $U\subset\overline{T^{*}X}$ if $WF\left(A-B\right)\cap U=\emptyset$. We also define by $\Psi_{\mathrm{cl}}^{c}\left(X;E\right)$ the class of pseudodifferential operators A with wavefront set $WF\left(A\right)\Subset T^{*}X$ compactly contained in the cotangent bundle. It is clear that $\Psi_{\mathrm{cl}}^{c}\left(X;E\right)\subset\Psi_{\mathrm{cl}}^{-\infty}\left(X;E\right)$.

An operator $A\in\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$ is said to be elliptic if $\left\langle \xi\right\rangle ^{m}\sigma\left(A\right)^{-1}$ exists and is uniformly bounded on $T^{*}X$. If $A\in\Psi_{\mathrm{cl}}^{m}\left(X;E\right)$, $m>0$, is formally self-adjoint such that $A+i$ is elliptic then it is essentially self-adjoint (with domain $C_{c}^{\infty}\left(X;E\right)$) as an unbounded operator on $L^{2}\left(X;E\right)$. Its resolvent $\left(A-z\right)^{-1}\in\Psi_{\mathrm{cl}}^{-m}\left(X;E\right)$, $z\in\mathbb{C}$, $\mathrm{Im}z\neq0$, now exists and is pseudodifferential by an application of Beals's lemma. The resolvent furthermore has an expansion $\left(A-z\right)^{-1}\sim\sum_{j=0}^{\infty}h^{j}\mathrm{Op}\left(a_{j}^{z}\right)$ in $\Psi_{\mathrm{cl}}^{-m}\left(X;E\right)$. Here each symbol appearing in the expansion has the form

$$\begin{aligned}a_{j}^{z} & = \left(\sigma\left(A\right)-z\right)^{-1}a_{j,1}^{z}\left(\sigma\left(A\right)-z\right)^{-1}\ldots\left(\sigma\left(A\right)-z\right)^{-1}a_{j,2j}^{z}\left(\sigma\left(A\right)-z\right)^{-1}\\ & \qquad\in S^{-m}\left(X;\mathrm{End}\left(E\right)\right),\end{aligned}$$

for polynomial in z symbols $a_{j,k}^{z}$, $k=1,\ldots,2j$. Given a Schwartz function $f\in\mathcal{S}\left(\mathbb{R}\right)$, the Helffer-Sjostrand formula now expresses the function $f\left(A\right)$ of such an operator in terms of its resolvent and an almost analytic continuation $\tilde{f}$ via

$$f\left(A\right)=\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}\left(z\right)\left(A-z\right)^{-1}dzd\bar{z}.$$

Plugging the resolvent expansion into the above formula then shows that the above lies in and has an expansion $f(A) \sim \sum_{j=0}^{\infty} h^j A_j^f$ in $\Psi_{\rm cl}^{-\infty}(X;E)$. Finally, one defines the classical λ-energy level of A via

$$\Sigma_\lambda^A = \{(x,\xi) \in T^*X \,|\, \det(\sigma(A)(x,\xi) - \lambda I) = 0\}.$$

Now, the form for the coefficients of the resolvent expansion also shows $WF(f(A)) \subset \Sigma_{\mathrm{spt}(f)}^A := \bigcup_{\lambda \in \mathrm{spt}(f)} \Sigma_\lambda^A$.

2.3.1. *The class $\Psi_\delta^m(X;E)$.* In Section 3 we shall need the more exotic class of symbols $S_\delta^m(\mathbb{R}^{2n};\mathbb{C})$ defined for each $0 < \delta < \frac{1}{2}$. A function $a : (0,1]_h \to C^\infty\left(\mathbb{R}^{2n}_{x,\xi};\mathbb{C}\right)$ is said to be in this class if and only if

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} \langle\xi\rangle^{-m+|\beta|} h^{-(|\alpha|+|\beta|)\delta} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\right| \tag{2.37}$$

is finite $\forall \alpha,\beta \in \mathbb{N}_0^n$. This class of operators is closed under the standard operations of composition, adjoint and changes of coordinates allowing the definition of the exotic pseudodifferential algebra $\Psi_\delta^m(X)$ on a compact manifold. The class $S_\delta^m(X)$ is a family of functions $a : (0,1]_h \to C^\infty(T^*X;\mathbb{C})$ satisfying the estimates (2.37) in every coordinate chart and induced trivialization. Such a family can be quantized to $a^W \in \Psi_\delta^m(X)$ satisfying $a^W b^W = (ab)^W + h^{1-2\delta}\Psi_\delta^{m+m'-1}(X)$ for another $b \in S_\delta^{m'}(X)$. The operators in $\Psi_\delta^0(X)$ are uniformly bounded on $L^2(X)$. Finally, the wavefront an operator $A \in \Psi_\delta^m(X;E)$ is similarly defined and satisfies the same basic properties as before.

2.3.2. *Fourier integral operators.* We shall also need the local theory of Fourier integral operators. Let $\kappa : U \to V$ be an exact symplectomorphism between two open subsets $U \subset T^*X$, $V \subset T^*Y$ inside cotangent spaces of manifolds of same dimension n. Assume that there exist local coordinates $(x_1,\ldots,x_n)$, $(y_1,\ldots y_n)$ on $\pi(U), \pi(V)$ respectively with induced canonical coordinates $(x,\xi), (y,\eta)$ on U,V. A function $S(x,\eta) \in C^\infty(\Omega)$ on an open subset $\Omega \subset \mathbb{R}^{2n}_{x,\eta}$ is said to be a generating function for the graph of κ if the Lagrangian submanifolds

$$\begin{aligned} (T^*X) \times (T^*Y)^- \supset \Lambda_\kappa &:= \{((x,\xi);\kappa(x,\xi)) \,|\, (x,\xi) \in U\} \\ &= \{(x,\partial_x S;\partial_\eta S,\eta) \,|\, (x,\eta) \in \Omega\} \end{aligned}$$

are equal. Such a generating function always exists locally near any point on Λ_κ. Letting $a : (0,1]_h \to C_c^\infty(\Omega \times \pi(V);\mathbb{C})$, which admits an expansion $a(x,y,\eta;h) \sim \sum_{k=0}^\infty h^k a_k(x,y,\eta)$, one may now define a Fourier integral operator associated to κ via

$$\begin{aligned} A : L^2(Y) &\to L^2(X) \\ (Af)(x) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(S(x,\eta) - y.\eta)} a(x,y,\eta;h) f(y)\, dy d\eta. \end{aligned}$$

The symbol of $\sigma(A) \in C_c^\infty(\Lambda_\kappa;\mathbb{C})$ is defined using the generating function via $\sigma(A)(x,\eta) = a_0(x,\partial_x S,\eta)$. The adjoint A^*, is again a Fourier integral operator associated to the symplectomorphism κ^{-1}. The wavefront set of A maybe defined as a subset $WF(A) \subset \overline{T^*X} \times \overline{T^*Y}$. A point $(x,\xi;y,\eta) \notin WF(A)$ if and only if there exist pseudodifferential operators $B \in \Psi^m_{\mathrm{cl}}(X), C \in \Psi^{m'}_{\mathrm{cl}}(Y)$ with $(x,\xi;y,\eta) \in WF(B) \times WF(C)$ such that $\|BAC\|_{H^s(Y)\to H^{s'}(X)} = O(h^\infty)$ for each $s,s' \in \mathbb{R}$. It can be shown that the wavefront set is in fact a compact subset $WF(A) \subset \Lambda_\kappa$. Given a pseudodifferential operator $B \in \Psi^m_{\mathrm{cl}}(X)$, Egorov's theorem says that the composite is a pseudodifferential operator $A^*BA \in \Psi^m_{\mathrm{cl}}(Y)$. Moreover its principal symbol is given via $\sigma(A^*BA) = (\kappa^{-1})^* |\sigma(A)|^2 \sigma(B) \in C^\infty_c(V)$, where we have again used the identification of V with Λ_κ given by the generating function. Finally one has the wavefront relation $WF(A^*BA) \subset WF(A) \cap WF(B)$ again using the identifications of U, V and Λ_κ.

An important special case arises when $\kappa = e^{tH_f}$ is the time t flow of a Hamiltonian $f \in S^m(T^*X)$. The operator $e^{\frac{it}{h}f^W}$, defined as a unitary operator via Stone's theorem, is now a Fourier integral operator associated to κ. Egorov's theorem now gives that the conjugation $e^{\frac{it}{h}f^W} A e^{-\frac{it}{h}f^W} \in \Psi^{m'}_{\mathrm{cl}}(X)$ is pseudodifferential for each $A \in \Psi^{m'}_{\mathrm{cl}}(X)$ with principal symbol $\sigma\left(e^{\frac{it}{h}f^W} A e^{-\frac{it}{h}f^W}\right) = \left(e^{tH_f}\right)^* \sigma(A)$.

3. First reductions

The trace expansion Theorem 1.3 will be proved in two steps based on the following two lemmas. Below, τ, T, T', f, θ and D are the same as before.

Lemma 3.1. *Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\vartheta \in C^\infty_c\left((T'h^\epsilon, T); [-1,1]\right)$. Then*

$$\begin{aligned} tr\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\vartheta\right)\left(\lambda\sqrt{h} - D\right)\right] &= \\ tr\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\vartheta}\left(\frac{\lambda\sqrt{h} - D}{h}\right)\right] &= O(h^\infty) \end{aligned}$$

for all$\lambda \in \mathbb{R}$.

We note that in the above lemma the function ϑ is allowed to depend on h, while its support and range are contained in h-independent intervals.

Lemma 3.2. *There exist smooth functions $u_j \in C^\infty(\mathbb{R})$ such that for each $\lambda \in \mathbb{R}$ and$\epsilon \in \left(0, \frac{1}{2}\right)$ one has a trace expansion*

$$\begin{aligned} tr\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta_\epsilon\right)\left(\lambda\sqrt{h} - D\right)\right] &= \\ tr\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h^{1-\epsilon}}\check{\theta}\left(\frac{\lambda\sqrt{h} - D}{h^{1-\epsilon}}\right)\right] &= h^{-m-1}\left(\sum_{j=0}^{N-1} c_j h^{j/2} + O\left(h^{N/2}\right)\right) \end{aligned}$$

where $\theta_\epsilon(x) := \theta\left(\frac{x}{h^\epsilon}\right)$.

We note that the trace expansion Theorem 1.3 follows from the above two lemmas on simply splitting $\theta(x) = \theta_\epsilon(x) + \underbrace{[\theta(x) - \theta_\epsilon(x)]}_{\vartheta(x)}$ and applying Lemma 3.2 and Lemma 3.1 to the first and second summands respectively. Lemma Lemma 3.2 is a relatively classical expansion proved via local index theory and will be deferred to Section 7. Our main occupation until then is in proving Lemma 3.1.

As a first step one chooses a microlocal partition of unity $A_\alpha \in \Psi^0_{\mathrm{cl}}(X)$, $0 \le \alpha \le N$, satisfying

$$\begin{aligned} \sum_{\alpha=0}^{N} A_\alpha \quad &= \quad 1 \\ WF(A_0) \subset \quad U_0 \subset \quad &\overline{T^*X} \setminus \Sigma^D_{(-\tau,\tau)} \\ WF(A_\alpha) \Subset \quad U_\alpha \subset \quad &\Sigma^D_{(-2\tau,2\tau)},\ 1 \le \alpha \le N. \end{aligned} \tag{3.1}$$

subordinate to an open cover $\{U_\alpha\}_{\alpha=0}^N$ of T^*X. Clearly, it suffices to prove

$$\operatorname{tr}\left[A_\alpha f\left(\frac{D}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)A_\beta\right] = O(h^\infty) \tag{3.2}$$

for $1 \le \alpha, \beta \le N$ with $WF(A_\alpha) \cap WF(A_\beta) \neq \emptyset$.

By the Helffer-Sjostrand formula we have the trace above is given by

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D) := \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\operatorname{tr}\ \left[A_\alpha\left(\frac{1}{\sqrt{h}}D - z\right)^{-1} A_\beta\right] dz d\bar{z}. \tag{3.3}$$

We note that the resolvent, the above trace as well as the left hand side of (3.2) are well defined for any essentially self-adjoint pseudodifferential operator in place of D. The next reduction step attempts to modify D without affecting the asymptotics of $\mathcal{T}^{\vartheta}_{\alpha\beta}(D)$. To this end, choose open subsets

$$WF(A_\alpha) \cup WF(A_\beta) \subset V_{\alpha\beta} \Subset T^*X, \tag{3.4}$$

for each such pair α, β with $WF(A_\alpha) \cap WF(A_\beta) \neq \emptyset$. With $d = \sigma(D) \in C^\infty(X; i\mathfrak{u}(S))$, define the required exit time

$$\begin{aligned} T_{\alpha\beta} &:= \frac{1}{\inf_{g\in\mathcal{G}_{\alpha\beta}} |H_g d|}, \qquad \text{where} \\ \mathcal{G}_{\alpha\beta} &:= \left\{ g \in C^\infty(T^*X; [0,1]) \mid g|_{WF(A_\alpha)\cap WF(A_\beta)} = 1,\ g|_{V^c_{\alpha\beta}} = 0 \right\}. \end{aligned} \tag{3.5}$$

If one were to use a scalar symbol $d \in C^\infty(X)$ instead in (3.5), the required exit time $T_{\alpha\beta}$ would have the following significance: any Hamiltonian trajectory $\gamma(t) = e^{tH_d}$ with $\gamma(0) \in WF(A_\alpha) \cap WF(A_\beta)$,$\gamma(T) \in V_{\alpha\beta}$, would have length $T \ge T_{\alpha\beta}$ atleast the required exit time. We now have the following.

Lemma 3.3. *Let $D' \in \Psi^1_{cl}(X;E)$ be essentially self-adjoint such that $D = D'$ microlocally on $V_{\alpha\beta}$. Then for $\vartheta \in C_c^\infty\left(\left(T'_{\alpha\beta}h^\epsilon, T_{\alpha\beta}\right);[0,1]\right)$, $0 < T'_{\alpha\beta} < T_{\alpha\beta}$, one has*

$$\mathcal{T}^\vartheta_{\alpha\beta}(D) = \mathcal{T}^\vartheta_{\alpha\beta}(D') \quad \text{mod } h^\infty.$$

Proof. Let $B \in \Psi^0_{\mathrm{cl}}(X)$ be a microlocal cutoff such that $B = 0$ on $WF(D - D')$ and $B = 1$ on $V_{\alpha\beta}$. Then $(1-B)A_\beta = 0$ microlocally implies

$$\begin{aligned}\left(z - \frac{1}{\sqrt{h}}D\right)B\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1}A_\beta = A_\beta - \left[\frac{1}{\sqrt{h}}D, B\right](z-D')^{-1}A_\beta \\ + B\left(\frac{1}{\sqrt{h}}D' - \frac{1}{\sqrt{h}}D\right)\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1}A_\beta \\ (\mathrm{mod}\ h^\infty)\end{aligned} \tag{3.6}$$

in trace norm. Next, multiplying through by $A_\alpha\left(z - \frac{1}{\sqrt{h}}D\right)^{-1}$ and using $A_\alpha B = B$ microlocally gives

$$\begin{aligned}A_\alpha\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1}A_\beta - A_\alpha\left(z - \frac{1}{\sqrt{h}}D\right)^{-1}A_\beta = \\ A_\alpha\left(z - \frac{1}{\sqrt{h}}D\right)^{-1}B\left(\frac{1}{\sqrt{h}}D' - \frac{1}{\sqrt{h}}D\right)\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1}A_\beta \\ - A_\alpha\left(z - \frac{1}{\sqrt{h}}D\right)^{-1}\left[\frac{1}{\sqrt{h}}D, B\right]\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1}A_\beta + O\left(|\mathrm{Im}z|^{-1}h^\infty\right)\end{aligned} \tag{3.7}$$

in trace norm. Now $B = 0$ on $WF(D - D')$ gives that the first term on the right hand side above is $O\left(|\mathrm{Im}z|^{-2}h^\infty\right)$.

We now estimate the second term. Let $S_{\alpha\beta} < S''_{\alpha\beta} < S'''_{\alpha\beta} < T_{\alpha\beta}$ and $S'_{\alpha\beta} > T'_{\alpha\beta}$ be such that $\vartheta \in C_c^\infty\left(\left[S'_{\alpha\beta}h^\epsilon, S_{\alpha\beta}\right];[0,1]\right)$. Let $g_0 \in \mathcal{G}_{\alpha\beta}$ with $|H_{g_0}(d)| \le \frac{1}{S'''_{\alpha\beta}}$. Set $g = \alpha_z g_0$, where

$$\alpha_z = \min\left(\frac{S''_{\alpha\beta}\mathrm{Im}z}{\sqrt{h}\log\frac{1}{h}}, N\right)$$

with the constant $N > 0$ to be specified later. We note that

$$G = \left(e^{g\log\frac{1}{h}}\right)^W \in h^{-N}\Psi^0_\delta(X)$$

for each $0 < \delta < \frac{1}{2}$. Since it has an elliptic symbol we may construct its inverse by symbolic calculus $G^{-1} \in h^N \Psi^0_\delta(X)$. Moreover

$$G\left(z - \frac{1}{\sqrt{h}} D_h\right) G^{-1} = \left(z - \frac{1}{\sqrt{h}} D_h\right) + i\left(\alpha_z \sqrt{h} \log \frac{1}{h}\right) \left(H_{g_0}(d)\right)^W + R^W, \quad \text{with} \tag{3.8}$$

$$R = O\left(h^{\frac{3}{2}} \alpha_z \log \frac{1}{h}\right) \quad \text{in } S^0_\delta(X). \tag{3.9}$$

Now, since $\left|\left(\alpha_z \sqrt{h} \log \frac{1}{h}\right) H_{g_0}(d)\right| \leq \frac{S''_{\alpha\beta}}{S'''_{\alpha\beta}} |\text{Im} z| < |\text{Im} z|$, the inverse $G\left(z - \frac{1}{\sqrt{h}} D_h\right)^{-1} G^{-1}$ of the above exists and is $O\left(|\text{Im} z|^{-1}\right)$ in operator norm for $\text{Im} z \neq 0$, and h sufficiently small.

Next, $G = e^{\alpha_z \log \frac{1}{h}}$ on $WF(A_\alpha)$,$G = G^{-1} = I$ on $WF(B) \setminus V_{\alpha\beta}$ and $[D_h, B] = 0$ on $V_{\alpha\beta}$ imply

$$\begin{aligned} e^{\alpha_z \log \frac{1}{h}} A_\alpha &\left(z - \frac{1}{\sqrt{h}} D_h\right)^{-1} \left[\frac{1}{\sqrt{h}} D_h, B\right] \\ &= A_\alpha G \left(z - \frac{1}{\sqrt{h}} D_h\right)^{-1} G^{-1} \left[\frac{1}{\sqrt{h}} D_h, B\right] + O\left(|\text{Im} z|^{-1} h^\infty\right) \end{aligned}$$

in trace norm. The above is now $O\left(|\text{Im} z|^{-1} h^{-n}\right)$ in trace norm. Hence

$$A_\alpha \left(z - \frac{1}{\sqrt{h}} D_h\right)^{-1} \left[\frac{1}{\sqrt{h}} D_h, B\right] = O\left(|\text{Im} z|^{-1} h^{-n} \max\left(h^N, e^{-\frac{S''_{\alpha\beta} \text{Im} z}{\sqrt{h}}}\right)\right)$$

in trace norm. This now estimates the second term of (3.7) and gives

$$\begin{aligned} A_\alpha &\left(z - \frac{1}{\sqrt{h}} D'_h\right)^{-1} A_\beta - A_\alpha \left(z - \frac{1}{\sqrt{h}} D_h\right)^{-1} A_\beta \\ &= O\left(|\text{Im} z|^{-2} h^{-n} \max\left(h^N, e^{-\frac{S''_{\alpha\beta} \text{Im} z}{\sqrt{h}}}\right)\right) \end{aligned} \tag{3.10}$$

in trace norm.

