Subleading asymptotics of ECH capacities

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arxiv 1811.00485

I Background & Motivation

(a) ECH spectrum

(Y^3, λ) closed contact & $\Gamma \in H_1(Y)$ $\rightsquigarrow$ M. Hutchings's Embedded contact Homology $ECH(Y, \underset{\ker\lambda}{\overset{\shortparallel}{\xi}}, \Gamma; \mathbb{Z}_2)$

Chain sp:

$$ECC(Y, \lambda, \Gamma; \mathbb{Z}_2) := \mathbb{Z}_2\left[\underset{\text{finite}}{\alpha = (\alpha_j, m_j)} \;\middle|\; \begin{array}{l} \alpha_j \text{ distinct (prime) Reeb orbits} \\ m_j = \begin{cases} 1 & \alpha_j \text{ hyperbolic} \\ \in \mathbb{N} & \alpha_j \text{ elliptic} \end{cases} \\ \sum m_j [\alpha_j] = \Gamma \end{array} \right]$$

Differential:

$$\partial \alpha := \sum_\beta \# \underbrace{\left[\mathcal{M}_1(\alpha, \beta)/\mathbb{R} \right]}_{\substack{\text{ECH index 1} \\ \text{hol. curves}}} \beta$$

Grading: $\underset{\text{torsion}}{c_1(\xi) + 2 P.D. \Gamma}$ $\Rightarrow$ $ECH(Y, \xi, \Gamma)$ is $\begin{cases} \text{abs. } \mathbb{Q} \text{ graded} \\ \text{rel. } \mathbb{Z}. \text{ graded} \end{cases}$

& $ECH_* = ECH_{*+2} \neq 0$; $* >> 0$.

Action filtration: $ECH(Y, \xi, \Gamma; \mathbb{Z}_2) = \varinjlim ECH^L(Y, \lambda, \Gamma; \mathbb{Z}_2)$

$$ECH^L(Y, \lambda, \Gamma; \mathbb{Z}_2) := \mathbb{Z}_2\left[\alpha \;\middle|\; \begin{array}{l} \text{=======} \\ \underset{\text{action}}{A(\alpha)} := \sum m_j \int_{\alpha_j} \lambda \leq L \end{array} \right]$$

ECH capacity: $\underset{(\sigma \in ECH)}{c_\sigma(\lambda)} := \inf \{ L \mid \sigma \in \mathrm{Im}(ECH^L \longrightarrow ECH) \}$

ECH zeta fn: $\zeta_{ECH}(s; Y, \lambda, \Gamma) := \sum_{\sigma \in ECH(Y, \xi, \Gamma; \mathbb{Z}_2)} c_\sigma^{-s}$; $s \in \mathbb{C}$, $Re(s) >> 0$

ECH counting fn: $\pi_{ECH}(x) := \# \{ \sigma \in ECH \mid c_\sigma \leq x \}$

Thm 1. (Cristofaro-Gardiner Hutchings & Ramos '15)

Let $c_1(\xi) + 2$ P.D.T torsion & $\sigma_j \in ECH_{*+2j}(Y, \xi, \Gamma)$, $j = 1, 2, 3 \ldots$

then $\boxed{c_{\sigma_j}(\lambda) \sim [2j \, \mathrm{vol}(Y, \lambda)]^{1/2}}$ — (*) as $j \to \infty$

Cor 1. (Cristofaro-Gardiner Hutchings '16)

(Y^3, λ) closed contact. $\exists$ atleast 2 distinct Reeb orbits.

Cor 2. (Irie '15)

λ generic contact. $\exists$ closed Reeb orbit through any open subset $U \subset Y$

(i.e. the union of closed Reeb orbits is dense in Y)

Q. What is the next term in (*) ?

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b) Geodesic length spectrum (analogous prob.)

