BERGMAN-SZEGŐ KERNEL ASYMPTOTICS IN WEAKLY PSEUDOCONVEX FINITE TYPE CASES

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Abstract. We construct a pointwise Boutet de Monvel-Sjöstrand parametrix for the Szegő kernel of a weakly pseudoconvex three dimensional CR manifold of finite type assuming the range of its tangential CR operator to be closed; thereby extending the earlier analysis of Christ [10, 11]. This particularly extends Fefferman's boundary asymptotics of the Bergman kernel [16] to weakly pseudoconvex domains in $\mathbb{C}^2$, in agreement with D'Angelo's example [13]. Finally our results generalize a three dimensional CR embedding theorem of Lempert [28].

1. Introduction

Cauchy Riemann (CR) manifolds are natural analogues of complex manifolds in odd dimensions. Their structure being modeled on that of a real-hypersurface inside a complex manifold, the natural question of when an abstract CR manifold can be embedded as such into complex space $\mathbb{C}^N$ has been long studied. In dimensions at least five a classical embedding theorem for strongly pseudo-convex CR manifolds was proved by Boutet de Monvel [5]; thereby leaving unresolved the cases of three dimensional manifolds and weakly pseudoconvex manifolds. In dimension three the problem is well known to be more subtle as there are examples of non-embeddable strongly pseudo-convex manifolds [38, 1]. However stronger conditions implying three dimensional embeddability are known; Kohn [26, 27] showed that embeddability of a strongly pseudoconvex CR manifold is equivalent its tangential Cauchy-Riemann operator $\bar{\partial}_b$ having closed range. Thereafter Lempert [28] (see also Epstein [15]) showed embeddability of a strongly pseudoconvex CR manifold assuming the existence of transversal CR circle action. In the weakly pseudoconvex case fewer results are known; Christ [10, 11] (see also Kohn [26]) showed embeddability of a weakly pseudoconvex CR three manifold of finite type assuming the range of its tangential Cauchy-Riemann operator $\bar{\partial}_b$ to be closed.

A closely related problem is to study the behavior of the Szegő kernel, the Schwartz kernel of the projector from smooth functions onto CR functions. When the manifold is the boundary of a strictly pseudoconvex domain the singularity of the Szegő kernel was described by Boutet de Monvel and Sjöstrand in [6], the Szegő projector is in this case is a Fourier integral operator with complex phase. Combined with the results of [5, 17, 26, 27] this description extends to strongly pseudo-convex manifolds whose tangential CR operator $\bar{\partial}_b$ has closed range; and in particular those of dimension at least five. In particular this description can be used to derive Fefferman's boundary asymptotics of the Bergman kernel [16] of a strongly pseudoconvex domain. The weakly pseudoconvex case analog of the problem has been studied by several authors before, with prior results including pointwise bounds on the kernels [10, 30, 32, 35] besides special cases of the asymptotics [13, 3].

In the present article we obtain a similar description for the pointwise Szegő kernel of weakly pseudoconvex CR three manifolds of finite type whose tangential CR operator $\bar{\partial}_b$ has closed

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range. Further we prove the boundary asymptotic expansion for the Bergman kernel for weakly pseudoconvex finite type domains in $\mathbb{C}^2$. Our results thereby extend the aforementioned analysis of Christ, the boundary Bergman kernel asymptotics of Fefferman and the embedding result of Lempert.

Let us now state our results more precisely. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension three. Thus $T^{1,0}X \subset T_{\mathbb{C}}X$ is a complex subbundle of dimension one satisfying $T^{1,0}X \cap T^{0,1}X = \emptyset$, $T^{0,1}X := \overline{T^{1,0}X}$. Denote by $HX := \mathrm{Re}\,(T^{1,0}X \oplus T^{0,1}X)$ the Levi-distribution and J its induced integrable almost complex structure. The Levi form is defined as

$$\begin{aligned} \mathscr{L} &\in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX) \\ \mathscr{L}(U,V) &:= [[U,V]] \in T_xX/H_xX \end{aligned} \tag{1.1}$$

for $U, V \in C^\infty(HX)$. Given a locally defined vector field $T \in C^\infty(TX)$ transversal to HX the Levi form can be thought of as a skew-symmetric bi-linear form on HX. We say that the point x is weakly/strongly pseudoconvex iff the corresponding bi-linear form $\mathscr{L}(., J.)$ is positive semi-definite/definite for some choice of orientation for T. The manifold is weakly/strongly pseudoconvex if each point $x \in X$ is weakly/strongly pseudoconvex. The CR manifold is said to be of finite type if the Levi-distribution HX is bracket generating: $C^\infty(HX)$ generates $C^\infty(TX)$ under the Lie bracket. More precisely, the type of a point $x \in X$ is the smallest integer $r(x)$ such that $HX_{r(x)} = TX$, where HX_j, $j = 1, \ldots$ are inductively defined by $HX_1 := HX$ and $HX_{j+1} := HX + [HX_j, HX]$, $\forall j \geq 1$. The function $x \mapsto r(x)$ is in general only an upper semi-continuous function. The finite type hypothesis is then equivalent to $r := \max_{x\in X} r(x) < \infty$. Note that the type of a strongly pseudoconvex point x is $r(x) = 2$. For points of higher type it shall be useful to analogously define the $r(x) - 2$ jet of the Levi-form at x

$$\begin{aligned} j^{r_x-2}\mathscr{L} &\in (HX^*)^{\otimes r_x} \otimes (T_xX/H_xX) \quad \text{by} \\ \left(j^{r_x-2}\mathscr{L}\right)(U_1, \ldots, U_{r_x}) &:= \left[\mathrm{ad}_{U_1}\mathrm{ad}_{U_2}\ldots\mathrm{ad}_{U_{r-1}}U_r\right] \in T_xX/H_xX \end{aligned} \tag{1.2}$$

for $U_j \in C^\infty(HX)$, $j = 1, \ldots, r$.

Next let $\bar\partial_b : \Omega^{0,*}(X) \to \Omega^{0,*+1}(X)$ denote the tangential CR operator and choose a smooth volume form μ on X. The Szegő kernel $\Pi(x,x')$ is by definition the Schwartz kernel of the L^2 projection $\Pi : L^2(X) \to \ker\left(\bar\partial_b\right)$. To describe our parametrix for Π, first recall the well known symbol class $S^m_{\rho,\delta}(\Omega \times \mathbb{R}_t)$, $m \in \mathbb{R}$, $\rho, \delta \in (0,1]$, $\Omega \subset \mathbb{R}^2$, of Hörmander [20]: these are smooth functions $a(x,t)$ satisfying the estimates $\partial_t^k \partial_x^\alpha a = O\left(t^{m-\rho k+\delta|\alpha|}\right)$, $\forall (k,\alpha) \in \mathbb{N}_0 \times \mathbb{N}_0^3$, as $t \to \infty$, uniformly on compact subsets of Ω. Further denote by the notation $S^m_\delta(\Omega \times \mathbb{R}_t)$ the special case when $\rho = 1$. We now introduce the subspace of classical symbols $S^m_{\delta,\mathrm{cl}}(\Omega \times \mathbb{R}_t) \subset S^m_\delta(\Omega \times \mathbb{R}_t)$ as those $a(x,t)$ for which there exist functions $a_j \in \mathcal{S}(\mathbb{R}^2)$, $j = 0, 1, 2, \ldots$, satisfying

$$a(x,t) - t^m\left[\sum_{j=0}^{N} t^{-\delta j} a_j\left(t^\delta x\right)\right] \in S^{m-\delta N}_\delta(\Omega \times \mathbb{R}_t), \quad \forall N \in \mathbb{N}_0. \tag{1.3}$$

Our first theorem is now the following.

Theorem 1. *Let X be a compact weakly pseudoconvex three dimensional CR manifold of finite type for which the range of the tangential CR operator $\bar\partial_b$ is closed. At any point $x' \in X$ of type $r = r(x')$, there exists a set of coordinates (x_1, x_2, x_3) centered at x' and a classical symbol $a \in S^{\frac{2}{r}}_{\frac{1}{r},cl}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$, with $a_0 > 0$, such that the pointwise Szegő kernel at x' satisfies*

$$\Pi(x,x') = \int_0^\infty dt\, e^{itx_3} a(x;t) + C^\infty(X). \tag{1.4}$$

We note again that the point $x' \in X$ above is fixed and thus our 'pointwise parametrix' is a distribution on the manifold X rather than the product. More can be said about the amplitude in (1.4): each coefficient $a_j \in \mathcal{S}(\mathbb{R}^2)$ in its symbolic expansion (1.3) is a linear combination of functions of the form $x_1^{\alpha_1} x_2^{\alpha_2} a_{j,\alpha}$, $\alpha_1 + \alpha_2 \leq 2jr$, with the functions $a_{j,\alpha} \in \mathcal{S}(\mathbb{R}^2)$ further depending only on the first jet of the Levi-form $j^{r_{x'}-2}\mathscr{L}$ at x' and the indices j, α. Furthermore at a strongly pseudoconvex point x' we may take each $a_{j,\alpha} = e^{-\left(x_1^2+x_2^2\right)}$ to be a Gaussian. Following this Theorem 1 is seen to recover the pointwise version of the Boutet de Monvel-Sjöstrand parametrix at strongly pseudoconvex points (see Remark 15 below). At points of higher type however the functions $a_{j,\alpha}$ are no longer Gaussians.

An important classical case arises when the CR manifold $X = \partial D$ is the boundary of a domain, i.e. a relatively compact open subset $D \subset \mathbb{C}^2$. The analogous Bergman kernel $\Pi_D(z, z')$ is the Schwartz kernel of the projector $\Pi_D : L^2(D) \to \ker\left(\bar{\partial}\right)$ onto the L^2-holomorphic functions in the interior. One is then interested in the on-diagonal behavior of the Bergman kernel as one approaches the boundary in terms of a boundary defining function $\rho \in C^\infty(\mathbb{C}^2)$, satisfying $D = \{\rho < 0\}$, $d\rho|_{\partial D} \neq 0$. This is given as below.

Theorem 2. *Let $D \subset \mathbb{C}^2$ be a domain with boundary $X = \partial D$ being smooth, weakly pseudoconvex of finite type. For any point $x' \in X = \partial D$ on the boundary, of type $r = r(x')$, the Bergman kernel satisfies the asymptotics*

$$\Pi_D(z, z) = \sum_{j=0}^{N} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{N} b_j (-\rho)^j \log(-\rho) + O\left((-\rho)^{\frac{N-2-2r}{r}}\right), \quad \forall N \in \mathbb{N},$$

as $z \to x'$ for some set of reals a_j, b_j with $a_0 > 0$.

Our description of Szegő kernel Theorem 1 becomes more concrete in the case when the CR manifold X is circle invariant. In this case one obtains an on diagonal expansion $\Pi_m(x, x)$, $m \to \infty$, for the mth Fourier mode of the Szegő kernel, we refer to Theorem 21 in Section 5 below for the precise statement. The asymptotics of these higher Fourier modes of the Szegő kernel allows one to construct a sufficient number of CR peak functions. These can be used to prove the following embedding theorem.

Theorem 3. *Let X be a compact weakly pseudoconvex three dimensional CR manifold of finite type admitting a transversal, CR circle action. Then it has an equivariant CR embedding into some $\mathbb{C}^N$, $N \in \mathbb{N}$.*

The Szegő kernel parametrix of Boutet de Monvel-Sjöstrand [6] has had a broad impact in complex analysis and geometry, we refer to [22] for a detailed account of this technique and its applications. Particularly this recovered the prior results of Fefferman [16] on the full boundary asymptotics for the Bergman kernel of a strongly pseudoconvex domain, which in turn refined its leading asymptotics by Hörmander [18]. The weakly pseudoconvex analog of the problem has also been considered by several authors before. Prior results have included pointwise upper [10, 30, 32, 35] and lower [7] bounds on the Bergman and Szegő kernels in low dimensions, besides particular special cases of the asymptotics for complex ovals [13], h-extendible/semiregular domains [3] and certain toric domains [24]. In higher dimensional weakly pseudoconvex cases the analogous bounds as well as the asymptotics of Theorem 1 and Theorem 2 are wide open, some known results in higher dimensions include weak estimates on the Bergman kernel [37] along with estimates on the Bergman metric [33] and distance [14].

In the presence of a transversal circle action a weakly pseudoconvex CR manifold is the unit circle bundle of a semi-positive holomorphic orbifold line bundle over a complex orbifold. When the action is free and the manifold strongly pseudoconvex, the Szegő kernel expansion Theorem

21corresponds to the Bergman kernel expansion of positive line bundles and was first obtained in [8, 40]. This was recently generalized to the Bergman kernel expansion of semipositive line bundles over a Riemann surface in [31]. The first author had earlier in [23] given a proof of the Szegő kernel expansion of a circle invariant weakly pseudoconvex CR manifold, although only on its strictly pseudoconvex part. For general non-free actions on strongly-pseudoconvex manifolds, the Szegő kernel expansion corresponds to the Bergman kernel expansion of positive orbifold line bundles and was first proved in [12], [29, Sec. 5.4].

As mentioned, the embeddability of strongly pseudoconvex CR manifolds equipped with a transversal CR circle action was shown in [28] and thus generalized by our last Theorem 3. In the weakly pseudoconvex case, [11] showed embeddability of a CR three manifold of finite type assuming the range of its tangential Cauchy-Riemann operator $\bar{\partial}_b$ to be closed.

The paper is organized as follows. In Section 2 we begin with some preliminaries in CR geometry including a construction of almost analytic coordinates adapted to the CR structure in 2.1. In Section 3 we construct an appropriate symbol calculus in 3.1 and construct the pointwise Szegő parametrix to prove Theorem 1. In Section 4 we consider the Bergman kernel of a weakly pseudoconvex domain in $\mathbb{C}^2$ and prove Theorem 2. In Section 5 we turn to the circle invariant case and prove Theorem 21. In the final section Section 6 we prove our embedding theorem Theorem 3.

2. CR Preliminaries

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension three. Thus $T^{1,0}X \subset T_{\mathbb{C}}X$ is a complex subbundle of dimension one satisfying $T^{1,0}X \cap T^{0,1}X = \emptyset$, $T^{0,1}X := \overline{T^{1,0}X}$. Let $HX := \mathrm{Re}\left(T^{1,0}X \oplus T^{0,1}X\right)$ be the Levi-distribution. This carries an almost complex structure $J : HX \to HX$

$$J\left(v+\bar{v}\right) := i\left(v-\bar{v}\right), \quad \forall v \in T^{1,0}X,$$

satisfying $J^2 = -1$ and the integrability condition

$$\begin{aligned} &[Jv, u] + [v, Ju] \in C^{\infty}\left(HX\right) \\ &[Jv, Ju] - [v, u] = J\left([Jv, u] + [v, Ju]\right) \end{aligned} \tag{2.1}$$

$\forall u, v \in C^{\infty}\left(HX\right)$. The antisymmetric Levi-form defined via (1.1) consequently satisfies $\mathscr{L}\left(Ju, v\right) = \mathscr{L}\left(u, Jv\right)$. The last contraction $\mathscr{L}\left(., J.\right)$ is equivalently thought of as a Hermitian form on $T^{1,0}X$ and denoted by the same notation via

$$\mathscr{L}\left(u, v\right) := \left[\frac{2}{i}\left[u, \bar{v}\right]\right] \in \left(TX/HX\right) \otimes \mathbb{C}, \tag{2.2}$$

$u, v \in T^{1,0}X$. The point $x \in X$ is strongly/weakly pseudoconvex if the Levi form above is positive definite/semi-definite at x for some choice of local orientation for TX/HX. Next one defines the flag of subspaces$HM_{1,x} \subset HM_{2,x} \subset \ldots$, at $x \in X$ inductively via

$$\begin{aligned} HM_{1,x} &:= HM_x \\ HM_{j+1,x} &:= HM_x + \left[HM_{j,x}, HM_x\right], \quad j \geq 1. \end{aligned}$$

The point x is said to be of finite type if $HM_{r(x),x} = TX$ for some $r\left(x\right) \in \mathbb{N}$, the minimum such integer being the type of the point x. The weight vector at the point x is defined to be $w\left(x\right) := \left(1, 1, r\left(x\right)\right)$. The CR structure is of finite type if each point is of finite type. Note that the type of a strongly pseudoconvex point $x \in X$ is $r\left(x\right) = 2$ by definition.

The set of *horizontal* paths of Sobolev regularity one connecting the two points $x, x' \in X$ is denoted by

$$\Omega_H(x,x') := \left\{ \gamma \in H^1([0,1];X) \,|\, \gamma(0) = x,\ \gamma(1) = x',\ \dot{\gamma}(t) \in HX_{\gamma(t)} \text{ a.e.} \right\}. \tag{2.3}$$

Fixing a metric g^{HX} on the Levi distribution HX, one define the obvious the obvious length functional $l(\gamma) := \int_0^1 |\dot{\gamma}|\, dt$ on the above path space Ω_H. By a classical theorem of Chow-Rashevsky the horizontal path space (2.3) is non-empty when the CR structure is of finite type; allowing the definition of a distance function

$$d^H(x,x') := \inf_{\gamma \in \Omega_H(x,x')} l(\gamma). \tag{2.4}$$

The weight $w(f)$ of a function f at the point x is defined to be the maximum integer $s \in \mathbb{N}_0$ for which $a + b = s$ implies that $\left(u^a v^b f\right)(x) = 0$; where u, v form a local frame for HX near x. Similarly the weight $w(P)$ of a differential operator P at the point $x \in X$ is the maximum integer for which $w(Pf) \geq w(P) + w(f)$ holds for each function $f \in C^\infty(X)$. It is known that there exists a set of coordinates (x_1, x_2, x_3) centered near a point $x \in X$ for which $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ forms a basis for HM_x of the canonical flag and moreover each coordinate function (x_1, x_2, x_3) has weight $w(x) := (1, 1, r(x))$ respectively; such a coordinate system is called privileged. In privileged coordinates near x the weight of a monomial x^α, $\alpha \in \mathbb{N}_0$, is thus $\alpha_1 + \alpha_2 + r\alpha_3$. The weight $w(f)$ of an arbitrary function $f \in C^\infty(X)$ is then the minimum weight of the monomials appearing in its Taylor series in these coordinates. The weight $w(V)$ of a vector field $V = \sum_{j=1}^3 f_j \partial_{x_j}$ is seen to be $w(V) := \min\{w(f_1) - 1, w(f_2) - 1, w(f_3) - r\}$. The distance function and volume (with respect to an arbitrary volume form μ) of a radius ε ball centered at the origin in such coordinates are known to satisfy

$$C\left(|x_1| + |x_2| + |x_3|^{1/r(x)}\right) \leq d^H(x,0) \leq C'\left(|x_1| + |x_2| + |x_3|^{1/r(x)}\right)$$
$$C\epsilon^{2+r(x)} \leq \mu(B(0;\varepsilon)) \leq C'\epsilon^{2+r(x)}; \quad \varepsilon \in (0,1).$$

Given a coordinate chart U as above, denote by $S_H^m(U)$ the space of smooth functions $p(x)$ on $U \setminus \{0\}$ satisfying the estimates

$$|\partial_x^\alpha p| \leq C_\alpha \left[d^H(x,0)\right]^{-m+\frac{2}{r}-\alpha.w_x} \mu\left(B\left(0; d^H(x,0)\right)\right)^{-1}, \tag{2.5}$$

$x \neq 0, \forall \alpha \in \mathbb{N}_0^3$. It was shown in [10, Sec.12], cf. [35] that the restriction to U of the Szegő kernel lies in the class

$$\Pi(x,0) \in S_H^{\frac{2}{r}}(U) \tag{2.6}$$

defined above.

2.1. **Construction of coordinates.** In [11, Prop. 3.2] it was shown that the privileged coordinate system near x maybe further chosen so that

$$\begin{aligned} T^{1,0}X &= \mathbb{C}[Z] \\ Z &= \frac{1}{2}\left[\partial_{x_1} + (\partial_{x_2} p)\,\partial_{x_3} + i\left(\partial_{x_2} - (\partial_{x_1} p)\,\partial_{x_3} + R\right)\right]; \end{aligned} \tag{2.7}$$

where $p(x_1, x_2)$ is a homogeneous real polynomial of degree/weight $r(x)$, and $R = \sum_{j=1}^3 r_j(x)\,\partial_{x_j}$ a real vector field of weight $w(R) \geq 0$. Furthermore, the pseudoconvexity of X gives $\Delta p = \left(\partial_{x_1}^2 + \partial_{x_2}^2\right) p \geq 0$. In this subsection we shall further show how to remove the remainder term R via almost analytic extension. We first have the following.

Lemma 4. *There exists a locally defined complex vector field T such that $T_x = \partial_{x_3}$ and $[T, Z]$ vanishes to infinite order at x.*

Proof. The desired equation for the components of $T = \sum_{j=1}^{3} t_j(x)\,\partial_{x_j}$ is seen to be one of the form

$$(\partial_{x_1} + i\partial_{x_2})\, t_j = -\left(\partial_{x_2}p - i\partial_{x_1}p\right)\partial_{x_3}t_j + \delta_{03}T\left(\partial_{x_2} - i\partial_{x_1}\right)p + Tr_j - Rt_j + O\left(|x|^\infty\right),$$

$j = 1, 2, 3$. As the component functions p, r_j have degree at least two and one respectively; the degree k homogeneous part on the left hand side above involves the Taylor coefficients of t_j for x^α, $|\alpha| = k-1$, while those on the right hand side involve those for $x^\alpha, |\alpha| < k-1$. We may hence solve the above recursively for the Taylor coefficients of t_j, beginning with $(t_1, t_2, t_3) = (0, 0, 1) + O(|x|)$, and apply Borel's construction. □

Next we complexify the open neighborhood of $x \in U \subset \mathbb{R}^3$ on which the above coordinates are defined to an open set $U^{\mathbb{C}} \subset \mathbb{C}^3$ such that $U^{\mathbb{C}} \cap \mathbb{R}^3 = U$. Denote by $z_j = x_j + iy_j$, $j = 1, 2, 3$, the corresponding complex coordinates. For a function $f \in C_c^\infty(\mathbb{C}^3)$, we write $f \sim 0$ if it vanishes to infinite order along the real plane: $|f(z)| = O\left(|\mathrm{Im} z|^\infty\right)$. A function $f \in C_c^\infty(\mathbb{C}^3)$, is said to be almost analytic iff $\partial_{\bar{z}_j} f \sim 0$, $j = 1, 2, 3$. A complex vector field $L = \sum_{j=1}^{3}\left[a_j\partial_{z_j} + b_j\partial_{\bar{z}_j}\right]$ is said to be almost analytic iff Lf is almost analytic and $L\bar{f} \sim 0$ for all almost analytic $f \in C_c^\infty(\mathbb{C}^3)$. This is seen to be equivalent to a_j being almost analytic and $b_j \sim 0$ for $j = 1, 2, 3$. For two complex vector fields we write $L_1 \sim L_2$ iff $L_1 - L_2$ is almost analytic. We choose almost analytic extensions $\tilde{T}, \tilde{Z}$ of T, Z respectively. We now have the next lemma.