Next, we have the Paley-Wiener estimate

$$\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) = \begin{cases} O\left(e^{\frac{S_{\alpha\beta}(\text{Im} z)}{\sqrt{h}}}\right); & \text{Im} z > 0 \\ O\left(e^{\frac{S'_{\alpha\beta}(\text{Im} z)}{h^{\frac{1}{2} - \epsilon}}}\right); & \text{Im} z < 0. \end{cases} \tag{3.11}$$

Introduce $\psi \in C^\infty(\mathbb{R}; [0,1])$ such that $\psi(x) = \begin{cases} 1; & x \leq 1 \\ 0; & x \geq 2 \end{cases}$. Setting $\psi_M(z) = \psi\left(\frac{\text{Im} z}{M\sqrt{h} \log \frac{1}{h}}\right)$, for another constant $M > 1$ yet to be chosen, we have the

estimate
(3.12)

$$\bar{\partial}\left(\psi_M \tilde{f}\right) = \begin{cases} O\left(\psi_M \left|\mathrm{Im}z\right|^N + \frac{1}{M\sqrt{h}\log\frac{1}{h}} 1_{[1,2]}\left(\frac{\mathrm{Im}z}{M\sqrt{h}\log\frac{1}{h}}\right)\right); & \mathrm{Im}z > 0 \\ O\left(\left|\mathrm{Im}z\right|^N\right); & \mathrm{Im}z < 0. \end{cases}$$

Finally, (3.10), (3.11) and (3.12) along with the observation $\psi_M \left|\mathrm{Im}z\right|^N = O\left(\left(M\sqrt{h}\log\frac{1}{h}\right)^N\right)$ gives

$$\begin{aligned} & \mathcal{T}^{\vartheta}_{\alpha\beta}\left(D'\right) - \mathcal{T}^{\vartheta}_{\alpha\beta}\left(D\right) \\ = \; & \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\left(\psi_M \tilde{f}\right) \check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\left[A_\alpha\left(z - \frac{1}{\sqrt{h}}D'_h\right)^{-1} A_\beta \right. \\ & \left. \qquad - A_\alpha\left(z - \frac{1}{\sqrt{h}}D_h\right)^{-1} A_\beta\right] dz d\bar{z} \\ = \; & O\left(h^\infty\right) + \\ & O\left[\int_{\left\{M\sqrt{h}\log\frac{1}{h} \leq \mathrm{Im}z \leq 2M\sqrt{h}\log\frac{1}{h}\right\}} \frac{h^{-n}}{\sqrt{h}\log\frac{1}{h}} \max\left(h^N e^{\frac{S_{\alpha\beta}(\mathrm{Im}z)}{\sqrt{h}}}, e^{-\frac{\left(S''_{\alpha\beta} - S_{\alpha\beta}\right)\mathrm{Im}z}{\sqrt{h}}}\right)\right] \\ = \; & O\left[\max\left(h^{N - 2MS_{\alpha\beta} - n}, h^{M\left(S''_{\alpha\beta} - S_{\alpha\beta}\right) - n}\right)\right]. \end{aligned}$$

Choosing $M \gg \frac{n}{\left(S''_{\alpha\beta} - S_{\alpha\beta}\right)}$ and furthermore $N \gg 2MS_{\alpha\beta} + n$ gives the result. ◻

In the proof above we have closely followed [10] Lemma 12.7. Again, the proof above avoids the use of an unknown parametrix for $e^{\frac{it}{h}D}$ which, following the significance of the required exit time $T_{\alpha\beta}$ noted before, maybe used to give an alternate proof in the case when d is scalar.

4. Reduction to $\mathbb{R}^n$

In this section we shall further reduce to the case of a Dirac operator on $\mathbb{R}^n$. First we cover X by a finite set of Darboux charts $\left\{\varphi_s : \Omega_s \to \Omega^0_s \subset \mathbb{R}^n\right\}_{s\in S}$ for the contact form a, centered at points $\left\{x_s\right\}_{s\in S} \in X$. By shrinking the partition of unity (3.1) we may assume that for each pair α, β, with $WF\left(A_\alpha\right) \cap WF\left(A_\beta\right) \neq \emptyset$, the open sets $V_{\alpha\beta} \subset T^*\Omega_s$ in (3.4) are contained in some Darboux chart. Now consider such a chart Ω_s with coordinates$(x_0, \ldots, x_{2m})$ centered at $x_s \in X$ and an orthonormal frame $\left\{e_j = w^k_j \partial_{x_k}\right\}, 0 \leq j \leq 2m$ for the tangent bundle on Ω_s. We hence have

$$w^k_j g_{kl} w^l_r = \delta_{jr}, \tag{4.1}$$

where g_{kl} is the metric in these coordinates and the Einstein summation convention is being used. Let Γ^l_{jk} be the Christoffel symbols for the Levi-Civita connection in the orthonormal frame e_i satisfying $\nabla_{e_j} e_k = \Gamma^l_{jk} e_l$. This orthonormal frame induces an orthonormal frame u_j, $1 \leq j \leq 2^m$, for the spin bundle S. We further choose a local orthonormal section $\mathtt{1}(x)$ for the Hermitian line bundle L and define via $\nabla^{A_0}_{e_j}\mathtt{1} = \Upsilon_j(x)\mathtt{1}$, $0 \leq j \leq 2m$ the Christoffel symbols of the unitary connection A_0 on L. In terms of the induced frame $u_j \otimes \mathtt{1}$, $1 \leq j \leq 2^m$, for $S \otimes L$ the Dirac operator (1.2) has the form (cf. [3] Section 3.3)

$$D = \gamma^j w^k_j P_k + h\left(\frac{1}{4}\Gamma^l_{jk}\gamma^j\gamma^k\gamma_l + \Upsilon_j\gamma^j\right), \quad \text{where} \tag{4.2}$$

$$P_k = h\partial_{x_k} + ia_k, \tag{4.3}$$

and

$$a(x) = a_k dx^k = dx_0 + \sum_{j=1}^{m}\left(x_j dx_{j+m} - x_{j+m} dx_j\right) \tag{4.4}$$

is the standard contact one form in these coordinates.

The expression in (4.2) is formally self-adjoint with respect to the Riemannian density $e^1 \wedge \ldots \wedge e^n = \sqrt{g}dx := \sqrt{g}dx^1 \wedge \ldots \wedge dx^n$ with $g = \det(g_{ij})$. To get an operator self-adjoint with respect to the Euclidean density dx one expresses the Dirac operator in the framing $g^{\frac{1}{4}}u_j \otimes \mathtt{1}, 1 \leq j \leq 2^m$. In this new frame the expression (4.2) for the Dirac operator needs to be conjugated by $g^{\frac{1}{4}}$ and hence the term $h\gamma^j w^k_j g^{-\frac{1}{4}}\left(\partial_{x_k} g^{\frac{1}{4}}\right)$ added. Hence, the Dirac operator in the new frame has the form

$$D = \left[\sigma^j w^k_j (\xi_k + a_k)\right]^W + hE \in \Psi^1_{\mathrm{cl}}\left(\Omega^0_s; \mathbb{C}^{2^m}\right),$$

with $\sigma^j = i\gamma^j$, for some self-adjoint endomorphism $E(x) \in C^\infty\left(\Omega^0_s; i\mathfrak{u}\left(\mathbb{C}^{2^m}\right)\right)$.

The one form a is extended to all of $\mathbb{R}^n$ by the same formula (4.4). The functions w^k_j are extended such that

$$\left(w^k_j \partial_{x_k} \otimes dx^j\right)\Big|_{(K^0_s)^c} = \partial_{x_0} \otimes dx^0 + \sum_{j=1}^{m} \mu_j^{\frac{1}{2}}\left(\partial_{x_j} \otimes dx^j + \partial_{x_{j+m}} \otimes dx^{j+m}\right)$$

(and hence $g|_{(K^0_s)^c} = dx_0^2 + \sum_{j=1}^m \mu_j\left(dx_j^2 + dx_{j+m}^2\right)$) outside a compact neighborhood $\Omega^0_s \Subset K^0_s$. These extensions may further be chosen such that the suitability assumption 1.1 holds globally on $\mathbb{R}^n$ and for an extended positive function $\nu \in C^\infty_c(\mathbb{R}^n)$ satisfying

$$\nu_0 \leq \mu_1\left(\inf_{\mathbb{R}^n} \nu\right). \tag{4.5}$$

The endomorphism $E(x) \in C_c^\infty\left(\mathbb{R}^n; i\mathfrak{u}\left(\mathbb{C}^{2^m}\right)\right)$ is extended to an arbitrary self-adjoint endomorphism of compact support. This now gives

$$D_0 = \left[\sigma^j w_j^k \left(\xi_k + a_k\right)\right]^W + hE \in \Psi_{\text{cl}}^1\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right) \tag{4.6}$$

as a well defined formally self adjoint operator on $\mathbb{R}^n$. Furthermore, the symbol of $D_0 + i$ is elliptic in the class $S^0(m)$ for the order function $m = \sqrt{1 + \sum_{k=0}^m \left(\xi_k + a_k\right)^2}$ and hence D_0 is essentially self adjoint (see [10] Ch. 8). Below $\vartheta \in C_c^\infty\left(\left(T'_{\alpha\beta} h^\epsilon, T_{\alpha\beta}\right); [0,1]\right)$, $0 < T'_{\alpha\beta} < T_{\alpha\beta}$, as before and we set $V_{\alpha\beta}^0 := (d\varphi_s)^* V_{\alpha\beta} \subset T^*\Omega_s^0$.

Proposition 4.1. *There exist $A_\alpha^0, A_\beta^0 \in \Psi_{cl}^0(\mathbb{R}^n)$, with $WF\left(A_\alpha^0\right) \cup WF\left(A_\beta^0\right) \Subset V_{\alpha\beta}^0 \subset T^*\tilde{\Omega}_s$, such that*

$$\mathcal{T}_{\alpha\beta}^\vartheta(D) = \underbrace{tr\left[A_\alpha^0 f\left(\frac{D_0}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h} - D_0}{h}\right)A_\beta^0\right]}_{:=\mathcal{T}_{\alpha\beta}^\vartheta(D_0)} \quad mod\ h^\infty.$$

Proof. Let $K'_{\alpha\beta}, K''_{\alpha\beta}$ and $V'_{\alpha\beta}, V''_{\alpha\beta}$ be compact and open subsets respectively satisfying $V_{\alpha\beta} \subset K'_{\alpha\beta} \subset V'_{\alpha\beta} \subset K''_{\alpha\beta} \subset V''_{\alpha\beta} \subset T^*\Omega_s$. Choose $D' \in \Psi_{\text{cl}}^0(X; S)$ self-adjoint such that $D = D'$ microlocally on $K'_{\alpha\beta}$ and

$$\Sigma_{(-\infty, 2\tau]}^{D'} \subset V'_{\alpha\beta} \tag{4.7}$$

and set $E = D' - 3\tau \in \Psi_{\text{cl}}^0(X; S)$. Pick a cutoff function $\chi(x; y, \eta) \in C_c^\infty\left(\pi\left(V''_{\alpha\beta}\right) \times (d\varphi_s)^* V''_{\alpha\beta}; [0,1]\right)$ such that $\chi = 1$ on $\pi\left(K''_{\alpha\beta}\right) \times (d\varphi_s)^* K''_{\alpha\beta}$. Now define the operator

$$\begin{aligned} U : L^2\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right) &\to L^2(X; S), \\ (Uf)(x) &= \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\varphi_s(x) - y).\eta}\chi(x; y, \eta) f(y)\, dy d\eta, \quad x \in X. \end{aligned}$$

The above is a semi-classical Fourier integral operator associated to symplectomorphism $\kappa = \left(d\varphi_s^{-1}\right)^*$ given by the canonical coordinates. Its adjoint $U^* : L^2(X; S) \to L^2\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$ is again a semi-classical Fourier integral operator associated to the symplectomorphism $\kappa^{-1} = (d\varphi_s)^*$. A simple computation gives the following compositions are pseudodifferential with

$$UU^* = I \quad \text{microlocally on } K''_{\alpha\beta} \quad \text{and} \tag{4.8}$$

$$U^*U = I \quad \text{microlocally on } \kappa\left(K''_{\alpha\beta}\right). \tag{4.9}$$

The composition

$$E' = E_0 := U^*EU \in \Psi_{\text{cl}}^0\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$$

is now a pseudodifferential operator by Egorov's theorem with symbol

$$\sigma(E_0) = (d\varphi_s)^* \chi^2.\sigma(E). \tag{4.10}$$

Similarly, $E_0' := UE_0U^* \in \Psi^0_{\mathrm{cl}}(X;S)$ and

$$\sigma\left(E_0'\right) = \left(d\varphi_s\right)^* \chi^4 . \sigma\left(E_0\right). \tag{4.11}$$

From (4.7), (4.10) and (4.11) we have $\Sigma^{E_0}_{(-\infty,-\tau]} \subset \kappa\left(V'_{\alpha\beta}\right)$ and $\Sigma^{E_0'}_{(-\infty,-\tau]} \subset V'_{\alpha\beta}$. Hence by proposition A.6 E, E', E_0 and E_0' all have discrete spectrum in $(-\infty,-\tau]$. We now select $g \in C_c^\infty(-5\tau,-\tau)$ such that $g=1$ on $[-4\tau,-2\tau]$. We have

$$WF\left(g\left(E\right)\right) \subset \Sigma^E_{\mathrm{spt}(g)} \subset \Sigma^E_{(-\infty,-\tau]} \subset V'_{\alpha\beta}.$$

Combined with (4.9) this gives $\left(U^*U - I\right) g\left(E\right) \in h^\infty \Psi^{-\infty}_{\mathrm{cl}}(X;S)$ and hence $\left\|\left(U^*U-I\right) g\left(E\right)\right\| = O\left(h^\infty\right)$ as an operator on $L^2(X;S)$. This in turn now gives

$$\left\|\left(U^*U - I\right)\Pi^E\right\|\left(\|E\|\,\|U\|+1\right) = O\left(h^\infty\right) \tag{4.12}$$

with $\Pi^E = \Pi^E_{[-4\tau,-2\tau]}$. Similarly, we get

$$\left\|\left(UU^* - I\right)\Pi^{E_0}\right\|\left(\|E_0\|\,\|U^*\|+1\right) = O\left(h^\infty\right). \tag{4.13}$$

Another easy computation gives $E = E_0'$ microlocally on $K''_{\alpha\beta}$ and we may similarly estimate similarly have

$$\left\|\left(E - E_0'\right)\Pi^{E_0'}\right\| = O\left(h^\infty\right). \tag{4.14}$$

Next we define $A^0_\alpha := U^* A_\alpha U$, $A^0_\beta := U^* A_\beta U \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$ and again note

$$\begin{aligned} UA^0_\alpha A^0_\beta U^* &= A_\alpha A_\beta \quad \text{microlocally on } K''_{\alpha\beta} \\ U^* A_\alpha A_\beta U &= A^0_\alpha A^0_\beta \quad \text{microlocally on } \kappa\left(K''_{\alpha\beta}\right). \end{aligned}$$

This again gives

$$\left\|\left[UA^0_\alpha A^0_\beta U^* - A_\alpha A_\beta\right]\Pi^E\right\| = O\left(h^\infty\right) \tag{4.15}$$

$$\left\|\left[U^* A_\alpha A_\beta U - A^0_\alpha A^0_\beta\right]\Pi^{E_0}\right\| = O\left(h^\infty\right). \tag{4.16}$$

Now using (4.12), (4.13), (4.14), (4.15), (4.16) and using the cyclicity of the trace we may apply A.5 of Section A with $\rho(x) = f\left(\frac{x+3\tau}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-3\tau-x}{h}\right)$ to get

$$\operatorname{tr}\left[A_\alpha f\left(\frac{D'}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D'}{h}\right)A_\beta\right] - \operatorname{tr}\left[A^0_\alpha f\left(\frac{D_0'}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D_0'}{h}\right)A^0_\beta\right]$$
$$= O\left(h^\infty\right)$$

for $D_0' := E_0 + 3\tau$. Finally observing $D = D'$ on $V_{\alpha\beta}$, $D_0 = D_0'$ on $V^0_{\alpha\beta}$ and using Lemma 3.3 completes the proof. □

5. Birkhoff normal form for the Dirac operator

In this section we derive a Birkhoff normal form for the Dirac operator (4.6) on $\mathbb{R}^n$. First consider the function

$$f_0 := -\frac{4x_0}{\pi} + \sum_{j=1}^{m} \left(x_j x_{j+m} + \xi_j \xi_{j+m}\right).$$

If H_{f_0} and $e^{tH_{f_0}}$ denote the Hamilton vector field and time t flow of f_0 respectively then it is easy to compute

$$\begin{aligned} e^{\frac{\pi}{4}H_{f_0}}\left(x_0,\xi_0\right) &= \left(x_0,\xi_0+1\right) \\ e^{\frac{\pi}{4}H_{f_0}}\left(x_j,\xi_j;x_{j+m}\xi_{j+m}\right) &= \left(\frac{x_j+\xi_{j+m}}{\sqrt{2}},\frac{-x_{j+m}+\xi_j}{\sqrt{2}};\frac{x_{j+m}+\xi_j}{\sqrt{2}},\frac{-x_j+\xi_{j+m}}{\sqrt{2}}\right). \end{aligned}$$

We abbreviate $(x',\xi') = (x_1,\ldots,x_m;\xi_1,\ldots,\xi_m)$, $(x'',\xi'') = (x_{m+1},\ldots,x_{2m};\xi_{m+1},\ldots,\xi_{2m})$ and $(x,\xi) = (x_0,x',x'';\xi_0,\xi',\xi'')$. Further, let $o_N \subset S^1_{\mathrm{cl}}\left(\mathbb{R}^{2n};\mathbb{C}^l\right)$ denote the subspace of self-adjoint symbols $a : (0,1]_h \to C^\infty\left(\mathbb{R}^{2n}_{x,\xi}; i\mathfrak{u}\left(2^m\right)\right)$ such that each of the coefficients a_k, $k = 0,1,2,\ldots$ in its symbolic expansion (2.35) vanishes to order N in (x_0,x',ξ'). We also denote by o_N the space of Weyl quantizations of such symbols.

Using Egorov's theorem, the operator (4.6) is conjugated to

$$e^{\frac{i\pi}{4h}f_0^W} D_0 e^{-\frac{i\pi}{4h}f_0^W} = d_0^W, \quad \text{with} \tag{5.1}$$

$$d_0 = \sqrt{2}\left(\sigma^j w^0_{j,f_0}\xi_0 + \sigma^j w^k_{j,f_0}\xi_k + \sigma^j w^{k+m}_{j,f_0}x_k\right) + ho_0 \tag{5.2}$$

$$\text{where } w^k_{j,f_0} = \left(e^{-\frac{\pi}{4}H_{f_0}}\right)^* w^k_j \tag{5.3}$$

A Taylor expansion of d_0 (5.2) now gives $r^0_j \in o_2$, $0 \leq j \leq 2m$, such that

$$d_0 = \sqrt{2}\sigma^j\left(\bar{w}^0_j\xi_0 + \bar{w}^k_j\xi_k + \bar{w}^{k+m}_j x_k\right) + \sigma^j r^0_j + ho_0$$

and where $\bar{w}^k_j\left(x_0,x'',\xi''\right) = w^k_j\left(x_0,-\frac{\xi''}{\sqrt{2}},\frac{x''}{\sqrt{2}}\right)$. On squaring using (4.1) we obtain

$$\begin{aligned} \left(d_0^W\right)^2 &= Q_0^W + ho_1 + o_3 + h^2 o_0, \quad \text{with} \\ Q_0 &= \begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} \begin{bmatrix} \bar{g}^{(k+m)(l+m)}\left(x_0,x'',\xi''\right) & \bar{g}^{(k+m)0}\left(x_0,x'',\xi''\right) & \bar{g}^{(k+m)l}\left(x_0,x'',\xi''\right) \\ \bar{g}^{0(l+m)}\left(x_0,x'',\xi''\right) & \bar{g}^{00}\left(x_0,x'',\xi''\right) & \bar{g}^{0l}\left(x_0,x'',\xi''\right) \\ \bar{g}^{k(l+m)}\left(x_0,x'',\xi''\right) & \bar{g}^{k0}\left(x_0,x'',\xi''\right) & \bar{g}^{kl}\left(x_0,x'',\xi''\right) \end{bmatrix} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix}. \end{aligned}$$

Here $\bar{g}^{kl}\left(x_0,x'',\xi''\right) = 2g^{kl}\left(x_0,-\frac{\xi''}{\sqrt{2}},\frac{x''}{\sqrt{2}}\right)$ and g^{kl} the components of the inverse metric on $T^*\mathbb{R}^n$.

