ASYMPTOTICS OF THE ETA INVARIANT

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ABSTRACT. We prove an asymptotic bound on the eta invariant of a family of coupled Dirac operators on an odd dimensional manifold. In the case when the manifold is the unit circle bundle of a positive line bundle over a complex manifold, we obtain precise formulas for the eta invariant.

1. Introduction

The eta invariant was introduced by Atiyah, Patodi and Singer in [1] as a correction term to an index theorem for manifolds with boundary. Consider a first order, elliptic and self-adjoint operator A on a compact manifold. Formally, the eta invariant $\eta(A)$ of this operator can be interpreted as its signature, or the difference between the number of positive and the number of negative eigenvalues of A. In reality, since A has infinitely many eigenvalues of each sign this needs to be defined via regularization (see Section 2).

A key feature of the invariant $\eta(A)$, much like the signature of a matrix, is that it is *not* in general a continuous function of the operator A. In particular consider a smooth one-parameter family of operators A_t. The corresponding eta invariant $\eta(A_t)$ is then in general a discontinuous function of the parameter t, making it difficult to understand how it behaves as t varies. In this paper we shall investigate how the eta invariant of such a one parameter family behaves asymptotically as the parameter gets large.

More precisely, consider a compact, oriented Riemannian manifold (Y, g^{TY}) of odd dimension $n = 2m+1$, equipped with a spin structure. Let S be the corresponding spin bundle on Y. Let L be a Hermitian line bundle on Y. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(Y; i\mathbb{R})$ be an imaginary one form on Y. This gives a family $A_r = A_0 + ra$ of unitary connections on L, with $r \in \mathbb{R}$ being a real parameter . Each connection in this family gives a coupled Dirac operator D_{A_r} acting on sections of $S \otimes L$. Our first result, regarding the asymptotics of the reduced eta invariant $\bar{\eta}^r = \bar{\eta}(D_{A_r})$, is the following.

The research leading to the results contained in this paper has received funding from the European Research Council (E.R.C.) under European Union's Seventh Framework Program (FP7/2007-2013)/ ERC grant agreement No. 291060.

Theorem 1.1. *As $r \to \infty$, the reduced eta invariant satisfies the asymptotics*

$$\bar{\eta}^{r} = o(r^{\frac{n}{2}}). \tag{1.1}$$

It is an interesting question as to what extent the little $o(r^{\frac{n}{2}})$ estimate of Theorem 1.1 can be improved.

In order to investigate this question we consider the eta invariant of such a family in the case where Y is the total space of a circle bundle. In particular, we shall let Y be the space of unit elements of a positive line bundle $\mathcal{L} \to X$ over a complex manifold X. We shall further equip Y with an adiabatic family of metrics g_{ε}^{TY} (see Section 5). Under an appropriate choice of the family of connections, this gives the corresponding eta invariant $\bar{\eta}^{r,\varepsilon}$, with now an additional dependence on the adiabatic parameter ε. Letting $\hat{A}(X)$ denote the $\hat{A}$-genus of X, we now prove the following more precise formula for the eta invariant (see theorem Theorem 5.3)

Theorem 1.2. *The eta invariant $\bar{\eta}^{r,\varepsilon}$ satisfies the asymptotics*

$$\bar{\eta}^{r,\varepsilon} = \sum_{a=0}^{m} \left\{ \left(\frac{r^{a+1}}{(a+1)!} - \sum_{k=1}^{\left[r+\frac{\varepsilon m}{2}\right]} \frac{k^a}{a!} \right) \int_X c_1(\mathcal{L})^a \left[\hat{A}(X)\right]^{m-a} \right\} + O(1).$$

as $r \to \infty$.

From this formula we observe that $\bar{\eta}^{r,\varepsilon}$, in this case, exhibits jump discontinuities at integer values of $r + \frac{\varepsilon m}{2}$. Furthermore, the size of the jumps is growing at the rate $r^{\frac{n-1}{2}}$ as $r \to \infty$. Hence this calculation demonstrates that Theorem 1.1 cannot be improved beyond an $O(r^{\frac{n-1}{2}})$ estimate on the eta invariant.

The eta invariant is a non-local quantity. That is, it cannot be written as an integral over the manifold of a canonical differential from obtained from the symbol of the operator. This makes it difficult to compute the eta invariant explicitly. In the final section of this paper we give an exact formula for the eta invariant $\bar{\eta}^{r,\varepsilon}$, assuming the value of the adiabatic parameter ε to be small, using the adiabatic limit technique of Bismut-Cheeger, Dai and Zhang [5, 9, 16]. We refer to Theorem 5.7 for the exact formula arising from the computation. A striking feature of this formula is that it expresses the eta invariant $\bar{\eta}^{r,\varepsilon}$ in purely topological terms on the base X. This generalizes a similar known computation in dimension three of Nicolaescu [11].

An asymptotic result of the form Theorem 1.1 was used by Taubes in [13, 14] in order to prove the Weinstein conjecture on the existence of Reeb orbits on three dimensional contact manifolds. Our results improve the estimates obtained therein and could lead to further information regarding Reeb orbits. The three dimensional case has been further explored, under certain hypotheses, by Tsai in [15].

In another direction, the asymptotics considered in this paper are closely related to the asymptotic results of Bismut-Vasserot from [4, 7]. In [7] the authors considered the Dolbeault Laplacian $\Delta^p_{\bar{\partial}_k}$ acting on p-forms, with values in a tensor power $\mathcal{L}^{\otimes k}$, of the positive line bundle $\mathcal{L}$ considered here earlier. They then derived an asymptotic formula for the holomorphic torsion of $\Delta^p_{\bar{\partial}_k}$ in the limit as $k \to \infty$. The asymptotics of the heat trace of $\Delta^p_{\bar{\partial}_k}$ were used in [4] to prove Demailly's asymptotic Morse inequalities. This Laplacian will arise in our computations in Section 5 and it would be interesting to explore this connection further.