Lemma 5. *There exist almost analytic complex coordinates $w_j = z_j + z_3 O(|z|)$, $j = 1, 2, 3$, on $U^{\mathbb{C}}$ such that $\tilde{T} - \partial_{w_3}$ vanishes to infinite order at x.*

Proof. Firstly we have $\tilde{T} \sim \sum_{j=1}^{3} t_j(z)\,\partial_{z_j}$ for some almost analytic functions t_j, $j = 1, 2, 3$, satisfying $(t_1, t_2, t_3) = (0, 0, 1) + O(|z|)$. Next we find an almost analytic function $w_1(z)$ satisfying $\sum_{j=1}^{3} t_j\partial_{z_j}w_1 = O\left(|z|^\infty\right)$ or equivalently

$$\partial_{z_3}w_1 = -\left[t_1\partial_{z_1}w_1 + t_2\partial_{z_2}w_1 + (t_3 - 1)\,\partial_{z_3}w_1\right] + O\left(|z|^\infty\right). \tag{2.8}$$

The degree k homogeneous part on the left hand side above involves the Taylor coefficients of w_1 for z^α, $|\alpha| = k$, while those on the right hand side involve those for $z^\alpha, |\alpha| < k$. We may hence again solve the above recursively for the Taylor coefficients of w_1, beginning with $w_1(z) = z_1 + O\left(|z|^2\right)$, and apply Borel's construction. Since solving the equation (2.8) involves integration in z_3, the higher order terms in the Taylor expansion $w_1(z) = z_1 + z_3 O(|z|)$ can be further taken to be multiples of z_3. In similar vein, we find almost an analytic functions $w_2(z) = z_2 + z_3 O(|z|)$, $w_3(z) = z_3(1 + O(|z|))$ satisfying $\sum_{j=1}^{3} t_j\partial_{z_j}w_2 = O\left(|z|^\infty\right)$ and $\sum_{j=1}^{3} t_j\partial_{z_j}w_3 = 1 + O\left(|z|^\infty\right)$ respectively. Thus (w_1, w_2, w_3) is the required coordinate system. □

We may now prove our main result of this subsection.

Theorem 6. *There exist almost analytic complex coordinates $\tilde{z}_j = p_j + iq_j$, $j = 1, 2, 3$ and an almost analytic function $\varphi(\tilde{z}_1, \tilde{z}_2)$ (of the first two new coordinates) on $U^{\mathbb{C}}$ such that*

1. $\tilde{z}_j = z_j + O\left(|z|^2\right)$, $j = 1, 2$, $\tilde{z}_3 = z_3 + z_3 O(|z|) + O\left(|z|^\infty\right)$,
2. $\tilde{Z} = \frac{1}{2}\left(\partial_{\tilde{z}_1} + i\partial_{\tilde{z}_2}\right) - \frac{i}{2}\left(\partial_{\tilde{z}_1}\varphi + i\partial_{\tilde{z}_2}\varphi\right)\partial_{\tilde{z}_3}$ *and* $\tilde{T} = \partial_{\tilde{z}_3} + O\left(|z|^\infty\right)$ *for*
3. $\varphi(\tilde{z}_1, \tilde{z}_2) = \varphi_0(\tilde{z}_1, \tilde{z}_2) + O\left(|z|^{r+1}\right)$ *with φ_0 a homogeneous polynomial with real coefficients satisfying* $\left(\partial_{p_1}^2 + \partial_{p_2}^2\right)\left(\varphi_0|_{q=0}\right) \geq 0$.

Proof. Firstly we have by definition $\tilde{Z} \sim \sum_{j=1}^{3} a_j \partial_{w_j}$, for some almost analytic functions a_j, $j = 1, 2, 3$ satisfying

$$a_1(0) = \frac{1}{2}, \quad a_2(0) = \frac{i}{2} \quad ,$$
$$a_3 = \frac{1}{2}\left(\partial_{w_2}\tilde{p} - i\partial_{w_1}\tilde{p}\right) + O\left(|w|^r\right)$$

and $\tilde{p}$ being an almost analytic extension of p. Furthermore, from the preceding Lemma 4, Lemma 5 we have $\left[\tilde{T}, \tilde{Z}\right] = \left[\partial_{w_3}, \tilde{Z}\right] + O\left(|w|^\infty\right)$ and may assume that a_j, $j = 1, 2, 3$, are independent of w_3. Next as in (2.8) we find almost analytic functions $\tilde{w}_j(w_1, w_2) = w_j + O\left(|w|^\infty\right)$, $j = 1, 2$, such that

$$a_1\partial_{w_1}\tilde{w}_1 + a_2\partial_{w_2}\tilde{w}_1 - \frac{1}{2} = O\left(|w|^\infty\right)$$
$$a_1\partial_{w_1}\tilde{w}_2 + a_2\partial_{w_2}\tilde{w}_2 - \frac{i}{2} = O\left(|w|^\infty\right).$$

Setting $\tilde{z}_3 = w_3$, we have then thus far achieved $\tilde{Z} = \frac{1}{2}\left(\partial_{\tilde{w}_1} + i\partial_{\tilde{w}_2}\right) + a_3\left(\tilde{w}_1, \tilde{w}_2\right)\partial_{\tilde{w}_3} + O\left(|\tilde{w}|^\infty\right)$. It is then easy to find $\varphi\left(\tilde{w}_1, \tilde{w}_2\right)$ satisfying $a_3 = -\frac{i}{2}\left(\partial_{\tilde{w}_1}\varphi + i\partial_{\tilde{w}_2}\varphi\right) + O\left(|\tilde{w}|^\infty\right)$ by a further application of the Borel construction giving $\tilde{Z} = \frac{1}{2}\left(\partial_{\tilde{w}_1} + i\partial_{\tilde{w}_2}\right) - \frac{i}{2}\left(\partial_{\tilde{w}_1}\varphi + i\partial_{\tilde{w}_2}\varphi\right)\partial_{\tilde{w}_3} + \tilde{Z}_\infty$ for some almost analytic vector field $\tilde{Z}_\infty = O\left(|\tilde{w}|^\infty\right)$.

Finally to remove this infinite order error term one applies the scattering trick of Nelson [36, Ch. 3]. Choose an almost analytic function $\chi \in C_c^\infty\left(U^{\mathbb{C}}\right)$, equal to one near zero, and set

$$\tilde{Z}_1 = \frac{1}{2}\left(\partial_{\tilde{w}_1} + i\partial_{\tilde{w}_2}\right) - \frac{i}{2}\left(\partial_{\tilde{w}_1}\varphi + i\partial_{\tilde{w}_2}\varphi\right)\partial_{\tilde{w}_3} + (1-\chi)\tilde{Z}_\infty. \tag{2.9}$$

It is clear that the almost analytic flows of $\tilde{Z}$, $\tilde{Z}_1$ starting at $U^{\mathbb{C}}$ exit $U^{\mathbb{C}}$ in uniformly finite time, outside which they are equal. Thus the limiting almost analytic map

$$W := \lim_{t\to\infty} e^{t\tilde{Z}} \circ e^{-t\tilde{Z}_1} \tag{2.10}$$

exists with the limit achieved in finite time. One then calculates

$$\frac{d}{dt}\left(e^{-t\tilde{Z}_1} \circ e^{t\tilde{Z}}\right)^* \tilde{w}_j = \left(e^{t\tilde{Z}}\right)^*\left(\tilde{Z} - \tilde{Z}_1\right)\left(e^{-t\tilde{Z}_1}\right)^* \tilde{w}_j = O\left(|\tilde{w}|^\infty\right)$$
$$\text{and thus} \quad \tilde{z}_j := W^*\tilde{w}_j = \tilde{w}_j + O\left(|\tilde{w}|^\infty\right). \tag{2.11}$$

This finally gives

$$\tilde{Z} = W_*\tilde{Z}_1$$
$$= \frac{1}{2}\left(\partial_{\tilde{z}_1} + i\partial_{\tilde{z}_2}\right) - \frac{i}{2}\left(\partial_{\tilde{z}_1}\varphi + i\partial_{\tilde{z}_2}\varphi\right)\partial_{\tilde{z}_3}$$

near zero from (2.9), (2.10), (2.11) proving the second part of the theorem. The last part follows by a Taylor expansion and the corresponding subharmonicity of $p(x_1, x_2)$ (2.7). □

We remark that although Nelson's method may also be used to linearize the almost analytic vector field $\tilde{Z}$ in some almost analytic coordinates, the resulting coordinates thereby will not satisfy the first property of the above Theorem 6 which shall be used later.

Before putting the above coordinates to use in the next section, we shall also need the construction of almost analytic continuations of functions in the class S_H^m defined in (2.5). These shall be defined on the region

$$R_{\delta,U} := \left\{(x, y) \in U \times \mathbb{R}^3 \middle|\, |y| \le \delta|x|^2, |y_3| \le \delta|x_3||x|\right\} \tag{2.12}$$

for any $\delta > 0$.

Lemma 7. *(Almost analytic continuations in S^m_H) For each $\delta > 0$ there exists almost analytic extension map $\mathcal{E} : S^m_H(U) \to C^\infty(R_{\delta,U} \setminus \{0\})$ satisfying*

$$\begin{aligned} &\mathcal{E}f|_{\mathbb{R}} = f, \\ &\partial_{\bar{z}}\mathcal{E}f = O(|Imz|^\infty) \text{ uniformly on } R_{\delta,U} \text{ and} \\ Lf \in C^\infty(U) \implies &\tilde{L}\mathcal{E}f \in C^\infty(R_{\delta,U}) \end{aligned} \tag{2.13}$$

for each $f \in S^m_H$ and vector field L with almost analytic extension $\tilde{L}$.

Proof. The map $\mathcal{E}$ is defined by the usual Borel-Hörmander construction. Namely with $\chi \in C_c^\infty(\mathbb{R})$ and equal to one near zero, set

$$(\mathcal{E}f)(x,y) := \sum_\alpha \frac{(iy)^\alpha}{\alpha!} f^{(\alpha)}(x)\, \chi\left(\lambda_{|\alpha|}|y|\right). \tag{2.14}$$

Note that on the given region $R_{\delta,U}$ (2.12) each successive term above satisfies the estimate $\frac{(iy)^\alpha}{\alpha!} f^{(\alpha)}(x) = O\left(|y|^{(|\alpha|-rm)/2}\right)$. For a suitable sequence constants $\lambda_k \to \infty$ sufficiently fast, the series above is then seen to be C^∞ convergent, and hence defining a smooth function, on compact subsets of $R_{\delta,U} \setminus \{0\}$. The first property in (2.13) then follows immediately from the above definition. The second property, follows easily on differentiating the definition (2.14) and applying the estimates (2.5) on the region $R_{\delta,U}$. Finally for the last property, note that $g := \tilde{L}\mathcal{E}f - \mathcal{E}Lf$ is an almost analytic continuation of zero in the sense $g|_{\mathbb{R}} = 0$ and $\partial_{\bar{z}} g = O(|\mathrm{Im}z|^\infty)$ uniformly on $R_{\delta,U}$. It furthermore satisfies estimates similar to (2.5) by definition. The Taylor expansion of g in the y-variable is seen to be

$$g(x,y) := \sum_{|\alpha| \le N} \frac{(iy)^\alpha}{\alpha!} \underbrace{g^{(\alpha)}(x)}_{=0} + O\left(|y|^{(N+1-rm)/2}\right),$$

giving $g = O(|\mathrm{Im}z|^\infty)$ and $g \in C^\infty(R_{\delta,U})$. Since $Lf \in C^\infty(U)$ is smooth and $\mathcal{E}Lf \in C^\infty(R_{\delta,U})$ by construction, the result follows. □

3. Szegő parametrix

In this section we shall prove our main Theorem 1. It shall first be useful to define a requisite symbol calculus below.

3.1. **Symbol spaces and calculus.** Below we denote by $x = (\hat{x}, x_3)$ local coordinates on $\mathbb{R}^3$, with $\hat{x} = (x_1, x_2)$ denoting the local coordinates on $\mathbb{R}^2$.

A smooth function $f(\hat{x}, \hat{y}) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is said to lie in the class $f \in \hat{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ if for each $\left(\hat{\alpha}, \hat{\beta}\right) \in \mathbb{N}_0^4$ there exists $N\left(\hat{\alpha}, \hat{\beta}\right) \in \mathbb{N}$ such that

$$\left|\partial_{\hat{x}}^{\hat{\alpha}} \partial_{\hat{y}}^{\hat{\beta}} f(\hat{x}, \hat{y})\right| \le C_{N,\hat{\alpha}\hat{\beta}} \frac{(1 + |\hat{x}| + |\hat{y}|)^{N\left(\hat{\alpha},\hat{\beta}\right)}}{(1 + |\hat{x} - \hat{y}|)^{-N}}, \tag{3.1}$$

$\forall (\hat{x}, \hat{y}, N) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{N}$. We note that for $f \in \hat{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ the functions

$$f(., \hat{y}),\ f(\hat{x}, .) \in \mathcal{S}\left(\mathbb{R}^2\right) \tag{3.2}$$

are Schwartz for fixed $\hat{y}$ and $\hat{x}$ respectively.

We now introduce some symbol spaces.

Definition 8. Let $r \in \mathbb{N}$, $r \geq 2$. A function $a(x,y,t) \in C^{\infty}\left(\mathbb{R}^{6}_{x,y} \times \mathbb{R}_{t}\right)$ is said to lie in the symbol class $\hat{S}^{m}_{\frac{1}{r}}$, $m \in \mathbb{R}$, if for each $(\alpha,\beta,\gamma) \in \mathbb{N}_{0}^{7}$ there exists $N(\alpha,\beta,\gamma) \in \mathbb{N}$ such that

$$\left|\partial_{x}^{\alpha}\partial_{y}^{\beta}\partial_{t}^{\gamma}a(x,y,t)\right| \leq C_{N,\alpha\beta\gamma}\left\langle t\right\rangle^{m-\gamma+\frac{1}{r}\left(\left|\hat{\alpha}\right|+\left|\hat{\beta}\right|\right)+\alpha_{3}+\beta_{3}}\frac{\left(1+\left|t^{\frac{1}{r}}\hat{x}\right|+\left|t^{\frac{1}{r}}\hat{y}\right|\right)^{N(\alpha,\beta,\gamma)}}{\left(1+\left|t^{\frac{1}{r}}\hat{x}-t^{\frac{1}{r}}\hat{y}\right|\right)^{-N}}, \tag{3.3}$$
$$\forall (x,y,t,N) \in \mathbb{R}^{6}_{x,y} \times \mathbb{R}_{t} \times \mathbb{N}.$$

We further set

$$\hat{S}^{m,k}_{\frac{1}{r}} := \bigoplus_{p+p'\leq k}(tx_{3})^{p}(ty_{3})^{p'}\hat{S}^{m}_{\frac{1}{r}}, \quad \forall (m,k) \in \mathbb{R}\times\mathbb{N}_{0}. \tag{3.4}$$

The subset $\hat{S}^{m}_{\frac{1}{r},\mathrm{cl}} \subset \hat{S}^{m}_{\frac{1}{r}}$ of classical symbols is those $a(x,y,t)$ for which there exist $a_{jpp'}(\hat{x},\hat{y}) \in \hat{S}\left(\mathbb{R}^{2}\times\mathbb{R}^{2}\right)$, $j,p,p' \in \mathbb{N}_{0}$, such that

$$a(x,y,t) - \sum_{j=0}^{N}\sum_{p+p'\leq j} t^{m-\frac{1}{r}j}(tx_{3})^{p}(ty_{3})^{p'}a_{jpp'}\left(t^{\frac{1}{r}}\hat{x},t^{\frac{1}{r}}\hat{y}\right) \in \hat{S}^{m-(N+1)\frac{1}{r},N+1}_{\frac{1}{r}} \tag{3.5}$$

$\forall N \in \mathbb{N}_{0}$. We also set

$$\hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}} := \bigoplus_{p+p'\leq k}(tx_{3})^{p}(ty_{3})^{p'}\hat{S}^{m}_{\frac{1}{r},\mathrm{cl}}.$$

The following inclusions are clear

$$\begin{aligned}
\partial_{t}\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m-1,k}_{\frac{1}{r}}\\
\partial_{\hat{x}}\hat{S}^{m,k}_{\frac{1}{r}},\ \partial_{\hat{y}}\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m+\frac{1}{r},k}_{\frac{1}{r}}\\
\partial_{x_{3}}\hat{S}^{m,k}_{\frac{1}{r}},\ \partial_{y_{3}}\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m+1,k}_{\frac{1}{r}}\\
\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m+1,k-1}_{\frac{1}{r}}, \quad k\geq 1,\\
\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m,k+1}_{\frac{1}{r}},\\
\hat{S}^{m,k}_{\frac{1}{r}} &\subset \hat{S}^{m',k}_{\frac{1}{r}}, \quad m<m',
\end{aligned} \tag{3.6}$$

with similar inclusions applying for $\hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$.

Next we set

$$\begin{aligned}
\hat{S}^{m,k,-\infty}_{\frac{1}{r}} &:= \cap_{j\in\mathbb{N}_{0}}\hat{S}^{m-\frac{j}{r},k+j}_{\frac{1}{r}}\\
\hat{S}^{-\infty}_{\frac{1}{r}} &:= \cup_{m,k}\hat{S}^{m,k,-\infty}_{\frac{1}{r}}.
\end{aligned}$$

Following a standard Borel construction, one has asymptotic summation: for any $a_{j} \in \hat{S}^{m-\frac{1}{r}j,k+j}_{\frac{1}{r}}$, $j=0,1,\ldots$, there exists $a \in \hat{S}^{m,k}_{\frac{1}{r}}$ such that

$$a - \left(\sum_{j=1}^{N}a_{j}\right) \in \hat{S}^{m-\frac{1}{r}(N+1),k+N+1}_{\frac{1}{r}}, \quad \forall N \in \mathbb{N}, \tag{3.7}$$

with a similar property being true for the classical symbols $\hat{S}^{m}_{\frac{1}{r},\mathrm{cl}}$. Moreover the symbol a (3.7) above is unique modulo $\hat{S}^{m,k,-\infty}_{\frac{1}{r}}$.

We now define the quantizations of the symbols in 8.

Definition 9. An operator $G : C_c^\infty(\mathbb{R}^3) \to C^{-\infty}(\mathbb{R}^3)$ is said to be in the class $G \in \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ if its distribution kernel satisfies

$$G(x,y) \equiv g^L := \int_0^\infty dt\, e^{it(x_3-y_3)} g(x,y,t) \tag{3.8}$$

for some $g \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}} + \hat{S}^{-\infty}_{\frac{1}{r}}$.

It is an easy exercise that for $G \in \hat{L}^{m}_{\frac{1}{r}}$, $m < -1-k$, the kernel $G(.,y) \in C^k$ for fixed y. We next have a reduction lemma showing that the show that the amplitude g in the quantization above (3.8) maybe chosen independent of x_3 or y_3.

Lemma 10. *For any $g \in \hat{S}^{m,k}_{\frac{1}{r},cl}$ there exist $g_1, g_2 \in \hat{S}^{m,k}_{\frac{1}{r},cl}$ independent of x_3, y_3 respectively such that $g^L = g_1^L = g_2^L$.*

Proof. By a Fourier transform, it is easy to see that

$$G = g_1^L \quad \text{for}$$
$$g_1(\hat{x}, x_3, \hat{y}, t) = \left[e^{i\partial_t \partial_{y_3}} g \right]_{y_3 = x_3}.$$

Here the above notation follows [21, Sec. 7.6] wherein the partial y_3, t Fourier transform $\mathcal{F}_{y_3,t}$ of $e^{i\partial_t\partial_{y_3}} g$ is given

$$e^{i\partial_t\partial_{y_3}} g = \mathcal{F}^{-1}_{y_3,t} e^{i\tau\eta_3} \mathcal{F}_{y_3,t} g$$

by multiplication by the exponential of the dual variables η_3, τ respectively. From [21, Thm. 7.6.5] and (3.6) it is then easy to see that $g_1 \in \hat{S}^{m+6,k}_{\frac{1}{r}}(U)$. In particular we have $g_1 \in \hat{S}^{-\infty}_{\frac{1}{r}}$ for $g \in \hat{S}^{-\infty}_{\frac{1}{r}}$.

Next for $g \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ we plug in its classical expansion (3.5) into (3.8). By writing $ty_3 = tx_3 + t(y_3 - x_3)$ and repeated integration by parts using $\partial_t e^{it(x_3-y_3)} = i(x_3 - y_3) e^{it(x_3-y_3)}$ we obtain $g_{1,N} \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$, $N \in \mathbb{N}_0$, independent of y_3 such that $g_1 - g_{1,N} \in \hat{S}^{m-(N+1)\frac{1}{r},k+N+1}_{\frac{1}{r}}$, $\forall N \in \mathbb{N}_0$. By asymptotic summation we find $g_1 \sim g_{1,1} + \sum_{N=1}^\infty (g_{1,N+1} - g_{1,N}) \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$, independent of y_3, which satisfies $g - g_1 \in \hat{S}^{m,k,-\infty}_{\frac{1}{r}} \subset \hat{S}^{-\infty}_{\frac{1}{r}}$. From this and the first part of the proof the Lemma follows. The construction of g_2 is similar. □

Following the above we shall define the principal symbol in $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ via

$$\sigma_L(G) = g_{000}(\hat{x}, \hat{y}) \in \hat{S}(\mathbb{R}^2 \times \mathbb{R}^2),$$

$G \in \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$, as the leading term in the symbolic expansion (3.5). The following symbol exact sequence is then clear

$$0 \to \hat{L}^{m-\frac{1}{r},k+1}_{\frac{1}{r},\mathrm{cl}} \to \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}} \xrightarrow{\sigma_L} \hat{S}(\mathbb{R}^2 \times \mathbb{R}^2) \to 0.$$

The class $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}(U)$ is clearly closed under adjoints. The symbol of the adjoint is furthermore easily computed

$$\sigma_L(G^*)(\hat{x},\hat{y}) = \overline{\sigma_L(G)(\hat{y},\hat{x})}. \tag{3.9}$$

We next have the composition of operators in $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$.