Next we consider another function f_1 of the form

$$f_1 = \frac{1}{2}\begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} \begin{bmatrix} \alpha_{m\times m}\left(x_0,x'',\xi''\right) & \gamma_{m\times m+1}\left(x_0,x'',\xi''\right) \\ \gamma^t_{m+1\times m}\left(x_0,x'',\xi''\right) & \beta_{m+1\times m+1}\left(x_0,x'',\xi''\right) \end{bmatrix} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix}$$

where α, β and γ are matrix valued functions of the given orders, with α, β being symmetric. An easy computation now shows

$$\begin{aligned} \left(e^{H_{f_1}}\right)^* \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix} &= e^{\Lambda} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix} + o_2 \quad \text{with} \\ \Lambda\left(x_0, x'', \xi''\right) &= \begin{bmatrix} 0 & -I_{m+1\times m+1} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{m\times m}\left(x_0, x'', \xi''\right) & \gamma_{m\times m+1}\left(x_0, x'', \xi''\right) \\ \gamma^t_{m+1\times m}\left(x_0, x'', \xi''\right) & \beta_{m+1\times m+1}\left(x_0, x'', \xi''\right) \end{bmatrix}. \end{aligned}$$

From the suitability assumption (1.3), we have that there exists a smooth matrix valued functions α, β and γ such that

$$\begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} e^{\Lambda^t} \begin{bmatrix} \bar{g}^{(k+m)(l+m)}\left(x_0, x'', \xi''\right) & \bar{g}^{(k+m)0}\left(x_0, x'', \xi''\right) & \bar{g}^{(k+m)l}\left(x_0, x'', \xi''\right) \\ \bar{g}^{0(l+m)}\left(x_0, x'', \xi''\right) & \bar{g}^{00}\left(x_0, x'', \xi''\right) & \bar{g}^{0l}\left(x_0, x'', \xi''\right) \\ \bar{g}^{k(l+m)}\left(x_0, x'', \xi''\right) & \bar{g}^{k0}\left(x_0, x'', \xi''\right) & \bar{g}^{kl}\left(x_0, x'', \xi''\right) \end{bmatrix} e^{\Lambda} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix}$$

$$= \xi_0^2 + \bar{\nu}\left[\sum_{j=1}^{m} \mu_j\left(x_j^2 + \xi_j^2\right)\right] + o_3$$

where

$$\bar{\nu}\left(x_0, x'', \xi''\right) = \nu\left(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}\right). \tag{5.4}$$

Letting

$$H_2 = \frac{1}{2}\sum_{j=1}^{m} \mu_j\left(x_j^2 + \xi_j^2\right),$$

Egorov's theorem now gives

$$\begin{aligned} e^{\frac{i}{h} f_1^W} d_0^W e^{-\frac{i}{h} f_1^W} &= \left(\sum_{j=0}^{2m} \sigma_j b_j\right)^W + h o_0 \quad \text{with} \\ \sum_{j=0}^{2m} b_j^2 &= \left(\xi_0^2 + 2\bar{\nu} H_2\right)^W + o_3. \end{aligned} \tag{5.5}$$

Another Taylor expansion in the variables (x', ξ_0, ξ') gives $A = \left(a_{jk}\left(x_0, x'', \xi''\right)\right) \in C^\infty\left(\mathbb{R}^n_{(x_0, x'', \xi'')}; \mathfrak{so}\left(n\right)\right)$ and $r_j \in o_2$, $j = 0, \ldots, 2m$, such that

$$e^{-A} \begin{bmatrix} b_0 \\ \vdots \\ b_{2m} \end{bmatrix} = \begin{bmatrix} \xi_0 \\ \left(2\bar{\nu}\mu_1\right)^{\frac{1}{2}} x_1 \\ \left(2\bar{\nu}\mu_1\right)^{\frac{1}{2}} \xi_1 \\ \vdots \\ \left(2\bar{\nu}\mu_m\right)^{\frac{1}{2}} x_m \\ \left(2\bar{\nu}\mu_m\right)^{\frac{1}{2}} \xi_m \end{bmatrix} + \begin{bmatrix} r_0 \\ \vdots \\ r_{2m} \end{bmatrix}.$$

We may now set $c_A = \frac{1}{i} a_{jk}\sigma^j\sigma^k \in C^\infty\left(\mathbb{R}^n_{(x_0,x'',\xi'')}; i\mathfrak{u}\left(2^m\right)\right)$ and compute

$$e^{ic_A^W} e^{\frac{i}{h}f_1^W} d_0^W e^{-\frac{i}{h}f_1^W} e^{-ic_A^W} = d_1^W, \quad \text{where} \tag{5.6}$$

$$d_1 = H_1 + \sigma^j r_j + h o_0, \quad \text{and} \tag{5.7}$$

$$H_1 := \xi_0\sigma_0 + (2\bar{\nu})^{\frac{1}{2}} \sum_{j=1}^{m} \mu_j^{\frac{1}{2}} \left(x_j\sigma_{2j-1} + \xi_j\sigma_{2j}\right). \tag{5.8}$$

5.1. **Weyl product and Koszul complexes.** We now derive a formal Birkhoff normal form for the symbol d_1 in (5.7). First denote by $R = C^\infty\left(x_0, x'', \xi''\right)$ the ring of real valued functions in the given $2m+1$ variables. Further define

$$S := R\left[\!\left[x', \xi_0, \xi'; h\right]\!\right]$$

the ring of formal power series in the further given $2m+2$ variables with coefficients in R. The ring $S \otimes \mathbb{C}$ is now equipped with the Weyl product

$$a * b := \left[e^{\frac{ih}{2}\left(\partial_{r_1}\partial_{s_2} - \partial_{r_2}\partial_{s_1}\right)}\left(a\left(s_1, r_1; h\right) b\left(s_2, r_2; h\right)\right)\right]_{x=s_1=s_2, \xi=r_1=r_2},$$

corresponding to the composition formula (2.36) for pseudodifferential operators, with

$$[a,b] \quad := \quad a*b - b*a$$

being the corresponding Weyl bracket. It is an easy exercise to show that for $a, b \in S$ real valued, the commutator $i\,[a,b] \in S$ is real valued.

Next, we define a filtration on S. Each monomial $h^k \xi_0 \left(x'\right)^\alpha \left(\xi'\right)^\beta$ in S is given the weight $2k + a + |\alpha| + |\beta|$. The ring S is equipped with a decreasing filtration

$$\begin{aligned} S = O_0 \quad &\supset \quad O_1 \supset \ldots \supset O_N \supset \ldots, \\ \bigcap_N O_N \quad &= \quad \{0\}, \end{aligned}$$

where O_N consists of those power series with monomials of weight N or more. It is an exercise to show that

$$\begin{aligned} O_N * O_M \quad &\subset \quad O_{N+M} \\ \left[O_N, O_M\right] \quad &\subset \quad ihO_{N+M-2}. \end{aligned}$$

The associated grading is given by

$$S = \bigoplus_{N=0}^{\infty} S_N$$

where S_N consists of those power series with monomials of weight exactly N . We also define the quotient ring $D_N := S/O_{N+1}$ whose elements may be identified with the set of homogeneous polynomials with monomials of weight at most N. The ring D_N is also similarly graded and filtered. In similar vein, we may also define the ring

$$S(m) = S \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$$

of $R \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$ valued formal power series in $(x', \xi_0, \xi'; h)$. The ring $S(m)$ is equipped with an induced product $*$ and decreasing filtration

$$\begin{aligned} O_0(m) &\supset O_1(m) \supset \ldots \supset O_N(m) \supset \ldots, \\ \bigcap_N O_N(m) &= \{0\}, \end{aligned}$$

where $O_N(m) = O_N \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$. It is again a straightforward exercise to show that for $a, b \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ self-adjoint, the commutator $i[a,b] \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ is self-adjoint.

5.1.1. *Koszul complexes.* Let us now again consider the $2m$ and $2m+1$ dimensional real inner product spaces $V = \mathbb{R}[e_1, \ldots, e_{2m}]$ and $W = \mathbb{R}[e_0] \oplus V$ from 2.2. Considering the chain groups $D_N \otimes \Lambda^k V$, $k = 0, 1, \ldots, n$, one may define four differentials

$$\begin{aligned} w_x^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(x_j e_{2j-1} \wedge + \xi_j e_{2j} \wedge \right) \\ i_x^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(x_j i_{e_{2j-1}} + \xi_j i_{e_{2j}} \right) \\ w_\partial^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} e_{2j-1} \wedge + \partial_{\xi_j} e_{2j} \wedge \right) \\ i_\partial^0 &= \sum_{j=1}^m \mu_j^{\frac{1}{2}} \left(\partial_{x_j} i_{e_{2j-1}} + \partial_{\xi_j} i_{e_{2j}} \right). \end{aligned}$$

We equip D_N with the $R[\![h]\!]$-valued inner products where the distinct monomials $\frac{1}{\sqrt{a!\alpha!\beta!}} \xi_0^a (x')^\alpha (\xi')^\beta$ are orthonormal. With these inner products w_x^0, i_∂^0 and w_∂^0, i_x^0 are respectively adjoints. The combinatorial Laplacians $\Delta^0 = w_x^0 i_\partial^0 + i_\partial^0 w_x^0 = w_\partial^0 i_x^0 + i_x^0 w_\partial^0$, are computed to be equal and act on basis elements $\xi_0^a (x')^\alpha (\xi')^\beta \left(\bigwedge e_j^{\gamma_j} \right)$ via multiplication by $\mu.(2(\alpha+\beta)+\gamma)$. It now follows that these have (co-)homology only in degree zero given by $R[\![h]\!]$.

Similarly, we may consider the chain groups $D_N \otimes \Lambda^k W$, $k = 0, 1, \ldots, n$, one may define four differentials

$$\begin{aligned} w_x &= \xi_0 e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} w_x^0 \\ i_x &= \xi_0 i_{e_0} + (2\bar{\nu})^{\frac{1}{2}} i_x^0 \\ w_\partial &= \partial_{\xi_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} w_\partial^0 \\ i_\partial &= \partial_{\xi_0} i_{e_0} + (2\bar{\nu})^{\frac{1}{2}} i_\partial^0. \end{aligned}$$

Again these complexes have cohomology only in degree zero given by $R[\![h]\!]$.

Next, we define twisted Koszul differentials on $D_N \otimes \Lambda^k V$ via

$$\begin{aligned}\tilde{w}^0_\partial &= \frac{i}{h}\sum_{j=1}^m \mu_j^{\frac{1}{2}}\left(\mathrm{ad}_{x_j} e_{2j-1}\wedge + \mathrm{ad}_{\xi_j} e_{2j}\wedge\right) = \sum_{j=1}^m \mu_j^{\frac{1}{2}}\left(\partial_{x_j} e_{2j}\wedge - \partial_{\xi_j} e_{2j-1}\wedge\right)\\ \tilde{i}^0_\partial &= \frac{i}{h}\sum_{j=1}^m \mu_j^{\frac{1}{2}}\left(\mathrm{ad}_{x_j} i_{e_{2j-1}} + \mathrm{ad}_{\xi_j} i_{e_{2j}}\right) = \sum_{j=1}^m \mu_j^{\frac{1}{2}}\left(\partial_{x_j} i_{e_{2j}} - \partial_{\xi_j} i_{e_{2j-1}}\right).\end{aligned}$$

We note that the above are symplectic adjoints to their untwisted counterparts with respect to the symplectic pairing $\sum_{j=1}^m e_{2j-1}\wedge e_{2j}$ on V.

Similar twisted Koszul differentials on $D_N \otimes \Lambda^k W$ are defined via

$$\begin{aligned}\tilde{w}_\partial &= \frac{i}{h}\mathrm{ad}_{\xi_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}}\,\tilde{w}^0_\partial = -\partial_{x_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}}\,\tilde{w}^0_\partial\\ \tilde{i}_\partial &= \frac{i}{h} i_{e_0}\mathrm{ad}_{\xi_0} + (2\bar{\nu})^{\frac{1}{2}}\,\tilde{i}^0_\partial = -\partial_{x_0} i_{e_0} + (2\bar{\nu})^{\frac{1}{2}}\,\tilde{i}^0_\partial.\end{aligned}$$

These twisted differentials correspond to the untwisted ones by a mere change of basis in V, W and hence also have (co-)homology only in degree zero given by $R[\![h]\!]$.

We now compute the twisted combinatorial Laplacian to be

$$\begin{aligned}\tilde{\Delta}^0 &= \tilde{w}^0_\partial i^0_x + i^0_x \tilde{w}^0_\partial\\ &= -\left(w^0_x \tilde{i}^0_\partial + \tilde{i}^0_\partial w^0_x\right)\\ &= \sum_{j=1}^m \mu_j\left[\xi_j\partial_{x_j} - x_j\partial_{\xi_j} + e_{2j} i_{e_{2j-1}} - e_{2j-1} i_{e_{2j}}\right].\end{aligned}$$

One may similarly define $\tilde{\Delta}$. Next, we define the space of twisted $\tilde{\Delta}^0$-harmonic, ξ_0- independent elements

$$\begin{aligned}\mathcal{H}^k_N &= \left\{\omega \in D_N \otimes \Lambda^k W \,|\, \tilde{\Delta}^0\omega = 0,\ \partial_{\xi_0}\omega = 0\right\}\\ \mathcal{H}^k &= \left\{\omega \in S \otimes \Lambda^k W \,|\, \tilde{\Delta}^0\omega = 0,\ \partial_{\xi_0}\omega = 0\right\}.\end{aligned}$$

We now prove a twisted version of the Hodge decomposition theorem.

Lemma 5.1. *The k-th chain group is spanned by the three subspaces*

$$D_N \otimes \Lambda^k W = \mathbb{R}\left[Im\left(i_x \tilde{w}_\partial\right), Im\left(\tilde{w}_\partial i_x\right), \mathcal{H}^k_N\right].$$

Proof. We first compute $\tilde{\Delta}$ in terms of $\tilde{\Delta}^0$ to be

$$\tilde{\Delta} = -\xi_0\partial_{x_0} + 2\bar{\nu}\tilde{\Delta}^0 - 2\left(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}\right) e_0 i^0_x.$$

Next, since $\tilde{\Delta}^0$ is skew-adjoint, we may decompose

$$D_N \otimes \Lambda^k W = E_0 \oplus \bigoplus_{\lambda>0}\left[E_{i\lambda} \oplus E_{-i\lambda}\right]$$

into its eigenspaces. We may now invert $\tilde{\Delta}$ on the non-zero eigenspaces of $\tilde{\Delta}^0$ above using the Volterra series

$$\tilde{\Delta}^{-1}=\left(2\bar{\nu}\tilde{\Delta}^0\right)^{-1}\sum_{j=0}^{\infty}\left[\left(2\bar{\nu}\tilde{\Delta}^0\right)^{-1}\left(\xi_0\partial_{x_0}+2\left(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}\right)e_0i_x^0\right)\right]^j.$$

The sum above is finite since $\xi_0\partial_{x_0}+2\left(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}\right)e_0i_x^0$ is nilpotent on $D_N\otimes\Lambda^kW$. Thus we have

$$\bigoplus_{\lambda>0}\left[E_{i\lambda}\oplus E_{-i\lambda}\right]\subset\operatorname{Im}\left(\tilde{\Delta}\right)\subset\mathbb{R}\left[\operatorname{Im}\left(i_x\tilde{w}_\partial\right),\operatorname{Im}\left(\tilde{w}_\partial i_x\right)\right].$$

Finally, we decompose

$$E_0=\bigoplus_{j=0}^{N}\xi_0^j\mathcal{H}_N^k$$

and write each $\omega\in\xi_0^j\mathcal{H}_N^k$, $j\geq1$, as

$$\begin{aligned}\omega &= \omega_0+\tilde{\Delta}\omega_1\\ \omega_0 &= \left[-2\left(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}\right)e_0i_x^0\xi_0^{-1}\int_0^{x_0}\right]^j\omega\in\mathcal{H}_N^k\\ \omega_1 &= -\left(\xi_0^{-1}\int_0^{x_0}\right)\sum_{l=0}^{j-1}\left[-2\left(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}\right)e_0i_x^0\xi_0^{-1}\int_0^{x_0}\right]^l\omega\end{aligned}$$

to complete the proof. □

5.2. **Formal Birkhoff normal form.** The importance of the Koszul complexes introduced in the previous subsection is in continuing the Birkhoff normal form procedure for the symbol d_1 in (5.7). The remaining steps in the procedure are formal.

First let us define the Clifford quantization of an element in $a\in S\otimes\Lambda^kW$, using (2.8) as an element in

$$c_0(a):=i^{\frac{k(k+1)}{2}}c(a)\in S(m).$$

It is clear from (2.10) and (2.11) this gives an isomorphism

$$c_0:S\otimes\Lambda^{\text{odd/even}}W\to S\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^m\right) \tag{5.9}$$

of real elements of the even or odd exterior algebra with self-adjoint elements in $S(m)$. It is clear from (5.7) that

$$d_1=H_1+c_0(r)+hS\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^m\right) \tag{5.10}$$

for $r:=\sum_{j=1}^n r_je_j\in O_2\otimes W$.

For $a\in\Lambda^{k}W$, we define $[a]:=\left[\frac{k}{2}\right]$. Now for $f\in O_{N}$, $N\geq3$ and $a\in O_{N}\otimes\Lambda^{\mathrm{even}}W$, $N\geq1$, we may compute the conjugations

(5.11)
$$e^{\frac{i}{h}f}H_{1}e^{-\frac{i}{h}f}=H_{1}+c_{0}\left(\tilde{w}\partial f\right)+O_{N}\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right)$$

(5.12)
$$e^{ic_{0}(a)}H_{1}e^{-ic_{0}(a)}=H_{1}+\left(-1\right)^{[a]+1}2c_{0}\left(i_{x}a\right)+hc_{0}\left(\tilde{w}\partial a\right)+O_{N+2}\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right)$$

in terms of the Koszul differentials.

We now come to the formal Birkhoff normal form for the symbol d_{1}.

Proposition 5.2. *There exist $f\in O_{3}$, $a\in O_{2}\otimes\Lambda^{even}W$ and $\omega\in\mathcal{H}^{odd}\cap O_{2}$ such that*

$$e^{ic_{0}(a)}e^{\frac{i}{h}f}d_{1}e^{-\frac{i}{h}f}e^{-ic_{0}(a)}=H_{1}+c_{0}\left(\omega\right). \tag{5.13}$$

Proof. We first prove that for each $N\geq1$, there exist $f_{N}\in O_{3}$, $a_{N}^{0}\in O_{1}\otimes\Lambda^{2}W$, $\omega_{N}^{0}\in\mathcal{H}^{1}\cap O_{2}$ and $r_{N}^{0}\in O_{N+1}\otimes W$ such that

(5.14)
$$\begin{aligned}e^{ic_{0}\left(a_{N}^{0}\right)}e^{\frac{i}{h}f_{N}}d_{1}e^{-\frac{i}{h}f_{N}}e^{-ic_{0}\left(a_{N}^{0}\right)}&=H_{1}+c_{0}\left(\omega_{N}^{0}\right)+c_{0}\left(r_{N}^{0}\right)+hS\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right),\\ f_{N+1}-f_{N}&\in O_{N+2},\\ a_{N+1}^{0}-a_{N}^{0}&\in O_{N},\\ \omega_{N+1}^{0}-\omega_{N}^{0}&\in O_{N+1}.\end{aligned}$$

The base case $N=1$ is given by (5.10) with $a_{1}^{0}=f_{1}=\omega_{1}^{0}=0$ and $r_{1}^{0}=r$. To complete the induction step we decompose

$$r_{N}^{0}=\underbrace{u_{N}^{0}}_{\in S_{N+1}\otimes W}+\underbrace{r_{N+1}^{0}}_{\in O_{N+2}\otimes W}. \tag{5.15}$$

Next we use Lemma 5.1 to find $b_{N},g_{N}\in O_{N+1}\otimes W$ and $\upsilon_{N}^{0}\in\mathcal{H}^{1}\cap S_{N+1}$ such that

$$u_{N}^{0}=\upsilon_{N}^{0}-i_{x}\tilde{w}\partial b_{N}^{0}-\tilde{w}\partial i_{x}g_{N}^{0}+O_{N+2} \tag{5.16}$$

Next, define $f_{N+1}=f_{N}+i_{x}g_{N}^{0}\in O_{3}$, $a_{N+1}^{0}=a_{N}^{0}+\frac{1}{2}\tilde{w}\partial b_{N}^{0}\in O_{1}\otimes\Lambda^{2}W$ and $\omega_{N+1}^{0}=\omega_{N}^{0}+\upsilon_{N}^{0}$. We now use (5.11), (5.12), (5.15) and (5.16) to compute

$$\begin{aligned}&e^{ic_{0}\left(a_{N+1}^{0}\right)}e^{\frac{i}{h}f_{N+1}}d_{1}e^{-\frac{i}{h}f_{N+1}}e^{-ic_{0}\left(a_{N+1}^{0}\right)}\\ =\;&e^{ic_{0}\left(\frac{1}{2}\tilde{w}\partial b_{N}^{0}\right)}e^{\frac{i}{h}i_{x}g_{N}^{0}}H_{1}e^{-\frac{i}{h}i_{x}g_{N}^{0}}e^{-ic_{0}\left(\frac{1}{2}\tilde{w}\partial b_{N}^{0}\right)}\\ &\qquad+c_{0}\left(\omega_{N}^{0}\right)+c_{0}\left(r_{N}^{0}\right)+hS\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right)\\ =\;&H_{1}+c_{0}\left(\omega_{N+1}^{0}\right)+c_{0}\left(r_{N+1}^{0}\right)+hS\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right)\end{aligned}$$

completing the induction step. Now setting $f=\lim_{N\to\infty}f_{N}$, $a_{0}=\lim_{N\to\infty}a_{N}^{0}$ and $\omega_{0}=\lim_{N\to\infty}\omega_{N}^{0}$ and letting $N\to\infty$ in (5.14) gives the relation

$$e^{ic_{0}(a_{0})}e^{\frac{i}{h}f}d_{1}e^{-\frac{i}{h}f}e^{-ic_{0}(a_{0})}=H_{1}+c_{0}\left(\omega_{0}\right)+hS\otimes i\mathfrak{u}_{\mathbb{C}}\left(2^{m}\right). \tag{5.17}$$