The paper is organized as follows. In Section 2 we begin with preliminary notations and facts used in the paper. In Section 3 we derive asymptotics of heat traces required in the proof of Theorem 1.1. In Section 4 we derive the asymptotics of the spectral measure of a rescaled Dirac operator and prove Theorem 1.1. In Section 5 we consider the eta invariant of the circle bundle. There we prove Theorem 5.3 and give the exact computation for the eta invariant of Theorem 5.7.

2. Preliminaries

Consider a compact, oriented, Riemannian manifold (Y, g^{TY}) of odd dimension n equipped with a spin structure. Let S be the corresponding spin bundle on Y. Let ∇^{TY} denote the Levi-Civita connection on TY. This lifts to the spin connection ∇^S on the spin bundle S. We denote the Clifford multiplication endomorphism by $c : T^*Y \to S \otimes S^*$ satisfying

$$c(a)^2 = -|a|^2, \quad \forall a \in T^*Y.$$

Let L be a Hermitian line bundle on Y. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(Y; i\mathbb{R})$ be an imaginary 1-form on Y. This gives a family $A_r = A_0 + ra$ of unitary connections on L. We denote by $\nabla^r = \nabla^S \otimes 1 + 1 \otimes A_r$ the tensor product connection on $S \otimes L$. Each such connection defines a coupled Dirac operator

$$D_{A_r} = c \circ \nabla^r : C^\infty(Y; S \otimes L) \to C^\infty(Y; S \otimes L).$$

Each Dirac operator D_{A_r} is elliptic and self-adjoint. It hence possesses a discrete spectrum of eigenvalues. Define the eta function of D_{A_r} by the formula

$$\eta(D_{A_r}, s) = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \mathrm{Spec}(D_{A_r})}} \mathrm{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}\left(D_{A_r} e^{-tD^2_{A_r}}\right) dt. \tag{2.1}$$

Here, and in the remainder of the paper, we use the convention that $\mathrm{Spec}(D_{A_r})$ denotes a multiset with each eigenvalue of D_r being counted with its multiplicity. The above series converges for $\mathrm{Re}(s) > n$. It was shown in [1, 2] that the eta function possesses a meromorphic continuation to the entire complex s-plane and has no pole at zero. Its value at zero is defined to be the eta invariant of the operator

$\eta(D_{A_r}) = \eta(D_{A_r}, 0)$. By including the zero eigenvalue in (2.1), with an appropriate convention, we may define a variant known as the reduced eta invariant by

$$\bar{\eta}(D_{A_r}) = \frac{1}{2}\left\{\dim \ker(D_{A_r}) + \eta(D_{A_r})\right\}.$$

We shall henceforth denote the reduced eta invariant by the shorthand $\bar{\eta}^r = \bar{\eta}(D_{A_r})$, and would like to investigate its asymptotics for large r. Our results will apply equally well to the unreduced version $\eta^r = \eta(D_{A_r})$.

Let L^r_t denote the Schwartz kernel of the operator $D_{A_r} e^{-tD^2_{A_r}}$ on the product $Y \times Y$. Denote by $\mathrm{tr}\left(L^r_t(x,x)\right)$ the pointwise trace of L^r_t along the diagonal. We may now analogously define the function

$$\eta(D_{A_r}, s, x) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}\left(L^r_t(x,x)\right) dt. \tag{2.2}$$

In [6] theorem 2.6, the authors showed that for $\mathrm{Re}(s) > -2$, the function $\eta(D_{A_r}, s, x)$ is holomorphic in s and smooth in x. From (2.2) it is clear that this is equivalent to

$$\mathrm{tr}(L^r_t) = O(t^{\frac{1}{2}}), \quad \text{as } t \to 0. \tag{2.3}$$

3. Asymptotics of the heat kernel

In order to control the eta invariant we shall need to find the asymptotics for the heat traces of D_{A_r}. We begin with an estimate on its heat kernel. We denote by dy the Riemannian volume form on (Y, g). All kernels will be calculated with respect to dy in what follows. Let i_g denote the injectivity radius of Y. Let $\rho(x,y)$ denote the geodesic distance function between two given points $x, y \in Y$. Define a function on $Y \times Y$ by the following formula

$$h_t(x,y) = \frac{e^{-\frac{\rho(x,y)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}.$$

Let $H^r_t(x,y)$ denote the kernel of $e^{-tD^2_{A_r}}$ for $t > 0$. We now have the following estimate.

Proposition 3.1. *There exist positive constants c_1, c_2 independent of r such that*

$$|H^r_t(x,y)| \leq c_1 h_{2t}(x,y) e^{c_2 r t} \tag{3.1}$$

for all $x, y \in Y$, $t > 0$ and $r \geq 1$.

Proof. Let ∇^S denote the spin connection on S and $\nabla^r = \nabla^S \otimes 1 + 1 \otimes A_r$ be the tensor product connection on $S \otimes L$. First observe that for fixed y the section $s_t(.) = H^r_t(., y)$ satisfies the heat equation $\partial_t s_t = -D^2_{A_r} s_t$. The Weitzenbock formula gives

$$D_{A_r}^2 = \nabla^{r*}\nabla^r + c(F_{A_0}) + rc(da) + \frac{\kappa}{4}$$

where F_{A_0}, κ denote the curvature of A_0 and the scalar curvature of g respectively. Using the Weitzenbock formula and the heat equation $\partial_t s_t = -D_{A_r}^2 s_t$, we now see that the function $f_t = |s_t|$ obeys the inequality

$$\partial_t f_t \leq -d^* df_t + c_1(r+1) f_t \tag{3.2}$$

for some constant $c_1 > 0$ independent of r. Hence the function $f_t^0 = e^{-c_1(r+1)t} f_t$ satisfies the inequality

$$\partial_t f_t^0 \leq -d^* df_t^0. \tag{3.3}$$

Let $\Phi_t(x,y)$ denote the heat kernel e^{-td^*d} for the Laplace operator acting on functions on Y. Now since $|H_t^r(x,y)|$ and $\Phi_t(x,y)$ have the same asymptotics as $t \to 0$, an application of the maximum principle for the heat equation gives

$$f_t^0 \leq \Phi_t(x,y) \tag{3.4}$$

for all time $t > 0$. Next we use the estimate

$$\Phi_t(x,y) \leq c_3 e^t h_{2t}(x,y), \quad \forall t > 0, \tag{3.5}$$

on the heat kernel. Equation (3.5) follows for large time since the heat kernel is bounded

$$\Phi_t(x,y) \leq c_4 \quad \text{for} \quad \forall x, y \in Y \text{ and } t \geq 1.$$