Proposition 11. *For any* $G \in \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$, $H \in \hat{L}^{m',k'}_{\frac{1}{r},\mathrm{cl}}$ *one has the composition* $G \circ H \in \hat{L}^{m+m'-\frac{2}{r},k+k'}_{\frac{1}{r},\mathrm{cl}}$. *Furthermore the leading symbol of the composition is given by*

$$\sigma_L (G \circ H)(\hat{x}, \hat{y}) = \int d\hat{u}\, \sigma_L (G)(\hat{x}, \hat{u})\, \sigma_L (H)(\hat{u}, \hat{y}) . \tag{3.10}$$

Proof. Write $G = g_1^L$, $H = h_2^L$ in terms of their x_3, y_3 independent quantizations respectively. From Fourier inversion it is easy to check that

$$\begin{aligned}
(G \circ H)(x, y) &= \int dt\, e^{it(x_3 - y_3)} (g \circ h)(x, y, t) \qquad \text{for} \\
(g \circ h)(x, y, t) &:= \int d\hat{u}\, g(\hat{x}, x_3, \hat{u}, t)\, h(\hat{u}, \hat{y}, y_3, t) \\
&= t^{-2\frac{1}{r}} \int dt\, e^{it(x_3 - y_3)} d\hat{v} g\left(\hat{x}, x_3, t^{-\frac{1}{r}}\hat{v}, t\right) h\left(t^{-\frac{1}{r}}\hat{v}, \hat{y}, y_3, t\right)
\end{aligned}$$

upon a change of variables $t^{\frac{1}{r}}\hat{u} = \hat{v}$. The $\hat{v}$ integral is seen to be convergent on account of (3.2), which also gives the necessary symbolic estimates for $g \circ h$. To obtain the symbolic expansion, we plug x_3, y_3 independent symbolic expansions for g, h respectively into the above to obtain a symbolic expansion for the composed symbol $g \circ h$ along with the formula (3.10) for the leading part. □

Finally we show that our algebra of operators is a module over the usual algebra of pseudo-differential operators.

Proposition 12. *Let* $G \in \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ *and let* $P \in \Psi^{m'}_{\mathrm{cl}}$ *be a classical pseudodifferential operator on* U *of order* k. *Then,* $PG \in \hat{L}^{m+m',k}_{\frac{1}{r},\mathrm{cl}}$ *with leading symbol*

$$\sigma_L (PG)(\hat{x}, \hat{y}) = \sigma(P)(0, 0; 0, 1)\, \sigma_L (G)(\hat{x}, \hat{y}) . \tag{3.11}$$

Proof. First write the kernels

$$\begin{aligned}
P(x, u) &= \frac{1}{(2\pi)^3} \int d\xi e^{i(x-u)\xi} p(x, \xi) \\
G(u, y) &= \int dt e^{i(u_3 - y_3)t} g(\hat{u}; \hat{y}, y_3, t)
\end{aligned}$$

$p \in S^{m'}_{\mathrm{cl}}$, $g \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ using an u_3 independent quantization for G. Then Fourier inversion gives the composition to be

$$(P \circ G)(x, y) = \int dt\, e^{i(x_3 - y_3)t} q(x, y, t)$$

$$q(x, y, t) := \frac{1}{(2\pi)^2} \int d\hat{u} d\hat{\xi} e^{i(\hat{x} - \hat{u})\hat{\xi}} p\left(x, \hat{\xi}, t\right) g(\hat{u}; \hat{y}, y_3, t) . \tag{3.12}$$

Again the above amplitude satisfies necessary symbolic estimates on account of (3.2).

To obtain the symbolic expansion, a change of variables $\hat{v} = t^{\frac{1}{r}}\hat{u}, \hat{\eta} = t^{-\frac{1}{r}}\hat{\xi}$ first gives

$$\begin{aligned}
q\left(t^{-\frac{1}{r}}\hat{x}, x_3, t^{-\frac{1}{r}}\hat{y}, y_3, t\right) &= \int d\hat{u} d\hat{\xi} e^{i\left(t^{-\frac{1}{r}}\hat{x} - \hat{u}\right)\hat{\xi}} p\left(t^{-\frac{1}{r}}\hat{x}, x_3, \hat{\xi}, t\right) g\left(\hat{u}; t^{-\frac{1}{r}}\hat{y}, y_3, t\right) \\
&= \int d\hat{v} d\hat{\eta} e^{i(\hat{x} - \hat{v})\hat{\eta}} p\left(t^{-\frac{1}{r}}\hat{x}, x_3, t^{\frac{1}{r}}\hat{\eta}, t\right) g\left(t^{-\frac{1}{r}}\hat{v}; t^{-\frac{1}{r}}\hat{y}, y_3, t\right) .
\end{aligned} \tag{3.13}$$

Next we plug in the symbolic expansion for g as well as

$$p\left(t^{-\frac{1}{r}}\hat{x},x_3,t^{\frac{1}{r}}\hat{\eta},t\right)\sim t^k\left[p_0\left(0,x_3,0,1\right)+\sum_{j=1}^{\infty}t^{-j/r}p_j\left(\hat{x},x_3,\hat{\eta}\right)\right],$$

obtained from the classical symbolic expansion for p, into (3.12), (3.13). A further Taylor expansion in x_3 for $p_0\left(0,x_3,0,1\right)\sim\sum_{j=0}^{\infty}t^{-j}\left(tx_3\right)^j\left(\partial_{x_3}^j p_0\right)(0,0,0,1)$ and each p_j plugged into the above completes the proof. □

Finally we need some mapping properties of operators in $\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$. To introduce the functional spaces first define

$$\begin{aligned}\hat{S}^m_{\frac{1}{r}}\left(\mathbb{R}^2\right)&\subset\hat{S}^m_{\frac{1}{r}}\left(\mathbb{R}^3\times\mathbb{R}^3\right)\\ \hat{S}^m_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^2\right)&\subset\hat{S}^m_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^3\times\mathbb{R}^3\right)\end{aligned}\tag{3.14}$$

as the subspace of x_3,y-independent elements in 8. Note that the above are included

$$\begin{aligned}\hat{S}^m_{\frac{1}{r}}\left(\mathbb{R}^2\right)&\subset S^m_{\frac{1}{r}}\left(\mathbb{R}^2\times\mathbb{R}_t\right)\\ \hat{S}^m_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^2\right)&\subset S^m_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^2\times\mathbb{R}_t\right)\end{aligned}\tag{3.15}$$

in the Hörmander symbol classes from the introduction.

We next define the space of partial t-Fourier transforms of the classes (3.14) below

$$\begin{aligned}S^m_H\left(\mathbb{R}^3\right)&:=\left\{p\in\mathcal{S}'\left(\mathbb{R}^3\right)|p=\int dte^{itx_3}a\left(t,\hat{x}\right),\ a\in\hat{S}^m_{\frac{1}{r}}\left(\mathbb{R}^2\right)\right\}\\ S^m_{H,\mathrm{cl}}\left(\mathbb{R}^3\right)&:=\left\{p\in\mathcal{S}'\left(\mathbb{R}^3\right)|p=\int dte^{itx_3}a\left(t,\hat{x}\right),\ a\in\hat{S}^m_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^2\right)\right\}.\end{aligned}\tag{3.16}$$

It is an easy exercise using Fourier transforms to see that elements of $S^m_H\left(\mathbb{R}^3\right)$ (3.16) above are smooth outside the origin. While the space $S^m_H\left(U\right)$ consists of restrictions to U of elements in the space $S^m_H\left(\mathbb{R}^3\right)$ defined above. It is further easy to see the inclusion

$$S^m_H\left(\mathbb{R}^3\right)\subset C^{\alpha}\left(\mathbb{R}^3\right),\quad m<-1-\alpha.\tag{3.17}$$

We now have the following.

Proposition 13. *For $G\in\hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{R}^3\right)$ and $p\in S^{m'}_H\left(\mathbb{R}^3\right)$ we have $Gp\in S_H^{m+m'-\frac{2}{r}}\left(\mathbb{R}^3\right)$. A similar property holds for $S^m_{H,\mathrm{cl}}\left(\mathbb{R}^3\right)$.*

Proof. Again using a y_3 independent quantization for G, the Fourier transform expression $p=\int dte^{itx_3}a\left(t,\hat{x}\right)$ and Fourier inversion gives

$$\begin{aligned}Gp\left(x\right)&=\int dte^{ix_3t}\left(g\circ a\right)\left(x,t\right)\\ \left(g\circ a\right)\left(x,t\right)&:=\int g\left(x,\hat{y},t\right)a\left(t,\hat{y}\right)d\hat{y}.\end{aligned}$$

Next we plugin the symbolic expansion for g into the above and use repeated integration by parts using $\partial_t e^{ix_3t}=ix_3e^{ix_3t}$ to obtain x_3-independence of the amplitude modulo C^{∞}. Plugging in a classical expansion for $p\in S^m_{H,\mathrm{cl}}\left(\mathbb{R}^3\right)$ gives a similar expansion for $g\circ a$. □

3.2. **Local Bergman kernels.** In this section we shall define certain local Bergman kernels using the coordinates introduced in Sec. 2.1. Furthermore these shall be shown to lie in the symbol classes introduced in the previous section.

First with the notation as in Theorem 6 one sets $V = U^{\mathbb{C}} \cap \{q = 0\} \subset \mathbb{R}^3_p$. With $\chi(p_1, p_2) \in C^\infty_c(\mathbb{R}^2)$ of sufficiently small support and equal to one near zero, the function

$$\varphi(p_1, p_2) \coloneqq \varphi_0|_{q=0} + \underbrace{\chi(\varphi - \varphi_0)|_{q=0}}_{=:\varphi_1} \tag{3.18}$$

is well defined on $\mathbb{R}^2$. This equals the restriction of φ to V near the origin, and hence we use the same notation. Next set

$$\hat{Z} \coloneqq \frac{1}{2}(\partial_{p_1} + i\partial_{p_2}) - \frac{i}{2}(\partial_{p_1}\varphi + i\partial_{p_2}\varphi)\partial_{p_3} \tag{3.19}$$

and define

$$\begin{aligned} \bar{\partial}_t :& \Omega^{0,0}(\mathbb{R}^2) \to \Omega^{0,1}(\mathbb{R}^2) \\ \bar{\partial}_t u \coloneqq& \left[\frac{1}{2}(\partial_{p_1} + i\partial_{p_2})u + \frac{1}{2}t(\partial_{p_1}\varphi + i\partial_{p_2}\varphi)u\right]d\bar{z} \end{aligned} \tag{3.20}$$

with $d\bar{z} = dp_1 - idp_2$. Define the Kodaira Laplacian via

$$\Box_t \coloneqq \bar{\partial}_t^* \bar{\partial}_t \tag{3.21}$$

acting on $\Omega^{0,0}$. We denote by

$$B_t : L^2(\mathbb{R}^2_p) \to \ker(\Box_t) \tag{3.22}$$

the local Bergman projector onto the kernel of $\Box_t$ and $B_t(p, p')$ it Schwartz kernel.

Replacing φ with its leading polynomial φ_0 in (3.19), (3.20), (3.21) one analogously defines

$$\bar{\partial}^0_t u \coloneqq \left[\frac{1}{2}(\partial_{p_1} + i\partial_{p_2})u + \frac{1}{2}t(\partial_{p_1}\varphi_0 + i\partial_{p_2}\varphi_0)u\right]d\bar{z} \tag{3.23}$$

$$\Box^0_t \coloneqq \left(\bar{\partial}^0_t\right)^* \bar{\partial}^0_t \tag{3.24}$$

as well as a corresponding Bergman projection B^0_t with kernel $B^0_t(\hat{p}, \hat{p}')$.

Theorem 14. *One has* $B^0_t(\hat{p}, \hat{p}')$, $B_t(\hat{p}, \hat{p}') \in \hat{S}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}}$, *with furthermore*

$$B^0_t(\hat{p}, \hat{p}') = t^{\frac{2}{r}} b_0\left(t^{\frac{1}{r}}\hat{p}, t^{\frac{1}{r}}\hat{p}'\right) \tag{3.25}$$

$$B_t(\hat{p}, \hat{p}') = t^{\frac{2}{r}} b_0\left(t^{\frac{1}{r}}\hat{p}, t^{\frac{1}{r}}\hat{p}'\right) + \hat{S}^{\frac{1}{r},0}_{\frac{1}{r},\mathrm{cl}} \tag{3.26}$$

for some $b_0(\hat{p}, \hat{p}') \in \hat{S}(\mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. Being symmetric and bounded below, the Kodaira Laplacians (3.21), (3.24) are essentially self-adjoint. Furthermore under the rescaling/dilation $\delta_{t^{-1/r}}(\hat{p}, \hat{p}') = \left(t^{-\frac{1}{r}}\hat{p}, t^{-\frac{1}{r}}\hat{p}'\right)$ these are seen to satisfy

$$\begin{aligned} \boxdot^0_t &\coloneqq t^{-2/r}\left(\delta_{t^{-1/r}}\right)_* \Box^0_t = \Box^0_1 \\ \boxdot_t &\coloneqq t^{-2/r}\left(\delta_{t^{-1/r}}\right)_* \Box_t = \Box^0_1 + t^{-1/r}E, \end{aligned} \tag{3.27}$$

where

$$E = a(\hat{p}, t)\bar{\partial}^0_1 + b(\hat{p}, t)\bar{\partial}^0_1 + c(\hat{p}, t)$$

is a self-adjoint operator with the coefficients $a(\hat{p},t)$, $b(\hat{p},t)$ and $c(\hat{p},t)$ being uniformly (in t) C^∞ bounded.

Next, as in [31, Sec. 4.1], see also Prop. 23 below, we have

$$\mathrm{Spec}\left(\Box_t^0\right)\subset\{0\}\cup\left[c_1t^{2/r},\infty\right) \tag{3.28}$$

$$\mathrm{Spec}\left(\Box_t\right)\subset\{0\}\cup\left[c_1t^{2/r}-c_2,\infty\right). \tag{3.29}$$

We remark the fact that φ here is complex valued makes no difference to the above formulas (3.28), (3.29) so far as the leading part φ_0 is real and sub-harmonic. This is because the higher order Taylor coefficients of φ appear at lower order $O\left(t^{-1/r}\right)$ after rescaling in (3.27). Hence for any $\chi\in C_c^\infty(-c_1,c_1)$ with $\chi=1$ near 0, the Bergman kernels equal

$$\begin{aligned}B_t^0(\hat{p},\hat{p}')&=\chi\left(t^{-2/r}\Box_t^0\right)(\hat{p},\hat{p}')=t^{2/r}\chi\left(\Box_1^0\right)\left(t^{1/r}p,t^{1/r}p'\right)\\B_t(\hat{p},\hat{p}')&=\chi\left(t^{-2/r}\Box_t\right)(\hat{p},\hat{p}')=t^{2/r}\chi\left(\boxdot_t\right)\left(t^{1/r}p,t^{1/r}p'\right)\end{aligned}$$

for $t\gg0$. By standard elliptic arguments, the Schwartz kernels of $\partial_p^\alpha\partial_{p'}^{\alpha'}\chi\left(\Box_1^0\right)$, $\partial_p^\alpha\partial_{p'}^{\alpha'}\chi\left(\boxdot_t\right)$, $\alpha,\alpha'\in\mathbb{N}_0^2$, are rapidly decaying off-diagonal. Regarding their on-diagonal behavior, the growth of $\chi\left(\Box_1^0\right)(\hat{p},\hat{p})$, $\chi\left(\boxdot_t\right)(\hat{p},\hat{p})$ as $\hat{p}\to\infty$ is controlled by the growth of the coefficient functions of the operators (3.27), which in turn have polynomial growth. This gives (3.25) with

$$\begin{aligned}b_0(\hat{p},\hat{p}')&=\chi\left(\Box_1^0\right)(\hat{x},\hat{y})\in\hat{S}\left(\mathbb{R}^2\times\mathbb{R}^2\right)\\B_t(\hat{p},\hat{p}')&\in\hat{S}_{\frac{1}{r}}^{\frac{2}{r},0}.\end{aligned} \tag{3.30}$$

To show the classical expansion for the above one may use a full expansion of the operator $\boxdot_t$ (3.27) as in Section 5 below. We shall however give a different proof consistent with the rest of this section. To this end, first begin with $\varphi=\varphi_0+\varphi_1$ from (3.18), where

$$\varphi_1(\hat{p})=O\left(|\hat{p}|^{r+1}\right), \tag{3.31}$$

$\varphi_1(\hat{p})\in C_0^\infty(\mathbb{R}^2,\mathbb{C})$. Next define the operator with distributional kernel

$$\begin{aligned}\tilde{B}_t:&L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)\\\tilde{B}_t(\hat{p},\hat{p}')&=e^{-t\varphi_1(\hat{p})}B_t^0(\hat{p},\hat{p}')\,e^{t\varphi_1(\hat{p}')}.\end{aligned} \tag{3.32}$$

It is clear that the above $\bar{\partial}_t\tilde{B}_t=0$, $\Box_t\tilde{B}_t=0$ lies in the kernels of (3.20), (3.21). This gives $B_t\tilde{B}_t=\tilde{B}_t$ and

$$\tilde{B}_t^*B_t=\tilde{B}_t^*, \tag{3.33}$$

where $\tilde{B}_t^*$ is the adjoint of $\tilde{B}_t$. Let $R_t:=\tilde{B}_t-\tilde{B}_t^*$ whose Schwartz kernel is computed to be

$$R_t(\hat{p},\hat{p}')=e^{-t\varphi_1(\hat{p})}B_t^0(\hat{p},\hat{p}')\,e^{t\varphi_1(\hat{p}')}-e^{t\bar{\varphi}_1(\hat{p})}B_t^0(\hat{p},\hat{p}')\,e^{-t\bar{\varphi}_1(\hat{p}')}. \tag{3.34}$$

Since $\left(\bar{\partial}+t\left(\bar{\partial}\varphi_0\right)\wedge\right)\left(e^{t\varphi_1}B_t\right)=0$, we have

$$\tilde{B}_tB_t=B_t. \tag{3.35}$$

From (3.33) and (3.35), we get $(I-R_t)B_t=\tilde{B}_t^*$ and hence

$$(I-R_t^N)B_t=(I+R_t+R_t^2+\cdots+R_t^{N-1})\tilde{B}_t^*,\quad\forall N\in\mathbb{N}. \tag{3.36}$$

From the first part (3.30), (3.31), (3.34) and a Taylor expansion, it is easy to see that $R_t\in\hat{S}_{\frac{1}{r},\mathrm{cl}}^{\frac{1}{r},0}$ and

$$R_t^j\in\hat{S}_{\frac{1}{r},\mathrm{cl}}^{(2-j)\frac{1}{r},0},\ \ \forall j\in\mathbb{N}, \tag{3.37}$$

by an argument similar to Prop. 11. From the above, (3.30), (3.36), (3.37) and $\tilde{B}_t^* \in \hat{S}^{\frac{2}{r}}_{\frac{1}{r},\mathrm{cl}}$, the theorem follows. □

Following the above we now prove one of our main theorems Theorem 1.

Proof of Theorem 1. Choose B as in (3.22) and $\chi \in C_c^\infty\left(\mathbb{R}^2_{\hat{p}'}\right)$ a cutoff equal to one near zero. Define the operator

$$\hat{B} : C_c^\infty\left(\mathbb{R}^3\right) \to C^{-\infty}\left(\mathbb{R}^3\right)$$

$$\hat{B} := \frac{1}{2\pi}\int_0^\infty dt\, e^{it\left(p_3 - p_3'\right)} B_t\left(p, p'\right)\chi\left(t^{\frac{1}{2r}}\hat{p}'\right). \tag{3.38}$$

Clearly $\hat{B} \in \hat{L}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}}$ by definition and Theorem 14. It follows from Theorem 14 that Schwartz kernel of the above $\hat{B}\left(p, p'\right)$ is smooth away from $p = 0$ and satisfies estimates similar to (2.5) in p. Next let $\tilde{B}\left(z, p'\right)$ denote the almost analytic continuation of the Schwartz kernel $\hat{B}\left(p, p'\right)$ in the p-variable given by Lemma 7. Now consider the coordinates (x_1, x_2, x_3) on a neighborhood U centered at the point $x' \in X$, along with (p_1, p_2, p_3) being (the real parts of) the corresponding almost analytic coordinates given by 6. Letting $\chi_1, \chi_2 \in C_c^\infty(U)$ be such that $\chi_1 = 1$ on spt (χ_2) and $\chi_2 = 1$ near zero we set

$$B\left(x, x'\right) := \chi_1(x)\left(\tilde{B}\Big|_{y,y'=0}\right)\chi_2\left(x'\right) \in C^{-\infty}\left(X \times X\right). \tag{3.39}$$

Since $\hat{Z}\hat{B} = 0$ we have $Z\left(\tilde{B}\Big|_{y,y'=0}\right) \in C^\infty$ by Lemma 7 from which is it easy to check that $\bar{\partial}_b B$ is smooth .