Next we claim that for each $N \geq 0$, there exist $a_N \in O_1 \otimes \Lambda^{\text{even}} W$, $\omega_N \in \mathcal{H}^* \cap O_2$ and such that

$$
\begin{aligned}
(5.18) \qquad e^{ic_0(a_N)} e^{\frac{i}{h}f} d_1 e^{-\frac{i}{h}f} e^{-ic_0(a_N)} &= H_1 + c_0(\omega_N) + hO_N \otimes i\mathfrak{u}_{\mathbb{C}}(2^m) \\
a_{N+1} - a_N &\in O_{N+1} \otimes \Lambda^{\text{even}} W \\
\omega_{N+1} - \omega_N &\in \mathcal{H}^{\text{odd}} \cap O_N
\end{aligned}
$$

The base case $N = 0$ is now provided by (5.17). To complete the induction step, we use the isomorphism (5.9) to decompose the remainder term in (5.18) above as

$$
c_0(u_N) + ihO_{N+1} \otimes \mathfrak{u}_{\mathbb{C}}(2^m)
$$

for $u_N \in S_N \otimes \Lambda^{\text{odd}} W$. Next we use Lemma 5.1 to find $b_N, g_N \in O_N \otimes \Lambda^{\text{odd}} W$ and $\upsilon_N \in \mathcal{H}^{\text{odd}} \cap S_N$ such that

$$
u_N = \upsilon_N - i_x \tilde{w}_\partial b_N - \tilde{w}_\partial i_x g_N + O_{N+1} \tag{5.19}
$$

Now define $a_{N+1} = a_N + i_x g_N + h\frac{(-1)^{[b_N]}}{2} \tilde{w}_\partial b_N \in O_1$ and $\omega_{N+1} = \omega_N + \upsilon_N$. We now use (5.11), (5.12), (5.15) and (5.19) to compute

$$
\begin{aligned}
& e^{ic_0(a_{N+1})} e^{\frac{i}{h}f} d_1 e^{-\frac{i}{h}f} e^{-ic_0(a_{N+1})} \\
= \; & H_1 + c_0(\omega_{N+1}) + ihO_{N+1} \otimes \mathfrak{u}_{\mathbb{C}}(2^m) .
\end{aligned}
$$

completing the induction step. Now setting $a = \lim_{N\to\infty} a_N$ and $\omega = \lim_{N\to\infty} \omega_N$ and letting $N \to \infty$ in (5.18) gives the proposition. □

Finally, we show how the Birkhoff normal form maybe used to perform a further reduction on the trace. First note that we may similarly use (2.8) to define a self-adjoint Clifford-Weyl quantization map

$$
c_0^W := \text{Op} \otimes c_0 : S^0_{\text{cl}}\left(\mathbb{R}^{2n}; \mathbb{C}\right) \otimes \Lambda^{\text{odd/even}} W \to \Psi^0_{\text{cl}}\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)
$$

which maps real valued symbols $S^0_{\text{cl}}\left(\mathbb{R}^{2n}; \mathbb{R}\right) \otimes \Lambda^{\text{odd/even}} W$ to self-adjoint operators in $\Psi^0_{\text{cl}}\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$. Similarly we define a space of real-valued, twisted $\tilde{\Delta}^0$-harmonic, ξ_0- independent symbols

$$
\mathcal{H}^k S^0_{\text{cl}} := \left\{ \omega \in S^0_{\text{cl}}\left(\mathbb{R}^{2n}; \mathbb{R}\right) \otimes \Lambda^k W | \, \tilde{\Delta}^0 \omega = 0, \, \partial_{\xi_0} \omega = 0 \right\} .
$$

Next, an application of Borel's lemma by virtue of (5.1), (5.6) and (5.13) gives the existence of

$$\bar{a} \sim \sum_{j=0}^{\infty} h^j \bar{a}_j \in S^0_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{R}\right) \otimes \Lambda^{\text{odd}} W$$

$$\bar{r} \sim \sum_{j=0}^{\infty} h^j \bar{r}_j \in S^0_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{R}\right) \otimes \Lambda^{\text{odd}} W$$

$$\bar{f} \sim \sum_{j=0}^{\infty} h^j \bar{f}_j \in S^0_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{R}\right)$$

$$\bar{\omega} \sim \sum_{j=0}^{\infty} h^j \bar{\omega}_j \in \mathcal{H}^{\text{odd}} S^0_{\text{cl}}$$

such that

$$e^{ic_0^W(\bar{a})} e^{\frac{i}{h}\bar{f}^W} d_0^W e^{-\frac{i}{h}\bar{f}^W} e^{-ic_0^W(\bar{a})} = \underbrace{H_1^W + c_0^W\left(\bar{\omega}\right)}_{:=\bar{D}} + c_0^W\left(\bar{r}\right) \tag{5.20}$$

on $\bar{V}_{\alpha\beta} := e^{X_{\bar{f}_0}}\left(V^0_{\alpha\beta}\right)$. Here $\left\{\bar{r}_j\right\}_{j\in\mathbb{N}_0}$, $\bar{f}_0$, $\bar{\omega}_0$ vanish to infinite, second and second order respectively along

$$\Sigma_0^{D_0} = \Sigma_0^{\bar{D}} = \Sigma_0^{\bar{D}+c_0^W(\bar{r})} = \left\{\xi_0 = x' = \xi' = 0\right\}.$$

Note that on account of (4.5) and (5.4) one again has

$$\nu_0 = \mu_1 \min_{x\in X} \nu\left(x\right) \leq \mu_1 \inf_{\mathbb{R}^n_{x_0,x'',\xi''}} \bar{\nu}$$

Furthermore, since $\bar{\omega}_0$ vanishes to second order we may choose $\bar{\omega}_0$ arbitrarily small satisfying the estimate

$$\left\|\bar{\omega}_0\right\|_{C^1} < \varepsilon, \tag{5.21}$$

for any $\varepsilon > 0$, while still satisfying (5.20).

We note that $\bar{D} \in \Psi^1_{\text{cl}}\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)$, with $\bar{D} + i$ having an elliptic symbol in the class $S^0\left(\langle\xi_0,\xi'\rangle\right)$, and is hence essentially self-adjoint as an unbounded operator on $L^2\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)$. The domain of its unique self-adjoint extension is $H^1\left(\mathbb{R}_{x_0}\right) \otimes L^2\left(\mathbb{R}^{n-1}_{x',x''};\mathbb{C}^{2^m}\right)$ (cf. Ch. 8 in [10]). We now set

$$\bar{A}_\alpha := e^{ic_0^W(\bar{a})} e^{\frac{i}{h}\bar{f}^W} A^0_\alpha e^{-\frac{i}{h}\bar{f}^W} e^{-ic_0^W(\bar{a})}$$

(5.22)

$$\mathcal{T}^\vartheta_{\alpha\beta}\left(\bar{D}\right) := \text{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D}}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-\bar{D}}{h}\right)\bar{A}_\beta\right]$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}\left(z\right)\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) \text{tr}\left[\bar{A}_\alpha\left(\frac{1}{\sqrt{h}}\bar{D} - z\right)^{-1}\bar{A}_\beta\right] dz d\bar{z}. \tag{5.23}$$

We next have the following proposition.

Proposition 5.3. *We have*

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) = \mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right) \quad mod\ h^{\infty}.$$

Proof. Since the conjugations in (5.1) and (5.20) are unitary and $WF\left(\bar{A}_\alpha\right), WF\left(\bar{A}_\beta\right) \subset \bar{V}_{\alpha\beta}$, we have

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\,\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) \mathrm{tr}\ \left[\bar{A}_\alpha\left(\frac{1}{\sqrt{h}}\left(\bar{D} + c_0^W(\bar{r})\right) - z\right)^{-1}\bar{A}_\beta\right] dz d\bar{z}.$$

It now remains to do away with the $c_0^W(\bar{r})$ above. Since this term vanishes to infinite order along $\Sigma_0^{\bar{D}} = \Sigma_0^{\bar{D}+c_0^W(\bar{r})}$, we may use symbolic calculus to find $P_N, Q_N \in \Psi^0_{\mathrm{cl}}\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$, $\forall N \geq 1$ such that

$$c_0^W(\bar{r}) = P_N\left(\bar{D} + c_0^W(\bar{r})\right)^N \tag{5.24}$$

$$c_0^W(\bar{r}) = Q_N\left(\bar{D}\right)^N. \tag{5.25}$$

Modifying $\bar{D}$ outside a neighborhood of $\bar{V}_{\alpha\beta}$ using Lemma 3.3 and A.6 we may assume that $\bar{D}, \bar{D} + c_0^W(\bar{r})$ have discrete spectrum in $\left(-\sqrt{2\nu_0}, \sqrt{2\nu_0}\right)$ and hence

$$\begin{aligned}
\mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right) &= \mathrm{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D}}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h} - \bar{D}}{h}\right)\bar{A}_\beta\right] \\
\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) &= \mathrm{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D} + c_0^W(\bar{r})}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h} - \bar{D} - c_0^W(\bar{r})}{h}\right)\bar{A}_\beta\right].
\end{aligned}$$

Next, with $\Pi^{\bar{D}} = \Pi^{\bar{D}}_{\left[-\sqrt{2\nu_0 h}, \sqrt{2\nu_0 h}\right]}$ and $\Pi^{\bar{D}+c_0^W(\bar{r})} = \Pi^{\bar{D}+c_0^W(\bar{r})}_{\left[-\sqrt{2\nu_0 h}, \sqrt{2\nu_0 h}\right]}$ denoting the spectral projections, (5.24) and (5.25) give

$$\begin{aligned}
\left\|c_0^W(\bar{r})\,\Pi^{\bar{D}}\right\| &= O\left(h^{\frac{N}{2}}\right) \\
\left\|c_0^W(\bar{r})\,\Pi^{\bar{D}+c_0^W(\bar{r})}\right\| &= O\left(h^{\frac{N}{2}}\right)
\end{aligned}$$

for each $N \geq 1$. Finally applying A.5 with $\rho(x) = f\left(\frac{x}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h} - x}{h}\right)$ and using the cyclicity of the trace gives $\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) - \mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right) = O\left(h^{-1}h^{\frac{N}{4096}}\right)$, $\forall N \geq 1$, completing the proof. □

6. Extension of a resolvent

In this section we complete the proof of Lemma 3.1. On account of the reductions in 4.1 and 5.3 in the previous sections, it suffices to now consider the trace $\mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right)$. First let $\bar{A}_\alpha = a_\alpha^W$, $\bar{A}_\beta = a_\beta^W$ for $a_\alpha, a_\beta \in S^0_{\mathrm{cl}}\left(\mathbb{R}^{2n}\right)$. The conjugations $e^{\frac{it}{h}x_0}\bar{A}_\alpha e^{-\frac{it}{h}x_0} = a^W_{\alpha,t}$ and $e^{\frac{it}{h}x_0}\bar{A}_\beta e^{-\frac{it}{h}x_0} = a^W_{\beta,t}$ are easily computed in terms of the one-parameter family of symbols $a_{\alpha,t}(\xi_0, \ldots) = a_\alpha(\xi_0 + t, \ldots)$, $a_\beta = a_\beta(\xi_0 + t, \ldots) \in S^0_{\mathrm{cl}}\left(\mathbb{R}^{2n}\right)$, $t \in \mathbb{R}$, obtained by translating in the ξ_0 direction. One now introduces almost analytic continuations of the symbols $a_{\alpha,t},\ a_{\beta,t}\in S_{\mathrm{cl}}^{0}\left(\mathbb{R}^{2n}\right)$, defined for $t\in\mathbb{C}$, such that all the Frechet semi-norms of $\bar{\partial}a_{\alpha,t},\ \bar{\partial}a_{\beta,t}$ are $O\left(\left|\mathrm{Im}t\right|^{\infty}\right)$. These maybe further chosen to have the property that their wavefront sets have uniform compact support when t is restricted to compact subsets of $\mathbb{C}$. Again one clearly has

$$a_{\alpha,t}^{W} = e^{-\frac{i\mathrm{Re}\,t}{h}x_{0}}\left(a_{\alpha,i\mathrm{Im}t}\right)^{W}e^{-\frac{i\mathrm{Re}\,t}{h}x_{0}},\quad\text{and}\tag{6.1}$$

$$a_{\beta,t}^{W} = e^{-\frac{i\mathrm{Re}\,t}{h}x_{0}}\left(a_{\beta,i\mathrm{Im}t}\right)^{W}e^{-\frac{i\mathrm{Re}\,t}{h}x_{0}}.\tag{6.2}$$

In similar vein we may define

(6.3)

$$\bar{D}_{t}\coloneqq e^{-\frac{it}{h}x_{0}}\bar{D}e^{\frac{it}{h}x_{0}}=H_{1,t}^{W}+c_{0}^{W}\left(\bar{\omega}\right)$$

$$H_{1,t}=\left(\xi_{0}+t\right)\sigma_{0}+\left(2\bar{\nu}\right)^{\frac{1}{2}}\sum_{j=1}^{m}\mu_{j}^{\frac{1}{2}}\left(x_{j}\sigma_{2j-1}+\xi_{j}\sigma_{2j}\right)\in S_{\mathrm{cl}}^{1}\left(\mathbb{R}^{2n}\right),\tag{6.4}$$

for $t\in\mathbb{R}$, on account of the ξ_{0}-independence of $\bar{\omega}$. An almost analytic continuation of $\bar{D}_{t}$ is easily introduced by simply allowing $t\in\mathbb{C}$ to be complex in (6.4) above. The resolvent $\left(\bar{D}_{t}-z\right)^{-1}:L^{2}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)\rightarrow L^{2}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)$ is well-defined and holomorphic in the region $\mathrm{Im}z>\left|\mathrm{Im}t\right|$.

In the lemma below we set $t=i\gamma\left(M,\delta\right)\coloneqq i2Mh^{\delta}\log\frac{1}{h}$, for $\delta=1-\epsilon\in\left(\frac{1}{2},1\right)$ with ϵ as in Lemma 3.1 and $M>1$. We now have the following.

Lemma 6.1. *For h sufficiently small and $\forall\varepsilon_{0}>0$, the resolvent*

$$\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma}-z\right)^{-1}:L^{2}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)\rightarrow L^{2}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)$$

extends holomorphically, and is uniformly $O\left(h^{-\frac{1}{2}}\right)$, in the region $Imz>-Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}$, $\left|Rez\right|\leq\sqrt{2\nu_{0}}-\varepsilon_{0}$.

Proof. We begin with the orthogonal Landau decomposition (2.31)

(6.5)

$$L^{2}\left(\mathbb{R}^{n};\mathbb{C}^{2^{m}}\right)=L^{2}\left(\mathbb{R}_{x_{0},x''}^{m+1}\right)\otimes\underbrace{\left(\mathbb{C}\left[\psi_{0,0}\right]\oplus\bigoplus_{\Lambda\in\mu.\left(\mathbb{N}_{0}^{m}\setminus0\right)}\left[E_{\Lambda}^{\mathrm{even}}\oplus E_{\Lambda}^{\mathrm{odd}}\right]\right)}_{=L^{2}\left(\mathbb{R}_{x'}^{m};\mathbb{C}^{2^{m}}\right)}\text{ where}$$

$$E_{\Lambda}^{\mathrm{even}}\coloneqq\bigoplus_{\substack{\tau\in\mathbb{N}_{0}^{m}\setminus0\\ \Lambda=\mu.\tau}}E_{\tau}^{\mathrm{even}}$$

$$E_{\Lambda}^{\mathrm{odd}}\coloneqq\bigoplus_{\substack{\tau\in\mathbb{N}_{0}^{m}\setminus0\\ \Lambda=\mu.\tau}}E_{\tau}^{\mathrm{odd}}$$

according to the eigenspaces of the squared magnetic Dirac operator $D^2_{\mathbb{R}^m}$ (2.21) on $\mathbb{R}^m$. It is clear from (6.4) that

$$H^W_{1,t} = (\xi_0 + t)\,\sigma_0 + \left[(2\bar{\nu})^{\frac{1}{2}}\right]^W \otimes D_{\mathbb{R}^m}$$

in terms of the above decomposition. Furthermore one has the commutation relations

$$\begin{aligned}\left[\sigma_0, D^2_{\mathbb{R}^m}\right] &= 0\\ \left[c^W_0(\bar{\omega}), D^2_{\mathbb{R}^m}\right] &= ihc^W_0\left(\tilde{\Delta}^0\bar{\omega}\right) = 0\end{aligned}$$

since $\bar{\omega}$ is $\tilde{\Delta}^0$-harmonic. The above and (6.3) show that the $\left(\frac{1}{\sqrt{h}}\bar{D}_t - z\right)$ preserves the eigenspaces in the decomposition (6.5) $\forall t \in \mathbb{C}$. It hence suffices to consider the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)$ to each eigenspace.

Let $E_0 \coloneqq \mathbb{C}\left[\psi_{0,0}\right]$, $E_\Lambda \coloneqq E^{\text{even}}_\Lambda \oplus E^{\text{odd}}_\Lambda$ and $\mathtt{P}_0, \mathtt{P}_\Lambda$ denote the projection onto the corresponding summands of (6.5). Define the restrictions

$$\begin{aligned}\Omega_0 \coloneqq \mathtt{P}_0 c^W_0(\bar{\omega})\,\mathtt{P}_0 &: L^2\left(\mathbb{R}^{m+1}_{x_0,x''}\right) \to L^2\left(\mathbb{R}^{m+1}_{x_0,x''}\right)\\ \Omega_\Lambda \coloneqq \mathtt{P}_\Lambda c^W_0(\bar{\omega})\,\mathtt{P}_\Lambda &: L^2\left(\mathbb{R}^{m+1}_{x_0,x''}; E^{\text{even}}_\Lambda \oplus E^{\text{odd}}_\Lambda\right) \to L^2\left(\mathbb{R}^{m+1}_{x_0,x''}; E^{\text{even}}_\Lambda \oplus E^{\text{odd}}_\Lambda\right),\ \Lambda > 0.\end{aligned}$$

Now $\bar{\omega} \sim \sum_{j=0}^\infty h^j \bar{\omega}_j \in \mathcal{H}^{\text{odd}} S^0_{\text{cl}}$ with ξ_0-independent $\bar{\omega}_0$ vanishing to second order along $\Sigma^{D_0}_0 = \Sigma^{\bar{D}}_0 = \{\xi_0 = x' = \xi' = 0\}$. Hence we may Taylor expand

$$\bar{\omega}_0 = \sum_{i\leq j}\left[a_{ij}z_i z_j + \bar{a}_{ij}\bar{z}_i\bar{z}_j + b_{ij}\bar{z}_i z_j + \bar{b}_{ij} z_i \bar{z}_j\right],$$

in terms of the complex coordinates $z_j = x_j + i\xi_j$, $\bar{z}_j = x_j - i\xi_j$,$1 \leq j \leq m$, with $a_{ij}, b_{ij} \in S^0_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{R}\right) \otimes \Lambda^{\text{odd}} W$. The self-adjoint Clifford-Weyl quantization now yields

$$\begin{aligned}c^W_0(\bar{\omega}_0) = &\sum_{i\leq j}\left[c^W_0(a_{ij})\,A_i A_j + A^*_j A^*_i c^W_0(\bar{a}_{ij}) + c^W_0(b_{ij})\,A^*_i A_j + A^*_j A_i c^W_0\left(\bar{b}_{ij}\right)\right]\\ &+ h\Psi^0_{\text{cl}}\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)\end{aligned}$$

in terms of the raising and lowering operators in (2.26). Since each lowering operator A_j annihilates $\psi_{0,0}$, this leads to the estimate

$$\|\Omega_0\| = O(h)\,. \tag{6.6}$$

Next, on account of (5.21) one may also expand $\bar{\omega}_0 = \sum_{j=1}^m\left[a_j z_j + \bar{a}_j \bar{z}_j\right]$, with $a_j \in S^0_{\text{cl}}\left(\mathbb{R}^{2n};\mathbb{R}\right) \otimes \Lambda^{\text{odd}} W$, satisfying $\|a_j\|_{C^0} \leq \varepsilon < 1$. On self-adjoint quantization this now gives

$$c^W_0(\bar{\omega}_0) = \sum_{j=1}^m\left[c^W_0(a_j)\,A_j + A^*_j c^W_0(\bar{a}_j)\right] + h\Psi^0_{\text{cl}}\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)$$

where

$$\left\|c_0^W(a_j)\right\|_{L^2\to L^2}, \left\|c_0^W(\bar{a}_j)\right\|_{L^2\to L^2} = \|a_j\|_{C^0} + O(h) \\ \leq \varepsilon + O(h).$$

Knowing the action of the lowering and raising operators A_j, A_j^* on each eigenstate (2.25) of $D^2_{\mathbb{R}^m}$ gives the estimate

$$\|\Omega_\Lambda\| \leq \varepsilon\sqrt{\Lambda h} + O(h) \tag{6.7}$$

with the $O(h)$ term above being uniform in Λ.