For small time, (3.5) follows from the heat kernel estimate of [8]. The proposition now follows from (3.4) and (3.5). □

Following this we shall prove a more refined estimate on the heat kernel comparing it with Mehler's kernel. We first recall the definition of the Mehler's kernel. Define an antisymmetric endomorphism A of TY via

$$ida(X,Y) = g(X, AY), \forall X, Y \in TY. \tag{3.6}$$

Let x, y be two points of Y such that $\rho(x,y) < i_g$. Let $v \in T_yY$ such that $x = \exp_v y$. Define a function on a geodesic neighborhood of the diagonal in $Y \times Y$ by

$$m_t^r(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \det{}^{\frac{1}{2}} \left(\frac{rtA_y}{\sinh rtA_y} \right) \exp\left\{ -\frac{1}{4t} g(v, rtA_y \coth(rtA_y) v) \right\} \tag{3.7}$$

Now let π_1 and π_2 denote the projections onto the two factors of $Y \times Y$ and define a section $e^{-tF_{A_r}}$ of $\pi_1^*(S \otimes L) \otimes \pi_2^*(S \otimes L)^*$, in a geodesic neighborhood of the diagonal.

This restricts to $e^{-tF_{A_r}}|_\Delta = e^{-tc(F_{A_r})}$ at the diagonal Δ and is parallel along geodesics $(\exp_{tv}(y), y)$. Consider a smooth cutoff function satisfying

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x| \geq i_g \\ 1 & \text{if} \quad |x| \leq \frac{i_g}{2}. \end{cases} \tag{3.8}$$

Mehler's kernel is now defined via

$$M_t^r(x,y) = \chi(\rho(x,y)) m_t^r(x,y) e^{-tF_{A_r}}. \tag{3.9}$$

Proposition 3.2. *There exist positive constants c_1 and c_2 independent of r such that*

$$|H_t^r(x,y) - M_t^r(x,y)| \leq c_1 h_{8t}(x,y) t^{\frac{1}{2}} e^{c_2 rt}, \tag{3.10}$$

for all $x, y \in Y$,$t > 0$ and $r \geq 1$.

Proof. First fix a point y and a set of geodesic coordinates centered at y. Now choose a basis s_α for S_y and a basis l for L_y. Parallel transport this basis along geodesics using the connections ∇^S, A_r to obtain trivializations $s_\alpha(x)$ and $l(x)$ of S and L respectively near y. Now define local orthonormal sections of $(S \otimes L) \otimes (S \otimes L)^*_y$ via

$$t_{\alpha\beta} = s_\alpha(x) \otimes l(x) \otimes s^*_\beta \otimes l^*. \tag{3.11}$$

The connection ∇^r can be expressed in this frame and these coordinates as

$$\nabla_i^r = \partial_i + A_i^r + \Gamma_i, \tag{3.12}$$

where each A_i^r is a Christoffel symbol of A_r (or $\dim(S \otimes L)_y$ copies of it) and each Γ_i is a Christoffel symbol of the spin connection on S. Since the section $l(x)$ is obtained via parallel transport along geodesics, the connection coefficient A_i^r maybe written in terms of the curvature F_{ij}^r of A^r via

$$A_i^r(x) = \int_0^1 d\rho (\rho x^j F_{ij}^r(\rho x)), \tag{3.13}$$

with the Einstein summation convention being used. The dependence of the curvature coefficients F_{ij}^r on the parameter r is seen to be linear $F_{ij}^r = F_{ij}^0 + r(da)_{ij}^0$ despite the fact that they are expressed in the r dependent frame l. This is because a gauge transformation from an r independent frame into l changes the curvature coefficient by conjugation. Since L is a line bundle this is conjugation by a function and hence does not change the coefficient. Next, using the Taylor expansion $(da)_{ij} = (da)_{ij}(0) + x^k a_{ijk}$, we see that the connection ∇^r has the form

$$\nabla_i^r = \partial_i + \frac{1}{2} r x^j (da)_{ij}(0) + x^j A_{ij}^0 + r x^j x^k A_{ijk} + \Gamma_i. \tag{3.14}$$

Here $A^0_{ij} = \int_0^1 d\rho(\rho F^0_{ij}(\rho x))$, $A_{ijk} = \int_0^1 d\rho(\rho a_{ijk}(\rho x))$ and Γ_i are all independent of r.