Let $\Pi : L^2(X) \to H_b^0(X) := \left\{u \in L^2(X) \,|\, \bar{\partial}_b u = 0\right\}$ denote the Szegő projection. Assuming $\bar{\partial}_b$ has closed range it was shown in [11, Prop. 4.1], [10] that there exists a bounded linear operator $G : \mathrm{Range}\left(\bar{\partial}_b\right) \to L^2(X)$ such that $\Pi = I - G\bar{\partial}_b$. Furthermore G is microlocal and it maps $G : \mathrm{Range}\left(\bar{\partial}_b\right) \cap H^s(X) \to H^{s+\frac{1}{r}}(X)$, $\forall s \in \mathbb{R}$. It now follows that Π is microlocal or that the Szegő kernel is smooth away from the diagonal. Furthermore $\Pi B = B - G\bar{\partial}_b B = B + C^\infty$ and

$$B^*\Pi = B^* + C^\infty. \tag{3.40}$$

Next, replace $\Pi(x, 0)$ by $\Pi_1(x) = \chi_1(x)\Pi(x, 0)$, which has the same singularities near $x = 0$, is compactly supported and satisfies similar bounds to (2.6). We almost analytically continue $\Pi_1(x, 0)$ in the x variable to define $\tilde{\Pi}_1(z, 0)$. We may further suppose that $\tilde{\Pi}_1$ is compactly supported by construction. The restriction $\tilde{\Pi}_1(p, 0) = \tilde{\Pi}_1\Big|_{q=0}$ is well-defined and we set $\tilde{\Pi}_{1,t}(p_1, p_2) := \int e^{-itp_3}\tilde{\Pi}_1(p, 0)\, dp_3$. Since $Z\Pi_1 \in C^\infty$ it follows that for the almost analytic extension $\tilde{Z}\tilde{\Pi}_1$ is smooth from Lemma 7. From Theorem 6 it follows that $\left[\frac{1}{2}\left(\partial_{\bar{z}_1} + i\partial_{\bar{z}_2}\right) - \frac{i}{2}\left(\partial_{\bar{z}_1}\varphi + i\partial_{\bar{z}_2}\varphi\right)\partial_{\bar{z}_3}\right]\tilde{\Pi}_1$ is smooth. Hence

$$\bar{\partial}_t\tilde{\Pi}_{1,t} := \left[\frac{1}{2}\left(\partial_{p_1} + i\partial_{p_2}\right) - \frac{i}{2}t\left(\partial_{p_1}\varphi + i\partial_{p_2}\varphi\right)\right]\tilde{\Pi}_{1,t} = O\left(t^{-\infty}\right)$$

in the Schwartz norm. From here it is clear that $B_t\tilde{\Pi}_{1,t} = \tilde{\Pi}_{1,t} + O\left(t^{-\infty}\right)$ in the Schwartz norm. Thus one has $\left(\hat{B}\tilde{\Pi}_1 - \tilde{\Pi}_1\right)(p, 0) \in C^\infty$ and hence by almost analytic continuation

$$\left(\tilde{B}\tilde{\Pi}_1 - \tilde{\Pi}_1\right)(x, 0) \in C^\infty. \tag{3.41}$$

Next for each $N \in \mathbb{N}$ define the operator with kernel

$$B_N(x,x') = \sum_{|\alpha|,|\beta|\le N} \frac{1}{\alpha!\beta!}\left(-\frac{\partial}{\partial x'}\right)^{\alpha}\left[(iy(x'))^{\alpha}(iq(x))^{\beta}\left(\frac{\partial}{\partial p}\right)^{\beta} B(p(x),p'(x'))\left|\frac{dp}{dx}(x')\right|\right]\chi_1(x')$$

where

$$\begin{aligned} p(x) &= p(x,0) = x + O(x^2) \\ q(x) &= q(x,0) = O(x^2) \\ y(x) &= y(p(x,0),0) = O(x^2) \end{aligned} \tag{3.42}$$

denote the coordinates coming from the change of variables Theorem 6, while the multiplication factor $\left|\frac{dp}{dx}\right|$ is the Jacobian for the change of variables with respect to the first. Following an integration by parts argument using (2.14), (3.41), Prop. 12, and Prop. 13 which motivates the construction of (3.42), it is easy to see that

$$(B_N\Pi - \Pi_1)(x,0) \in S_H^{\frac{2-N}{r}} \tag{3.43}$$

$\forall N \in \mathbb{N}$. Furthermore writing (3.42) in the p, p' coordinates gives

$$\begin{aligned} &B_{N+1} - B_N \in \hat{L}^{\frac{2-N}{r},N}_{\frac{1}{r},\mathrm{cl}} \\ \text{hence}\quad &B_\infty := B_0 + \sum_{N=0}^{\infty}(B_{N+1}-B_N) \in \hat{L}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}} \end{aligned} \tag{3.44}$$

is well-defined by asymptotic summation. The last two equations (3.43), (3.44) then give

$$(B_\infty\Pi - \Pi)(x,0) \in C^\infty. \tag{3.45}$$

Next the Schwartz kernel of the $\mu = e^{g(x)}dx$ adjoint B^* of B is calculated to be

$$B^*(x,x') = e^{g(x')-g(x)}\chi_2(x')\,\overline{\tilde{B}(p(x')+iq(x'),p(x))}\chi_1(x). \tag{3.46}$$

Following Theorem 6, a Taylor expansion, and writing in p, p' coordinates, it is easy to see $B^* \in \hat{L}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}}$. Furthermore from the above and (3.42) it has the same principal symbol as B_∞

$$\begin{aligned} &\sigma_L(B^*) = \sigma_L(B_\infty) \quad \text{hence} \\ &R := B_\infty - B^* \in \hat{L}^{\frac{1}{r},1}_{\frac{1}{r},\mathrm{cl}}. \end{aligned} \tag{3.47}$$

Finally combining (3.40) and (3.45) we have

$$\begin{aligned} \Pi(x,0) &= [B_\infty\Pi](x,0) + C^\infty \\ &= (B^* + R)\Pi(x,0) \\ &= [B^* + R\Pi](x,0) + C^\infty \quad \text{hence} \end{aligned} \tag{3.48}$$

$$\left[\left(I - R^N\right)\Pi\right](x,0) = \left[1 + R + R^2 + \ldots + R^{N-1}\right]B^*(x,0) \tag{3.49}$$

$\forall N \in \mathbb{N}$. Following 11, (3.47) we have $R^j \in \hat{L}^{(2-j)\frac{1}{r},j}_{\frac{1}{r},\mathrm{cl}}$, $j = 1,2,\ldots$. Hence by asymptotic summation $\exists P \in \hat{L}^{\frac{2}{r},0}_{\frac{1}{r},\mathrm{cl}}$ such that

$$P_N := P - \left[1 + R + R^2 + \ldots + R^{N-1}\right] \in \hat{L}^{(2-N)\frac{1}{r},N}_{\frac{1}{r},\mathrm{cl}} \tag{3.50}$$

$\forall N \in \mathbb{N}$. Since $B^*(x,0) \in S^{\frac{2}{r}}_{H,\mathrm{cl}}$ by definition, we have

$$\Pi(x,0) = [PB^*](x,0) + S_H^{\frac{2-N}{r}} \tag{3.51}$$

$\forall N \in \mathbb{N}$, from (2.6), (3.49), (3.50) and Prop. 13. Choosing N large gives $\Pi(x,0) \in S^{\frac{2}{r}}_{H,\mathrm{cl}}$ using (3.17) and Prop. 13 which completes the proof on account of (3.14) and (3.15). □

The next remark shows that our parametrix Theorem 1 recovers the Boutet de Monvel-Sjöstrand parametrix at strongly pseudoconvex points.

Remark 15. (Strongly pseudoconvex points) Here we show that our main Theorem 1 recovers the Boutet de Monvel-Sjöstrand description of the Szegő kernel at strongly pseudoconvex points $x' \in X$. As noted before, the type of a strongly pseudoconvex point is $r_{x'} = 2$. The two degree 2 homogeneous polynomials in (2.7) and Theorem 6 can be further taken to be $p(x_1,x_2) = x_1^2+x_2^2$, $\varphi_0(\tilde{z}_1,\tilde{z}_2) = \tilde{z}_1^2 + \tilde{z}_2^2$ respectively. Following these, the model Bergman kernel is computed to be an exponential $B_t^0(\hat{p},\hat{p}') = tb_0\left(t^{\frac{1}{2}}p, t^{\frac{1}{2}}p'\right)$

$$\begin{aligned} B_t^0(\hat{p},\hat{p}') &= tb_0\left(t^{\frac{1}{2}}p, t^{\frac{1}{2}}p'\right) = te^{-t\Phi_0(\hat{p},\hat{p}')} \\ b_0(\hat{p},\hat{p}') &= e^{-\Phi_0(\hat{p},\hat{p}')} \\ \Phi_0(\hat{p},\hat{p}') &\coloneqq \frac{1}{4}\left(p_1^2+p_2^2+(p_1')^2+(p_2')^2+2p_1p_1'+2p_2p_2'+2ip_1'p_2-2ip_1p_2'\right) \end{aligned} \tag{3.52}$$

[29, Sec. 4.1.6]. And hence

$$\tilde{B}_t(p,p') = te^{-t[\Phi_0(\hat{p},\hat{p}')+\varphi_1(\hat{p})-\varphi_1(\hat{p}')]}$$

Next the local Bergman kernel B_t (3.22) is by 3.36 modulo C^N a finite a sum of terms of the form

$$\tilde{B}_t^*\left(\tilde{B}_t\tilde{B}_t^*\right)^k \quad \text{or} \quad \left(\tilde{B}_t\tilde{B}_t^*\right)^k.$$

Applying the complex stationary phase formula of Melin-Sjöstrand [34, Sec. 2], the kernels of the above take the form $a_k(\hat{p}',\hat{p},t)\,e^{-t\Phi_k(\hat{p}',\hat{p})}$, where $a_k \in S^1_{0,\mathrm{cl}}\left(\mathbb{R}^4_{p,p'} \times \mathbb{R}_t\right)$ is a classical symbol and $\Phi_k = \Phi_0 + O\left(|(p,p')|^2\right)$ a phase function agreeing with (3.52) at leading order. To get the Boutet de Monvel-Sjöstrand description however one needs to ensure that all phase functions Φ_k agree. To this end, one may replace $\tilde{B}_t$ (3.32) with the alternate approximation for the local Bergman kernel B_t given by

$$\begin{aligned} \tilde{B}_t^1(\hat{p},\hat{p}') &\coloneqq e^{t\Phi_1(\hat{p},\hat{p}')}B_t^0(\hat{p},\hat{p}') \\ \Phi_1(\hat{p},\hat{p}') &\coloneqq \varphi_1(\hat{p}) + \varphi_1(\hat{p}') - 2\sum_{\alpha,\beta}\left(\partial_\zeta^\alpha\partial_{\bar{\zeta}}^\beta\varphi_1\right)(0)\,\frac{\zeta^\alpha\bar{\zeta}'^\beta}{\alpha!\beta!}, \end{aligned}$$

$\zeta \coloneqq p_1 + ip_2$. The proof of Thm. Theorem 14, Theorem 1 all carry through with $\tilde{B}_t$ replaced by $\tilde{B}_t^1$. The above further has the advantage of being self-adjoint

$$\begin{aligned} \tilde{B}_t^1(\hat{p},\hat{p}') &= \overline{\tilde{B}_t^1(\hat{p}',\hat{p})} \quad \text{and equals} \\ \tilde{B}_t^1(\hat{p},\hat{p}') &= e^{t[\Phi_0+\Phi_1]} \end{aligned}$$

in the strongly pseudoconvex case again using (3.52). Furthermore, the composition of complex Fourier integral operators and the complex stationary phase formula of Melin-Sjöstrand [34, Sec. 2] in this case gives

$$\left(\tilde{B}_t^1\right)^2 = a\left(\hat{p},\hat{p}',t\right)e^{-t\Phi(\hat{p},\hat{p}')} \tag{3.53}$$

with the same phase function Φ for $a \in S^1_{0,\mathrm{cl}}\left(\mathbb{R}^4_{p,p'} \times \mathbb{R}_t\right)$ a classical symbol. Following this and repeating the argument for Theorem 14 with $\tilde{B}_t$ replaced by $\tilde{B}_t^1$, the equations (3.32), 3.34 and 3.36 are seen to give a similar form as (3.53) for the local Bergman kernel $B_t = a\left(\hat{p},\hat{p}',t\right)e^{-t\Phi(\hat{p},\hat{p}')}$, $a \in S^1_{0,\mathrm{cl}}\left(\mathbb{R}^4_{p,p'} \times \mathbb{R}_t\right)$. Plugging this form for the local Bergman kernel into the equations (3.39), (3.42), (3.46), (3.49) and (3.51) within the proof of Theorem 1, and another use of the Melin-Sjöstrand formula gives

$$\Pi\left(x,0\right) = \int_0^\infty dt\, a\left(\hat{p},t\right)e^{itp_3 - t\Phi(\hat{p},0)} \tag{3.54}$$

for some $a \in S^1_{0,\mathrm{cl}}\left(\mathbb{R}^2_{\hat{p}} \times \mathbb{R}_t\right)$ which is the pointwise version of the Boutet de Monvel-Sjöstrand form for the parametrix at strongly pseudoconvex points.

We finally note that the reduction to the form (3.54) above is possible on account of the explicit knowledge of the model Bergman kernel B_t^0 (3.52), related to Mehler's formula for the harmonic oscillator $\boxdot_t^0$, at a strongly pseudoconvex point. At points of higher type the model kernel to contend with is less explicit, modeled on anharmonic oscillators, and one has to live with the description (1.4).

4. Pseudoconvex domains

We now consider the special case when the CR manifold is the boundary of a domain D in $\mathbb{C}^2$. Thus $D \subset \mathbb{C}^2$ is a relatively compact open subset with smooth boundary $X = \partial D$. The CR structure on the boundary is simply obtained by restriction $T^{1,0}X = T^{1,0}\mathbb{C}^2 \cap T_{\mathbb{C}}X$ of the complex tangent space of $\mathbb{C}^2$.

We fix a Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}T\mathbb{C}^2$ so that $T^{1,0}\mathbb{C}^2 \perp T^{0,1}\mathbb{C}^2$. The Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}T\mathbb{C}^2$ induces by duality, Hermitian metrics $\langle\cdot|\cdot\rangle$ on $\oplus_{0\leq p,q\leq 2}T^{*p,q}\mathbb{C}^2$, where $T^{*p,q}\mathbb{C}^2$ denote the bundles of (p,q) forms. With dv being the induced volume form on $\mathbb{C}^2$ let $(\cdot|\cdot)_D$ and $(\cdot|\cdot)_{\mathbb{C}^2}$ be the inner products on $\Omega^{0,q}(\overline{D})$ and $\Omega_0^{0,q}(\mathbb{C}^2)$ defined by

$$\begin{aligned}(f\,|\,h)_D &= \int_D \langle f\,|\,h\rangle dv, \quad f,h \in \Omega^{0,q}(\overline{D}),\\ (f\,|\,h)_{\mathbb{C}^2} &= \int_{\mathbb{C}^2} \langle f\,|\,h\rangle dv, \quad f,h \in \Omega_0^{0,q}(\mathbb{C}^2).\end{aligned} \tag{4.1}$$

Also denote by $\|\cdot\|_D$ and $\|\cdot\|_{\mathbb{C}^2}$ be the corresponding norms and by $L^2\left(D\right)$, $L^2_{(0,q)}\left(D\right)$ the corresponding spaces of square integrable functions. Let $\rho \in C^\infty(\mathbb{C}^2,\mathbb{R})$ be a defining function of X satisfying $\rho = 0$ on X, $\rho < 0$ on D and $d\rho|_X \neq 0$. This maybe further chosen to satisfy $\|d\rho\| = 1$ on X.

Let $\bar{\partial} : \Omega^{0,q}(\mathbb{C}^2) \to \Omega^{0,q+1}(\mathbb{C}^2)$ be the exterior differential operator and consider its formal adjoint

$$\begin{aligned}\overline{\partial}_f^* :&\Omega^{0,1}(\mathbb{C}^2) \to C^\infty(\mathbb{C}^2) \quad \text{satisfying}\\ (\bar{\partial}f\,|\,h)_{\mathbb{C}^2} &= (f\,|\,\overline{\partial}_f^* h)_{\mathbb{C}^2}, \quad f \in C_c^\infty(\mathbb{C}^2),\ h \in \Omega^{0,1}(\mathbb{C}^2).\end{aligned}$$

Also denote by $\bar{\partial}^* : L^2_{(0,1)}(D) \to L^2(D)$ the L^2 adjoint of $\bar{\partial}$, as an unbounded operator, with respect to $(\cdot|\cdot)_D$. The Bergman kernel of the domain is the distributional kernel $\Pi_D\left(z,z'\right) \in$

$C^{-\infty}(D\times D)$ of the orthogonal projection

$$\Pi_D : L^2(D) \to \operatorname{Ker}\bar\partial \subset L^2(D)$$

with respect to $(\cdot|\cdot)_D$. The goal of this section is to establish an asymptotic expansion for $\Pi_D(z,z)$ as $z\to x'$ approaches a point on the boundary $x'\in X$.

This shall use the relation of the Bergman kernel with the Szegő kernel of the boundary via the Poisson operator [6, Sec. 3b]. To state this, let

$$\Box_f = \bar\partial\,\bar\partial_f^* + \bar\partial_f^*\,\bar\partial : C^\infty(\mathbb{C}^2) \to C^\infty(\mathbb{C}^2)$$

denote the complex Laplace-Beltrami operator on functions and γ the operator of restriction to the boundary X. The Poisson operator is the solution operator to the Dirichlet problem on D defined by

$$P : C^\infty(X) \to C^\infty(\overline{D}) \tag{4.2}$$

$$\Box_f Pu = 0,\ \ \gamma Pu = u, \quad \forall u\in C^\infty(X). \tag{4.3}$$

Its adjoint is defined to satisfy

$$P^* : C^\infty(\overline{D}) \to C^{-\infty}(X) \tag{4.4}$$

$$(\,P^*u\,|\,v\,)_X = (\,u\,|\,Pv\,)_D,\quad u\in C^\infty(\overline{D}),\ v\in C^\infty(X) \tag{4.5}$$

The microlocal structure of P was described by Boutet de Monvel [4]. Firstly, from [4, pg. 29] the operators P,P^* extend continuously

$$\begin{aligned} P &: H^s(X) \to H^{s+\frac12}(\overline{D}),\\ P^* &: H^s(\overline{D}) \to H^{s+\frac12}(X),\ \forall s\in\mathbb{R}, \end{aligned}$$

and in particular map smooth functions onto smooth ones. Furthermore, $P^*P : C^\infty(X)\to C^\infty(X)$ is an injective continuous operator and its inverse $(P^*P)^{-1}$ is a classical elliptic pseudodifferential operator of order one on X. Its principal symbol is given by

$$\sigma_{(P^*P)^{-1}} = \sigma_{\left(2\sqrt{-\triangle_X}\right)}, \tag{4.6}$$

[22] where $\sigma_{\left(2\sqrt{-\triangle_X}\right)}$ denotes the principal symbol of the square root of the Laplace-Beltrami operator. Next, there is a continuous operator

$$G : H^s(\overline{D}) \to H^{s+2}(\overline{D}),\ \forall s\in\mathbb{R}, \quad \text{satisfying} \tag{4.7}$$

$$G\Box_f + P\gamma = I \ \text{ on } C^\infty(\overline{D}). \tag{4.8}$$

It furthermore follows from the methods of [4] that the Schwartz kernel of G satisfies the estimates

$$\left|\partial_z^\alpha\partial_w^\beta G(z,w)\right| \le C\,|z-w|^{-5-|\alpha|-|\beta|} \tag{4.9}$$

$\forall z,w\in\overline{D}$, $\alpha,\beta\in\mathbb{N}_0^4$, along the diagonal.

With

$$\begin{aligned} \Gamma^\wedge &: T^{*0,q}\mathbb{C}^2 \to T^{*0,q+1}\mathbb{C}^2\\ \Gamma^{\wedge,*} &: T^{*0,q+1}\mathbb{C}^2 \to T^{*0,q}\mathbb{C}^2, \quad \forall\Gamma\in T^{*0,1}\mathbb{C}^2, \end{aligned}$$

denoting the wedge and contraction, adjoint with respect to $\langle\cdot|\cdot\rangle$, operators, one has

$$\begin{aligned} I &= 2(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*} + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge, \quad \text{on } \Omega^{0,q}(\mathbb{C}^2),\\ \bar\partial_b &= 2\gamma(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\bar\partial P \ \text{ on } C^\infty(X). \end{aligned} \tag{4.10}$$

In using the above to describe the behavior of Π_D near a boundary point $x' \in X$ one uses the parametrix construction for the Szegő kernel Π from Theorem 1. Let (x_1, x_2, x_3) be the local coordinates on an open set $x' \in U \subset X$ on the boundary as in the proof of Theorem 1 with B the operator (3.39) therein. Since B is smoothing away the diagonal we may again assume that B is properly supported on U. It is well-known that $\Box_b$ is elliptic outside its characteristic variety $\Sigma := HX^{\perp} \subset T^*X$ given by the annihilator of the Levi distribution HX. The characteristic variety Σ carries an orientation given by $J^t d\rho$, with J^t denoting the dual complex structure on $T^*\mathbb{C}^2$, and we denote by Σ^- its negatively oriented part. By construction (3.38) we have

$$WF\,(Bu) \subset \Sigma^- \cap T^*U, \quad \forall u \in C^{-\infty}\,(U)\,.$$

Let $\tilde{U}$ be an open set of $\mathbb{C}^2$ with $\tilde{U} \cap \overline{D} = U$. We then have the next lemma.

Lemma 16. *The Schwartz kernel* $\bar{\partial} PB\,(z,x) \in C^\infty\left(\left(\tilde{U} \times U\right) \cap \left(\overline{D} \times X\right)\right)$

Proof. From (4.10), we have

$$2\gamma(\bar{\partial}\rho)^{\wedge,*}(\bar{\partial}\rho)^{\wedge}\bar{\partial}PB = \bar{\partial}_b B \in C^\infty.$$

Combining this with $\left(\bar{\partial}_f^*\bar{\partial}\right) P = 0$, $P\gamma\bar{\partial}P = \bar{\partial}P$, we have

$$\begin{aligned} 0 &= \bar{\partial}_f^*\bar{\partial}PB \\ &= \bar{\partial}_f^* P\gamma\bar{\partial}PB \\ &= \bar{\partial}_f^* P\gamma\left(I - 2(\bar{\partial}\rho)^{\wedge,*}(\bar{\partial}\rho)^{\wedge}\right)\bar{\partial}PB + C^\infty \\ &= \bar{\partial}_f^* P\gamma\left(2(\bar{\partial}\rho)^{\wedge}(\bar{\partial}\rho)^{\wedge,*}\right)\bar{\partial}PB + C^\infty. \end{aligned} \tag{4.11}$$

From (4.11), we deduce that

$$\gamma(\bar{\partial}\rho)^{\wedge}\bar{\partial}_f^* P\gamma(2(\bar{\partial}\rho)^{\wedge}(\bar{\partial}\rho)^{\wedge,*})\bar{\partial}PB \in C^\infty.$$

It is known that [22, Propsition 4.2] that

$$\gamma(\bar{\partial}\rho)^{\wedge}\bar{\partial}_f^* P : C^\infty(X, I^{0,2}T^*\mathbb{C}^2) \to C^\infty(X, I^{0,2}T^*\mathbb{C}^2)$$

is elliptic near Σ^-, where $I^{0,2}T^*\mathbb{C}^2$ is the vector bundle over $\mathbb{C}^2$ with fiber

$$I^{0,2}T^*\mathbb{C}^2 = \left\{(\bar{\partial}\rho)(z) \wedge g;\, g \in T_z^{*0,1}\mathbb{C}^2\right\}.$$

Since WF $(Bu) \subset \Sigma^- \cap T^*U$, $u \in C^{-\infty}\,(U)$, we get $\gamma(2(\bar{\partial}\rho)^{\wedge}(\bar{\partial}\rho)^{\wedge,*})\bar{\partial}PBu$ is smooth and hence

$$\gamma(2(\bar{\partial}\rho)^{\wedge}(\bar{\partial}\rho)^{\wedge,*})\bar{\partial}PB \in C^\infty$$

as required. □

In view of [22, Lemmas 4.1, 4.2], we see that P and $(P^*P)^{-1}P^*$ are smoothing away from the diagonal. Hence, they maybe replaced by continuous properly supported operators

$$\begin{aligned} L &: C_c^\infty(\tilde{U} \cap \overline{D}) \to C_c^\infty(U), \\ \hat{P} &: C_c^\infty(U) \to C_c^\infty(\tilde{U} \cap \overline{D}) \end{aligned}$$

such that

$$\begin{aligned} &L - (P^*P)^{-1}P^* \equiv 0 \mod C^\infty((U \times \tilde{U}) \cap (X \times \overline{D})), \\ &\hat{P} - P \equiv 0 \mod C^\infty((\tilde{U} \times U) \cap (\overline{D} \times X)). \end{aligned} \tag{4.12}$$

We now set

$$A := \hat{P}BL : C^\infty(\tilde{U} \cap \overline{D}) \to C^\infty(\tilde{U} \cap \overline{D}). \tag{4.13}$$

From Lemma 16, we see that

$$\bar{\partial}A \equiv 0 \mod C^\infty((\tilde{U}\times\tilde{U})\cap(\overline{D}\times\overline{D})). \tag{4.14}$$

Since the boundary X is of finite type, it was proved by Kohn [25] that there is a pseudolocal continuous operator

$$N : L^2(M) \to L^2_{(0,1)}(M)\cap \mathrm{Dom}\,\bar{\partial}^*,$$
$$N : C^\infty(\overline{D}) \to \Omega^{0,1}(\overline{D}),$$

such that

$$\Pi_D = I - \bar{\partial}^* N\bar{\partial}. \tag{4.15}$$

From (4.14) and (4.15), we deduce that

$$\Pi_D A \equiv A \qquad \mod C^\infty((\tilde{U}\times\tilde{U})\cap(\overline{D}\times\overline{D})). \tag{4.16}$$

Next we take $\tilde{U}$ small enough enough so that $z=(x,\rho)$ form local coordinates of $\tilde{U}$.