Next we compute the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)$ to the E_0 eigenspace in (6.5) using (2.6) to be

$$D_{i\gamma,0}(z) := \mathtt{P}_0\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)\mathtt{P}_0 = \frac{1}{\sqrt{h}}\left[-\xi_0 - i\gamma - z\sqrt{h} + \Omega_0\right]. \tag{6.8}$$

The above is again understood as a closed unbounded operator on $L^2\left(\mathbb{R}^{m+1}_{x_0,x''}\right)$ with domain $H^1(\mathbb{R}_{x_0}) \otimes L^2\left(\mathbb{R}^m_{x''}\right)$. Set $R_{i\gamma,0}(z) = [r_{i\gamma,0}(z)]^W$, with

$$r_{i\gamma,0}(z) = \frac{\sqrt{h}}{-\xi_0 - i\gamma - z\sqrt{h}},$$

which is well defined for $\mathrm{Im} z > -\frac{\gamma}{2\sqrt{h}} = -Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}$, and compute

$$R_{i\gamma,0}(z)\, D_{i\gamma,0}(z) = I + O\left(h^{1-\delta}\right) \\ D_{i\gamma,0}(z)\, R_{i\gamma,0}(z) = I + O\left(h^{1-\delta}\right)$$

using (6.6). This shows that the inverse $D_{i\gamma,0}(z)^{-1}$ exists and is $O(R_{i\gamma,0}(z)) = O\left(h^{\frac{1}{2}-\delta}\right)$.

Next, we compute the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)$ to the E_Λ, $\Lambda > 0$, eigenspace in (6.5). Using (2.32), (2.33) this has the form

$$\begin{aligned} D_{i\gamma,\Lambda}(z) &:= \mathtt{P}_\Lambda\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)\mathtt{P}_\Lambda \\ &= \frac{1}{\sqrt{h}}\begin{bmatrix} -\xi_0 - i\gamma - z\sqrt{h} & \left(\sqrt{2\bar{\nu}\Lambda h}\right)^W \\ \left(\sqrt{2\bar{\nu}\Lambda h}\right)^W & \xi_0 + i\gamma - z\sqrt{h} \end{bmatrix} + \frac{1}{\sqrt{h}}\Omega_\Lambda \end{aligned}$$

with respect to the $\mathbb{Z}_2$- grading $E_\Lambda = E_\Lambda^{\mathrm{even}} \oplus E_\Lambda^{\mathrm{odd}}$. Here we leave the identification $\mathtt{i}_\tau$ in (2.32) between the odd and even parts as being understood.

Set $R_{i\gamma,\Lambda}(z) = [r_{i\gamma,\Lambda}(z)]^W$

$$r_{i\gamma,\Lambda}(z) := \frac{\sqrt{h}\begin{bmatrix} -\xi_0 - i\gamma - z\sqrt{h} & \left(\sqrt{2\bar{\nu}\Lambda h}\right) \\ \left(\sqrt{2\bar{\nu}\Lambda h}\right) & \xi_0 + i\gamma - z\sqrt{h}\end{bmatrix}}{z^2 h - (\xi_0 + i\gamma)^2 - 2\bar{\nu}\Lambda h}$$

which is well defined for $|\mathrm{Re} z| \leq \sqrt{2\nu_0} - \varepsilon_0 < \inf_{\mathbb{R}^n} \sqrt{2\bar{\nu}\Lambda}$, and h sufficiently small. We now compute

$$\begin{aligned} \|R_{i\gamma,\Lambda}(z) D_{i\gamma,\Lambda}(z) - I\| &\leq C\varepsilon + O(h) \\ \|D_{i\gamma,\Lambda}(z) R_{i\gamma,\Lambda}(z) - I\| &\leq C\varepsilon + O(h) \end{aligned}$$

using (6.7) with the constants above being uniform in Λ. Choosing ε sufficiently small in (5.21) shows that the inverse $D_{i\gamma,\Lambda}(z)^{-1}$ exists and is $O(R_{i\gamma,\Lambda}(z)) = O\left(h^{-\frac{1}{2}}\right)$ uniformly. □

We now finally finish the proof of Lemma 3.1.

Proof of Lemma 3.1. As noted in the beginning of the section, on account of (3.2), (3.3) and the reductions 4.1 and 5.3, it suffices to show $\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) = O(h^{\infty})$. We now define the trace

$$\tau_{\alpha\beta,t}(z) := \mathrm{tr}\left[a^W_{\alpha,t}\left(\frac{1}{\sqrt{h}}\bar{D}_t - z\right)^{-1} a^W_{\beta,t}\right], \quad \mathrm{Im} z > |\mathrm{Im} t|, \tag{6.9}$$

in terms of the almost analytic continuations. We clearly have

$$\begin{aligned} \tau_{\alpha\beta,t}(z) &= O\left(h^{-n}|\mathrm{Im} z|^{-1}\right) \\ \frac{\partial}{\partial \bar{t}}\tau_{\alpha\beta,t}(z) &= O\left(h^{-n}|\mathrm{Im} t|^{\infty}|\mathrm{Im} z|^{-2}\right). \end{aligned}$$

Furthermore, by (6.1), (6.2) and (6.3) $\tau_{\alpha\beta,t}(z)$ only depends on $\mathrm{Re} t$ and we have

$$\tau_{\alpha\beta,i\,\mathrm{Im} t}(z) = \tau_{\alpha\beta,0}(z) + O\left(h^{-n}|\mathrm{Im} t|^{\infty}|\mathrm{Im} z|^{-2}\right). \tag{6.10}$$

As before, we again introduce $\psi \in C^{\infty}(\mathbb{R};[0,1])$ such that $\psi(x) = \begin{cases} 1; & x \leq 1 \\ 0; & x \geq 2 \end{cases}$ and set $\psi_M(z) = \psi\left(\frac{\mathrm{Im} z}{M\sqrt{h}\log\frac{1}{h}}\right)$. The estimates (3.11), (3.12) along with the observation $\psi_M |\mathrm{Im} z|^N = O\left(\left(M\sqrt{h}\log\frac{1}{h}\right)^N\right)$ now give

$$\begin{aligned} \mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) &= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\left(\psi_M \tilde{f}\right)\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,0}(z)\, dz d\bar{z} \\ &= O(h^{\infty}) + \\ &\quad \frac{1}{\pi}\int_{\left\{M\sqrt{h}\log\frac{1}{h} \leq \mathrm{Im} z \leq 2M\sqrt{h}\log\frac{1}{h}\right\}} \bar{\partial}\left(\psi_M \tilde{f}\right)\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,0}(z)\, dz d\bar{z}. \end{aligned}$$

Using (6.10) and $\gamma = 2Mh^{\delta}\log\frac{1}{h}$, $\delta \in \left(\frac{1}{2},1\right)$, the above now equals

$$\begin{aligned}\mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right) &= O\left(h^{\infty}\right)+\\ &\quad \frac{1}{\pi}\int_{\left\{M\sqrt{h}\log\frac{1}{h}\leq \mathrm{Im}z\leq 2M\sqrt{h}\log\frac{1}{h}\right\}}\bar{\partial}\left(\psi_M\tilde{f}\right)\check{\vartheta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\tau_{\alpha\beta,i\gamma}\left(z\right)dzd\bar{z}.\end{aligned}$$

Since the resolvent $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma}-z\right)^{-1}$, and hence the trace $\tau_{\alpha\beta,i\gamma}\left(z\right)$, extends holomorphically to $\mathrm{Im}z > -Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}, \left|\mathrm{Re}z\right|\leq\sqrt{2\nu_0}-\varepsilon_0$ by Lemma 6.1 we may replace the integral in the last line above

$$\begin{aligned}\mathcal{T}^{\vartheta}_{\alpha\beta}\left(\bar{D}\right) &= O\left(h^{\infty}\right)+\\ &\quad \frac{1}{\pi}\int_{\left\{-\frac{1}{2}Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}\leq \mathrm{Im}z\leq -\frac{1}{4}Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}\right\}}\bar{\partial}\left(\psi_M\tilde{f}\right)\check{\vartheta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\tau_{\alpha\beta,i\gamma}\left(z\right)dzd\bar{z}\\ &= O\left(h^{\infty}\right)+\\ &\quad O\left[\int_{\left\{-\frac{1}{2}Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}\leq \mathrm{Im}z\leq -\frac{1}{4}Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}\right\}}\frac{h^{-n-\frac{1}{2}}}{\sqrt{h}\log\frac{1}{h}}e^{\frac{S'_{\alpha\beta}(\mathrm{Im}z)}{h^{\frac{1}{2}-\epsilon}}}dzd\bar{z}\right]\\ &= O\left[h^{\frac{M}{4}\left(S'_{\alpha\beta}\right)-n-\frac{1}{2}}\right]\end{aligned}$$

using (3.11) and $O\left(h^{-\frac{1}{2}}\right)$ estimate on the resolvent $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma}-z\right)^{-1}$. Choosing M sufficiently large now gives the result. □

7. Local trace expansion

In this section we prove Lemma 3.2. This is a relatively classical trace expansion. A parametrix construction for the operator $e^{\frac{it}{h}D_h^2}$ may potentially be employed in its proof since the principal symbol of D_h^2 is Morse-Bott critical as in [6]. However Lemma 3.2 would require an understanding of the large time behaviour of parametrix left open in [6] (see [7, 19]). Here we prove the expansion using the alternate methods of local index theory. The expansion is analogous to the heat trace expansions arising in the analysis of the Bergman kernel [4, 20]. Here we adopt a modification of the approach in [20] Ch. 1, 4.

First, fix a point $p\in X$. On account of 1.1 there is on orthonormal basis $e_{0,p}=R_p, e_{j,p}$, $e_{j+m,p}$, $j=1,\ldots,m$ of T_pX consisting of eigenvectors of $\mathfrak{J}_p$ with eigenvalues $0,\pm\lambda_{j,p}\left(:=\pm i\mu_j\nu\left(p\right)\right)$, $j=1,\ldots,m$, such that

$$da\left(p\right)=\sum_{j=1}^{m}\lambda_j\left(p\right)e^*_{j,p}\wedge e^*_{j+m,p}. \tag{7.1}$$

Using the parallel transport from this basis fix a geodesic coordinate system $(x_0,\ldots,x_{2m})$ on an open neighborhood of $p\in\Omega$. Let $e_j=w_j^k\partial_{x_k}$, $0\leq j\leq 2m$, be the local orthonormal frame of TX obtained by parallel transport of $e_{j,p}=\partial_{x_j}\big|_p$,$0\leq j\leq 2m$, along geodesics. Hence we again have

$w_j^k g_{kl} w_r^l = \delta_{jr}$; $w_j^k\Big|_p = \delta_j^k$ with g_{kl} being the components of the metric in these coordinates. Choose an orthonormal basis $\{s_{j,p}\}_{j=1}^{2m}$for S_p in which Clifford multiplication

$$c\left(e_j\right)|_p = \gamma_j \tag{7.2}$$

is standard. Choose an orthonormal basis $\mathtt{l}_p$ for L_p. Parallel transport the bases $\{s_{j,p}\}_{j=1}^{2m}$, $\mathtt{l}_p$ along geodesics using the spin connection ∇^S and unitary family of connections $\nabla^h = A_0 + \frac{i}{h}a$ to obtain trivializations $\{s_j\}_{j=1}^{2m}$, $\mathtt{l}$ of S, L on Ω. Since Clifford multiplication is parallel, the relation (7.2) now holds on Ω. The connection $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ can be expressed in this frame and these coordinates as

$$\nabla^{S\otimes L} = d + A_j^h dx^j + \Gamma_j dx^j, \tag{7.3}$$

where each A_j^h is a Christoffel symbol of ∇^h and each Γ_j is a Christoffel symbol of the spin connection ∇^S. Since the section l is obtained via parallel transport along geodesics, the connection coefficient A_j^h maybe written in terms of the curvature $F_{jk}^h dx^j \wedge dx^k$ of ∇^h via

$$A_j^h(x) = \int_0^1 d\rho\left(\rho x^k F_{jk}^h\left(\rho x\right)\right). \tag{7.4}$$

The dependence of the curvature coefficients F_{jk}^h on the parameter h is seen to be linear in $\frac{1}{h}$ via

$$F_{jk}^h = F_{jk}^0 + \frac{i}{h}\left(da\right)_{jk} \tag{7.5}$$

despite the fact that they are expressed in the h dependent frame $\mathtt{l}$. This is because a gauge transformation from an h independent frame into $\mathtt{l}$ changes the curvature coefficient by conjugation. Since L is a line bundle this is conjugation by a function and hence does not change the coefficient. Furthermore, the coefficients in the Taylor expansion of (7.5) at 0 maybe expressed in terms of the covariant derivatives $\left(\nabla^{A_0}\right)^l F_{jk}^0$, $\left(\nabla^{A_0}\right)^l (da)_{jk}$ evaluated at p. Next, using the Taylor expansion

$$(da)_{jk} = (da)_{jk}(0) + x^l a_{jkl}, \tag{7.6}$$

we see that the connection $\nabla^{S\otimes L}$ has the form

$$\nabla^{S\otimes L} = d + \left[\frac{i}{h}\left(\frac{x^k}{2}\left(da\right)_{jk}(0) + x^k x^l A_{jkl}\right) + x^k A_{jk}^0 + \Gamma_j\right] dx^j \tag{7.7}$$

where

$$\begin{aligned} A_{jk}^0 &= \int_0^1 d\rho\left(\rho F_{jk}^0\left(\rho x\right)\right) \\ A_{jkl} &= \int_0^1 d\rho\left(\rho a_{jkl}\left(\rho x\right)\right) \end{aligned}$$

and Γ_j are all independent of h. Finally from (7.2) and (7.7) may write down the expression for the Dirac operator (1.2) also given as $D = hc \circ \left(\nabla^{S\otimes L}\right)$ in terms of the chosen frame and coordinates to be

(7.8)
$$D = \gamma^r w_r^j \left[h\partial_{x_j} + i\frac{x^k}{2} (da)_{jk}(0) + ix^k x^l A_{jkl} + h\left(x^k A^0_{jk} + \Gamma_j\right)\right]$$

(7.9)
$$= \gamma^r \left[w_r^j h\partial_{x_j} + iw_r^j \frac{x^k}{2} (da)_{jk}(0) + \frac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}\left(g^{\frac{1}{2}} w_r^j\right)\right] + \gamma^r \left[iw_r^j x^k x^l A_{jkl} + hw_r^j \left(x^k A^0_{jk} + \Gamma_j\right) - \frac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}\left(g^{\frac{1}{2}} w_r^j\right)\right] \in \Psi^1_{\mathrm{cl}}\left(\Omega^0_s; \mathbb{C}^{2^m}\right)$$

In the second expression above both square brackets are self-adjoint with respect to the Riemannian density $e^1 \wedge \ldots \wedge e^n = \sqrt{g}dx := \sqrt{g}dx^1 \wedge \ldots \wedge dx^n$ with $g = \det(g_{ij})$. Again one may obtain an expression self-adjoint with respect to the Euclidean density dx in the framing $g^{\frac{1}{4}} u_j \otimes \mathtt{1}, 1 \leq j \leq 2^m$, with the result being an addition of the term $h\gamma^j w_j^k g^{-\frac{1}{4}}\left(\partial_{x_k} g^{\frac{1}{4}}\right)$.

Let i_g be the injectivity radius of g^{TX} . Define the cutoff $\chi \in C_c^\infty(-1,1)$ such that $\chi = 1$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We now modify the functions w_j^k, outside the ball $B_{i_g/2}(p)$, such that $w_j^k = \delta_j^k$ (and hence $g_{jk} = \delta_{jk}$) are standard outside the ball $B_{i_g}(p)$ of radius i_g centered at p. This again gives

(7.10)
$$\mathbb{D} = \gamma^r \left[w_r^j h\partial_{x_j} + iw_r^j \frac{x^k}{2} (da)_{jk}(0) + \frac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}\left(g^{\frac{1}{2}} w_r^j\right)\right] + \chi\left(|x|/i_g\right) \gamma^r \left[iw_r^j x^k x^l A_{jkl} + hw_r^j \left(x^k A^0_{jk} + \Gamma_j\right) - \frac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}\left(g^{\frac{1}{2}} w_r^j\right)\right] \in \Psi^1_{\mathrm{cl}}\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$$

as a well defined operator on $\mathbb{R}^n$ formally self adjoint with respect to $\sqrt{g}dx$. Again $\mathbb{D} + i$ being elliptic in the class $S^0(m)$ for the order function

$$m = \sqrt{1 + g^{jl}\left(\xi_j + \frac{x^k}{2}(da)_{jk}(0)\right)\left(\xi_l + \frac{x^r}{2}(da)_{lr}(0)\right)},$$

the operator $\mathbb{D}$ is essentially self adjoint.

Proposition 7.1. *There exist tempered distributions $u_j \in \mathcal{S}'(\mathbb{R}_s)$, $j = 0, 1, 2, \ldots$, such that one has a trace expansion*

(7.11)
$$tr\, \phi\left(\frac{D}{\sqrt{h}}\right) = h^{-n/2}\left(\sum_{j=0}^N u_j(\phi)\, h^{j/2}\right) + h^{(N+1-n)/2} O\left(\sum_{k=0}^{n+1} \left\| \langle \xi \rangle^N \hat{\phi}^{(k)} \right\|_{L^1}\right)$$

for each $N \in \mathbb{N}$, $\phi \in \mathcal{S}(\mathbb{R}_s)$.

Proof. We begin by writing $\phi = \phi_0 + \phi_1$, with

$$\begin{aligned}\phi_0(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi s}\hat{\phi}(\xi)\,\chi\left(\frac{2\xi\sqrt{h}}{i_g}\right)d\xi\\ \phi_1(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi s}\hat{\phi}(\xi)\left[1-\chi\left(\frac{2\xi\sqrt{h}}{i_g}\right)\right]d\xi\end{aligned}$$

via Fourier inversion.

First considering ϕ_1, integration by parts gives the estimate

$$\left|s^{n+1}\phi_1(s)\right| \leq C_N h^{\frac{N-1}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^N\hat{\phi}^{(k)}\right\|_{L^1}\right),$$

$\forall N \in \mathbb{N}$. Hence,

$$\left\|D^{n+1-a}\phi_1\left(\frac{D}{\sqrt{h}}\right)D^a\right\|_{L^2\to L^2} = C_N h^{\frac{n+N}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^N\hat{\phi}^{(k)}\right\|_{L^1}\right),$$

$\forall N \in \mathbb{N}$, $\forall a = 0,\ldots,n+1$. Semi-classical elliptic estimate and Sobolev's inequality now give the estimate

$$\left|\phi_1\left(\frac{D}{\sqrt{h}}\right)\right|_{C^0(X\times X)} \leq C_N h^{\frac{n+N}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^N\hat{\phi}^{(k)}\right\|_{L^1}\right) \tag{7.12}$$

$\forall N \in \mathbb{N}$, on the Schwartz kernel.

Next, considering ϕ_0, we first use the change of variables $\alpha = \xi\sqrt{h}$ to write

$$\phi_0\left(\frac{D}{\sqrt{h}}\right) = \frac{1}{2\pi\sqrt{h}}\int_{\mathbb{R}} e^{i\alpha\left(D_{A_0}+ih^{-1}c(a)\right)}\hat{\phi}\left(\frac{\alpha}{\sqrt{h}}\right)\chi\left(\frac{2\alpha}{i_g}\right)d\alpha.$$

Now since $D = \mathbb{D}$ on $B_{i_g/2}(p)$, we may use the finite propagation speed of the wave operators $e^{i\alpha h^{-1}D}$, $e^{i\alpha h^{-1}\mathbb{D}}$ (cf. D.2.1 in [20]) to conclude

$$\phi_0\left(\frac{D}{\sqrt{h}}\right)(p,\cdot) = \phi_0\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(0,\cdot). \tag{7.13}$$

The right hand side above is defined using functional calculus of self-adjoint operators, with standard local elliptic regularity arguments implying the smoothness of its Schwartz kernel. By virtue of (7.12), a similar estimate for $\phi_1\left(\frac{\mathbb{D}}{\sqrt{h}}\right)$, and (7.13) it now suffices to consider $\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)$.