Now using Weitzenbock's formula, we note that the operator $D^2_{A_r}$ has the form

(3.15)
$$D^2_{A_r} = \mathcal{H} + E, \quad \text{with}$$
(3.16)
$$\mathcal{H} = -\partial_i^2 - r(da)_{ij}(0)x^j\partial_i - \frac{r^2}{4}x^ix^j\left(\sum_k (da)_{ik}(0)(da)_{jk}(0)\right) + c\left(F_{A_r}\right) \quad \text{and}$$
(3.17)
$$E = P_{ijkl}x^kx^l\partial_i\partial_j + Q_{ijk}rx^jx^k\partial_i + R_i\partial_i + S_{ijk}r^2x^ix^jx^k + T_irx^i + U.$$

Here P, Q, R, S, T and U are each smooth endomorphisms of $S \otimes L$ independent of r. Since $(\partial_t + D^2_{A_r})H_t = 0$ we now have

$$\left(\partial_t + D^2_{A_r}\right)\left(H^r_t - M^r_t\right) = -\left(\partial_t + \mathcal{H}\right)M^r_t - EM^r_t. \tag{3.18}$$

Note that the right hand side of (3.18) is zero for $\rho(x, y) > i_g$ since M^r_t is supported in a geodesic neighborhood of the diagonal, by (3.9). From the defining equations (3.7) and (3.9), Mehler's kernel is given in geodesic coordinates via

(3.19)
$$M^r_t(x, y) = \chi(\rho(x, y))m^r_t(x, y)e^{-tF_{A_r}}$$
(3.20)
$$= \chi(|x|)\frac{1}{(4\pi t)^{\frac{n}{2}}}\det{}^{\frac{1}{2}}\left(\frac{rtA_y}{\sinh rtA_y}\right)\exp\left\{-\frac{1}{4t}\left\langle x, rtA_y\coth(rtA_y)x\right\rangle\right\}e^{-tc(F_{A_r})}.$$

We now differentiate (3.20) using (3.15)-(3.17) to compute the right hand side of (3.18). By Mehler's formula, see section 4.2 in [3], we have $(\partial_t + \mathcal{H})\left\{m^r_t(x, y)e^{-tF_{A_r}}\right\} = 0$ for $d(x, y) < \frac{i_g}{2}$. Differentiating the rest, we observe that the right hand side of (3.18) has the form of a finite sum

$$-\left(\partial_t + \mathcal{H}\right)M^r_t - EM^r_t = \sum_{(k,d,I)} t^k r^d x^I P_{k,d,I}(x)g_{k,d,I}(rt)M^r_t, \text{ where} \tag{3.21}$$

- each $(k, d, I) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0^n$ and satisfies the inequality

$$d \leq k + \frac{|I|}{2} + \frac{1}{2}, \tag{3.22}$$

- each $P_{k,d,I}$ appearing in (3.21) denotes a smooth endomorphism of $S \otimes L$, independent of r, and supported on $\rho(x,y) < i_g$
- and $g_{k,d,I}$ in (3.21) denote functions, coming from the matrix entries of $rtA_y \coth(rtA_y)$, each satisfying an exponential bound

$$|g_{k,d,I}(x)| < c_1 e^{c_2 x}. \tag{3.23}$$

Now since the kernels H_t^r and M_t^r both have the same asymptotics as $t \to 0$, Duhamel's principle, using (3.18), gives

$$H_t^r - M_t^r = \int_0^t e^{-(t-s)D_{A_r}^2} \left\{ -(\partial_s + \mathcal{H}) M_s^r - E M_s^r \right\} ds. \tag{3.24}$$

Now we substitute (3.21) into (3.24). Following this substitution, we use the heat kernel bound (3.1), (3.23) and the bound

$$|M_t^r(x,y)| \leq c_3 e^{c_4 rt} h_t(x,y) \tag{3.25}$$

for constants c_3 and c_4 independent of r. These bounds can be used to estimate the right hand side of (3.24) by a sum of finitely many terms of the form

$$c_5 e^{c_6 rt} \int_0^t ds \left(\int_Y h_{t-s}(z,x) s^k r^d \rho(x,y)^I h_s(x,y) dx \right), \tag{3.26}$$

with each multi-index (k,d,I) above satisfying (3.22). Finally, (3.22) and the inequalities

$$\rho(x,y)^I h_t(x,y) \leq C t^{\frac{1}{2}|I|} h_{2t}(x,y), \tag{3.27}$$

$$\int_0^t s^{-\frac{1}{2}} ds \left(\int_Y dx h_{2(t-s)}(z,x) h_{2s}(x,y) \right) \leq C t^{\frac{1}{2}} h_{8t}(z,y), \tag{3.28}$$

(see Section A for a proof of (3.28)) give (3.10). □

3.1. **Bound on the trace of $D_{A_r} e^{-tD_{A_r}^2}$.** We now turn to bound the pointwise $\mathrm{tr}(D_{A_r} e^{-tD_{A_r}^2})$. To this end, first consider the expansion for the heat kernel $H_t^r(x,y)$ given by

$$H_t^r(x,y) \sim \chi(\rho(x,y)) h_t(x,y) \left(b_0^r(x,y) + b_1^r(x,y)t + b_2^r(x,y)t^2 + \dots \right). \tag{3.29}$$

Here the coefficients b_k^r are smooth sections defined on the neighborhood $\rho(x,y) < i_g$ of the diagonal in $Y \times Y$. They are generated by solving a recursive system of transport equations along geodesics as in chapter 7 of [12]. The kernel $L_t^r(x,y)$ of $D_{A_r} e^{-tD_{A_r}^2}$ is simply $L_t^r = D_{A_r} H_t^r$. It hence has an expansion given by

(3.30)
$$L_t^r(x,y) \sim h_t(x,y)\left(c(d\chi)+c\left(-\frac{\rho d\rho}{2t}\right)\right)\left\{b_0^r(x,y)+b_1^r(x,y)t+b_2^r(x,y)t^2+\dots\right\}$$
$$+\, h_t(x,y)\left\{D_{A_r}b_0^r(x,y)+D_{A_r}b_1^r(x,y)t+D_{A_r}b_2^r(x,y)t^2+\dots\right\}.$$

By (3.8), $c(d\chi)$ is an endomorphism of the spin bundle supported in the region where $\frac{i_g}{2}\le\rho(x,y)\le i_g$. By (2.3) and (3.30), the pointwise trace $\mathrm{tr}(D_{A_r}e^{-tD_{A_r}^2})$ along the diagonal has an expansion starting with a leading term of order $t^{\frac{1}{2}}$. Since the restriction to the diagonal of the $c(d\chi)+c\left(-\frac{\rho d\rho}{2t}\right)$ term in (3.30) is zero, this implies that

$$\mathrm{tr}\left(D_{A_r}b_k^r(x,x)\right)=0,\quad \text{for } k<\frac{n+1}{2}, \tag{3.31}$$

at each point on the diagonal. To bound the trace of L_t^r we will need a lemma giving a schematic form for the coefficients $b_k^r(x,y)$. We again work in geodesic coordinates centered at a point $y\in Y$. Each heat kernel coefficient can be written in terms of the frame (3.11) as

$$b_k^r=\sum_{\alpha\beta}f_{\alpha\beta,k}^r t_{\alpha\beta} \tag{3.32}$$

for some functions $f_{\alpha\beta,k}^r$.