The Bergman kernel Π_D is then known to satisfy the bounds

$$\left|\partial_\rho^\gamma\partial_{\rho'}^{\gamma'}\partial_x^\alpha \Pi_D\left((x,\rho),(0,\rho')\right)\right| \leq C_{\alpha\gamma\gamma'}\left(|\rho|+|\rho'|+d^H(x)\right)^{-w.\alpha-\gamma-\gamma'-r(x')-2}, \tag{4.17}$$

$\forall\,(\alpha,\gamma,\gamma')\in\mathbb{N}_0^5$, similar to (2.5) (see [32, 35]). This gives corresponding estimates for the kernel $\gamma\Pi_D\left((x,\rho),(0,\rho')\right)$ which satisfies $\bar{\partial}_b\gamma\Pi_D=0$. Following these, we can repeat the procedure in the proof of Theorem 1 to conclude

$$(B_\infty\gamma\Pi_D)((x,0),(0,\rho')) \equiv \gamma\Pi_D((x,0),(0,\rho')) \mod C^\infty((\tilde{U}\times\tilde{U})\cap(X\times\overline{D})), \tag{4.18}$$

as with eqn. (3.45).Next $\Pi_D = P\gamma\Pi_D$ gives $P^*\Pi_D = P^*P\gamma\Pi_D$ and hence $\left(P^*P\right)^{-1}P^*\Pi_D = \gamma\Pi_D$. This combines with (4.18) to give

$$\begin{aligned}(A_\infty\Pi_D)(z,(0,\rho')) &\equiv \Pi_D(z,(0,\rho')) \mod C^\infty((\tilde{U}\times\tilde{U})\cap(\overline{D}\times\overline{D}))\\ &\text{for}\quad A_\infty := \hat{P}B_\infty L.\end{aligned} \tag{4.19}$$

We now have the following proposition.

Lemma 17. *One has*

$$\Pi_D(z,(0,\rho)) = (PQP^*)(z,(0,\rho)) + C^\infty\left(\tilde{U}\times\mathbb{R}_\rho\right)$$

for some properly supported $Q\in\hat{L}^{1+\frac{2}{r}}_{\frac{1}{r},\mathrm{cl}}$.

Proof. Denote by

$$A^* : C_c^\infty(\tilde{U}\cap\overline{D}) \to C^\infty(\tilde{U}\cap\overline{D}), \tag{4.20}$$

be the adjoint of A with respect to $(\cdot\,|\,\cdot)$. From (4.16), we have

$$A^*\Pi_D \equiv A^* \mod C^\infty((\tilde{U}\times\tilde{U})\cap(\overline{D}\times\overline{D})). \tag{4.21}$$

Thus

$$\begin{aligned}&(A_\infty\Pi_D)(z,(0,\rho'))\\ &= (A^*\Pi_D)(z,(0,\rho')) + ((A_\infty - A^*)\Pi_D)(z,(0,\rho'))\\ &\equiv A^*(z,(0,\rho')) + (R\Pi_D)(z,(0,\rho')) \mod C^\infty((\tilde{U}\times\tilde{U})\cap(\overline{D}\times\overline{D})),\end{aligned} \tag{4.22}$$

for

$$R := A_\infty - A^*$$
$$= PB_\infty(P^*P)^{-1}P^* - P(P^*P)^{-1}B^*P^* \mod C^\infty((\tilde U\times\tilde U)\cap(\overline D\times\overline D))$$
$$(4.23)\qquad = P(P^*P)^{-1}\underbrace{\Big((P^*P)B_\infty(P^*P)^{-1} - B^*\Big)}_{E:=}P^* \mod C^\infty((\tilde U\times\tilde U)\cap(\overline D\times\overline D)).$$

From (4.19) and (4.22), we get

$$(I-R)\Pi_D(z,(0,\rho')) \equiv A^*(z,(0,\rho)) \mod C^\infty((\tilde U\times\mathbb{R}_\rho)\cap(\overline D\times\mathbb{R}_\rho))$$

(4.24)
$$\Big((I-R^N)\Pi_D\Big)(z,(0,\rho')) \equiv \Big((I+R+R^2+\cdots+R^{N-1})A^*\Big)(z,(0,\rho)) \mod C^\infty((\tilde U\times\mathbb{R}_\rho)\cap(\overline D\times\mathbb{R}_\rho)).$$

$\forall N\in\mathbb{N}$.

From (3.9) and (3.11) one has $E\in \hat L^{\frac1r,1}_{\frac1r,\mathrm{cl}}$ and thus $E^N\in \hat L^{(2-N)\frac1r,N}_{\frac1r,\mathrm{cl}}$, $\forall N\in\mathbb{N}$. By Prop. 12, Prop. 11 and asymptotic summation $\exists Q\in \hat L^{1+\frac2r}_{\frac1r,\mathrm{cl}}$ such that

$$Q-(P^*P)^{-1}\Big((I+E+E^2+\cdots E^N)B_1^*\Big)\in \hat L^{1+(2-N)\frac1r,N}_{\frac1r,\mathrm{cl}}.$$

Thus for each l, $\exists N_l\in\mathbb{N}$ such that

$$PQP^* - \Big((I+R+R^2+\cdots+R^{N-1})A^*\Big)(z,(0,\rho))$$
$$(4.25)\qquad = P\left[Q-(P^*P)^{-1}\Big((I+E+E^2+\cdots+E^{N-1})B_1^*\Big)\right]P^*\in C^l((\tilde U\times\mathbb{R}_\rho)\cap(\overline D\times\mathbb{R}_\rho))$$

$\forall N\ge N_l$. Finally from the kernel estimates (4.17), for each $l\ge 0$, $\exists N_l'\in\mathbb{N}$ such that

$$(4.26)\qquad (R^N\Pi_D)((x,0),(0,\rho))\in C^\ell((\tilde U\times\mathbb{R}_\rho)\cap(\overline D\times\mathbb{R}_\rho)).$$

$\forall N\ge N_l'$. From 4.24, (4.25) and 4.26 the Lemma follows. □

Next let

$$\Box_f = \bar\partial\,\bar\partial_f^* + \bar\partial_f^*\,\bar\partial : C^\infty(\mathbb{C}^2)\to C^\infty(\mathbb{C}^2)$$

denote the complex Laplace-Beltrami operator on functions and we denote q_0 the principle symbol of $\Box_f$. Repeating the proof of [22, Prop. 7.6] one has the following.

Lemma 18. *There exists a smooth function $\phi(z,y)\in C^\infty((\tilde U\times U)\cap(\overline D\times X))$ such that*

$$\phi(x,y) = x_3 - y_3,$$
$$(4.27)\qquad \phi(z,y) = x_3 - y_3 - i\rho\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))} + O(|\rho|^2),$$
$$q_0(z,d_z\phi) \text{ vanishes to infinite order on } \rho=0,$$

where $\mathrm{Re}\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))}>0,\ z=(x,\rho)$.

For our next result we shall need an extension of the symbol spaces Definition 8. Namely one may similarly define the classes

$$(4.28)\qquad \hat S^m_{\frac1r}\left(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t\right),\ \hat S^{m,k}_{\frac1r}\left(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t\right),\ \hat S^{m,k}_{\frac1r,\mathrm{cl}}\left(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t\right)$$
$$(4.29)\qquad \hat S^m_{\frac1r}\left(\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{R}_t\right),\ \hat S^{m,k}_{\frac1r}\left(\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{R}_t\right),\ \hat S^{m,k}_{\frac1r,\mathrm{cl}}\left(\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{R}_t\right)$$

as functions depending on additional ρ or ρ,ρ' variables. The additional variables appear in a fashion similar to the x_3, y_3 variables in the symbolic estimates and expansions. That is the equations 3.3, (3.4) and (3.5) are replaced by

$$\left|\partial_x^\alpha \partial_\rho^{\alpha_4} \partial_y^\beta \partial_t^\gamma a(x,y,t)\right| \le C_{N,\alpha\beta\gamma} \langle t\rangle^{m-\gamma+\frac{1}{r}\left(|\hat{\alpha}|+|\hat{\beta}|\right)+\alpha_3+\alpha_4+\beta_3} \frac{\left(1+\left|t^{\frac{1}{r}}\hat{x}\right|+\left|t^{\frac{1}{r}}\hat{y}\right|\right)^{N(\alpha,\beta,\gamma)}}{\left(1+\left|t^{\frac{1}{r}}\hat{x}-t^{\frac{1}{r}}\hat{y}\right|\right)^{-N}}, \quad \forall (x,y,t,N) \in \mathbb{R}^6_{x,y} \times \mathbb{R}_t \times \mathbb{N}. \tag{4.30}$$

$$\hat{S}^{m,k}_{\frac{1}{r}} := \bigoplus_{p+q+p'\le k} (tx_3)^p (t\rho)^q (ty_3)^{p'} \hat{S}^m_{\frac{1}{r}}, \quad \forall (m,k) \in \mathbb{R}\times\mathbb{N}_0. \tag{4.31}$$

$$a(x,y,t) - \sum_{j=0}^N \sum_{p+q+p'\le j} t^{m-\frac{1}{r}j} (tx_3)^p (t\rho)^q (ty_3)^{p'} a_{jpp'}\left(t^{\frac{1}{r}}\hat{x}, t^{\frac{1}{r}}\hat{y}\right) \in \hat{S}^{m-(N+1)\frac{1}{r},N+1}_{\frac{1}{r}} \tag{4.32}$$

in defining $\hat{S}^m_{\frac{1}{r}}(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t)$, $\hat{S}^{m,k}_{\frac{1}{r}}(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t)$, $\hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t)$ respectively. And similarly for the classes (4.29).

We now have the following

Lemma 19. *Let $H = h^L \in \hat{L}^{m,k}_{\frac{1}{r},\mathrm{cl}}$ be an operator in the class 9 with distribution kernel*

$$H(x,y) = h^L(x,y) = \int_0^\infty e^{i(x_3-y_3)t} h(x,y,t)\,dt.$$

Then there exists $\alpha(z;y,t) \in \hat{S}^{m,k}_{\frac{1}{r},\mathrm{cl}}(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t)$, with $\alpha(x,0;y,t) = h(x,y,t)$, such that

$$\Lambda(z,y) = \int_0^\infty e^{i\phi(z,y)t}\alpha(z,y,t)\,dt \quad \textit{with}$$
$$(PH-\Lambda)((0,\rho),y) \in C^\infty(\mathbb{R}_\rho \times U).$$

Proof. Denote the Riemannian metric on $T\mathbb{C}^2$, induced from the Hermitian metric $\langle\cdot|\cdot\rangle$, by

$$g = \sum_{j,k=1}^4 g_{j,k}(z) dx_j \otimes dx_k, \quad dx_4 = d\rho$$

and let $(g_{j,k}(z))^{-1}_{1\le j,k\le 4} = \left(g^{j,k}(z)\right)_{1\le j,k\le 4}$ be the inverse metric on $T^*\mathbb{C}^2$. In the local coordinates $z=(x,\rho)$ chosen one has

$$\Box_f = -\frac{1}{2}\Big(a^{4,4}(z)\frac{\partial^2}{\partial\rho^2} + 2\sum_{j=1}^3 a^{4,j}(z)\frac{\partial^2}{\partial\rho\partial x_j} + T(\rho)\Big) + \text{first order}, \quad \text{where} \tag{4.33}$$

$$T(\rho) = \sum_{j,k=1}^3 a^{j,k}(z)\frac{\partial^2}{\partial x_j \partial x_k}. \tag{4.34}$$

Furthermore, $T(0) - \triangle_X$ is a first order operator on the boundary with

$$\begin{aligned} &a^{4,4}(x) = 1, \\ &a^{4,j}(x) = 0, \quad j = 1,\ldots,3. \end{aligned} \tag{4.35}$$

From the above (4.27), (4.33), (4.34) and (4.35) we now compute

$$\Box_f\left[\int_0^\infty e^{i\phi(z,y)t}\beta(x,y,t)dt\right]\equiv\int_0^\infty e^{i\phi(z,y)t}\left[\frac{t}{2}\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))}\partial_\rho\beta+L\beta\right]dt \quad (4.36)$$
$$\mod C^\infty((\tilde U\times\tilde U)\cap(\overline{D}\times X)),\quad \text{where}$$
$$L=\left(t\rho b_1\left(z,y\right)+b_2\left(z,y\right)\right)\partial_\rho+L_{2,x}+tL_{1,x}$$

for some smooth $t-$independent functions b_1, b_2 and second/first order differential operators $L_{2,x}$, $L_{1,x}$ respectively in the (x_1, x_2, x_3) variables. It is easy to check that the above maps

$$\frac{\rho}{t}L:\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}\to\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}$$
$$\left(\frac{\rho}{t}L\right)^N:\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}\to\rho^{N-k}\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}+\hat S^{m-\frac{N}{r},k+N}_{\frac{1}{r},\mathrm{cl}},\quad N\geq k.$$

Next setting

$$h_1(z,y,t):=h(x,y,t)-\frac{2\rho}{t\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))}}Lh$$
$$=h(x,y,t)-\frac{2\rho}{t\sqrt{-\sigma_{\triangle_X}(x,(0,0,1))}}\left(L_{2,x}+tL_{1,x}\right)h\in\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}$$

and following (4.36) one computes

$$\Box_f\left[\int_0^\infty e^{i\phi(z,y)t}h_1(z,y,t)dt\right]\equiv\int_0^\infty e^{i\phi(z,y)t}r_1(z,y,t)dt\quad\mod C^\infty((\tilde U\times\tilde U)\cap(\overline{D}\times X)),$$
$$r_1\in\rho^{1-k}\hat S^{m+2,k}_{\frac{1}{r},\mathrm{cl}}+\hat S^{m+2-\frac{1}{r},k+1}_{\frac{1}{r},\mathrm{cl}},\quad 1\geq k.$$

Continuing in this way, we can find $h_N(z,y,t)\in\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}$ such that

$$h_{N+1}-h_N\in\rho^{N-k}\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}+\hat S^{m-\frac{N}{r},k+N}_{\frac{1}{r},\mathrm{cl}}\qquad\text{and}$$
$$\Box_f\left[\int_0^\infty e^{i\phi(z,y)t}h_N(z,y,t)dt\right]\equiv\int_0^\infty e^{i\phi(z,y)t}r_N(z,y,t)dt\quad\mod C^\infty((\tilde U\times\tilde U)\cap(\overline{D}\times X))$$
$$r_N\in\rho^{N-k}\hat S^{m+2,k}_{\frac{1}{r},\mathrm{cl}}+\hat S^{m+2-\frac{N}{r},k+N}_{\frac{1}{r},\mathrm{cl}},\quad N\geq k.$$

By asymptotic summation we can find $\alpha:=h_1+\sum_{N=1}^{\infty}\left(h_{N+1}-h_N\right)\in\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}$ satisfying

$$\Box_f\left[\underbrace{\int_0^\infty e^{i\phi(z,y)t}\alpha(z,y,t)dt}_{=\Lambda}\right]\equiv\int_0^\infty e^{i\phi(z,y)t}r_\infty(z,y,t)dt\quad\mod C^\infty((\tilde U\times\tilde U)\cap(\overline{D}\times X))$$
$$r_\infty\in\rho^\infty\hat S^{m+2,k}_{\frac{1}{r},\mathrm{cl}}+\hat S^{m+2,k,-\infty}_{\frac{1}{r},\mathrm{cl}}$$
$$\gamma\Lambda=H.$$

Finally we apply the Green's operator G (4.7) to both sides of the above equation to get

$$G\Box_f\left(\left(0,\rho\right),y\right)=\int dw\int_0^\infty dtG\left(\left(0,\rho\right),w\right)e^{i\phi(w,y)t}r_\infty(w,y,t)dt\,.$$

Writing $w=(u_1,u_2,u_3,\rho')$ and repeated integration by parts using $e^{iu_3t}=\frac{1}{it}\partial_{u_3}e^{iu_3t}$, (4.9) and (4.27) gives the above

$$G\Box_f\left((0,\rho),y\right)\in C^\infty\left(\mathbb{R}_\rho\times U\right)\qquad\text{and hence}$$
$$\left[PH-\Lambda\right]\left((0,\rho),y\right)\in C^\infty\left(\mathbb{R}_\rho\times U\right)$$

from (4.8) as required. □

Similarly, as Lemma 18, we can find $\Phi(z,w)\in C^\infty((\tilde U\times\tilde U)\cap(\overline{D}\times\overline{D}))$ such that $\Phi(z,y)=\phi(z,y)$ and

$$\begin{aligned}&\Phi(z,w)=\phi(z,y)-i\rho'\sqrt{-\sigma_{\triangle_X}(y,(0,0,1))}+O(|\rho'|^2),\\&q_0(w,-\overline{d}_w\Phi)\text{ vanishes to infinite order on }\rho'=0.\end{aligned}\tag{4.37}$$

A similar argument to Lemma 19 then gives the following.

Lemma 20. *Let $H=h^L\in\hat L^{m,k}_{\frac{1}{r},\mathrm{cl}}$ be an operator in the class 9 with distribution kernel*

$$H(x,y)=h^L\left(x,y\right)=\int_0^\infty e^{i(x_3-y_3)t}h\left(x,y,t\right)dt.$$

Then there exists $\alpha(z;w,t)\in\hat S^{m,k}_{\frac{1}{r},\mathrm{cl}}\left(\mathbb{C}^2\times\mathbb{R}^3\times\mathbb{R}_t\right)$, with $\alpha\left(x,0;y,0,t\right)=h\left(x,y,t\right)$, such that

$$\Lambda(z,w)=\int_0^\infty e^{i\Phi(z,w)t}\alpha\left(z,w,t\right)dt\qquad\textit{with}$$
$$\left(PHP^*-\Lambda\right)\left((0,\rho),y\right)\in C^\infty\left(\mathbb{R}_\rho\times\mathbb{R}_{\rho'}\right).$$

Finally from the above we can prove one of our the main theorems Theorem 2. Setting $z=(0,\rho)$, $w=(0,\rho)$ in 17 and the above 20 gives

$$\Pi_D((0,\rho),(0,\rho))=\int_0^\infty e^{i\Phi((0,\rho),(0,\rho))t}\alpha((0,0,\rho)),(0,0,\rho);t)dt+C^\infty\left(\mathbb{R}_\rho\right).$$

for $\alpha\in\hat S^{1+\frac{2}{r}}_{\frac{1}{r},\mathrm{cl}}(\tilde U\times\tilde U\times\mathbb{R}_+)$. Plugging the classical symbolic expansion for α into the above and using

$$\Phi((0,\rho),(0,\rho))=-2i\rho\sqrt{-\sigma_{\triangle_X}(0,(0,0,1))}+O(|\rho|^2)$$

gives

$$\Pi_D((0,\rho),(0,\rho))=\sum_{j=0}^N\frac{1}{\left(-\rho\right)^{2+\frac{2}{r}-\frac{1}{r}j}}a_j+\sum_{j=0}^N b_j\left(-\rho\right)^j\log\left(-\rho\right)+O\left(\left(-\rho\right)^{\frac{N-2-2r}{r}}\right),$$

$\forall N\in\mathbb{N}$, proving Theorem 2.

5. S^1 invariant case

In this section we investigate the Szegő kernel parametrix in the circle invariant case obtaining a more concrete version of our main theorem Theorem 1.

Thus we now assume that X is equipped with a CR S^1-action which is transversal; that is the generator T of the S^1action satisfies

$$\left[T,C^\infty\left(T^{1,0}X\right)\right]\subset C^\infty\left(T^{1,0}X\right)\tag{5.1}$$

$$\mathbb{C}\left[T\right]\oplus T^{1,0}X\oplus T^{0,1}X=TX\otimes\mathbb{C}.\tag{5.2}$$

Denote by $s_x := |S_x|$ cardinality of the stabilizer $S_x := \left\{e^{i\theta} \in S^1 | e^{i\theta}x = x\right\}$ of the point $x \in X$ with respect to the circle action. For a locally free circle action as above the function $x \mapsto s_x$ is also upper semi-continuous with the compactness of X implying $s := \max_{x\in X} s_x < \infty$. We further set $s_0 := \min_{x\in X} s_x$, $X_{i,j} := \{x \in X | s_x = j,\, r_x = i\}$ to obtain a decomposition of the manifold $X = \bigcup_{i=1,j=2}^{s,r} Y_{i,j}$ where each $X_{\leq i,\leq j} := \bigcup_{i'=1,j'=2}^{i,j} Y_{i,j}$ is open and $X_{s_0,\leq r} \subset X$ is dense. The m-th Fourier mode of the Szegő kernel $\Pi_m(x,x')$, $m \in \mathbb{Z}$, taken with respect to an S^1-invariant volume form μ, is now a smooth function on the product. What corresponds to the singularity in (1.4) is its asymptotic behavior as $m \to \infty$. This is described below and is the main theorem of this section.

Theorem 21. *Let X be a compact pseudoconvex three dimensional CR manifold of finite type admitting a transversal, CR circle action. The m-th Fourier mode of the Szegő kernel has the pointwise expansion on diagonal*

$$\Pi_m(x,x) = \phi_m(s_x)\left[m^{2/r_x}\sum_{j=0}^{N} c_j(x)\, m^{-2j/r_x} + O\left(m^{-2N/r_x}\right)\right], \quad \forall N \in \mathbb{N}, \tag{5.3}$$

as $m \to \infty$. Here each c_j is a smooth function on X, with the leading term $c_0 = \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0) > 0$ given in terms of certain model Bergman kernels on the Levi-distribution HX at x and the phase factor $\phi_m(s_x) := \sum_{l=0}^{s_x-1} e^{i\frac{2\pi l m}{s_x}} = \begin{cases} s_x; & s_x \mid m, \\ 0; & s_x \nmid m.\end{cases}$

We first begin with some requisite CR geometry in the circle invariant case.