We now introduce the rescaling operator $\mathscr{R}: C^\infty\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right) \to C^\infty\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)$; $(\mathscr{R}s)(x) := s\left(\frac{x}{\sqrt{h}}\right)$. Conjugation by $\mathscr{R}$ amounts to the rescaling of coordinates $x \to x\sqrt{h}$. A Taylor expansion in (7.10) now gives the existence of classical (h-independent) self-adjoint, first-order differential operators $\mathtt{D}_j = a_j^k(x)\,\partial_{x_k} + b_j(x)$, $j = 0,1\ldots$, with polynomial coefficients (of degree at most $j+1$) as well as h-dependent self-adjoint, first-order differential operators $\mathrm{E}_j = \sum_{|\alpha|=N+1} x^\alpha\left[c^k_{j,\alpha}(x;h)\,\partial_{x_k} + d_{j,\alpha}(x;h)\right]$, $j = 0,1\ldots$, with uniformly C^{∞} bounded coefficients $c_{j,\alpha}^{k}$, $d_{j,\alpha}$ such that

$$\mathscr{R}\mathbb{D}\mathscr{R}^{-1} = \sqrt{h}\mathtt{D} \quad \text{with} \tag{7.14}$$

$$\mathtt{D} = \left(\sum_{j=0}^{N} h^{j/2}\mathtt{D}_{j}\right) + h^{(N+1)/2}\mathrm{E}_{N+1}, \ \forall N. \tag{7.15}$$

The coefficients of the polynomials $a_{j}^{k}\left(x\right)$, $b_{j}\left(x\right)$ again involve the covariant derivatives of the curvatures F^{TX}, F^{A_0} and da evaluated at p. Furthermore, the leading term in (7.15) is easily computed

$$\mathtt{D}_{0} = \gamma^{j}\left[\partial_{x_{j}} + i\frac{x^{k}}{2}\left(da\right)_{jk}\left(0\right)\right] \tag{7.16}$$

$$= \gamma^{0}\partial_{x_{0}} + \underbrace{\gamma^{j}\left[\partial_{x_{j}} + \frac{i\lambda_{j}\left(p\right)}{2}x_{j+m}\right] + \gamma^{j+m}\left[\partial_{x_{j+m}} - \frac{i\lambda_{j}\left(p\right)}{2}x_{j}\right]}_{:=\mathtt{D}_{00}} \tag{7.17}$$

using (7.1), (7.6). It is now clear from (7.14) that

$$\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)\left(x,x'\right) = h^{-n/2}\phi\left(\mathtt{D}\right)\left(\frac{x}{\sqrt{h}},\frac{x'}{\sqrt{h}}\right). \tag{7.18}$$

Next, let $I_{j} = \left\{k = \left(k_{0},k_{1},\ldots\right)|k_{\alpha}\in\mathbb{N},\ \sum k_{\alpha} = j\right\}$ denote the set of partitions of the integer j and set

$$\mathtt{C}_{j}^{z} = \sum_{k\in I_{j}}\left(z-\mathtt{D}_{0}\right)^{-1}\left[\Pi_{\alpha}\mathtt{D}_{k_{\alpha}}\left(z-\mathtt{D}_{0}\right)^{-1}\right]. \tag{7.19}$$

Local elliptic regularity estimates again give $\left(z-\mathtt{D}\right)^{-1} = O_{L_{\mathrm{loc}}^{2}\to L_{\mathrm{loc}}^{2}}\left(\left|\mathrm{Im}z\right|^{-1}\right)$ and $\mathtt{C}_{j}^{z} = O_{L_{\mathrm{loc}}^{2}\to L_{\mathrm{loc}}^{2}}\left(\left|\mathrm{Im}z\right|^{-j-1}\right)$, $j = 0,1,\ldots$. A straightforward computation using (7.15) then yields

$$\left(z-\mathtt{D}\right)^{-1} - \left(\sum_{j=0}^{N} h^{j/2}\mathtt{C}_{j}^{z}\right) = O_{L_{\mathrm{loc}}^{2}\to L_{\mathrm{loc}}^{2}}\left(\left(\left|\mathrm{Im}z\right|^{-1}h^{\frac{1}{2}}\right)^{N+1}\right). \tag{7.20}$$

A similar expansion as (7.15) for the operator $\left(1+\mathtt{D}^{2}\right)^{(n+1)/2}\left(z-\mathtt{D}\right)$ also gives the bounds

$$\left(1+\mathtt{D}^{2}\right)^{-(n+1)/2}\left(z-\mathtt{D}\right)^{-1} - \left(\sum_{j=0}^{N} h^{j/2}\mathtt{C}_{j,n+1}^{z}\right) = O_{H_{\mathrm{loc}}^{s}\to H_{\mathrm{loc}}^{s+n+1}}\left(\left(\left|\mathrm{Im}z\right|^{-1}h^{\frac{1}{2}}\right)^{N+1}\right) \tag{7.21}$$

$\forall s\in\mathbb{R}$, for classical (h-independent) Sobolev spaces H_{loc}^{s}. Here each $\mathtt{C}_{j,n+1}^{z} = O_{H_{\mathrm{loc}}^{s}\to H_{\mathrm{loc}}^{s+n+1}}\left(\left|\mathrm{Im}z\right|^{-j-1}\right)$ with the leading term

$$\mathtt{C}_{0,n+1}^{z} = \left(1+\mathtt{D}_{0}^{2}\right)^{-(n+1)/2}\left(z-\mathtt{D}_{0}\right)^{-1}.$$

Finally, plugging the expansion (7.21) into the Helffer-Sjostrand formula

$$\phi(\mathtt{D}) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\varrho}(z)\left(1+\mathtt{D}^2\right)^{-(n+1)/2}(z-\mathtt{D})^{-1}\,dz d\bar{z},$$

with $\varrho(x) := \langle x\rangle^{n+1}\phi(x)$, gives

$$\phi(\mathtt{D})(0,0) = \left(\sum_{j=0}^{N} h^{j/2}U_{j,p}(\phi)\right) + h^{(N+1)/2}O\left(\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^N \hat{\phi}^{(k)}\right\|_{L^1}\right) \tag{7.22}$$

using Sobolev's inequality. Here each

$$U_{j,p}(\phi) = -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{\varrho}(z)\,\mathtt{C}^z_{j,n+1}(0,0)\,dz d\bar{z} \in \mathrm{End}S_p^{TX} \tag{7.23}$$

defines a smooth family (in $p \in X$) of distributions U_j and the remainder term in (7.22) comes from the estimate $\bar{\partial}\tilde{\varrho} = O\left(|\mathrm{Im}z|^{N+1}\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^N\hat{\phi}^{(k)}\right\|_{L^1}\right)$ on the almost analytic continuation (cf. [28] Sec. 3.1). Integrating the trace of (7.22) over X and using (7.18) gives (7.11). □

Next we would like to understand the structure of the distributions u_j appearing in (7.11). Clearly,

$$\begin{aligned} u_j &= \int_X u_{j,p} \quad \text{with} \\ u_{j,p} &:= \mathrm{tr}\, U_{j,p} \in C^\infty\left(X;\mathcal{S}'(\mathbb{R}_s)\right) \end{aligned} \tag{7.24}$$

being the smooth family of tempered distributions parametrized by X defined via the point-wise trace of (7.23). Letting $H(s) \in \mathcal{S}'(\mathbb{R}_s)$ denote the Heaviside distribution, we now define the following elementary tempered distributions

$$v_{a;p}(s) := s^a,\ a \in \mathbb{N}_0 \tag{7.25}$$

$$v_{a,b,c,\Lambda;p}(s) := \partial_s^a\left[|s|\,s^b\left(s^2-2\nu_p\Lambda\right)^{c-\frac{1}{2}}H\left(s^2-2\nu_p\Lambda\right)\right], \quad (a,b,c;\Lambda) \in \mathbb{N}_0\times\mathbb{Z}\times\mathbb{N}_0\times\mu.\left(\mathbb{N}_0^m\setminus 0\right). \tag{7.26}$$

We now have the following.

Proposition 7.2. *For each j, the distribution* (7.24) *can be written in terms of* (7.25), (7.26)

$$u_{j,p}(s) = \sum_{a\le 2j+2} c_{j;a}(p)\,s^a + \sum_{\substack{\Lambda\in\mu.\left(\mathbb{N}_0^m\setminus 0\right).\\ a,|b|,c\le 4j+4}} c_{j;a,b,c,\Lambda}(p)\,v_{a,b,c,\Lambda;p}(s). \tag{7.27}$$

Moreover, the coefficient functions $c_{j;a}$, $c_{j;a,b,c,\Lambda} \in C^\infty(X)$ above are evaluations at p of polynomials in the covariant derivatives (with respect to $\nabla^{TX}\otimes 1 + 1\otimes\nabla^{A_0}$) of the curvatures F^{TX}, F^{A_0} of the Levi-Civita connection ∇^{TX}, ∇^{A_0} and da.

Proof. It suffices to consider the restriction of u_j to the interval $\left(-\sqrt{2\nu M}, \sqrt{2\nu M}\right)$ for each $0 < M \notin \mu.\left(\mathbb{N}_0^m \setminus 0\right)$. We begin by finding the spectrum of the operator $\mathtt{D}_{00}$ in (7.17). To this end, define the unitary operator $\mathtt{U}_\lambda : C^\infty\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right) \to C^\infty\left(\mathbb{R}^n; \mathbb{C}^{2^m}\right)$

$$\begin{aligned}
\left(\mathtt{U}_\lambda s\right)(x_0, x_1, x_2, \ldots) &= \left(\Pi_{j=1}^m \lambda_j\right) s\left(x_0, \lambda_1^{-\frac{1}{2}} x_1, \lambda_1^{-\frac{1}{2}} x_2, \lambda_2^{-\frac{1}{2}} x_3, \lambda_2^{-\frac{1}{2}} x_4, \ldots\right) \\
\text{and } f &= \sum_{j=1}^m \left(x_j x_{j+m} + \xi_j \xi_{j+m}\right) \in C^\infty\left(\mathbb{R}^{2m}\right).
\end{aligned}$$

Next, as in (5.1) we compute the conjugate

$$e^{\frac{i\pi}{4} f_0^W} \mathtt{U}_\lambda \mathtt{D}_{00} \mathtt{U}_\lambda^* e^{-\frac{i\pi}{4} f_0^W} = \left[2\nu\left(p\right)\right]^{\frac{1}{2}} D_{\mathbb{R}^m}|_{h=1}$$

of the operator in (7.17) in terms of the magnetic Dirac operator on $\mathbb{R}^m$ (2.21) evaluated at $h = 1$. Hence the eigenspaces of $\mathtt{D}_{00}$ are

$$\begin{aligned}
&\mathtt{U}_\lambda^* e^{-\frac{i\pi}{4} f_0^W}\left(E_0 \otimes L^2\left(\mathbb{R}^{m+1}_{x_0, x''}\right)\right), \\
&\mathtt{U}_\lambda^* e^{-\frac{i\pi}{4} f_0^W}\left(E_\Lambda^\pm \otimes L^2\left(\mathbb{R}^{m+1}_{x_0, x''}\right)\right); \qquad \Lambda \in \mu.\left(\mathbb{N}_0^m \setminus 0\right),
\end{aligned}$$

with eigenvalues $0, \pm\sqrt{2\nu\Lambda}$ respectively, where

$$\begin{aligned}
E_0 &:= \mathbb{C}\left[\psi_{0,0}|_{h=1}\right] \\
E_\Lambda^\pm &= \bigoplus_{\substack{\tau \in \mathbb{N}_0^m \setminus 0 \\ \Lambda = \mu.\tau}} E_\tau^\pm|_{h=1},
\end{aligned}$$

are as in (6.5). We again let $\mathtt{P}_0, \mathtt{P}_\Lambda^\pm$ denote the respective projections onto the eigenspaces of $\mathtt{D}_{00}$ and $\mathtt{P}_\Lambda = \mathtt{P}_\Lambda^+ \oplus \mathtt{P}_\Lambda^-$. We also denote by $\mathtt{P}_{>M} = \oplus_{\Lambda > M} \mathtt{P}_\Lambda$ the projection onto eigenspaces with eigenvalue greater than $\sqrt{2\nu M}$ in absolute value.

Now, since expansions in L^2_{loc} are unique it suffices to work with the resolvent expansion (7.20) in the computation of u_j. The jth term in the expansion is of the form

$$\mathtt{C}_j^z = \sum_{k \in I_j} \left(z - \mathtt{D}_0\right)^{-1}\left[\Pi_\alpha \mathtt{D}_{k_\alpha}\left(z - \mathtt{D}_0\right)^{-1}\right] \tag{7.28}$$

where each $\mathtt{D}_{k_\alpha}$ is a differential operator with polynomial coefficients involving the covariant derivatives of the curvatures F^{TX}, F^{A_0} and da. Now using (7.17) we decompose each resolvent term above according to the eigenspaces of $\mathtt{D}_{00}$

$$(7.29)\quad \begin{aligned}(z-\mathtt{D}_0)^{-1} &= \mathtt{P}_0\left(\frac{1}{z-\gamma^0\partial_{x_0}}\right)\mathtt{P}_0\oplus\\ &\bigoplus_{\Lambda\in\mu.\mathbb{N}_0^m\cap(0,M)}\mathtt{P}_\Lambda\left(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial_{x_0}^2-2\nu\Lambda}\right)\mathtt{P}_\Lambda\\ &\oplus\mathtt{P}_{>M}\left(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}.\end{aligned}$$

Next, we plug (7.29) into (7.28). This gives an expansion for $\mathtt{C}_j^z$ with some of the terms given by

$$\begin{aligned}T^z\left[\Pi_\alpha\mathtt{D}_{k_\alpha}T^z\right]\quad &;\quad \text{where}\\ T^z &= \mathtt{P}_{>M}\left(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}\end{aligned}$$

and being holomorphic for $\mathrm{Re}z\in\left(-\sqrt{2\nu M},\sqrt{2\nu M}\right)$. For the rest of the terms in $\mathtt{C}_j^z$, we use the commutation relations

$$\begin{aligned}\left[\gamma^0,\mathtt{P}_0\right] &= \left[\gamma^0,\mathtt{P}_\Lambda\right]=\left[\gamma^0,\mathtt{P}_{>M}\right]=0\\ \left[\partial_{x_0},\mathtt{P}_0\right] &= \left[\partial_{x_0},\mathtt{P}_\Lambda\right]=\left[\partial_{x_0},\mathtt{P}_{>M}\right]=0\\ \left[\partial_{x_0},\mathtt{D}_{00}\right] &= 0\\ \left[\left(z^2+\partial_{x_0}^2-2\nu\Lambda\right)^{-1},x_j\right] &= \delta_{0j}\left(z^2+\partial_{x_0}^2-2\nu\Lambda\right)^{-2}\partial_{x_0}\\ \left[\left(z^2+\partial_{x_0}^2-2\nu\Lambda\right)^{-1},\partial_{x_J}\right] &= 0\end{aligned}$$

as well as the Clifford relations (2.7). This now gives a finite sum of terms of the form

(7.30)

$$T_0^z\left[\Pi_{k=1}^K S_k T_k^z\right]\times\left[\Pi_{\Lambda\in\mu.\mathbb{N}_0^m\cap(0,M)}\frac{1}{\left(z^2+\partial_{x_0}^2-2\nu\Lambda\right)^{a_\Lambda}}\right]\left(z-\gamma^0\partial_{x_0}\right)^{-a_0}z^{b_1}x_0^{b_2}\partial_{x_0}^{b_3},$$

$a_0+\Sigma a_\Lambda\leq 2j+2$; $b_1,b_2,b_3\leq j+1$, where each S_k is a differential operator in $(x'x'')$ (i.e. independent of x_0) with polynomial coefficients and each

$$(7.31)\quad T_k^z=\begin{cases}\mathtt{P}_0 & \text{or}\\ \mathtt{P}_\Lambda, & \text{or}\\ \mathtt{P}_\Lambda\mathtt{D}_{00}\mathtt{P}_\Lambda, & \text{or}\\ \mathtt{P}_{>M}\left(\frac{1}{z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}, & \text{or}\\ \mathtt{P}_{>M}\left(\frac{\mathtt{D}_{00}}{z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M} & \end{cases}$$

with at least one occurrence of $\mathtt{P}_0, \mathtt{P}_\Lambda$ or $\mathtt{P}_\Lambda \mathtt{D}_{00} \mathtt{P}_\Lambda$ in (7.30). Now using partial fractions, (7.30) may be written as a sum of terms of the form

$$T_0^z \left[\Pi_{k=1}^K S_k T_k^z\right] \times \left(z - \gamma^0 \partial_{x_0}\right)^{-a_0} z^{b_1} x_0^{b_2} \partial_{x_0}^{b_3},$$

(7.32)

$$T_0^z \left[\Pi_{k=1}^K S_k T_k^z\right] \times \left(z^2 + \partial_{x_0}^2 - 2\nu\Lambda\right)^{-a_\Lambda} z^{b_1} x_0^{b_2} \partial_{x_0}^{b_3}; \, \Lambda \in \mu.\mathbb{N}_0^m \cap (0, M),$$

$a_0, a_\Lambda \leq 2j+2$; $b_1, b_2, b_3 \leq j+1$. Next, we plug (7.32) into the Helffer-Sjostrand formula and use the holomorphicity of $\mathtt{P}_{>M}\left(\frac{1}{z^2+\partial_{x_0}^2 - \mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}$ and $\mathtt{P}_{>M}\left(\frac{\mathtt{D}_{00}}{z^2+\partial_{x_0}^2 - \mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}$ for $\mathrm{Re} z \in \left(-\sqrt{2\nu M}, \sqrt{2\nu M}\right)$. This gives

$$U_{j,p}(\phi) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\phi}(z)\, \mathtt{C}_j^z(0,0)\, dz d\bar{z},$$

for $\phi \in C_c^\infty\left(-\sqrt{2\nu M}, \sqrt{2\nu M}\right)$, as a sum of terms of the form

$$\left(T_0^0 \left[\Pi_{k=1}^K S_k T_k^0\right] \times x_0^{b_2} \partial_{x_0}^{b_3} \phi_0\left(\gamma^0 \partial_{x_0}\right)\right)(0,0),$$

(7.33)

$$\left(T_0^0 \left[\Pi_{k=1}^K S_k T_k^0\right] \times x_0^{b_2} \partial_{x_0}^{b_3} \phi_\Lambda\left(-\partial_{x_0}^2 + 2\nu\Lambda\right)\right)(0,0), \, \Lambda \in \mu.\mathbb{N}_0^m \cap (0, M);$$

where

$$T_k^0 = \begin{cases} \mathtt{P}_0, & \text{or} \\ \mathtt{P}_\Lambda, & \text{or} \\ \mathtt{P}_\Lambda \mathtt{D}_{00} \mathtt{P}_\Lambda, & \text{or} \\ \mathtt{P}_{>M}\left(\frac{1}{2\nu\Lambda - \mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}, & \text{or} \\ \mathtt{P}_{>M}\left(\frac{\mathtt{D}_{00}}{2\nu\Lambda - \mathtt{D}_{00}^2}\right)\mathtt{P}_{>M}; & \end{cases}$$

and

$$\begin{aligned} \phi_0(s) &= \frac{(-1)^{a_0-1}}{(a_0-1)!} x^{b_1} \phi(s) \\ \phi_\Lambda\left(s^2\right) &= \frac{(-1)^{a_\Lambda-1}}{(a_\Lambda-1)!}\left\{\left[\partial_r^{a_\Lambda-1}\left(\frac{r^{b_1}\phi(r)}{(r-s)^{a_\Lambda}}\right)\right]\Bigg|_{r=-s} \right. \\ &\quad \left. - \left[\partial_r^{a_\Lambda-1}\left(\frac{r^{b_1}\phi(r)}{(r+s)^{a_\Lambda}}\right)\right]\Bigg|_{r=s}\right\}. \end{aligned}$$

At least one occurrence of $\mathtt{P}_0$,$\mathtt{P}_\Lambda$ and $\mathtt{P}_\Lambda \mathtt{D}_{00} \mathtt{P}_\Lambda$ in (7.33) gives the smoothness of the kernel.