Lemma 3.3. *Each function $f_{\alpha\beta,k}^r$ appearing in (3.32) can be written as a finite sum*

$$f_{\alpha\beta,k}^r=\sum_{(d,I)}r^d x^I f_{d,I} \tag{3.33}$$

for some functions $f_{d,I}$, independent of r. Moreover, each multi-index $(d,I)\in\mathbb{N}_0\times\mathbb{N}_0^n$ appearing in (3.33) satisfies the inequality

$$d\le k+\frac{1}{2}|I|. \tag{3.34}$$

Proof. The heat kernel coefficients $b_k^r(x,y)$ are given, as in chapter 7 of [12], by the recursion

$$b_0^r(x,y) = \sum_\alpha g^{-1/4}(x)t_{\alpha\alpha}, \tag{3.35}$$

$$b_k^r(x,y) = -\frac{1}{g^{1/4}(x)}\int_0^1 \rho^{k-1}g^{1/4}(\rho x)D_{A_r}^2 b_{k-1}(\rho x)d\rho,\quad \text{for}\quad k\ge 1, \tag{3.36}$$

where g denotes the determinant of the metric on Y. Hence b_0^r is clearly seen to be of the form (3.33). Equations (3.15)-(3.17) and (3.36) imply that b_k^r has the form (3.33) assuming it to be true for b_{k-1}^r. The lemma now follows by induction on k. □

Following this we are ready to bound the pointwise trace $\text{tr}(D_{A_r}e^{-tD_{A_r}^2})$. The above lemma will play an important role in the proposition below.

Proposition 3.4. *There exist constants c_1, c_2, independent of r, such that the pointwise trace $tr(D_{A_r}e^{-tD_{A_r}^2})$ satisfies the estimate*

$$\|tr(D_{A_r}e^{-tD_{A_r}^2})\|_{C^0} \leq c_1 r^{\frac{n}{2}} e^{c_2 rt}, \tag{3.37}$$

for all $t > 0$, $r \geq 1$.

Proof. Consider the remainder obtained after subtracting the first $\frac{n-1}{2}$ terms of the kernel expansion (3.30)

$$L_t^{r,\frac{n-1}{2}} = L_t^r - D_{A_r}(\chi h_t(b_0^r(x,y) + \ldots + t^{\frac{n-1}{2}} b_{\frac{n-1}{2}}^r)). \tag{3.38}$$

From (3.30) and (3.31) we see that

$$\text{tr}(L_t^r) = \text{tr}(L_t^{r,\frac{n-1}{2}}), \tag{3.39}$$

and it hence suffices to bound $L_t^{r,\frac{n-1}{2}}$. We clearly have $L_t^{r,\frac{n-1}{2}} = D_{A_r} H_t^{r,\frac{n-1}{2}}$ with $H_t^{r,\frac{n-1}{2}}$ being the analogous remainder in the kernel expansion for the heat trace

$$H_t^{r,\frac{n-1}{2}} = H_t^r - \chi h_t(b_0^r(x,y) + \ldots + t^{\frac{n-1}{2}} b_{\frac{n-1}{2}}^r). \tag{3.40}$$

Let us denote

$$S_t^{r,\frac{n-1}{2}} = h_t(b_0^r(x,y) + \ldots + t^{\frac{n-1}{2}} b_{\frac{n-1}{2}}^r). \tag{3.41}$$

The result of applying the heat operator to (3.38) is then

$$(\partial_t + D_{A_r}^2)(L_t^{r,\frac{n-1}{2}}) = Q_t^{r,\frac{n-1}{2}} + R_t^{r,\frac{n-1}{2}}, \quad \text{where} \tag{3.42}$$

$$Q_t^{r,\frac{n-1}{2}} = -\chi(\partial_t + D_{A_r}^2) D_{A_r} S_t^{r,\frac{n-1}{2}} \quad \text{and} \tag{3.43}$$

$$R_t^{r,\frac{n-1}{2}} = -D_{A_r}^2\left\{c(d\chi)S_t^{r,\frac{n-1}{2}}\right\} - D_{A_r}\left\{c(d\chi)D_{A_r}S_t^{r,\frac{n-1}{2}}\right\} - \left\{c(d\chi)D_{A_r}^2 S_t^{r,\frac{n-1}{2}}\right\}. \tag{3.44}$$

In other words, $R_t^{r,\frac{n-1}{2}}$ is the sum of the terms obtained when some derivative differentiates the cutoff function χ in (3.38). Now since $L_t^{r,\frac{n-1}{2}} \to 0$ as $t \to 0$, Duhamel's principle applied to (3.42) gives

$$L_t^{r,\frac{n-1}{2}} = E_t^{r,\frac{n-1}{2}} + F_t^{r,\frac{n-1}{2}}, \quad \text{where} \tag{3.45}$$

$$E_t^{r,\frac{n-1}{2}}(z,y) = \int_0^t ds \left(\int_Y dx\, H_{t-s}^r(z,x) Q_s^{r,\frac{n-1}{2}}(x,y) \right) \quad \text{and} \tag{3.46}$$

$$F_t^{r,\frac{n-1}{2}}(z,y) = \int_0^t ds \left(\int_Y dx\, H_{t-s}^r(z,x) R_s^{r,\frac{n-1}{2}}(x,y) \right). \tag{3.47}$$