5.1. **S^1 invariant CR geometry.** Let $HX := \mathrm{Re}\left(T^{1,0}X \oplus T^{0,1}X\right) \subset TX$ be the real Levi distribution. The volume form μ is further assumed to be S^1 invariant. We let $h^{T^{1,0}X}$ be an S^1 invariant Hermitian metric on $T^{1,0}X$ and denote by $h^{T^{0,1}X}$ the invariant Hermitian metric on $T^{0,1}X$. This gives one h^{TX} on $TX \otimes \mathbb{C}$ for which (5.1) is an orthogonal decomposition with $|T| = 1$. We also denote by g^{TX} the induced Riemannian metric on the real tangent space and by $\langle , \rangle$ the corresponding $\mathbb{C}$-bilinear form on $TX \otimes \mathbb{C}$. It is easy to see that $h^{T^{0,1}X}$ may be chosen so that the volume form μ arises as the Riemannian volume of such an invariant metric.

Such a S^1-invariant CR manifold is locally the unit circle bundle of a Hermitian, holomorphic line bundle $\left(L, h^L\right)$

$$X = S^1 L \xrightarrow{\pi} Y; \tag{5.4}$$

on a complex manifold Y, $\dim_{\mathbb{C}} Y = 1$. To describe the equivalence, choose a local hypersurface $Y \subset X$ through a given point $x \in X$ transversal to generator $T \pitchfork TY$ satisfying $T_x Y = (HX)_x$. The map $TY \hookrightarrow TX \to TX/\mathbb{R}[T] \cong HX$ is then an isomorphism inducing a integrable complex structure on TY and a corresponding Hermitian metric $h^{T^{1,0}Y}$ on its complex tangent space from $h^{T^{1,0}X}$. We choose $S^1 \times Y =: X_0 \to Y$ to be the trivial circle bundle with the map $\iota : X_0 \to X$; $\iota\left(y, e^{i\theta}\right) = y e^{i\theta}$ being a local diffeomorphism between collar neighborhoods

$$\begin{aligned} &\iota : \left(-\frac{\pi}{s_x}, \frac{\pi}{s_x}\right) \times Y \xrightarrow{\sim} NY \subset X \\ &s_x := |S_x| \end{aligned} \tag{5.5}$$

of the zero section of X_0 and $Y \subset X$. The Levi-distribution then defines a horizontal distribution on X_0 giving corresponding connections ∇^L on X_0 and the associated Hermitian line bundle $L := \mathbb{C} \times Y \to Y$ corresponding to the trivial representation of S^1. By the integrability condition the curvature of the corresponding connection is a $(1,1)$ form on Y and hence the

$(0,1)$ part of the connection prescribes a holomorphic structure on L. It is now clear that X_0 it the unit circle bundle of L with ι being the required CR isomorphism by definition. We also note that pseudoconvexity of the CR structure corresponds to semi-positivity of the curvature R^L of ∇^L.

We may also obtain a local coordinate expression for the CR structure. To this end, start with a local orthonormal basis $\{e_1, e_2 = Je_1\}$ of $T_xY = (HX)_x$. Using the exponential map obtain a geodesic coordinate system on a geodesic ball $B_{2\varrho}(x)$ centered at $x \in Y$. The point x corresponding to point $(y, \mathtt{1}_y) \in (S^1L)_y$ in the fiber above y, one may parallel transport $\mathtt{1}_y$ along geodesic rays in Y to obtain a local orthonormal frame $\mathtt{1}$ for L. In such a parallel frame the connection on the tensor product is of the following form

$$\begin{aligned}
\nabla^{\Lambda^{0,*}\otimes L^m} &= d + a^{\Lambda^{0,*}} + ma^L \\
a_j^{\Lambda^{0,*}} &= \int_0^1 d\rho \left(\rho y^k R_{jk}^{\Lambda^{0,*}}(\rho x)\right) \\
a_j^L &= \int_0^1 d\rho \left(\rho y^k R_{jk}^L(\rho x)\right)
\end{aligned} \tag{5.6}$$

in terms of the respective curvatures of $\nabla^{T^{1,0}X}$, ∇^L see [31, Sec. 4]. The connection one form may further be written

$$\begin{aligned}
a_1^L + ia_2^L &= \partial_z\varphi \\
&= \frac{1}{r_x}\bar{z}\left(\sum_{|\alpha|=r_x-2} R_\alpha^L y^\alpha\right) + O\left(y^{r_x}\right)
\end{aligned} \tag{5.7}$$

in terms of a potential function and a Taylor expansion with the tensor R_α^L denoting the first non-vanishing jet of the curvature at x. In some local coordinate system $(\theta, y) \in \left(-\frac{\pi}{s_x}, \frac{\pi}{s_x}\right) \times B_{2\varrho}(x)$, $s_x := |S_x|$, on an open set $U \subset X$ the CR structure on X is then locally given by

$$\begin{aligned}
T^{1,0}X &= \mathbb{C}\left[\partial_z + i\left(\partial_z\varphi\right)\partial_\theta\right], \\
T &= \partial_\theta,
\end{aligned} \tag{5.8}$$

$z = y_1 + iy_2$, by construction. A trivialization/coordinate system in which the CR looks as above (5.8) is referred to as a *BRT trivialization* [2, Thm II.1].

Following its local description as a unit circle bundle, several notions/formulas from complex geometry carry over to the S^1-invariant CR geometry of X. Firstly, the tangential CR operator $\bar{\partial}_b$ locally corresponds to Dolbeault differential $\bar{\partial}$ on Y (5.4) under pullback $\bar{\partial}_b(\pi^*\omega) = \pi^*\bar{\partial}\omega$, $\omega \in \Omega^{0,*}(Y)$ and similarly for their adjoints. An analog of the Chern connection then exists on $T^{0,1}X$. This is the unique (S^1-invariant) connection $\nabla^{T^{0,1}X}$ compatible with $\langle , \rangle$ satisfying $\nabla_T^{T^{0,1}X} = \mathcal{L}_T$ and whose $(0,1)$-component agrees with the tangential CR operator

$$\nabla_U^{T^{0,1}X} = i_U\bar{\partial}_b, \quad U \in T^{0,1}X.$$

Locally, $\nabla_T^{T^{0,1}X}$ is just the pullback of the Chern connection $\nabla^{T^{0,1}Y}$ from Y. Complex conjugation then defines a connection $\nabla^{T^{1,0}X}$ on $\left(T^{1,0}X\right)^*$. We denote by the same notation dual connections on $T^{1,0}X/T^{0,1}X$ and set $\tilde{\nabla}^{TX} := d \oplus \nabla^{T^{1,0}X} \oplus \nabla^{T^{0,1}X}$ to be a connection on $TX \otimes \mathbb{C}$ with d denoting the trivial connection on $\mathbb{C}[T]$ under the decomposition (5.1). The connection $\tilde{\nabla}^{TX}$ preserves the real tangent space $TX \subset TX \otimes \mathbb{C}$ and we denote by $\mathcal{T}$ its torsion. The torsion $\mathcal{T}$ maps $T^{1,0}X \otimes T^{1,0}X$ into $T^{1,0}X$ (respectively for $T^{0,1}X$), $T^{1,0}X \otimes T^{0,1}X$ into $\mathbb{C}[T]$

and vanishes on $HX \otimes \mathbb{C}[T]$. Indeed its components involving the generator T are

$$\mathcal{T}(.,T) = \mathcal{T}(T,.) = 0$$
$$\left\langle \mathcal{T}\left(U,\bar{V}\right),T\right\rangle = i\mathscr{L}\left(U,\bar{V}\right), \quad U,V \in T^{1,0}X.$$

Next with ∇^{TX} being the Levi-Civita connection and P^{HX} being the horizontal projection onto HX, define a new connection on TX via $\nabla^{HX} := d \oplus P^{HX}\nabla^{TX}$ with respect to the decomposition $TX = \mathbb{R} \oplus HX$. Locally; ∇^{HX} is the pullback of the Levi-Civita connection ∇^{TY} from Y (5.4). The torsion $\mathcal{T}^{HX}$ of ∇^{HX} thus has

$$\mathcal{T}^{HX}(U,V) = i\mathscr{L}\left(U,\bar{V}\right)T, \quad U,V \in HX,$$

as its only non-vanishing component. The difference $S := \tilde{\nabla}^{TX} - \nabla^{HX}$ maybe computed

$$\langle S(U)V,W\rangle = 0, \left\langle S(U)\bar{V},W\right\rangle = -\frac{1}{2}\left\langle \mathcal{T}\left(U,\bar{V}\right),W\right\rangle; \tag{5.9}$$

$$\left\langle S(T)U,\bar{V}\right\rangle = \frac{i}{2}\mathscr{L}\left(U,\bar{V}\right), \left\langle S(U)\bar{V},T\right\rangle = -\left\langle S(U)T,\bar{V}\right\rangle = 0, \tag{5.10}$$

$U,V,W \in T^{1,0}X$, in terms of the torsion and Levi forms.

One next defines the Bismut connection ∇^B on TX via $\nabla^B := \nabla^{HX} + S^B$;

$$\left\langle S^B(T)U,\bar{V}\right\rangle := \frac{i}{2}\mathscr{L}\left(U,\bar{V}\right)$$
$$\left\langle S^B(U)V,W\right\rangle := 0,$$

$U,V,W \in T^{1,0}X$. Its horizontal projection $P^{HX}\nabla^B$ locally agrees with the pullback of the Bismut connection of Y [29, Def. 1.2.9]. The connection ∇^B preserves the decomposition (5.1) and hence induces a connection on $T^{1,0}X$, $(T^{0,1}X)^*$ and their exterior powers which we again denote by ∇^B. Finally set

$$\nabla^{B,\Lambda^{0,*}} := \nabla^B + \langle S(.)w,\bar{w}\rangle, \tag{5.11}$$

$w \in T^{1,0}X$, $|w| = 1$. We now define the Clifford multiplication endomorphism

$$c : (TX)^* \to \text{End}\left(\Lambda^* T^{0,1}X\right)$$
$$c(v) := \sqrt{2}\left(v^{1,0} \wedge -i_{v^{0,1}}\right), \quad \forall v \in (HX)^*,$$
$$c(\theta)\omega := \pm\omega, \quad \forall \omega \in \Lambda^{\text{even/odd}}T^{0,1}X.$$

Next, the Kohn-Dirac operator

$$\begin{aligned} D_b &:= \sqrt{2}\left(\bar{\partial}_b + \bar{\partial}_b^*\right) \\ &= c \circ \nabla^{B,\Lambda^{0,*},H} \end{aligned} \tag{5.12}$$

maybe written as the composition of Clifford multiplication with the horizontal component of the Bismut connection

$$\nabla^{B,\Lambda^{0,*},H} := \pi^H \circ \nabla^{B,\Lambda^{0,*}} : C^\infty\left(X;\Lambda^{0,*}\right) \to C^\infty\left(X;H^*X \otimes \Lambda^{0,*}\right)$$

(cf. [29, Thm 1.4.5]). We then have the following Lichnerowicz formula.

Theorem 22. *The Kohn Laplacian* (5.12) *satisfies*

$$2\Box_b = D_b^2 = \underbrace{\left(\nabla^{B,\Lambda^{0,*},H}\right)^* \nabla^{B,\Lambda^{0,*},H}}_{\Delta^{B,\Lambda^{0,*}} :=} + \frac{1}{2}r^X \bar{w} i_{\bar{w}} + \mathscr{L}(w,\bar{w})\left[2\bar{w}i_{\bar{w}} - 1\right] i\mathcal{L}_T \tag{5.13}$$

where $\frac{1}{2}r^X := R^{T^{1,0}X}(w,\bar{w})$ *for* $w \in T^{1,0}X$, $|w| = 1$.

Proof. On account of S^1 invariance, both sides of the formula commute with the generator $\mathcal{L}_T$. It then suffices to check their equality on sections that are locally of the form $s\left(y, e^{i\theta}\right) = s_0\left(y\right)e^{im\theta}$, $s_0 \in \Omega^{0,*}\left(Y\right)$, eigenspaces of $\mathcal{L}_T$, on the unit circle bundle (5.4). Further one locally has the correspondence

$$C^\infty_m\left(X\right) \cong C^\infty\left(Y; L^m\right) \tag{5.14}$$

between sections on X that are m-eigenspaces of $\mathcal{L}_T$ and sections of L^m on Y for example. Under this correspondence D_b, $\nabla^{B,\Lambda^{0,*},H}$ act by the Dolbeault-Dirac operators and Bismut connection on tensor powers L^m. The Lie derivative $\mathcal{L}_T$ acts by multiplication by m while the curvature of L is identified with the Levi form by definition. The curvatures r^X, $R^{T^{1,0}X}$ are pulled back from the scalar and Chern curvatures on Y while the horizontal components of Θ agree with the components of the corresponding tensor on Y. With these identifications, the Lichnerowicz formula (5.13) is locally the same as [29, Thm 1.4.7]. □

The (horizontal) Bochner Laplacian appearing in (5.13) can be written

$$\Delta^{B,\Lambda^{0,*}} := \sum_{j=1}^{2m}\left[-\left(\nabla^{B,\Lambda^{0,*}}_{U_j}\right)^2 s + \left(\text{div}U_j\right)\nabla^{B,\Lambda^{0,*}}_{U_j} s\right],$$

in terms of a real orthonormal basis $\{U_j\}_{j=1}^{2m}$ for HX. It is a sub-Riemannian Laplacian associated to the metric (bracket generating) distribution HX and the natural Riemannian volume see [31, Sec. 2]. From the above expression it is clearly a hypoelliptic operator of Hörmander type [19]. It satisfies a hypoelliptic estimate: $\exists C > 0$ such that

$$\left\langle \Delta^{B,\Lambda^{0,*}} u, u\right\rangle + \|u\|^2 \geq C\,\|u\|^2_{H^{1/r}} \tag{5.15}$$

$\forall u \in \Omega^{0,*}\left(X\right)$ (see [39]). Here r is the maximal type of a point on X. This is also referred to as the step or degree of non-holonomy of the Levi distribution HX in sub-Riemannian geometry.

5.2. **Spectral gap and closed range.** In this subsection we show that the spectral gap property for the Kohn Laplacian as well as the closedness of the range for $\bar\partial_b$ in three dimensions automatically follow in the circle invariant case.

First, each $\Omega^{0,q}\left(X\right)$ has an orthogonal decomposition (with respect to the chosen invariant metric) into the Fourier modes for the S^1 action

$$\begin{aligned} \Omega^{0,q}\left(X\right) &= \oplus_{m\in\mathbb{Z}}\Omega^{0,q}_m\left(X\right) \qquad \text{where} \\ \Omega^{0,q}_m\left(X\right) &:= \left\{\omega \in \Omega^{0,q}_m\left(X\right) | \mathcal{L}_T\omega = im\omega\right\}. \end{aligned} \tag{5.16}$$

Indeed the orthogonal projection of any $\omega \in \Omega^{0,q}\left(X\right)$ onto its mth Fourier mode is given by

$$\begin{aligned} P_m :&\Omega^{0,q}\left(X\right) \to \Omega^{0,q}_m\left(X\right) \\ \left(P_m\omega\right)(x) &:= \int_{S^1} d\theta\,\omega\left(x.e^{i\theta}\right)e^{-im\theta} \end{aligned} \tag{5.17}$$

Since the S^1 action is assumed to be CR, we have $\left[\bar\partial_b, T\right] = 0$. Hence the tangential CR operator preserves the Fourier modes (5.16) $\left[\bar\partial_b, P_m\right] = 0$, $\bar\partial_b : \Omega^{0,q}_m\left(X\right) \to \Omega^{0,q+1}_m\left(X\right)$. One may then define the m-th equivariant Kohn-Rossi cohomology

$$H^*_{b,m}\left(X\right) := \frac{\ker\left[\bar\partial_b : \Omega^{0,q}_m\left(X\right) \to \Omega^{0,q+1}_m\left(X\right)\right]}{\text{Im}\left[\bar\partial_b : \Omega^{0,q}_m\left(X\right) \to \Omega^{0,q+1}_m\left(X\right)\right]}.$$

Its adjoint $\bar{\partial}_b^*$ with respect to the invariant metric further commutes $\left[T, \bar{\partial}_b^*\right] = 0$ with the generator. A similar commutation then applies to the Kohn Laplacian $[T, \Box_b] = 0$, thus giving a decomposition

$$\Box_b = \oplus_{q=0,1} \oplus_{m \in \mathbb{Z}} \Box_{b,m}^q$$
$$\Box_{b,m}^q := \Box_b|_{\Omega_m^{0,q}} : \Omega_m^{0,q}(X) \to \Omega_m^{0,q}(X).$$

The equivariant version of the Hodge theorem holds

$$\ker\left(\Box_{b,m}^q\right) = H_{b,m}^*(X) \tag{5.18}$$

[9, Thm 3.7].

The S^1-invariant operator $\Box^X := -T^2 + \Box_b$ is elliptic and self-adjoint with respect to $\langle , \rangle$, dx. There is thus a complete orthonormal basis $\varphi_{j,m}^q$, $j = 0, 1, \ldots$, $m \in \mathbb{Z}$, of $L^2\left(X; \Lambda^{0,q}\right)$ consisting of joint eigenvectors $\Box^X \varphi_{j,m}^q = \lambda_{j,m}^q \varphi_{j,m}^q$; $T\varphi_{j,m}^q = im\varphi_{j,m}^q$ with

$$0 \leq m^2 \leq \lambda_{0,m}^q \leq \lambda_{1,m}^q \leq \ldots \nearrow \infty,$$

$\forall m \in \mathbb{Z}$. Thus, for each fixed $m \in \mathbb{Z}$ the set $\left\{\varphi_{j,m}^q\right\}_{j=0}^\infty$ is then an orthonormal basis of $L_m^2\left(X; \Lambda^{0,q}\right) := \left\{\omega \in L_m^2\left(X; \Lambda^{0,q}\right) | T\omega = im\omega\right\}$ of eigenvectors for $\Box_{b,m}^q$. Note that $\Box_{b,m}^q$ is an unbounded operator on $L_m^2\left(X; \Lambda^{0,q}\right)$ with domain $\text{Dom}\left(\Box_{b,m}^q\right) = \left\{\omega \in L_m^2\left(X; \Lambda^{0,q}\right) | \Box_{b,m}^q \omega \in L_m^2\left(X; \Lambda^{0,q}\right)\right\}$ Similarly one also has

$$\text{Dom}\left(\bar{\partial}_b\right) = \left\{\omega \in L^2\left(X; \Lambda^{0,*}\right) | \bar{\partial}_b \omega \in L^2\left(X; \Lambda^{0,*}\right)\right\},$$
$$\text{Dom}\left(\bar{\partial}_b^*\right) = \left\{\omega \in L^2\left(X; \Lambda^{0,*}\right) | \bar{\partial}_b^* \omega \in L^2\left(X; \Lambda^{0,*}\right)\right\},$$

as unbounded operators on L^2.

We now have the following spectral gap property for $\Box_{b,m}^q$.

Proposition 23. *There exists a constants $c_1, c_2 > 0$ such that*

$$\textit{Spec}\left(\Box_{b,m}^0\right) \subset \{0\} \cup \left[c_1 |m|^{2/r} - c_2, \infty\right) \tag{5.19}$$

$$\textit{Spec}\left(\Box_{b,m}^1\right) \subset \left[c_1 |m|^{2/r} - c_2, \infty\right) \tag{5.20}$$

for each $m \in \mathbb{Z}$.

Proof. The Lichnerowicz formula (5.13) when restricted for the restriction to the m-th Fourier mode $\Omega_m^{0,1}(X)$ gives

$$2\Box_{b,m}^1 = \underbrace{\Delta_m^{B,\Lambda^{0,1}}}_{=: \Delta^{B,\Lambda^{0,*}}|_{\Omega_m^{0,1}(X)}} + \frac{1}{2} r^X + \mathscr{L}(w, \bar{w})\, m$$

in terms of an invariant orthonormal frame $w \in T^{1,0}X$. Following the subelliptic estimate (5.15) and pseudoconvexity gives

$$\begin{aligned} 2\lambda_{j.m}^q = \left\langle 2\Box_{b,m}^1 \varphi_{j,m}, \varphi_{j,m}\right\rangle &= \left\langle \left[\Delta_m^{B,\Lambda^{0,1}} + \frac{1}{2} r^X + \mathscr{L}(w, \bar{w})\, m\right] \varphi_{j,m}, \varphi_{j,m}\right\rangle \\ &\geq c_1 \left\|\varphi_{j,m}\right\|_{H^{1/r}}^2 - c_2 \left\|\varphi_{j,m}\right\|^2 \\ &= c_1 \left\|T^{1/r} \varphi_{j,m}\right\|^2 - \left\|\varphi_{j,m}\right\|^2 = c_1 m^{2/r} - c_2 \end{aligned}$$

for $m \geq 0$. Thus $\text{Spec}\left(\Box_{b,m}^1\right) \subset \left[c_1 m^{2/r} - c_2, \infty\right)$ for $m \geq 0$ proving the second part (5.20). Similarly one proves $\text{Spec}\left(\Box_{b,m}^0\right) \subset \left[c_1 |m|^{2/r} - c_2, \infty\right)$ for $m \leq 0$. The first part (5.19) follows on noting that the Dirac operator $D_{b,m} := D_b|_{\Omega^{0,*}_m(X)}$ gives an isomorphism between the non-zero eigenspaces of $\Box^0_{b,m}$ and $\Box^1_{b,m}$ for each $m \in \mathbb{Z}$. $\Box$

It follows immediately from (5.18), (5.20) that $H^1_{b,m}(X) = 0$ for $m \gg 0$. This gives

$$\begin{aligned} d_m := \dim H^0_{b,m}(X) &= \phi_m(s_0) \int_X \mathrm{Td}_b\left(T^{1,0}X\right) e^{-md\theta} \wedge \theta \\ &= \phi_m(s_0)\, m \left[\int_X d\theta \wedge \theta\right] + O(1) \end{aligned}$$

following the index theorem of [9, Cor. 1.13], where $\mathrm{Td}_b(T^{1,0}X)$ denotes the tangential/invariant Todd class of $T^{1,0}X$ [9, Sec. 2.3] and $\theta(T) = 1$, $\theta(HX) = 0$.

As another application of Proposition 23 one has the estimate $\mathrm{Spec}^+\left(D^2_{b,m}\right) \subset \left[c_1 |m|^{2/r} - c_2, \infty\right)$ for each $m \in \mathbb{Z}$ on the positive spectrum of the Dirac operator. One then sees

$$\left\|D^2_b \omega\right\| \geq c_1 \left\|D_b \omega\right\|, \quad \forall \omega \in \Omega^{0,1}(X),$$

on its decomposition into Fourier modes. The last inequality is rewritten $\left\|\bar\partial_b \bar\partial^*_b \omega\right\| \geq c_1 \left\|\bar\partial^*_b \omega\right\|$, $\forall \omega \in \Omega^{0,1}(X)$ and hence

$$\left\|\bar\partial_b \omega\right\| \geq c_1 \left\|\omega\right\|, \forall \omega \in \mathrm{Dom}\left(\bar\partial_b\right) \cap \overline{\mathrm{Range}\left(\bar\partial^*_b\right)}, \tag{5.21}$$

which is equivalent to the closed range property for $\bar\partial_b$ (see [18] Sec. 1).