Finally, an elementary computation involving Laplace transforms using the knowledge of the heat kernel $e^{t\partial_{x_0}^2}(x_0,y_0)=\frac{1}{\sqrt{4\pi t}}e^{-|x_0-y_0|^2/4t}$ gives

$$x_0^{b_2}\partial_{x_0}^{b_3}\phi_0\left(\gamma^0\partial_{x_0}\right)(0,0)=\frac{\left(-\frac{1}{2}\right)^{\left[\frac{b_3+1}{2}\right]}}{\sqrt{\pi}\Gamma\left(\left[\frac{b_3+1}{2}\right]+\frac{1}{2}\right)}\delta_{0b_2}v_{b_3;p}\left(\phi_0\right)$$

$$x_0^{b_2}\partial_{x_0}^{b_3}\phi_\Lambda\left(-\partial_{x_0}^2+2\nu\Lambda\right)(0,0)=\begin{cases}\frac{\left(-\frac{1}{2}\right)^{\frac{b_3}{2}}}{4\pi\Gamma\left(\frac{b_3}{2}-\frac{1}{2}\right)}\delta_{0b_2}v_{0,0,\frac{b_3}{2},\Lambda;p}\left(\phi_\Lambda\left(s^2\right)\right); & b_3 \text{ even}\\ 0; & b_3 \text{ odd,}\end{cases}$$

completing the proof. □

As an immediate corollary of the above proposition 7.2 we have that the distributions u_j are smooth near 0.

Corollary 7.3. *For each j,*

$$singspt\left(u_j\right)\subset\mathbb{R}\setminus\left(-\sqrt{2\nu_0},\sqrt{2\nu_0}\right).$$

Proof. This follows immediately from (7.24), (7.25), (7.26) and (7.27) on noting that the distributions $v_{a;p}$ are smooth while $v_{a,b,c,\Lambda;p}=0$ on $\mathbb{R}\setminus\left(-\sqrt{2\nu_0},\sqrt{2\nu_0}\right)$ for each $p\in X$. □

We next give the exact computation for the first coefficient u_0 of 7.1. In the computation below , recall that $Z_\tau=|I_\tau|$ (2.13) denotes the number of non-zero components of $\tau\in\mathbb{N}_0^m\setminus 0$.

Proposition 7.4. *The first coefficient u_0 of (7.11) is given by*

$$u_{0,p} = c_{0;0}+\sum_{\Lambda\in\mu.\left(\mathbb{N}_0^m\setminus 0\right)}c_{0;0,0,0,\Lambda}(p)\,v_{0,0,0,\Lambda;p}(s),\quad \textit{where} \tag{7.34}$$

$$c_{0;0} = \frac{\nu_p^m\left(\Pi_{j=1}^m\mu_j\right)}{(4\pi)^m}\quad\textit{and}$$

$$c_{0;0,0,0,\Lambda}(p) = \frac{\nu_p^m\left(\Pi_{j=1}^m\mu_j\right)}{(4\pi)^m}\,\textit{dim}\,\left(E_\Lambda\right)$$

$$= \frac{\nu_p^m\left(\Pi_{j=1}^m\mu_j\right)}{(4\pi)^m}\left(\sum_{\substack{\tau\in\mathbb{N}_0^m\setminus 0\\ \mu.\tau=\Lambda}}2^{Z_\tau}\right). \tag{7.35}$$

Proof. First note that the square of (7.16) gives the harmonic oscillator

$$\mathtt{D}_0^2=-\delta^{jk}\partial_{x_j}\partial_{x_k}-i\,(da)^{\;j}_{\;k}(0)\,x^k\partial_{x_j}+\frac{1}{4}x^kx_l\,(da)^{\;j}_{\;k}(0)\,(da)^{\;l}_{\;j}(0)+\frac{i}{2}\gamma^j\gamma^k\,(da)_{jk}(0).$$

The heat kernel $e^{-t\mathtt{D}_0^2}$ of the above is given by Mehler's formula (cf. [3] section 4.2)

(7.36)

$$e^{-t\mathtt{D}_0^2}(x,y)=\frac{1}{(4\pi t)^{m}}\det{}^{\frac{1}{2}}\left(\frac{itda(0)}{\sinh itda(0)}\right)$$
$$\times\exp\left\{-\frac{1}{4t}\left\langle (x-y),itda(0)\coth\left(itda(0)\right)(x-y)\right\rangle\right\}e^{-tc(ida(0))}$$

Next, using (7.1) we compute

$$e^{-tc(ida(0))}=\Pi_{j=1}^{m}\left[\cosh\left(t\lambda_j\right)-ic\left(e_j\right)c\left(e_{j+m}\right)\sinh\left(t\lambda_j\right)\right]. \tag{7.37}$$

For $I\subset\{2,\ldots,m\}$ and $\omega_I=\bigwedge_{j\in I}\left(e_j\wedge e_{j+m}\right)$, the commutation

$$c\left(e_1\right)c\left(e_{m+1}\right)c\left(\omega_I\right)=\frac{1}{2}\left[c\left(e_1\right),c\left(e_{m+1}\right)c\left(\omega_I\right)\right]$$

shows that the only traceless terms in (7.37) are the constants. Hence, Mehler's formula (7.36) gives

$$\begin{aligned}
\mathrm{tr}\ e^{-t\mathtt{D}_0^2}(0,0) &= \frac{1}{(4\pi t)^{m}}\det{}^{\frac{1}{2}}\left(\frac{itda(0)}{\tanh itda(0)}\right)\\
&= \frac{t^{-\frac{1}{2}}}{(4\pi)^m}\left(\Pi_{j=1}^{m}\frac{\lambda_j}{\tanh t\lambda_j}\right)\\
&= \frac{t^{-\frac{1}{2}}}{(4\pi)^m}\left[\Pi_{j=1}^{m}\lambda_j\left(1+2e^{-2t\lambda_j}+2e^{-4t\lambda_j}+\ldots\right)\right]\\
&= \frac{t^{-\frac{1}{2}}}{(4\pi)^m}\left(\Pi_{j=1}^{m}\lambda_j\right)\left(\sum_{\tau\in\mathbb{N}_0^m}2^{Z_\tau}e^{-2t\tau.\lambda}\right)\\
&= \frac{\nu_p^m\left(\Pi_{j=1}^{m}\mu_j\right)}{(4\pi)^m}\left(t^{-\frac{1}{2}}\sum_{\tau\in\mathbb{N}_0^m}2^{Z_\tau}e^{-2t\tau.\lambda}\right)\\
&= u_{0,p}\left(e^{-ts^2}\right) \qquad (7.38)
\end{aligned}$$

with $u_{0,p}$ as in (7.34) and the last line above following from an easy computation of Laplace transforms (see [25] section 4). Furthermore, differentiating Mehler's formula using (7.16) gives

$$\mathrm{tr}\mathtt{D}_0 e^{-t\mathtt{D}_0^2}(0,0)=0=u_{0,p}\left(se^{-ts^2}\right) \tag{7.39}$$

since the right hand side of (7.34) is an even distribution. From (7.38) and (7.39) we have that the evaluations of both sides of (7.34) on e^{-ts^2},se^{-ts^2} are equal. Differentiating with respect to t and setting $t=1$ gives that the two sides of (7.34) evaluate equally on $s^ke^{-s^2}$, $\forall k\in\mathbb{N}_0$. The proposition now follows from the density of this collection in $\mathcal{S}\left(\mathbb{R}_s\right)$. □

We now complete the proof of Lemma 3.2.

Proof of Lemma 3.2. We begin by writing

$$\begin{aligned} & \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h^{1-\epsilon}}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h^{1-\epsilon}}\right)\right] \\ = & \frac{h^{-\frac{1}{2}}}{2\pi}\int dt \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right]\theta\left(th^{\frac{1}{2}-\epsilon}\right). \end{aligned} \tag{7.40}$$

Next, the expansion 7.1, with $\phi(x) = f(x)e^{it(\lambda-x)}$, combined with the smoothness of u_j on $\operatorname{spt}(f) \subset \left(-\sqrt{2\nu_0}, \sqrt{2\nu_0}\right)$ 7.3 gives

$$\begin{aligned} \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right] &= e^{it\lambda}h^{-n/2}\left(\sum_{j=0}^{N}h^{j/2}\widehat{fu_j}(t)\right) \\ &\quad +h^{(N+1-n)/2}\underbrace{O\left(\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^N\hat{\phi}^{(k)}(\xi-t)\right\|_{L^1}\right)}_{=O\left(\langle t\rangle^N\right)}. \end{aligned} \tag{7.41}$$

Finally, plugging (7.41) into (7.40) and using $\theta\left(th^{\frac{1}{2}-\epsilon}\right) = 1 + O(h^{\infty})$ gives via Fourier inversion

$$\begin{aligned} & \frac{h^{-\frac{1}{2}}}{2\pi}\int dt \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right]\theta\left(th^{\frac{1}{2}-\epsilon}\right) \\ = & \; h^{-m-1}\left(\sum_{j=0}^{N}h^{j/2}f(\lambda)u_j(\lambda)\right) + O\left(h^{\epsilon(N+1)-m-1}\right) \end{aligned}$$

as required. □

8. Asymptotics of spectral invariants

In this section we prove theorem Theorem 1.2 on the asymptotics of the spectral invariants.

Proof of Theorem 1.2. To prove the local Weyl law (1.5), we choose $\theta \in C_c^{\infty}((-T,T);[0,1])$ such that $\theta(x) = 1$ on $(-T', T')$, $T' < T$, $\check{\theta}(\xi) \geq 0$ and $\check{\theta}(\xi) \geq 1$ for $|\xi| \leq c$ in Theorem 1.3. Choosing $f(x) \geq 0$ with $f(0) = 1$, the trace expansion (1.7) with $\lambda = 0$ now gives

$$\frac{1}{h}N(-ch, ch)\left(1 + O\left(\sqrt{h}\right)\right) \leq \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{-D}{h}\right)\right] = O\left(h^{-m-1}\right)$$

proving (1.5).

To prove the estimate (1.6) on the eta invariant, we first use its invariance under positive scaling (2.2) and the formula (2.5) to write

$$\eta_h=\eta\left(\frac{D}{\sqrt{h}}\right) = \int_0^\infty dt\frac{1}{\sqrt{\pi t}}\text{ tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]$$
$$= \int_0^1 dt\frac{1}{\sqrt{\pi t}}\text{ tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]+\int_1^\infty dt\frac{1}{\sqrt{\pi t}}\text{ tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]. \tag{8.1}$$

Next, the equation 4.5 pg. 859 of [25] with $r=\frac{1}{h}$ translates to the estimate

$$\text{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]=O\left(h^{-m}e^{ct}\right). \tag{8.2}$$

Plugging, (8.2) into the first integral of (8.1) gives

$$\eta_h=O\left(h^{-m}\right)+\text{tr }E\left(\frac{D}{\sqrt{h}}\right) \tag{8.3}$$

where

$$E(x)=\text{sign}(x)\text{erfc}(|x|)=\text{sign}(x)\cdot\frac{2}{\sqrt{\pi}}\int_{|x|}^\infty e^{-s^2}ds$$

with the convention $\text{sign}(0)=0$. The function $E(x)$ above is rapidly decaying with all derivatives, odd and smooth on $\mathbb{R}_x\setminus 0$. We may hence choose functions $f\in C_c^\infty\left(-\sqrt{2\nu_0},\sqrt{2\nu_0}\right)$, $g\in C_c^\infty\left(\mathbb{R}_{<0}\right)$ such that

$$f(x)+g(x) = E(x) \text{ for } x\leq 0.$$

Define the spectral measure $\mathfrak{M}_f(\lambda'):=\sum_{\lambda\in\text{Spec}\left(\frac{D}{\sqrt{h}}\right)}f(\lambda)\,\delta(\lambda-\lambda')$. It is clear that the expansion (1.7) to its first term may be written as

$$\mathfrak{M}_f*\left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda)=h^{-m-\frac{1}{2}}\left(f(\lambda)\,u_0(\lambda)+O\left(h^{1/2}\right)\right)$$

where $\theta_{\frac{1}{2}}(x)=\theta\left(\frac{x}{\sqrt{h}}\right)$ as before. Both sides above involving Schwartz functions in λ, the remainder maybe replaced by $O\left(\frac{h^{1/2}}{\langle\lambda\rangle^2}\right)$. One may then integrate the equation to obtain

$$\begin{aligned}&\int_{-\infty}^0 d\lambda\int d\lambda'\left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda-\lambda')\,\mathfrak{M}_f(\lambda')\\ &=h^{-m-\frac{1}{2}}\left(\int_{-\infty}^0 d\lambda f(\lambda)\,u_0(\lambda)+O\left(h^{1/2}\right)\right).\end{aligned} \tag{8.4}$$

Next we observe

$$\int_{-\infty}^0 d\lambda\left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda-\lambda') = \int_{-\infty}^0 dt\check{\theta}\left(t-\frac{\lambda'}{\sqrt{h}}\right)$$
$$= 1_{(-\infty,0]}(\lambda')+O\left(\left\langle\frac{\lambda'}{\sqrt{h}}\right\rangle^{-\infty}\right). \tag{8.5}$$

While the local Weyl law yields

$$\int d\lambda' \mathfrak{M}_f(\lambda') O\left(\left\langle \frac{\lambda'}{\sqrt{h}} \right\rangle^{-\infty}\right) = O\left(h^{-m}\right). \tag{8.6}$$

Substituting (8.5) and (8.6) into (8.4) gives

$$\sum_{\substack{\lambda \leq 0 \\ \lambda \in \mathrm{Spec}\left(\frac{D}{\sqrt{h}}\right)}} f(\lambda) = h^{-m-\frac{1}{2}} \left(\int_{-\infty}^{0} d\lambda f(\lambda) u_0(\lambda)\right) + O\left(h^{-m}\right).$$

This combined with

$$\mathrm{tr}\ g\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}} u_0(g) + O\left(h^{-m}\right)$$

then gives

$$\sum_{\substack{\lambda \leq 0 \\ \lambda \in \mathrm{Spec}\left(\frac{D}{\sqrt{h}}\right)}} E(\lambda) = h^{-m-\frac{1}{2}} \left(\int_{-\infty}^{0} d\lambda E(\lambda) u_0(\lambda)\right) + O\left(h^{-m}\right)$$

where the integral makes sense from the formula (7.34) for u_0. A similar formula for

$$\sum_{\substack{\lambda \geq 0 \\ \lambda \in \mathrm{Spec}\left(\frac{D}{\sqrt{h}}\right)}} E(\lambda)$$

now gives

$$\mathrm{tr}\ E\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}} \left(\int_{-\infty}^{\infty} d\lambda E(\lambda) u_0(\lambda)\right) + O\left(h^{-m}\right).$$

Since E is odd and u_0 is even from (7.34), the integral above is zero and hence $\eta_h = \mathrm{tr}\ E\left(\frac{D}{\sqrt{h}}\right) = O\left(h^{-m}\right)$ from (8.3) as required. □

8.1. **Sharpness of the result.** Here, we finally show that the result Theorem 1.2 is sharp. The worst case example was already noted in [25] Section 5 for η_h. To recall, we let Y be a complex manifold of dimension $2m$ with complex structure J and a Riemannian metric g^{TY}. Fix a positive, holomorphic, Hermitian line bundle $\mathcal{L} \to Y$. The curvature $F^{\mathcal{L}}$ of the Chern connection is thus a positive $(1,1)$ form. Let X be the total space of the unit circle bundle $S^1 \to X \xrightarrow{\pi} Y$ of $\mathcal{L}$. The Chern connection gives a splitting of the tangent bundle

$$TX = TS^1 \oplus \pi^* TY \tag{8.7}$$

where TS^1 is the vertical tangent space spanned by the generator e of the S^1 action. Define a metric g^{TS^1} on TS^1 via $\|e\|_{g^{TS^1}} = 1$. A metric on X can now be given using the splitting (8.7) via

$$g^{TX} = g^{TS^1} \oplus \varepsilon^{-1} \pi^* g^{TY},$$

for any $\varepsilon > 0$. A spin structure on Y corresponds to a holomorphic, Hermitian square root $\mathcal{K}$ of the canonical line bundle $K_Y = \mathcal{K}^{\otimes 2}$. Fixing such a spin structure as well as the trivial spin structure on TS^1 gives a spin structure on X. Finally the one form $a = e^* \in \Omega^1(X)$ while the auxiliary is chosen to be trivial $L = \mathbb{C}$ with the family of connections $\nabla^h = d + \frac{i}{h}a$. We now have the required family of Dirac operators D_h (1.2). One may check that $\left(X^{2m+1}, a, g^{TX}, J\right)$ here gives a metric contact structure (1.4) and hence the assumption 1.1 is satisfied.

Denote by $\Delta^p_{\bar{\partial}_k} : \Omega^{0,p}\left(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right) \longrightarrow \Omega^{0,p}\left(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right)$ the Hodge Laplacian acting on $(0,p)$ forms on X. Its null-space is given by the cohomology $H^p\left(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right)$ of the tensor product via Hodge theory. Let $e^{p,k}_\mu$ denote the dimension of a each positive eigenspace with eigenvalue $\frac{1}{2}\mu^2 \in \mathrm{Spec}^+\left(\Delta^p_{\bar{\partial}_k}\right)$. The spectrum of D_h was now computed in Proposition 5.2 of [25].

Proposition 8.1. *The spectrum of D_h is given by*

(1) Type 1:

$$\lambda = (-1)^p h\left(k + \left(\varepsilon - \frac{m}{2}\right) - \frac{1}{h}\right), \tag{8.8}$$

$0 \leq p \leq m, k \in \mathbb{Z}$, *with multiplicity* $dim H^p\left(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right)$.

(2) Type 2:

$$\lambda = h\left[\frac{(-1)^{p+1}\varepsilon \pm \sqrt{(2k + \varepsilon(2p - m) - \frac{2}{h} + 1)^2 + 4\mu^2\varepsilon}}{2}\right], \tag{8.9}$$

$0 \leq p \leq m, k \in \mathbb{Z}$, $\frac{1}{2}\mu^2 \in$ *Spec*$^+\left(\Delta^p_{\bar{\partial}_k}\right)$ *with multiplicity* $d^{p,k}_\mu := e^{p,k}_\mu - e^{p-1,k}_\mu + \ldots + (-1)^p e^{0,k}_\mu$.

As observed in [25] on choosing

$$\varepsilon < \inf_{k,p}\left\{\frac{1}{2}\mu^2 \in \mathrm{Spec}^+\left(\Delta^p_{\bar{\partial}_k}\right)\right\}$$

the eigenvalues of Type 2 are either positive or negative depending on the sign appearing in (8.9). Hence the dimension of the kernel k_h of D_h is now given by the type 1 eigenvalues

$$k_h = \begin{cases} \dim\ H^*\left(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right); & \frac{1}{h} = k + \left(\varepsilon - \frac{m}{2}\right) \\ 0 & \text{otherwise.} \end{cases} \tag{8.10}$$

Now by a combination of Kodaira vanishing and Hirzebruch-Riemann-Roch

$$\begin{aligned} \dim H^*\left(X;\mathcal{K}\otimes\mathcal{L}^{\otimes k}\right) &= \dim H^0\left(X;\mathcal{K}\otimes\mathcal{L}^{\otimes k}\right) \\ &= \chi(X,\mathcal{K}\otimes\mathcal{L}^{\otimes k}) \\ &= \int_X \mathrm{ch}(\mathcal{K}\otimes\mathcal{L}^{\otimes k})\mathrm{td}(X) \end{aligned} \tag{8.11}$$

for $k \gg 0$, where $\chi(X,\mathcal{K}\otimes\mathcal{L}^{\otimes k}), \mathrm{ch}(\mathcal{K}\otimes\mathcal{L}^{\otimes k})$ and $\mathrm{td}(X)$ denote Euler characteristic, Chern character and Todd genus respectively. Hence 8.10, (8.11) show that the kernel and hence the counting function are discontinuous of order $O\left(h^{-m}\right) = k_h \leq N\left(-ch, ch\right)$ in this example. A similar discontinuity of the eta invariant of $O\left(h^{-m}\right)$ was proved in Theorem 5.3 of [25].

Appendix A. Some spectral estimates

In this appendix we prove some important spectral estimates used in Section 4 and Section 5.