We first bound $F_t^{r,\frac{n-1}{2}}$, again working in geodesic coordinates and the frame (3.11). Using (3.14) and the fact that Clifford multiplication is parallel, the Dirac operator is seen to be of the form

$$D_A = A_i \partial_i + r x_i B_i + C, \tag{3.48}$$

where A_i, B_i and C are endomorphisms of $S \otimes L$ independent of r. Using (3.32),(3.41),(3.44) and (3.48) we may write

$$R_t^{r,\frac{n-1}{2}} = h_t \sum_{k=0}^{\frac{n-1}{2}} t^k \left(\sum f_{\alpha\beta,k}^{r,0} t_{\alpha\beta} \right) \tag{3.49}$$

in the frame (3.11) for some coefficient functions $f_{\alpha\beta}^{r,0}$. By (3.33) each $f_{\alpha\beta}^{r,0}$ can be written as a finite sum

$$f_{\alpha\beta}^{r,0} = \sum_{(d,I)} r^d x^I f_{d,I}^0 \tag{3.50}$$

for some functions $f_{d,I}^0$ independent of r. Since $d\chi = 0$ in a neighborhood of the diagonal, (3.44) implies each $f_{\alpha\beta}^{r,0}$ vanishes to infinite order near the diagonal. Hence by (3.34) we may assume that each multi-index (d,I) in (3.50) satisfies $d \le k + \frac{1}{2}|I| + 1$. We now substitute (3.49) and (3.50) in (3.47). Using this substitution along with the heat kernel bound (3.1), we may bound $|F_t^{r,\frac{n-1}{2}}(y,y)|$ by a sum of terms of the form (3.26) each satisfying $d \le k + \frac{1}{2}|I| + 1$. The inequalities (3.27)-(3.28) then give the estimate

$$|F_t^{r,\frac{n-1}{2}}(y,y)| \le c_1 r^{\frac{n}{2}} e^{c_2 r t}. \tag{3.51}$$

Next we estimate $E_t^{r,\frac{n-1}{2}}$. Following (3.41) and using the cancellations in the kernel expansion, resulting from the transport equation, we see that

$$Q_t^{r,\frac{n-1}{2}} = -\chi(\partial_t + D_{A_r}^2) D_{A_r} S_t^{r,\frac{n-1}{2}} \tag{3.52}$$

$$= \chi h_t t^{\frac{n-1}{2}} \left\{ -D_{A_r}^3 b_{\frac{n-1}{2}}^r + c\left(\frac{\rho d\rho}{2t}\right) D_{A_r}^2 b_{\frac{n-1}{2}}^r \right\}. \tag{3.53}$$

Equation (3.46) now gives

$$E_t^{r,\frac{n-1}{2}}(y,y) = \int_0^t ds \left(\int_Y dx\, H_{t-s}^r(y,x) \chi h_s(x,y) s^{\frac{n-1}{2}} \left\{ -D_{A_r}^3 b_{\frac{n-1}{2}}^r(x,y) + c\left(\frac{\rho d\rho}{2s}\right) D_{A_r}^2 b_{\frac{n-1}{2}}^r(x,y) \right\} \right). \tag{3.54}$$

We denote by $U_t^{r,\frac{n-1}{2}}$ and $V_t^{r,\frac{n-1}{2}}$ the kernels obtained by replacing H_{t-s}^r in (3.54) by $(H_{t-s}^r - M_{t-s}^r)$ and M_{t-s}^r respectively

$$U_t^{r,\frac{n-1}{2}}(y,y) = \int_0^t ds \left(\int_Y dx\, \left(H_{t-s}^r(y,x) - M_{t-s}^r(y,x)\right) \chi h_s(x,y) s^{\frac{n-1}{2}} \left\{ -D_{A_r}^3 b_{\frac{n-1}{2}}^r(x,y) + c\left(\frac{\rho d\rho}{2s}\right) D_{A_r}^2 b_{\frac{n-1}{2}}^r(x,y) \right\} \right), \tag{3.55}$$

$$V_t^{r,\frac{n-1}{2}}(y,y) = \int_0^t ds \left(\int_Y dx\, M_{t-s}^r(y,x) \chi h_s(x,y) s^{\frac{n-1}{2}} \left\{ -D_{A_r}^3 b_{\frac{n-1}{2}}^r(x,y) + c\left(\frac{\rho d\rho}{2s}\right) D_{A_r}^2 b_{\frac{n-1}{2}}^r(x,y) \right\} \right). \tag{3.56}$$

It is clear that

$$E_t^{r,\frac{n-1}{2}} = U_t^{r,\frac{n-1}{2}} + V_t^{r,\frac{n-1}{2}}. \tag{3.57}$$

We first bound $U_t^{r,\frac{n-1}{2}}$, again working in geodesic coordinates and the frame (3.11). In terms of the orthonormal frame we may write

$$D_{A_r}^3 b_{\frac{n-1}{2}}^r = \sum f_{\alpha\beta}^{r,1} t_{\alpha\beta}, \quad D_{A_r}^2 b_{\frac{n-1}{2}}^r = \sum f_{\alpha\beta}^{r,2} t_{\alpha\beta} \tag{3.58}$$

for some coefficient functions $f^{r,1}_{\alpha\beta}$ and $f^{r,2}_{\alpha\beta}$. Using (3.33) and (3.48) these can be expressed as finite sums

$$f^{r,1}_{\alpha\beta} = \sum_{(d,I)\in S_1} x^I r^d f^1_{d,I}, \tag{3.59}$$

$$f^{r,2}_{\alpha\beta} = \sum_{(d,I)\in S_2} x^I r^d f^2_{d,I}, \tag{3.60}$$

where $f^1_{d,I}, f^2_{d,I}$ are functions independent of r and S_1, S_2 are finite subsets of $\mathbb{N}_0 \times \mathbb{N}_0^n$. Moreover, (3.34) now gives

$$d \le \frac{n-1}{2} + \frac{1}{2}|I| + \frac{3}{2} \qquad \forall (d,I) \in S_1 \quad \text{and} \tag{3.61}$$

$$d \le \frac{n-1}{2} + \frac{1}{2}|I| + 1 \qquad \forall (d,I) \in S_2. \tag{3.62}$$