5.3. **Szegő kernel expansion.** We now investigate the asymptotic expansion of the Szegő kernel on diagonal. In the presence of a locally free circle action, its m-th Fourier component of the Szegő kernel $\Pi_{b,m}(x, x') := \frac{1}{2\pi} \int \Pi_b\left(x, x' e^{i\theta}\right) e^{im\theta} d\theta$ is given as the Schwartz kernel of the orthogonal projector

$$\Pi_{b,m} := \Pi_b \circ P_m : L^2(X) \to \ker\left(\Box^0_{b,m}\right) \subset L^2(X). \tag{5.22}$$

It is also written in terms of orthonormal zero eigenfunctions $\left\{\psi^m_1, \ldots, \psi^m_{d_m}\right\} = \left\{\varphi^0_{j,m} \in L^2(X) \,|\, \lambda^0_{j,m} = 0\right\}$ of $\Box^0_{b,m}$ via

$$\Pi_{b,m}(x, x') := \sum_{j=1}^{d_m} \psi^m_j(x) \overline{\psi^m_j(x')} \tag{5.23}$$

of $\ker\left(\Box^0_{b,m}\right)$. The singularity of the Szegő kernel Theorem 1 corresponds to the on-diagonal asymptotics of the Fourier component $\Pi_{b,m}(x, x)$ as $m \to \infty$ in the circle invariant case and we wish to describe it in this subsection.

We shall first localize the problem. We use the local description (5.4) of X as the unit circle bundle of $X = S^1 L \xrightarrow{\pi} Y$ of a Hermitian holomorphic line bundle $\left(L, \nabla^L, h^L\right)$ over a complex Hermitian manifold Y. Finally and as partly noted before under the identification (5.14) one has the mth Fourier component of the Kohn Laplacian

$$\Box^0_{b,m} = \Box_m := \frac{1}{2} D^2_m \tag{5.24}$$

is locally given in terms of the Kodaira Laplacian on tensor powers $C^\infty(Y; L^m)$. We now define a modify the frame $\{e_1, e_2\}$, used in the expressions (5.6), and define the frame $\{\tilde e_1, \tilde e_2\}$ on $\mathbb{R}^2$ which agrees with $\{e_1, e_2\}$ on $B_\varrho(y)$ and with $\{\partial_{x_1}, \partial_{x_2}\}$ outside $B_{2\varrho}(x)$. Also define the modified metric $\tilde g^{TY}$ and almost complex structure $\tilde J$ on $\mathbb{R}^2$ to be standard in this frame and hence agreeing with g^{TY}, J on $B_{\varrho}(x)$. The Christoffel symbol of the corresponding modified induced connection on $\Lambda^{0,*}$ now satisfies

$$\tilde{a}^{\Lambda^{0,*}} = 0 \quad \text{outside } B_{2\varrho}(x).$$

Being identified with the Levi form, the curvature R^L is semi-positive by assumption with its order of vanishing $\text{ord}_x\left(R^L\right) = r_x - 2 \in 2\mathbb{N}_0$ being given in terms of the type of the point x.

We may then Taylor expand the curvature

$$R^L = \underbrace{\sum_{|\alpha|=r_x-2} R^L_{\alpha} y^{\alpha} dy_1 dy_2}_{=R^L_0} + O\left(y^{r_x-1}\right) \quad \text{with} \tag{5.25}$$

$$iR^L_0\left(e_1, e_2\right) \geq 0. \tag{5.26}$$

Now define the modified connection on L via

$$\tilde{\nabla}^L = d + \left[\underbrace{\int_0^1 d\rho\, \rho y^k \left(\tilde{R}^L\right)_{jk}(\rho y)}_{=\tilde{a}^L_j}\right] dy_j, \quad \text{where}$$

$$\tilde{R}^L = \chi\left(\frac{|y|}{2\varrho}\right) R^L + \left[1 - \chi\left(\frac{|y|}{2\varrho}\right)\right] R^L_0. \tag{5.27}$$

which agrees with ∇^L on $B_{\varrho}(y)$. Note that the curvature $\tilde{R}^L$ of $\tilde{\nabla}^L$ above is also semi-positive by definition. Furthermore one also has $\tilde{R}^L = R^L_0 + O\left(\varrho^{r_y-1}\right)$ and that the (r_x-2)-th derivative/jet of $\tilde{R}^L$ is non-vanishing at all points on $\mathbb{R}^2$ for

$$0 < \varrho < c\left|j^{r_y-2} R^L(y)\right|. \tag{5.28}$$

Here c is a uniform constant on the manifold. We then define the modified Kodaira Dirac operator on $\mathbb{R}^2$ by the similar formula

$$\tilde{D}_m = c \circ \tilde{\nabla}^{\Lambda^{0,*} \otimes L^m} \tag{5.29}$$

which agrees with D_m on $B_{\varrho}(y)$. The above satisfies a similar Lichnerowicz formula for the corresponding Kodaira Laplacian

$$2\tilde{\Box}_m := \tilde{D}^2_m = \left(\tilde{\nabla}^{\Lambda^{0,*}\otimes L^m}\right)^* \tilde{\nabla}^{\Lambda^{0,*}\otimes L^m} + m\tilde{R}^L(w, \bar{w})\left[2\bar{w} i_{\bar{w}} - 1\right] + \frac{1}{2}\tilde{r}^X \bar{w} i_{\bar{w}} \tag{5.30}$$

where $w = \frac{1}{\sqrt{2}}\left(\tilde{e}_1 - i\tilde{e}_2\right)$, $\tilde{r}^X := \tilde{R}^{T^{1,0}X}(w, \bar{w})$, the adjoint being taken with respect to the metric $\tilde{g}^{TY}$ and corresponding volume form. The above (5.30) again agrees with

$$\tilde{\Box}_m = \Box_m \quad \text{on } B_{\varrho}(y) \tag{5.31}$$

while the endomorphism $\tilde{r}^X$ vanishes outside $B_{\varrho}(y)$. Being semi-bounded below (5.30) is essentially self-adjoint. A similar argument as Corollary 23 gives a spectral gap

$$\text{Spec}\left(\tilde{\Box}_m\right) \subset \{0\} \cup \left[c_1 m^{2/r_x} - c_2, \infty\right). \tag{5.32}$$

Thus for $m \gg 0$, the resolvent $\left(\tilde{\Box}_m - z\right)^{-1}$ is well-defined in a neighborhood of the origin in the complex plane. From local elliptic regularity, the Bergman projector

$$\tilde{B}_m : L^2\left(\mathbb{R}^2; L^{\otimes m}\right) \to \ker\left(\tilde{\Box}_m\right) \tag{5.33}$$

then has a smooth Schwartz kernel with respect to the Riemannian volume of $\tilde{g}^{TY}$.

Now we choose a set of such BRT trivializations $\left\{U_j = \left(-\frac{\pi}{s_{x_j}}, \frac{\pi}{s_{x_j}}\right) \times B_{2\varrho}(x_j)\right\}_{j=1}^N$ (5.8) centered at $\{x_j \in X\}_{j=1}^N$ with corresponding modified Laplacians $\{\tilde{\Box}_{m,j}\}_{j=1}^N$ and Bergman projectors $\left\{\tilde{B}_{m,j}\right\}_{j=1}^N$ such that $\left\{U_j^0 := \left(-\frac{\pi}{2s_{x_j}}, \frac{\pi}{2s_{x_j}}\right) \times B_\varrho(x_j)\right\}_{j=1}^N$ cover X. Choose a partition of unity $\left\{\chi_j \in C_c^\infty\left(U_j^0; [0,1]\right)\right\}_{j=1}^N$ subordinate to the the BRT cover, $j = 1, \ldots, N$. Further choose $\psi_j \in C_c^\infty(U_j; [0,1])$ such that $\psi_j = 1$ on $\mathrm{spt}(\chi_j)$ and

$$\sigma_j \in C_c^\infty\left(-\frac{\pi}{s_{x_j}}, \frac{\pi}{s_{x_j}}\right)_\theta \quad \text{with} \quad \int \sigma_j d\theta = 1 \tag{5.34}$$

in each such trivialization. We note that a finite propagation argument as in [29, Sec. 1.6] gives

$$\chi_j \tilde{B}_{m,j} \psi_j = \chi_j \tilde{B}_{m,j} \mod O\left(m^{-\infty}\right) \tag{5.35}$$

thus the right hand side above maybe assumed to be properly supported mod $O(m^{-\infty})$.

Now define the approximate Szegő kernel via

$$\tilde{\Pi}_m := \left(\sum_{j=1}^\infty \int dy' d\theta' \, \chi_j(y,\theta) e^{im\theta} \tilde{B}_{m,j}(y,y') e^{-im\theta'} \psi_j(y',\theta') \sigma_j(\theta')\right) \tag{5.36}$$

$$\tilde{\Pi}_{b,m} := \tilde{\Pi}_m \circ P_m. \tag{5.37}$$

We now have the following localization lemma.

Lemma 24. *The approximate Szegő kernel* (5.36) *satisfies*

$$\tilde{\Pi}_{b,m} - \Pi_{b,m} = O\left(m^{-\infty}\right) \tag{5.38}$$

in the C^∞ norm on the product $X \times X$.

Proof. We first show that by direct computation that

$$\tilde{\Pi}_{b,m} \Pi_{b,m} = \Pi_{b,m} \mod O\left(m^{-\infty}\right). \tag{5.39}$$

To this end, let $f \in C^\infty(X)$ and $g := \Pi_{b,m} f$. Then $g \in \ker\left(\Box_{b,m}^0\right)$ and thus $g\left(ye^{i\theta}\right) = g_0(y) e^{im\theta}$ with $\tilde{\Box}_{m,j} g_0 = 0$ on each $B_\varrho(x_j)$ by (5.24), (5.31). With

$$\tilde{B}_{m,j} g_0 = g_0 \mod O\left(m^{-\infty}\right) \text{ on } B_\varrho(x_j) \tag{5.40}$$

following from a finite propagation argument, we may calculate

$$\begin{aligned}
\tilde{\Pi}_{b,m} g &= \tilde{\Pi}_m g \\
&= \sum_{j=1}^\infty \int dy' d\theta' \, \chi_j(x) e^{im\theta} \tilde{B}_{m,j}(y,y') e^{-im\theta'} \sigma_j(\theta') g_0(y') e^{im\theta'} \\
&= \sum_{j=1}^\infty \chi_j(x) g \mod O\left(m^{-\infty}\right) \\
&= g \mod O\left(m^{-\infty}\right)
\end{aligned}$$

using (5.34), (5.36) and (5.40) showing (5.39).

In similar vein, with $P_m f = g \in C_m^\infty(X)$ satisfying $g\left(ye^{i\theta}\right) = g_0(y)\, e^{im\theta}$ on each $B_\varrho(x_j)$ as before we calculate

$$
\begin{aligned}
\tilde{\Pi}_{b,m}\Box_b f &= \tilde{\Pi}_m \circ P_m \circ \Box_b f \\
&= \tilde{\Pi}_m \circ \Box_{b,m} g \\
&= \sum_{j=1}^{\infty} \int dy' d\theta'\, \chi_j(x)\, e^{im\theta} \tilde{B}_{m,j}(y,y')\, e^{-im\theta'} \sigma_j(\theta')\, \tilde{\Box}_{m,j} g_0(y')\, e^{im\theta'} \quad \mod O\left(m^{-\infty}\right) \\
&= 0 \quad \mod O\left(m^{-\infty}\right)
\end{aligned} \tag{5.41}
$$

using (5.24), (5.31) and another finite propagation argument.

Finally letting

$$
\begin{aligned}
N_m : & L_m^2(X) \to \mathrm{Dom}\left(\Box_{b,m}^0\right), \\
N_m f &= \begin{cases} 0; & f \in \ker\left(\Box_{b,m}^0\right), \\ \Box_{b,m}^{-1} f; & f \in \ker\left(\Box_{b,m}^0\right)^{\perp} \end{cases}
\end{aligned} \tag{5.42}
$$

denote the partial inverse of $\Box_{b,m}^0$ we calculate

$$
\begin{aligned}
\tilde{\Pi}_{b,m}^* &= P_m \tilde{\Pi}_{b,m}^* \\
&= \left(N_m \Box_b P_m + \Pi_{b,m}\right) \tilde{\Pi}_{b,m}^* \\
&= \Pi_{b,m} \tilde{\Pi}_{b,m}^* \quad \mod O\left(m^{-\infty}\right) \\
&= \Pi_{b,m} \quad \mod O\left(m^{-\infty}\right)
\end{aligned}
$$

following (5.39), (5.41) and proving the proposition on account of the self-adjointness of $\Pi_{b,m}$. □

We note that the on-diagonal asymptotic expansion for the local Bergman kernel $\tilde{B}_m(y,y)$ follows in a similar fashion as Theorem 14. A slight difference here is that $\tilde{B}_m(y,y)$ is defined with respect to a more general metric while the metric in Theorem 14 is flat. This however makes little difference to the argument and gives the following (cf. [31, Thm. 3]).

Theorem 25. *For any differential operator P of order l, the derivative of the local Bergman kernel has the pointwise expansion on diagonal*

$$
P\Pi^{\tilde{\Box}_m}(y,y) = m^{(2+l)/r_y} \left[\sum_{j=0}^{N} c_j(P,y)\, m^{-2j/r_y} + O\left(m^{l-2N+1}\right)\right], \tag{5.43}
$$

$\forall N \in \mathbb{N}_0$. *The leading term, for $P = 1$, is given $c_{0,0}(1,y) = \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0) > 0$ in terms of the Bergman kernel of the model Kodaira Laplacian on HX (see* [31, Sec. A]*).*

Following this and the localization property of the Szegő kernel just proved now implies Theorem 21 as below.

Proof of Theorem 21. By the localization property (5.38) it suffices to show the pointwise expansion of the approximate Szegő kernel (5.37). In showing the expansion at $x \in X$, we may further assume the BRT cover and partition of unity defining (5.37) is chosen so that

$\chi_j = \begin{cases} 1, & j = 1 \\ 0, & j > 1 \end{cases}$, near x. We then compute

$$
\begin{aligned}
\tilde{\Pi}_{b,m} &= \int_0^{2\pi} d\theta\, \tilde{\Pi}_m\left(x, xe^{i\theta}\right) e^{im\theta} \\
&= \sum_{l=0}^{s_x-1} \int_{\frac{2\pi}{s_x}l}^{\frac{2\pi}{s_x}(l+1)} d\theta\, \tilde{\Pi}_m\left(x, xe^{i\theta}\right) e^{im\theta} \\
&= \sum_{l=0}^{s_x-1} e^{i\frac{2\pi lm}{s_x}} \int_0^{\frac{2\pi}{s_x}l} d\theta\, \tilde{\Pi}_m\left(x, xe^{i\theta}\right) e^{im\theta} \\
&= \sum_{l=0}^{s_x-1} e^{i\frac{2\pi lm}{s_x}} \int_0^{\frac{2\pi}{s_x}l} d\theta\, \tilde{\Pi}_m\left(x, xe^{i\theta}\right) e^{im\theta} \\
&= \left(\sum_{l=0}^{s_x-1} e^{i\frac{2\pi lm}{s_x}}\right)\left(\int d\theta\, \sigma_1(\theta)\right) \tilde{B}_m(y,y) \\
&= \left(\sum_{l=0}^{s_x-1} e^{i\frac{2\pi lm}{s_x}}\right) m^{2/r_y}\left[\sum_{j=0}^{N} c_j(x)\, m^{-2j/r_y} + O\left(m^{-(2N+1)/r_y}\right)\right], \quad \forall N \in \mathbb{N}, \qquad (5.44)
\end{aligned}
$$

from (5.37), Theorem 25 to prove (5.3). ☐

As noted in [31, Rem. 24] the expansion for the local Bergman kernels $\tilde{B}_m(y,y)$ is the same as the positive case on $X_{2,s}$ (the strongly pseudoconvex points) and furthermore uniform in any C^l-topology on compact subsets of $X_{2,s}$ cf. [29, Theorem 4.1.1]. In particular the first two coefficients for $y \in Y_2$ are given by

$$
\begin{aligned}
c_0(y) &= \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0) = \frac{1}{2\pi}\tau^L \\
c_1(y) &= \frac{1}{16\pi}\tau^L\left[\kappa - \Delta \ln \tau^L\right]. \qquad (5.45)
\end{aligned}
$$

The derivative expansion on $X_{2,s}$ is also known to satisfy $c_0 = c_1 = \ldots = c_{\left[\frac{l-1}{2}\right]} = 0$ (i.e. begins at the same leading order m).

In the next section we shall also need uniform estimates on the local Bergman kernels as below.

Theorem 26. *The local Bergman kernel satisfies the estimate*

(5.46)
$$
\left[\inf_{x \in X_{r,\leq s}} \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0)\right]\left[1 + o(1)\right] m^{2/r} \leq \tilde{B}_m(y,y) \leq \left[\sup_{x \in X} \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0)\right]\left[1 + o(1)\right] m
$$

with the $o(1)$ terms being uniform in $x \in X$.

Furthermore, there exists constants C_l, $l = 0, 1, \ldots$, uniform in $y \in Y$, such that for any differential operator P_l of order l, the derivative of the local Bergman kernel satisfies the estimate

$$
\left|P_l \tilde{B}_m(y,y)\right| \leq m^{l/3} C_l \tilde{B}_m(y,y). \qquad (5.47)
$$

Proof. Note that theorem Theorem 25 already shows

$$
\Pi_m(y,y) \geq C_{r_y}\left(\left|j^{r_y-2}R^L\right| m\right)^{2/r_y} - c_y \qquad (5.48)
$$

$\forall y\in Y$, with $c_y = c\left(\left|j^{r_y-2}R^L(y)\right|^{-1}\right) = O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$ being a (y-dependent) constant given in terms of the norm of the first non-vanishing jet. The norm of this jet affects the choice of ϱ needed for (5.28); which in turn affects the C^∞-norms of the coefficients of (5.43) via (5.27). We first show that this estimate extends to a small ($\left|j^{r_y-2}R^L(y)\right|$- dependent) size neighborhood of y. To this end, for any $\varepsilon>0$ there exists a uniform constant c_ε depending only on ε and $\left\|R^L\right\|_{C^r}$ such that

$$\left|j^{r_y-2}R^L(\mathsf{y})\right| \geq (1-\varepsilon)\left|j^{r_y-2}R^L(y)\right|, \tag{5.49}$$

$\forall \mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$.

We begin by rewriting the model Kodaira Laplacian $\tilde{\square}_m$ (5.30) near y in terms of geodesic coordinates centered at y. In the region

$$\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\left\{C_0\left(\left|j^0R^L(\mathsf{y})\right|m\right)\geq m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\right\}$$

a rescaling of $\tilde{\square}_m$ by $\delta_{m^{-1/2}}$, now centered at y, shows

$$\begin{aligned}\Pi_m(\mathsf{y},\mathsf{y}) &= m\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^0R^L,J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &= m\left|j^0R^L(\mathsf{y})\right|\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^0R^L}{\left|j^0R^L(\mathsf{y})\right|},J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &\geq m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (5.50)\end{aligned}$$

as in (5.48). Now, in the region

$$\begin{aligned}\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\Big\{&C_1\left(\left|j^1R^L(\mathsf{y})/j^0R^L(\mathsf{y})\right|m\right)^{2/3}\\ &\geq m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\geq C_0\left(\left|j^0R^L(\mathsf{y})\right|m\right)\Big\}\end{aligned}$$

a rescaling of $\tilde{\square}_m$ by $\delta_{m^{-1/3}}$ centered at y similarly shows

$$\begin{aligned}\Pi_m(\mathsf{y},\mathsf{y}) &= m^{2/3}\left[1+O\left(m^{2/r-2/3}\right)\right]\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L,J_{\mathsf{y}}^{TY}}(0,0)\\ &\quad+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &= m^{2/3}\left[1+O\left(m^{2/r-2/3}\right)\right]\left|j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L\right|^{2/3}\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L}{\left|j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L\right|},J_{\mathsf{y}}^{TY}}(0,0)\\ &\quad+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (5.51)\\ &\geq(1-\varepsilon)\,m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (5.52)\end{aligned}$$

Next, in the region

$$\begin{aligned}&\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\Big\{C_2\left(\left|j^2R^L(\mathsf{y})/j^1R^L(\mathsf{y})\right|m\right)^{1/2}\\ &\geq m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\geq\max\left[C_0\left(\left|j^0R^L(\mathsf{y})\right|m\right),C_1\left(\left|j^1R^L(\mathsf{y})/j^0R^L(\mathsf{y})\right|m\right)^{2/3}\right]\Big\}\end{aligned}$$

a rescaling of $\tilde{\Box}_m$ by $\delta_{m^{-1/4}}$ centered at y shows

$$\begin{aligned}\Pi_m(\mathsf{y},\mathsf{y}) &= m^{1/2}\left[1+O\left(m^{2/r-1/2}\right)\right]\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L,J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &= m^{1/2}\left[1+O\left(m^{2/r-1/2}\right)\right]\left|j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L\right|^{1/2}\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L}{\left|j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L\right|},J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &\geq (1-\varepsilon)\,m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\end{aligned} \tag{5.53}$$

Continuing in this fashion, we are finally left with the region

$$\begin{aligned}\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\Big\{&m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\\ &\geq\max\left[C_0\left(\left|j^0R^L(\mathsf{y})\right|m\right),\ldots,C_{r_y-3}\left(\left|j^{r_y-3}R^L(\mathsf{y})/j^{r_y-4}R^L(\mathsf{y})\right|m\right)^{2/(r_y-1)}\right]\Big\}.\end{aligned}$$

In this region we have

$$\left|j^{r_y-2}R^L(\mathsf{y})/j^{r_y-3}R^L(\mathsf{y})\right|\geq(1-\varepsilon)\left|j^{r_y-2}R^L(y)\right|+O\left(m^{2/r_y-2/(r_y-1)}\right)$$

following (5.49) with the remainder being uniform. A rescaling by $\delta_{m^{-1/r_y}}$ then giving a similar estimate in this region, we have finally arrived at

$$\Pi_m(\mathsf{y},\mathsf{y})\geq(1-\varepsilon)\,m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$$

$\forall\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$.

Finally a compactness argument finds a finite set of points $\{y_j\}_{j=1}^N$ such that the corresponding $B_{c_\varepsilon\left|j^{r_{y_j}-2}R^L\right|}(y_j)$'s cover Y. This gives a uniform constant $c_{1,\varepsilon}>0$ such that

$$\Pi_m(y,y)\geq(1-\varepsilon)\left[\inf_{y\in Y_r}\Pi^{g_y^{TY},j_y^{r-2}R^L,J_y^{TY}}(0,0)\right]m^{2/r}-c_{1,\varepsilon}$$

$\forall y\in Y$, $\varepsilon>0$ proving the lower bound of (5.46). The argument for the upper bound is similar.