Let H be a separable Hilbert space. Let $A : H \to H$ be a bounded self-adjoint operator. The resolvent set and the spectrum of A are defined to be

$$\begin{aligned} R\left(A\right) &= \left\{\lambda \in \mathbb{C} | A - \lambda I \text{ is invertible}\right\} \\ \mathrm{Spec}\left(A\right) &= \mathbb{C} \setminus R\left(A\right). \end{aligned}$$

Since A is self-adjoint, $\mathrm{Spec}\left(A\right) \subset \mathbb{R}$. We may now define the following subsets of the spectrum

$$\begin{aligned} \mathrm{EssSpec}\left(A\right) &= \left\{\lambda \in \mathbb{C} | A - \lambda I \text{ is not Fredholm}\right\} \\ \mathrm{DiscSpec}\left(A\right) &= \mathrm{Spec}\left(A\right) \setminus \mathrm{EssSpec}\left(A\right). \end{aligned}$$

We shall consider $\mathrm{DiscSpec}\left(A\right)$ above as a multiset with the multiplicity function $m^A : \mathrm{DiscSpec}\left(A\right) \to \mathbb{N}_0$ defined by $m^A\left(\lambda\right) = \dim \ker\left(A\right)$. We may then find a countable set of orthonormal eigenvectors $v_1^A, v_2^A, v_3^A, \ldots$, with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \lambda_3^A \leq \ldots$ such that $\mathrm{DiscSpec}\left(A\right) = \left\{\lambda_1^A, \lambda_2^A, \ldots\right\}$ as multisets. Now let $[a,b] \subset \mathbb{R}$ be a finite closed interval such that $\mathrm{EssSpec}\left(A\right) \cap [a,b] = \emptyset$ (i.e. A has discrete spectrum in $[a,b]$). Then

$$H^A_{[a,b]} = \bigoplus_{\lambda \in \mathrm{Spec}(A) \cap [a,b]} \ker\left(A - \lambda\right)$$

is a finite dimensional vector subspace of H. We let $\Pi^A_{[a,b]} : H \to H^A_{[a,b]} \subset H$ denote the orthogonal projection onto $H^A_{[a,b]}$. We denote by $N^A_{[a,b]}$ the dimension of $H^A_{[a,b]}$. The operator $\rho\left(A\right) : H \to H$ may now be defined for any function $\rho \in C^0_c\left([a,b]\right)$ by functional calculus.

Lemma A.1. *Let $v \in H$ and $\lambda \in [a,b]$. Assume there exists $\varepsilon > 0$ such that A has discrete spectrum in $[a-\sqrt{\varepsilon}, b+\sqrt{\varepsilon}]$ and $\|(A-\lambda)v\| \leq \varepsilon \|v\|$. Then*

$$\left\| \Pi^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]} v - v \right\| \leq \sqrt{\varepsilon}\, \|v\| \quad \text{and} \tag{A.1}$$

$$\|(\rho(A) - \rho(\lambda))v\| \leq 3\sqrt{\varepsilon}\, \|\rho\|_{C^{0,1}} \|v\| \tag{A.2}$$

for any Holder continuous function $\rho \in C_c^{0,1}([a,b])$.

Proof. We abbreviate $\Pi = \Pi^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]}$. Let $H_0 := H^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]} = \Pi H$ which by assumption is a finite dimensional vector space. Let $H_0^\perp$ be the orthogonal complement of H_0. By assumption, $\operatorname{Spec}\left((A-\lambda)^2\Big|_{H_0^\perp} \right) \cap [-\varepsilon, \varepsilon] = \emptyset$. Hence by the mini-max principle for self-adjoint operators bounded from below (cf. Lemma 4.21 in [10]), we have $\varepsilon \leq (A-\lambda)^2\Big|_{H_0^\perp}$. Hence

$$\begin{aligned} \|\Pi v - v\|^2 \varepsilon &\leq \|(A-\lambda)(\Pi v - v)\|^2 \\ &\leq \|(A-\lambda)(\Pi v - v)\|^2 + \|(A-\lambda)\Pi v\|^2 = \|(A-\lambda)v\|^2 \leq \varepsilon^2 \|v\|^2 \end{aligned}$$

since $(A-\lambda)(\Pi v - v)$ and $(A-\lambda)\Pi v$ are orthogonal. This gives

$$\|\Pi v - v\| < \sqrt{\varepsilon}\, \|v\|. \tag{A.3}$$

To prove (A.2) first note that $\|\Pi' v - v\| < \sqrt{\varepsilon}\, \|v\|$, for $\Pi' = \Pi^A_{[\lambda-\sqrt{\varepsilon},\lambda+\sqrt{\varepsilon}]}$,- by the same argument. We now have

$$\begin{aligned} \|(\rho(A) - \rho(\lambda))v\| &\leq \left\|(\rho(A)-\rho(\lambda))\left(\Pi' v - v\right)\right\| + \left\|(\rho(A)-\rho(\lambda))\Pi' v\right\| \\ &\leq 2\sqrt{\varepsilon}\, \|\rho\|_{C^{0,1}} \|v\| + \sqrt{\varepsilon}\, \|\rho\|_{C^{0,1}} \|v\|. \end{aligned}$$

□

Before stating the next lemma we need the following definition.

Definition A.2. Given $0 < \varepsilon < 1$, a set of vectors $w_1, w_2, \ldots, w_N \in H$ is called an ε-almost orthonormal set of eigenvectors (ε-AOSE for short) of A if

(1) $\left| \|w_j\|^2 - 1 \right| < \varepsilon$ for all j
(2) $|\langle w_j, w_k \rangle| < \varepsilon$ for all $j \neq k$
(3) $\|(A - \mu_j) w_j\| < \varepsilon$ for some $\mu_j \in \mathbb{R}$, for all j.

Now we have another lemma.

Lemma A.3. *Assume $H_0 \subset H$ has finite dimension M and is mapped onto itself by A. Let $w_1, w_2, \ldots, w_N \in H_0$ be an ε-AOSE of A for some $\varepsilon < \frac{1}{2(M+1)}$. Then there exist orthonormal $w_1', w_2', \ldots, w_{M-N}' \in H_0$ such that $\left\| \left(A - \mu_j'\right) w_j' \right\| < 4M\varepsilon$ for some $\mu_j' \in \mathbb{R}$, for all j. Furthermore $\langle w_j, w_k' \rangle = 0$ for each j, k.*

Proof. It follows from $\varepsilon < \frac{1}{2(M+1)}$ that $w_1, w_2, \ldots, w_N$ are linearly independent. Let W denote their span and $W^{\perp} \subset H_0$ its orthogonal complement. Let $\Pi, \Pi^{\perp}$ be the orthogonal projections onto $W, W^{\perp}$ and consider the operator $A_0 := \Pi^{\perp} A \Pi^{\perp} : W^{\perp} \to W^{\perp}$. Let $w_1', w_2', \ldots, w_{M-N}' \in W^{\perp}$ be an orthogonal basis of eigenvectors of A_0. Hence

$$\Pi^{\perp} A w_j' = \mu_j' w_j'$$

for some $\mu_j' \in \mathbb{R}$, for all j. Also

$$\left|\left\langle A w_j', w_k \right\rangle\right| = \left|\left\langle w_j', (A - \mu_k)\, w_k \right\rangle\right| < \varepsilon.$$

It then follows that $\left\| \Pi A w_j' \right\| \leq 2M\varepsilon\sqrt{1+\varepsilon} < 4M\varepsilon$ giving the result. □

Now we prove another lemma.

Lemma A.4. *Given $N \in \mathbb{N}$, let $0 < \varepsilon < \left(\frac{1}{\|A\|+|a|+|b|+N+1}\right)^4$. Let $w_1, w_2, \ldots, w_N \in H$ be an ε-AOSE for A. Assume that A has discrete spectrum in $\left[a - \varepsilon^{\frac{1}{8}}, b + \varepsilon^{\frac{1}{8}}\right]$. Then there exist orthonormal vectors $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_N \in H$, which span the same subspace of H as $w_1, w_2, \ldots, w_N$. Moreover $\|w_j - \overline{w}_j\| < \sqrt{\varepsilon}$ and $\|(\rho(A) - \rho(\mu_j))\, \overline{w}_j\| \leq 3\varepsilon^{\frac{1}{8}} \|\rho\|_{C^{0,1}}$ for $1 \leq j \leq N$, and any Holder continuous function $\rho \in C_c^{0,1}([a,b])$.*

Proof. Again it follows easily that the vectors w_j,$1 \leq j \leq N$, are linearly independent. Let $W \subset H$ be their span and choose an orthonormal basis e_i, $1 \leq j \leq N$, for W. We write

$$w_j = \sum_{k=1}^{N} m_{jk} e_k.$$

If we consider the matrix $M = [m_{jk}]$, then assumptions 1 and 2 of Definition A.2 are equivalent to $|M^* M - I| < \varepsilon$. Consider the polar decomposition $M = UP$ where U is unitary and P is a positive semi-definite Hermitian matrix. We have $|P^* P - I| < \varepsilon$ and hence $\|P^* P - I\| < N\varepsilon$. Thus any eigenvalue λ^P of P, being nonnegative, satisfies $\left|\lambda^P - 1\right| < \varepsilon$ and we have $\|P - I\| < N\varepsilon$. Thus $\|M - U\| = \|UP - U\| < N\varepsilon$. If we now let $U = [u_{jk}]$ and $\overline{w}_j = \sum_{k=1}^{N} u_{jk} e_k$, then the $\overline{w}_j$ are clearly orthonormal and satisfy $\|w_j - \overline{w}_j\| < \sqrt{\varepsilon}$. This last inequality along with assumption 3 of Definition A.2 easily gives

$$\|(A - \mu_j)\, \overline{w}_j\| < \varepsilon^{\frac{1}{4}}.$$

Now Lemma A.1 gives

$$\|\Pi \overline{w}_j - \overline{w}_j\| < \varepsilon^{\frac{1}{8}} \quad \text{and} \tag{A.4}$$

$$\|(\rho(A) - \rho(\mu_j))\, \overline{w}_j\| < 3\varepsilon^{\frac{1}{8}} \|\rho\|_{C^{0,1}}. \tag{A.5}$$

□

Next, let H' be another separable Hilbert space. Let $U : H \to H'$ be a bounded operator. Let $B, D : H' \to H'$ and $C : H \to H$ be bounded self-adjoint operators. Define $A' = UAU^* : H' \to H'$, $B' = U^*BU : H \to H$, $C' = UCU^* : H' \to H'$ and $D' = U^*DU : H \to H$. In the next proposition we assume that there exists $\delta > 0$ such that A, A', B and B' have discrete spectrum in $[a-\delta, b+\delta]$. We also abbreviate $N^A = N^A_{[a-\delta,b+\delta]}$ and $\Pi^A = \Pi^A_{[a-\delta,b+\delta]}$ and similarly define $N^{A'}, N^B, N^{B'}, \Pi^{A'}, \Pi^B, \Pi^{B'}$.

Proposition A.5. *Suppose there exists* $0 < \varepsilon < L^{-2048}$, *with*

$$L = 25\Big\{ \|A\| + \|A'\| + \|B\| + \|B'\| + \|C\| + \|D\| + N^A + N^{A'} + N^B + N^{B'} + |a| + |b| + \delta^{-1} + 1 \Big\},$$

such that

(1) $\left\| (U^*U - I)\, \Pi^A \right\| (\|A\| \|U\| + 1) < \varepsilon$ *and* $\left\| (UU^* - I)\, \Pi^B \right\| (\|B\| \|U^*\| + 1) < \varepsilon$

(2) $\left\| (A' - B)\, \Pi^{A'} \right\| < \varepsilon$ *and* $\left\| (A - B')\, \Pi^{B'} \right\| < \varepsilon$

(3) $\left\| (C' - D)\, \Pi^A \right\| < \varepsilon$ *and* $\left\| (C - D')\, \Pi^B \right\| < \varepsilon$.

Then we have

$$|tr\,[C\rho(A)] - tr\,[D\rho(B)]| \le \varepsilon^{\frac{1}{2048}} \|\rho\|_{C^1}$$

for any $\rho \in C^1_c([a,b])$.

Proof. Let $(\mathrm{DiscSpec}(A), m^A) \cap [a,b] = \{\lambda^A_{a_1}, \lambda^A_{a_2}, \ldots, \lambda^A_{a_N}\}$, with $N = N^A_{[a,b]}$, as multisets. Let $\rho^+(x) = \frac{\rho(x)+|\rho(x)|}{2}$ and $\rho^-(x) = \frac{\rho(x)-|\rho(x)|}{2}$. We then have $\rho^+, \rho^- \in C^{0,1}_c([a,b])$ with $\|\rho^+\|_{C^{0,1}} \le \|\rho\|_{C^1}, \|\rho^-\|_{C^{0,1}} \le \|\rho\|_{C^1}$. We further decompose $C = C^+ + C^-$, $D = D^+ + D^-$ into their positive and non-positive parts. Clearly

$$\mathrm{tr}\left[C^+\rho^+(A)\right] = \sum_{j=1}^{N} \rho^+\left(\lambda_{a_j}\right) \left\langle v_{a_j}, C^+ v_{a_j} \right\rangle.$$

Next we consider $w_j = Uv_{a_j} \in H'$. From assumption 1 we have

$$\left\| \left(A' - \lambda_{a_j}\right) w_j \right\| = \left\| \left(UAU^* - \lambda_{a_j}\right) Uv_{a_j} \right\| \le \left\| (U^*U - I)\, \Pi^A_{[a,b]} \right\| \|A\| \|U\| < \varepsilon.$$

Similar estimates give $\left| \|w_j\|^2 - 1 \right| < \varepsilon$, and $|\langle w_j, w_k\rangle| < \varepsilon$ for $j \ne k$. Now by Lemma A.1 we have $\|\Pi w_j - w_j\| < (2\varepsilon)^{\frac{1}{2}}$ with $\Pi = \Pi^{A'}_{[a-\sqrt{2\varepsilon}, b+\sqrt{2\varepsilon}]}$. Following this and using assumption 3 we have

$$\begin{aligned} \left\| \left(B - \lambda_{a_j}\right) w_j \right\| &\le \left\| \left(A' - \lambda_{a_j}\right) w_j \right\| + \left\| (B - A')\, \Pi w_j \right\| + \left\| (B - A')\, (\Pi w_j - w_j) \right\| \\ &\le \varepsilon + \varepsilon\sqrt{1+\varepsilon} + (2\varepsilon)^{\frac{1}{2}} \left(\|A'\| + \|B\| \right) \\ &< \varepsilon^{\frac{1}{4}} \le \varepsilon^{\frac{1}{8}} \|w_j\|. \end{aligned}$$

Next define $w_j^0 := \Pi^B_{\left[a-\varepsilon^{\frac{1}{16}}, b+\varepsilon^{\frac{1}{16}}\right]} w_j$. By Lemma A.1

$$\left\| w_j^0 - w_j \right\| \leq \varepsilon^{\frac{1}{16}} \left\| w_j \right\| . \tag{A.6}$$

From here it follows immediately that $w_1^0, w_2^0, \ldots, w_N^0$ form an $\varepsilon^{\frac{1}{64}}$-ASOE of B. If we let $H_0 = H^B_{\left[a-\varepsilon^{\frac{1}{16}}, b+\varepsilon^{\frac{1}{16}}\right]}$, then by Lemma A.4 there exist orthonormal $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_N \in H_0$ which span the same subspace of H_0 as the w_j^0's. Furthermore

$$\left\| w_j^0 - \overline{w}_j \right\| < \varepsilon^{\frac{1}{128}} \tag{A.7}$$

and $\left\| \left(\rho^+ (B) - \rho^+ \left(\lambda_{a_j}\right)\right) \overline{w}_j \right\| \leq 3 \left\| \rho \right\|_{C^1} \varepsilon^{\frac{1}{512}}$. From (A.6) and (A.7) we also have $\left\| w_j - \overline{w}_j \right\| < \varepsilon^{\frac{1}{256}}$. From Lemma A.3 there exist orthonormal $w_1', w_2', \ldots, w_{M-N}'$ with $M = N^B_{\left[a-\varepsilon^{\frac{1}{16}}, b+\varepsilon^{\frac{1}{16}}\right]}$ such that $\langle w_i', \overline{w}_j \rangle = 0$ and $\left\| \left(B - \mu_j'\right) w_j' \right\| < 4M\varepsilon^{\frac{1}{64}} < \varepsilon^{\frac{1}{128}}$. Hence Lemma A.1 $\left\| \left(\rho^+ (B) - \rho^+ \left(\mu_j'\right)\right) w_j' \right\| \leq 3 \left\| \rho \right\|_{C^1} \varepsilon^{\frac{1}{256}}$. We now have

$$\begin{aligned}
\operatorname{tr}\left[D^+ \rho^+ (B)\right] &= \sum_{j=1}^{N} \left\langle \overline{w}_j, D^+ \rho^+ (B) \overline{w}_j \right\rangle + \sum_{j=1}^{M-N} \left\langle w_j', D^+ \rho^+ (B) w_j' \right\rangle \\
&\geq \sum_{j=1}^{N} \rho^+ \left(\lambda_{a_j}\right) \left\langle \overline{w}_j, D^+ \overline{w}_j \right\rangle + \sum_{j=1}^{M-N} \rho^+ \left(\mu_j'\right) \left\langle w_j', D^+ w_j' \right\rangle \\
&\qquad - 3\varepsilon^{\frac{1}{512}} M \left\| D \right\| \left\| \rho \right\|_{C^1} \\
&\geq \sum_{j=1}^{N} \rho^+ \left(\lambda_{a_j}\right) \left\langle \overline{w}_j, D^+ \overline{w}_j \right\rangle - 3\varepsilon^{\frac{1}{512}} M \left\| D \right\| \left\| \rho \right\|_{C^1} \\
&\geq \sum_{j=1}^{N} \rho^+ \left(\lambda_{a_j}\right) \left\langle w_j, D^+ w_j \right\rangle - 6\varepsilon^{\frac{1}{512}} M \left\| D \right\| \left\| \rho \right\|_{C^1} \\
&\geq \sum_{j=1}^{N} \rho^+ \left(\lambda_{a_j}\right) \left\langle v_{a_j}, C^+ v_{a_j} \right\rangle - 6\varepsilon^{\frac{1}{512}} M \left(\left\| D \right\| + 1\right) \left\| \rho \right\|_{C^1} \\
&\geq \operatorname{tr}\left[C^+ \rho^+ (A)\right] - \varepsilon^{\frac{1}{1024}} \left\| \rho \right\|_{C^1} .
\end{aligned}$$

Reversing the roles of H and H' gives

$$\left| \operatorname{tr}\left[D^+ \rho^+ (B)\right] - \operatorname{tr}\left[C^+ \rho^+ (A)\right] \right| \leq \varepsilon^{\frac{1}{1024}} \left\| \rho \right\|_{C^1} .$$

Similar estimates with $C^+ \rho^- (A)$,$C^- \rho^+ (A)$ and $C^- \rho^- (A)$ give the result. □

Finally, we now give a criterion implying the discreteness of spectrum for pseudodifferential operators required by the preceding propositions in this appendix.

Proposition A.6. *Let $A \in \Psi_{cl}^{m}\left(\mathbb{R}^{n};\mathbb{C}^{l}\right)$ and $I=[a,b]\subset\mathbb{R}$ a closed interval such that the I energy band*

$$\Sigma_{I}^{A}:=\bigcup_{\lambda\in I}\Sigma_{\lambda}^{A}$$

is bounded. Then for $h<h_{0}$ sufficiently small

$$EssSpec\left(A\right)\cap I=\emptyset.$$

Proof. Let $\sigma\left(A\right)=a\left(x,\xi\right)\in C^{\infty}\left(\mathbb{R}^{2n}\right)$ and $\Sigma_{I}\left(a\right)\subset B_{R}$ some open ball of finite radius R around the origin. For $\lambda\in I$ and $\left(x,\xi\right)\notin B_{R}$, we hence have that $a_{-1}:=\left(a\left(x,\xi\right)-\lambda\right)^{-1}$ exists. Let $\chi\in C_{c}^{\infty}\left(-4R,4R\right)$ such that $\chi\left(x\right)=1$ for $x<2R$. Set $\phi\left(x\right)=1-\chi\left(x\right)$ and define

$$A_{-1}=\left[\phi\left(\left|\left(x,\xi\right)\right|\right)a_{-1}\left(x,\xi\right)\right]^{W}\in\Psi_{\mathrm{cl}}^{0}\left(\mathbb{R}^{n};\mathbb{C}^{l}\right).$$

Then since it has vanishing symbol, we have

$$\left(A-\lambda\right)A_{-1}-\left(I-\chi\left(\left|\left(x,\xi\right)\right|\right)^{W}\right)=hR\in h\Psi_{\mathrm{cl}}^{0}\left(\mathbb{R}^{n};\mathbb{C}^{l}\right).$$

Next, we clearly have $I+hR$ is invertible for $h<h_{0}$ sufficiently small. Also, $\chi\left(\left|\left(x,\xi\right)\right|\right)^{W}$ is trace class by [16] Lemma 19.3.2. Hence if $S:=A_{-1}\left(I+hR\right)^{-1}$, then $\left(A-\lambda\right)S-I$ is trace class. By a similar argument, $S\left(A-\lambda\right)-I$ is trace class. Hence by Proposition 19.1.14 of [16], $A-\lambda$ is Fredholm. □

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