Again we substitute (3.58)-(3.60) into (3.55). This substitution, the bound (3.10), along with the inequalities (3.27)-(3.28) and (3.59)-(3.60) now give the estimate

$$|U_t^{r,\frac{n-1}{2}}(y,y)| \le c_3 r^{\frac{n}{2}} e^{c_4 rt}. \tag{3.63}$$

Next we estimate $V_t^{r,\frac{n-1}{2}}$. First we use a Taylor expansion to write

$$f^1_{d,I} = g^1_{d,I} + x_i h^1_{d,I,i} \quad \text{and} \quad f^2_{d,I} = g^2_{d,I} + x_i h^2_{d,I,i} \tag{3.64}$$

where each of $g^1_{d,I}$ and $g^2_{d,I}$ is an even function in these coordinates. We now let

$$\bar{S}_1 = \left\{ (d,I) \in S_1 | d = \frac{n-1}{2} + \frac{1}{2}|I| + \frac{3}{2} \right\}, \tag{3.65}$$

$$\bar{S}_2 = \left\{ (d,I) \in S_2 | d = \frac{n-1}{2} + \frac{1}{2}|I| + 1 \right\}, \tag{3.66}$$

and define

$$\bar{f}^{r,1}_{\alpha\beta} = \sum_{(d,I)\in \bar{S}_1} x^I r^d g^1_{d,I}, \quad \tilde{f}^{r,1}_{\alpha\beta} = \sum_{(d,I)\in S_1\setminus \bar{S}_1} x^I r^d g^1_{d,I} + \sum_{(d,I)\in S_1} x^I r^d \left(x_i h^1_{d,I,i} \right), \tag{3.67}$$

$$\bar{f}^{r,2}_{\alpha\beta} = \sum_{(d,I)\in \bar{S}_2} x^I r^d g^2_{d,I}, \quad \tilde{f}^{r,2}_{\alpha\beta} = \sum_{(d,I)\in S_2\setminus \bar{S}_2} x^I r^d g^2_{d,I} + \sum_{(d,I)\in S_2} x^I r^d \left(x_i h^2_{d,I,i} \right). \tag{3.68}$$

Clearly, by (3.59)-(3.60) and (3.64)-(3.68),

$$f^{r,1}_{\alpha\beta} = \bar{f}^{r,1}_{\alpha\beta} + \tilde{f}^{r,1}_{\alpha\beta} \quad \text{and} \quad f^{r,2}_{\alpha\beta} = \bar{f}^{r,2}_{\alpha\beta} + \tilde{f}^{r,2}_{\alpha\beta}.$$

Next we claim that the contribution of $\bar{f}^{r,1}_{\alpha\beta}$ to $V_t^{r,\frac{n-1}{2}}$ is zero. To see this, first observe that $(d,I)\in\bar{S}_1$ implies $|I|$ is odd by (3.65). Hence $\bar{f}^{r,1}_{\alpha\beta}$ is an odd function, using (3.67) and the fact that $g^1_{d,I}$ is even. Hence the integral corresponding to $\bar{f}^{r,1}_{\alpha\beta}$ in (3.56) is zero, being the integral of an odd function in these coordinates. Similarly, we claim that the contribution of $\bar{f}^{r,2}_{\alpha\beta}$ to $V_t^{r,\frac{n-1}{2}}$ is zero. This time, $(d,I)\in\bar{S}_2$ implies $|I|$ is even by (3.66). Hence $\bar{f}^{2}_{\alpha\beta}$ is an even function using (3.68). However the integral corresponding to $\bar{f}^{r,2}_{\alpha\beta}$ in (3.56) is still the integral of an odd function in these coordinates, because of the $c\left(\frac{\rho d\rho}{2s}\right)$ term in (3.56).

Following this the contribution of $\tilde{f}^{r,1}_{\alpha\beta}$ to $V_t^{r,\frac{n-1}{2}}(y,y)$ can be bounded by a finite sum of terms of the form

$$c_1 e^{c_2 rt}\int_0^t ds\left(\int_Y h_{2(t-s)}(y,x)s^{\frac{n-1}{2}}r^d\rho(x,y)^I h_s(x,y)dx\right),\quad \text{with}\quad d\leq\frac{n-1}{2}+\frac{1}{2}|I|+1.$$

Again using the inequalities (3.27)-(3.28), and estimating the contribution of $\tilde{f}^{r,2}_{\alpha\beta}$ in similar fashion, gives the estimate

$$|V_t^{\frac{n-1}{2}}(y,y)|\leq c_5 r^{\frac{n}{2}}e^{c_6 rt}. \tag{3.69}$$

Following (3.63), (3.69) and (3.57) we obtain the estimate

$$|E_t^{\frac{n-1}{2}}(y,y)|\leq c_7 r^{\frac{n}{2}}e^{c_8 rt}, \tag{3.70}$$

for constants c_5 and c_6 independent of r. Equations (3.45), (3.51) and (3.70) then give

$$|L_t^{\frac{n-1}{2}}(y,y)|\leq c_9 r^{\frac{n}{2}}e^{c_{10} rt}, \tag{3.71}$$

for constants c_9 and c_{10} independent of r. The proposition now follows from (3.39) and (3.71). □

4. Asymptotics of the spectral measure

We now consider the rescaled operator $D_r=\frac{1}{\sqrt{r}}D_{A_r}$. Here we shall find the asymptotics of its spectral measure using the heat trace estimates of the previous section. We first consider the heat traces of D_r.