(X^n, g) Riem mfld $\sec(g) < 0$

Reeb flow on S^*X is Anosov (& nondeg)

$$\pi(x) := \# \{ \gamma \text{ prime closed geodesic} \mid \ell_\gamma := \text{length}(\gamma) \leq \ln x \}$$

Ruelle zeta fn: $\zeta_{\text{Ruelle}}(s; X, g) := \prod_{\substack{\gamma \text{ prime} \\ \text{closed geod}}} (1 - e^{-s \ell_\gamma})^{-1} = \sum_{\substack{(m_j, \gamma_j) \\ \in \mathbb{N} \;\; \text{prime geod}}} e^{-s \sum m_j \ell_{\gamma_j}}$; $\mathrm{Re}(s) \gg 0$, $s \in \mathbb{C}$.

(Margulis '69) $\pi(x) \sim \dfrac{x^h}{\ln x}$ h = top. entropy

(Giulietti - Liverani Pollicott '13) Assuming $\sec(g)$ is $1/4$ pinched: $\exists \, \delta > 0$ s.t.

$$\pi(x) = \frac{x^h}{\ln x} + O(x^{h-\delta}) \quad (\text{sharp } \delta \text{ unknown})$$

Prime number conjecture: $\underset{\substack{\uparrow \\ \text{prime \# counting}}}{\pi(x)} = \underset{\substack{? \\ \frac{x}{\ln x}}}{li(x)} + \begin{cases} O(x^{\frac{1}{2}+\varepsilon}) & \forall \varepsilon > 0 \\ O(x^{\frac{1}{2}} \ln x) & \text{— equiv to RH} \end{cases}$

(Ruelle '78, Parry - Pollicott '83) $\zeta_{\text{Ruelle}}(s)$ extends meromorphically to $\mathrm{Re}(s) > h - \{h\}$ with single (simple) pole at $s = h$.

(GLP '13, Dyatlov - Zworski '16) $\zeta_{\text{Ruelle}}(s)$ extends meromorphically to $\mathbb{C}$.

(II) Main Results

Thm 2 (Cristofaro-Gardiner, S '18)

Let $c_1(\xi) + 2\,\mathrm{P.D.}\,\Gamma$ be torsion & $\sigma_j \in ECH_{*+2j}(Y, \xi, \Gamma; \mathbb{Z}_2)$ $j = 1, 2, \dots$

$$e(j) := c_{\sigma_j}(\lambda) - [2j \operatorname{vol}(Y, \lambda)]^{1/2} = O(j^{2/5}) \quad \text{as } j \to \infty$$

(W. Sun '18) $\qquad e(j) = O(j^{125/252})$

(Hutchings '19) if $Y = \partial X$, $X \subset \mathbb{R}^4$ domain $\qquad e(j) = O(j^{1/4})$

Viterbo conjecture: if X convex : $e(1) \leq 0$.

Cor 3 (C.G-S) $\pi_{ECH}(x) = \dfrac{2^d - 1}{\operatorname{vol}(Y, \lambda)} x^2 + O(x^{9/5})$; $d := \dim ECH_* + \dim ECH_{*+1}$, $* >> 0$.

$\zeta_{ECH}(s)$ continues meromorphically to $Re(s) > \frac{5}{3}$.

with single (simple) pole at $s = 2$ & $Res_{s=2}\, \zeta_{ECH} = \dfrac{2^d - 1}{\operatorname{vol}}$.

Conjecture: $e(j) = \begin{cases} O(1) \\ -\frac{1}{2} Ru(X) + o(1); & \text{if } \lambda \text{ generic} \end{cases}$

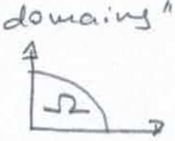
$\zeta_{ECH}(s)$ continues meromorphically to $\mathbb{C}$

$s = 1$ is a simple pole with $Res_{s=1}\, \zeta_{ECH} =$ "$Ru(X)$"

(Ruelle invariant)

(Hutchings '19) Conjecture holds for "strictly convex/concave toric domains".

i.e. $Y = \partial X_\Omega$; $X_\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid (|z_1|^2, |z_2|^2) \in \Omega\}$



(III) Proof sketch

Uses isomorphism

$$ECH(Y, \xi, \mathrm{P.D.}\, c_1(E)) \cong \check{HM}(Y, S^E)$$

E: cpx line bundle; S^E: spin^c str.