The proof of the uniform estimate on the derivative (5.47) is similar. Given $\varepsilon>0$ we find a uniform c_ε such that (5.49) holds for each $y\in Y$ and $\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$. Then rewrite the model Kodaira Laplacian $\tilde{\Box}_m$ (5.30) near y in terms of geodesic coordinates centered at y. In the region

$$\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\left\{C_0\left(\left|j^0R^L(\mathsf{y})\right|m\right)\geq m^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\right\}$$

a rescaling of $\tilde{\Box}_m$ by $\delta_{m^{-1/2}}$, now centered at y, shows

$$\partial^\alpha\Pi_m(\mathsf{y},\mathsf{y})=\frac{m}{2\pi}\left(\partial^\alpha\tau^L(\mathsf{y})\right)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$$

following (5.45) as $r_{\mathsf{y}}=2$. Diving the above by (5.50) gives

$$\begin{aligned}\frac{\left|\partial^\alpha\Pi_m(\mathsf{y},\mathsf{y})\right|}{\Pi_m(\mathsf{y},\mathsf{y})}&\leq\frac{\left|\partial^\alpha\tau^L(\mathsf{y})\right|}{\tau^L(\mathsf{y})}+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{-1}\right)\\ &\leq m^{|\alpha|/3}\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}(0,0)}\right]\Pi_m(\mathsf{y},\mathsf{y})\\ &\quad+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{-1}\right)\end{aligned}$$

Next, in the region

$$\mathsf{y} \in B_{c_\varepsilon \left|j^{r_y-2}R^L\right|}(y) \cap \Big\{ C_1 \left(\left| j^1 R^L(\mathsf{y}) / j^0 R^L(\mathsf{y}) \right| m \right)^{2/3} \geq m^{2/r_y} \Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) \geq C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| m \right) \Big\}$$

a rescaling of $\tilde{\Box}_m$ by $\delta_{m^{-1/3}}$ centered at y similarly shows

$$\partial^\alpha \Pi_m(\mathsf{y},\mathsf{y}) = m^{(2+|\alpha|)/3} \left[1 + O\left(m^{2/r-2/3}\right)\right] \left[\partial^\alpha \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{(1+|\alpha|)/3}\right).$$

Dividing this by (5.52) gives

$$\begin{aligned}\frac{\left|\partial^\alpha \Pi_m(\mathsf{y},\mathsf{y})\right|}{\Pi_m(\mathsf{y},\mathsf{y})} &\leq m^{|\alpha|/3}(1+\varepsilon) \frac{\left|\left[\partial^\alpha \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\left[\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)} + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{(|\alpha|-1)/3}\right) \\ &\leq m^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y} \frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}\right] + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{(|\alpha|-1)/3}\right).\end{aligned}$$

Continuing in this fashion as before eventually gives

$$\frac{\left|\partial^\alpha \Pi_m(\mathsf{y},\mathsf{y})\right|}{\Pi_m(\mathsf{y},\mathsf{y})} \leq m^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y} \frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}\right] + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(m^{(|\alpha|-1)/3}\right)$$

$\forall y \in Y$, $\mathsf{y} \in B_{c_\varepsilon \left|j^{r_y-2}R^L\right|}(y)$, $\forall \alpha \in \mathbb{N}_0^2$. By compactness one again finds a uniform $c_{1,\varepsilon}$ such that

$$\frac{\left|\partial^\alpha \Pi_m(y,y)\right|}{\Pi_m(y,y)} \leq m^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y} \frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}\right] + c_{1,\varepsilon}$$

$\forall y \in Y$, proving the lemma. □

6. Equivariant CR Embedding

In this section we construct the CR embedding for X required to prove Theorem 3. Firstly, setting $m = p.(s!) \in (s!).\mathbb{N}_0$ in (5.18), (5.23), (5.46) and (5.44) the base locus

$$\mathrm{Bl}_p(X) := \left\{ x \in X | s(x) = 0, \forall s \in H^0_{b,p.(s!)}(X) \right\} = \emptyset \tag{6.1}$$

is empty for $p \gg 0$. Thus the subspace

$$\Phi_{p,x} := \left\{ s \in H^0_{b,p.(s!)}(X) | s(x) = 0 \right\} \subset H^0_{b,p.(s!)}(X) \tag{6.2}$$

is a hyperplane for each $x \in X$. Identifying the Grassmanian $\mathbb{G}\left(d_{p.(s!)} - 1; H^0_{b,p.(s!)}(X)\right)$, $d_{p.(s!)} = \dim H^0_{b,p.(s!)}(X)$, with the projective space $\mathbb{P}\left[H^0_{b,p.(s!)}(X)^*\right]$, by sending a non-zero element of $H^0_{b,p.(s!)}(X)^*$ to its kernel, gives a well-defined Kodaira map

$$\Phi_p : X \to \mathbb{P}\left[H^0_{b,p.(s!)}(X)^*\right] \tag{6.3}$$

for $p \gg 0$.

In terms of the basis $\left\{\psi_1^{p.(s!)}, \ldots, \psi_{d_{p.(s!)}}^{p.(s!)}\right\}$ of $\ker\left(\Box^0_{b,p.(s!)}\right) = H^0_{b,p.(s!)}(X)$, with $\bar{\partial}_b \psi_j = 0$, $j = 1, \ldots, d_{p.(s!)}$, and corresponding dual basis of $H^0_{b,p.(s!)}(X)^*$ the map is written

$$\Phi_p(x) \coloneqq \left(\psi_1^{p.(s!)}(x), \ldots, \psi_{d_{p.(s!)}}^{p.(s!)}(x)\right) \in \mathbb{C}^{d_{p.(s!)}} \tag{6.4}$$

and is seen to be CR.

We now define the augmented Kodaira map

$$\begin{aligned}
\Psi_p :& X \to \mathbb{C}^N, \\
\Psi_p \coloneqq& \left(\Phi_p, \Psi_p^1, \ldots, \Psi_p^s\right), \\
\Psi_p^k \coloneqq& \left(\underbrace{\psi_1^{p.k}, \ldots, \psi_{d_{p.k}}^{p.k}}_{=:\Psi_p^{k,0}}; \underbrace{\psi_1^{(p+1).k}, \ldots, \psi_{d_{(p+1).k}}^{(p+1).k}}_{=:\Psi_p^{k,1}}\right), \quad 1 \le k \le s, \\
N \coloneqq& d_{p.(s!)} + \sum_{k=1}^{s} \left(d_{p.k} + d_{(p+1).k}\right),
\end{aligned} \tag{6.5}$$

which is again CR. We shall now show that the above augmented map is an embedding for $p \gg 0$. We first show that it is an immersion, whereby it suffices to show that its first component Φ_p (6.4) defines an immersion; the augmented components of (6.5) are required to separate further points.

Theorem 27. *The Kodaira map Φ_p (6.3) is an immersion for $p \gg 0$.*

Proof. We work in a BRT trivialization (5.8), (5.8) near $x \in X$. We choose $\chi \in C_c^\infty((-\varepsilon, \varepsilon); [0,1])$, $\chi = 1$ on $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, $\sigma_0(\theta) \in C_c^\infty\left(-\frac{2\pi}{s}, \frac{2\pi}{s}\right)$, $\int \theta \sigma_0(\theta)\, d\theta = 1$ and set

$$\begin{aligned}
u_1 &= y_2 \chi\left(m^{1/r_x} y\right) \sigma(\theta) e^{im\theta} \\
u_2 &= y_1 \chi\left(m^{1/r_x} y\right) \sigma(\theta) e^{im\theta} \\
u_3 &= \chi\left(m^{1/r_x} y\right)(m\theta)\, \sigma_0(m\theta) e^{im\theta} \quad \text{and} \\
v_j &= \Pi_{m,b} u_j, \qquad j = 1, 2, 3,
\end{aligned} \tag{6.6}$$

with $m = p.(s!)$ and $x = (y, \theta)$ being BRT coordinates.

The equations $\bar{\partial}_b \Pi_{m,b}(., x) = 0$, $\bar{\partial}_b^* \Pi_{m,b}(., x) = 0$ written in the BRT chart give

$$\begin{aligned}
\partial_{y_1} \Pi_{m,b}(x', x) &= m\left[\frac{1}{r_x} y_2' \left(\sum_{|\alpha| = r_x - 2} R_\alpha^L y'^\alpha\right) + O\left(y'^{r_x}\right)\right] \Pi_{m,b}(x', x) \\
\partial_{y_2} \Pi_{m,b}(x', x) &= m\left[-\frac{1}{r_x} y_1' \left(\sum_{|\alpha| = r_x - 2} R_\alpha^L y'^\alpha\right) + O\left(y'^{r_x}\right)\right] \Pi_{m,b}(x', x)
\end{aligned} \tag{6.7}$$

from (5.8), (5.7). Further note that

$$\begin{aligned} \overline{\Pi_{m,b}u_j\left(x\right)} &= \int dx'\, \overline{\Pi_{m,b}\left(x,x'\right)u_j\left(x'\right)} \\ &= \int dx'\, \Pi_{m,b}\left(x',x\right)\overline{u_j\left(x'\right)} \\ &= s_x \int dx'\, \tilde{\Pi}_m\left(x',x\right)\overline{u_j\left(x'\right)} \mod O\left(m^{-\infty}\right) \end{aligned} \tag{6.8}$$

from (5.38). We now estimate the derivative of the CR functions (6.6). Below $c_{ij}\left(\left|j^{r_x-2}\mathcal{L}\right|\right)$, $1 \le i, j \le 3$, continuous positive functions of the norms of the jet of the Levi form. We then have

$$\begin{aligned} \partial_{y_1}\overline{v_1} &= s_x m \int dy' d\theta \left[\frac{1}{r_x}\left(y_2'\right)^2 \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha\right) + O\left(y'^{r_x+1}\right)\right] \times \\ &\qquad \tilde{\Pi}_m\left(x',x\right)\chi\left(m^{1/r_x}y'\right)\sigma\left(\theta\right)e^{im\theta} + O\left(m^{-\infty}\right) \\ &= s_x \int dy' d\theta \left[\frac{1}{r_x}\left(y_2'\right)^2 \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha\right) + O\left(m^{-1/r_x}y'^{r_x+1}\right)\right] \times \\ &\qquad \tilde{\Pi}_m\left(m^{-1/r_x}x',0\right)\chi\left(y'\right)\sigma\left(\theta\right)e^{im\theta} \\ &= s_x \int dy' d\theta \left[\frac{1}{r_x}\left(y_2'\right)^2 \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha\right) + O\left(m^{-1/r_x}y'^{r_x+1}\right)\right] \times \\ &\qquad \left[\Pi^{g^{HX}_x, j^{r_x-2}\mathscr{L}, J^{HX}}\left(x',0\right) + O\left(m^{-1/r_x}\right)\right]\chi\left(y'\right)\sigma\left(\theta\right)e^{im\theta} \\ &\ge \varepsilon^{r_x+2}c_{11}\left(\left|j^{r_x-2}\mathcal{L}\right|\right) + O_x\left(m^{-1/r_x}\right) \end{aligned} \tag{6.9}$$

using (6.7) and (6.8).

And similarly,

$$\begin{aligned}
\partial_{y_2}\overline{v_1} &= s_x m \int dy' d\theta \left[\frac{1}{r_x} y_1' y_2' \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha \right) + O\left(y'^{r_x+1} \right) \right] \times \\
&\qquad \tilde{\Pi}_m\left(x',0\right) \chi\left(m^{1/r_x} y'\right) \sigma\left(\theta\right) e^{im\theta} + O\left(m^{-\infty}\right) \\
&= s_x \int dy' d\theta \left[\frac{1}{r_x} y_1' y_2' \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha \right) + O\left(m^{-1/r_x} y'^{r_x+1} \right) \right] \times \\
&\qquad \tilde{\Pi}_m\left(m^{-1/r_y} x',0\right) \chi\left(y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&= s_x \int dy' d\theta \left[\frac{1}{r_x} y_1' y_2' \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y'^\alpha \right) + O\left(m^{-1/r_x} y'^{r_x+1} \right) \right] \times \\
&\qquad \left[\Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}\left(x',0\right) + O\left(m^{-1/r_x}\right) \right] \chi\left(y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&= s_x \int dy' d\theta \left[O\left(y'^{r_x+1}\right) + O\left(m^{-1/r_x}\right) \right] \chi\left(y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&\leq \varepsilon^{r_y+3} c_{12}\left(\left| j^{r_x-2}\mathcal{L} \right| \right) + O_x\left(m^{-1/r_x}\right) \qquad (6.10)
\end{aligned}$$

using a Taylor expansion $\Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(y,0) = \Pi_0 + y_1\Pi_1 + y_2\Pi_2$; $\Pi_0 = \Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}(0,0)$ on spt (χ). Finally we compute

$$\begin{aligned}
\partial_\theta \overline{v_1} &= s_x m \int dy' d\theta \tilde{\Pi}_m\left(x',x\right) y_1' \chi\left(m^{1/r_x} y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&= s_x m^{1-1/r_x} \int dy' d\theta m^{-2/r_x} \tilde{\Pi}_m\left(m^{-1/r_y} x',0\right) y_1' \chi\left(y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&= s_x m^{1-1/r_x} \int dy' d\theta \left[\Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}\left(x',0\right) + O\left(m^{-1/r_x}\right) \right] y_1' \chi\left(y'\right) \sigma\left(\theta\right) e^{im\theta} \\
&\leq m^{1-1/r_x} \varepsilon^4 c_{13}\left(\left| j^{r_x-2}\mathcal{L} \right| \right) + O_x\left(m^{1-2/r_x}\right).
\end{aligned}$$

We have similar estimates on derivatives of v_2

$$\begin{aligned}
\partial_{y_1} v_2 &\leq \varepsilon^{r_y+2} c_{21}\left(\left| j^{r_y-2} R^L \right| \right) + O_x\left(m^{-1/r_x}\right) \\
\partial_{y_2} v_2 &\geq \varepsilon^{r_y+2} c_{22}\left(\left| j^{r_y-2} R^L \right| \right) + O_x\left(m^{-1/r_x}\right) \\
\partial_\theta v_2 &\leq m^{1-1/r_x} \varepsilon^4 c_{23}\left(\left| j^{r_x-2}\mathcal{L} \right| \right) + O_x\left(m^{1-2/r_x}\right). \qquad (6.11)
\end{aligned}$$

for two further constants $c_{21}\left(\left|j^{r_y-2}R^L\right|\right)$, $c_{22}C\left(\left|j^{r_y-2}R^L\right|\right)\left(\left|j^{r_y-2}R^L\right|\right)$ depending only on the norm of jet of the Levi tensor at x.

Finally, and in similar vein, we estimate the derivative of v_3

$$\begin{aligned}
\partial_\theta v_3 &= m \int dz d\theta \tilde{\Pi}_{b,m}\left(z,0\right) \chi\left(m^{1/r_x} y\right) \left(m\theta\right) \sigma_0\left(m\theta\right) \\
&= \int dz d\theta \left[\Pi^{g_x^{HX}, j^{r_x-2}\mathscr{L}, J^{HX}}\left(y,0\right) + O\left(m^{-1/r_x}\right) \right] \chi\left(y\right) \theta \sigma_0\left(\theta\right) \\
&\geq \varepsilon^2 c_{33}\left(\left| j^{r_y-2} R^L \right| \right) + O_x\left(m^{-1/r_x}\right) \qquad (6.12)
\end{aligned}$$

and

$$\begin{aligned}\partial_{y_1} v_3 &= m \int dz d\theta \tilde{\Pi}_{b,m}(z,0)\, \chi\left(m^{1/r_x} y\right)(m\theta)\, \sigma_0(m\theta) \\ &= m \int dz d\theta \left[\frac{1}{r_x} y_2 \left(\sum_{|\alpha|=r_x-2} R^L_\alpha y^\alpha\right) + O\left(y^{r_x}\right)\right] \times \\ &\qquad \tilde{\Pi}_{b,m}(z,0)\, \chi\left(m^{1/r_x} z\right)(m\theta)\, \sigma_0(m\theta) \\ &= \int dz d\theta \left[O\left(m^{-1} m^{1/r_r} y^{r_x-1}\right)\right] \tilde{\Pi}_{b,m}\left(m^{-1/r_y} z, 0\right) \chi(z)\, \theta \sigma(\theta) \\ &\leq m^{-1+1/r_x} \varepsilon^{r_x+1} c_{13}\left(\left|j^{r_y-2} R^L\right|\right) + O_x\left(m^{-1+2/r_x}\right). \end{aligned} \tag{6.13}$$

and similarly for$\partial_{y_2} v_3$. Following these estimates there exists $C\left(\left|j^{r_y-2}R^L\right|\right)$ such that the differential of $x \mapsto (v_1, v_2, v_3)$, and thus of Φ_p, $m = p.(s!)$, is invertible at x for $\varepsilon < C\left(\left|j^{r_y-2}R^L\right|\right)$ and $p > C\left(\left|j^{r_y-2}R^L\right|\right)$. Thus for some $C_1\left(\left|j^{r_y-2}R^L\right|\right)$ the differential of Φ_p is invertible on a $C_1\left(\left|j^{r_y-2}R^L\right|\right)$ ball centered at x for $p > C_1\left(\left|j^{r_y-2}R^L\right|\right)$; which completes the proof following a compactness argument. □

Next to show the Kodaira map is injective, one needs the following definition.

Definition 28. The peak function $S^p_{x_0} \in H^0_{b,p.(s!)}(X)$ at $x_0 \in X$ is the unit norm element of the orthogonal complement to $\Phi^\perp_{p,x_0} \subset H^0_{b,p.(s!)}(X)$ (6.2).

Clearly, if the orthonormal basis $\left\{\psi_1, \ldots, \psi_{d_{p.(s!)}}\right\}$ of $\ker\left(\square^0_{b,p.(s!)}\right) = H^0_{b,p.(s!)}(X)$ is chosen so that $\psi_j(x_0) = 0$, $1 \leq j \leq d_{p.(s!)} - 1$, one has $S^p_{x_0} = \psi_{d_{p.(s!)}}$. From (5.23) one then has

$$\begin{aligned}\left|S^p_{x_0}(x_0)\right|^2 &= \Pi_{b,p.(s!)}(x_0, x_0) \\ S^p_{x_0}(x) &= \frac{1}{\Pi_{b,p.(s!)}(x_0,x_0)} \Pi_{b,p.(s!)}(x, x_0) . S^p_{x_0}(x_0). \end{aligned} \tag{6.14}$$

Theorem 29. *The augmented Kodaira map Ψ_p (6.3) is injective for $p \gg 0$.*

Proof. We assume to the contrary that there are two sequences of points $x^1_{p_j}$, $x^2_{p_j}$, $j = 1, 2, \ldots$, such that $p_j \to \infty$ as $j \to \infty$ and

$$\begin{aligned} x^1_{p_j} &\neq x^2_{p_j} \\ \Psi_{p_j}\left(x^1_{p_j}\right) &= \Psi_{p_j}\left(x^2_{p_j}\right), \quad \forall j. \end{aligned} \tag{6.15}$$

By compactness we may further suppose $x^1_{p_j} \to x^1$, $x^2_{p_j} \to x^2$ as $j \to \infty$.

Case i: Suppose $e^{i\theta} x^1 \neq x^2$, $\forall e^{i\theta} \in S^1$, i.e. the limit points do not lie on the same S^1 orbit. The equation (6.15) in particular implies $\Phi_{p_j}\left(x^1_{p_j}\right) = \Phi_{p_j}\left(x^2_{p_j}\right)$ by definition (6.5). Thus (6.14) implies

$$\Pi_{b,p_j.(s!)}\left(x^1_{p_j}, x^1_{p_j}\right) \Pi_{b,p_j.(s!)}\left(x^2_{p_j}, x^2_{p_j}\right) = \left|\Pi_{b,p_j.(s!)}\left(x^1_{p_j}, x^2_{p_j}\right)\right|^2.$$

The left hand side above is uniformly bounded below by $cp_j^{4/r}$ on account of (5.36), (5.38), (5.46). While the right hand side can be seen to be $O\left(p_j^{-\infty}\right)$ on choosing the S^1 orbits of the BRT charts containing x^1, x^2 in defining (5.36) to be disjoint.

Case ii: Suppose $e^{i\theta} x^1 = x^2$, for some $e^{i\theta} \in S^1$. We now again consider a BRT chart (5.8) of the form $U = \left(-\frac{\pi}{s_{x^1}}, \frac{\pi}{s_{x^1}}\right) \times B_{2\varrho}(x^1)$ containing the point x^1. As before this is obtained as the unit circle bundle $S^1L \to Y$ over a hypersurface $Y \subset X$ containing the point x^1. For each j we denote by $\left[x^1_{p_j}\right]$, $\left[x^2_{p_j}\right] \in Y$ the unique points satisfying $x^1_{p_j} \in S^1.\left[x^1_{p_j}\right]$, $x^2_{p_j} \in S^1\left[x^2_{p_j}\right]$. The inclusion further defines a local holomorphic map $\Phi_p : Y \to \mathbb{P}\left[H^0_{b,p.(s!)}(X)^*\right]$ satisfying $\Phi_{p_j}\left(\left[x^1_{p_j}\right]\right) = \Phi_{p_j}\left(\left[x^2_{p_j}\right]\right)$, $j = 1, 2, \ldots$. Via the Noetherian property for analytic sets as in [29, Sec. 5.1] this gives $\left[x^1_{p_j}\right] = \left[x^2_{p_j}\right]$ or $x^2_{p_j} = e^{i\theta}x^1_{p_j} \in S^1.x^1_{p_j}$ for $j \gg 0$. Thus $x^1_{p_j}$, $x^2_{p_j}$ lie on the same orbit and with $k = s_{x^1_{p_j}} = s_{x^2_{p_j}}$ we have

$$\begin{array}{ccc} \Psi^k_p\left(x^1_{p_j}\right) & = & \Psi^k_p\left(x^2_{p_j}\right) \\ \| & & \| \\ \left(\Psi^{k,0}_p\left(x^1_{p_j}\right), \Psi^{k,1}_p\left(x^1_{p_j}\right)\right) & & \left(e^{ip.k\theta}\Psi^{k,0}_p\left(x^1_{p_j}\right), e^{i(p+1).k\theta}\Psi^{k,1}_p\left(x^1_{p_j}\right)\right). \end{array}$$

It now follows that $\theta \in \frac{2\pi}{k}\mathbb{Z}$ implying $x^1_{p_j} = x^2_{p_j}$ and contradicting (6.15). □

We note that following the closed range property (5.21) for $\bar{\partial}_b$ of 5.2 our embedding theorem Theorem 3, aside from the equivariance, can be obtained from the main theorem of [11].

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