Hutchings; Taubes; Monopole-Floer homology, Kronheimer-Mrowka

$S^E = (\mathbb{C} \oplus \xi) \otimes E$; $c : T^*Y \to \operatorname{End}(S^E)$

rk 2 v.bundle; clifford multiplication.

$\bar{\omega}$:= harmonic representative of $c_1(\xi)$

$\mathcal{G}(Y) := C^\infty(Y, S^1)$ gauge group.

Chain gp: $\check{HC}(Y, s^E) = \mathbb{Z}_2\Big[\underset{\mathrm{Conn}(E)}{A}, \underset{C^\infty(S^E)}{\Psi}) \,\Big|\, \begin{array}{ll} c(*F_A + *i\bar{\omega} + ir\lambda) = \Psi\otimes\Psi^* - \frac{1}{2}|\Psi|^2 \\ D_A\Psi = 0 \end{array}\Big]$

$\oplus\ \mathbb{Z}_2\Big[(A,\Psi) \,\Big|\, \begin{array}{ll} F_A + i\bar{\omega} + ir\lambda = 0 \\ D_A\Psi = \lambda\Psi \text{ eigenvector } (\lambda > 0) \end{array}\Big] / \mathcal{G}($

variational equations for:

$$\underset{\text{Chern-Simons Dirac functional}}{\mathfrak{L}_r}(A,\Psi) := \underbrace{\frac{1}{2}\int_Y (A - A_0)\wedge(F_A + F_{A_0} - i\bar{\omega})}_{=:\, cs(A)} + \frac{1}{2}\int\langle D_A\Psi, \Psi\rangle \mathrm{vol} - \frac{ir}{2}\underbrace{\int\lambda\wedge F_A}_{=:\, e_\lambda(A)}$$

Floer homology is indep of $r > 0$

Choose a sequence $\sigma = [(A_r, \Psi_r)]$ rep. given class $\sigma \in \check{HM}(Y, s^E)$

$$(\Psi_r^\sigma)^{-1}(0) \longrightarrow (\alpha_j, m_j) \quad \text{Reeb orbit set.}$$

$$e_\lambda(A_r) \longrightarrow c_\sigma(\lambda) \quad \text{capacity} \quad \text{as } r \to \infty$$

$$gr(A_r, \Psi_r) = -\frac{1}{2\pi^2} cs(A_r) - \underset{\text{APS eta invariant}}{\eta}(\underset{\text{Hessian of } \mathfrak{L}_r}{H_{A_r,\Psi_r}})$$

Reg. estimates:

i) $cs(A) \leq c\, r^{2/3} e_\lambda(A)^{4/3}$

ii) $\dfrac{d\,cs}{dr} = r\dfrac{de_\lambda}{dr}$; $r > r_\sigma$ (threshold) } Taubes

iii) $\eta(H_{A,\Psi}) = O(r^{3/2})$ } (S'14)

In terms of $F(r) := \frac{1}{2} r_\sigma^2 \mathrm{vol} + \int_{r_\sigma}^{r} e_\lambda(A_s)\,ds$

$$e_\lambda(A) = F'(r) \quad \& \quad cs(A) = rF' - F$$

playing with (i) & (ii) (ODI's for F)

$$R\Big[1 - \frac{c}{r_0^{1/3}} R^{1/3}\Big] \leq F'(r) \leq \underbrace{\Big[\frac{gr(\sigma)}{r_\sigma} + r_0^{1/2}\Big]}_{=:\, R}\Big[1 + \frac{c}{r_0^{1/3}} R^{1/3}\Big] \quad \text{for } r_0 \geq O(gr(\sigma))^{1/2}$$

Use $gr(\sigma_j) = q + 2j$; $r_{\sigma_j} = \dfrac{j^{1/2}}{\mathrm{vol}^{1/2}} + O(1)$ (S'18) & set $r_0 = j^{4/5}$