Theorem 4.1. *For any $t>0$,*

$$\lim_{r\to\infty} r^{-\frac{n}{2}} tr(e^{-tD_r^2}) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_Y \det{}^{\frac{1}{2}}\left(\frac{tA_y}{\tanh tA_y}\right) dy \quad \text{and} \tag{4.1}$$

$$\lim_{r\to\infty} r^{-n/2} tr\left(D_r e^{-tD_r^2}\right) = 0, \tag{4.2}$$

where A is as defined by (3.6)*. The convergences above are uniform in compact intervals of $t\in\mathbb{R}_{>0},\mathbb{R}_{\geq 0}$ respectively.*

Proof. If H'_t denotes the kernel of $e^{-tD_r^2}$ it is clear that $H'_t = H_{\frac{t}{r}}$ after rescaling. Hence proposition 3.2 gives the estimate

$$|H'_t - M_{\frac{t}{r}}| \leq c_1 h_{\frac{8t}{r}} t^{\frac{1}{2}} r^{-\frac{1}{2}} e^{c_2 t} \tag{4.3}$$

for some constants c_1, c_2 independent of r. Hence $\mathrm{tr}(e^{-tD_r^2}) - \mathrm{tr}(M_{\frac{t}{r}}) = O(r^{\frac{n-1}{2}})$, $\forall t > 0$. It now remains to compute $\mathrm{tr}(M_{\frac{t}{r}})$ in order to prove (4.1). By (3.9), the highest order part in r of $\mathrm{tr}(M_{\frac{t}{r}})$ is given by

$$\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_Y \det{}^{\frac{1}{2}}\left(\frac{tA_y}{\sinh tA_y}\right)\mathrm{tr}\left(e^{-tc(da)}\right) dy.$$

Since A is an antisymmetric endomorphism it maybe diagonalized to give an orthonormal basis $e, e_1, f_1, \ldots, e_m, f_m$ of T_yY with eigenvalues $0, \pm\lambda_1, \ldots, \pm\lambda_m$ ($\lambda_i \geq 0$) respectively. We hence have $da = i\left(\lambda_1 e_1\wedge f_1 + \ldots + \lambda_m e_m \wedge f_m\right)$ and

$$e^{-tc(da)} = \prod_{k=1}^{m}\left(\cosh(t\lambda_k) - ie_k f_k \sinh(t\lambda_k)\right). \tag{4.4}$$

Now if $I \in \{2, \ldots, m\}$, we have the commutation

$$e_1 f_1 \left(\prod_{i\in I} e_k f_k\right) = \frac{1}{2}\left[e_1, f_1\left(\prod_{i\in I} e_k f_k\right)\right].$$

This shows that the only traceless terms in the expansion of (4.4) are the constants and hence $\mathrm{tr}\left(e^{-tc(da)}\right) = \cosh(tA_y)$. Equation (4.1) now follows. For the second part of the theorem note that proposition 3.4 gives the estimate

$$|\mathrm{tr}(D_r e^{-tD_r^2})| \leq c_1 r^{\frac{n-1}{2}} e^{c_2 t} \tag{4.5}$$

for uniform constants c_1, c_2 independent of r. From this equation, (4.2) follows. The uniformity of the limits (4.1),(4.2) is also an easy consequence of the estimates (4.3) and (4.5). □

The above theorem also follows from the rescaling argument as described in section c) of [7].

Next we let $N_r(\sigma)$ denote the number of eigenvalues of D_r in the interval $[-\sigma,\sigma]$. We also use the notation $\langle x\rangle=\sqrt{1+|x|^2}$ for any $x\in\mathbb{R}^d$. We shall need the following estimates.

Proposition 4.2. *Let $\varphi\in\mathcal{S}$ be a Schwartz function. Then we have the estimates*

$$N_r(\sigma) \leq c_1 r^{\frac{n}{2}}(1+\sigma^2)^{\frac{n}{2}} \tag{4.6}$$
$$tr\,\varphi(D_r) \leq c_2 r^{\frac{n}{2}}\left\|\langle x\rangle^{n+2}\varphi\right\|_{C^0} \tag{4.7}$$

for constants c_1,c_2 independent of r. In case the function φ is odd, we have

$$\lim_{r\to\infty} r^{-\frac{n}{2}}tr\,\varphi(D_r)=0. \tag{4.8}$$

Proof. We begin with

$$N_r(\sigma)e^{-t\sigma^2}\leq \mathrm{tr}(e^{-tD_r^2})=\mathrm{tr}(H_{\frac{t}{r}})\leq c_1\left(\frac{r}{t}\right)^{\frac{n}{2}}e^{c_2t}$$

using proposition 3.1. This gives $N_r(\sigma)\leq c_1\left(\frac{r}{t}\right)^{\frac{n}{2}}e^{(c_2+1)(\sigma^2+1)t}$ from which(4.6) follows on substituting $t=\frac{1}{\sigma^2+1}$.

For the second part we estimate

$$\begin{aligned}|\mathrm{tr}\,\varphi(D_r)| &= \left|\sum_{k=-\infty}^{\infty}\sum_{\lambda\in Spec(D_r)\cap[k,k+1)}\varphi(\lambda)\right|\\ &\leq c_1\sum_{k=-\infty}^{\infty}\sum_{\lambda\in Spec(D_r)\cap[k,k+1)}\frac{\langle\lambda\rangle^{n+2}}{\langle k+1\rangle^{n+2}}|\varphi(\lambda)|\\ &\leq c_1\left\|\langle x\rangle^{n+2}\varphi\right\|_{C^0}\left(\sum_{k=-\infty}^{\infty}\frac{N_r(k+1)}{\langle k+1\rangle^{n+2}}\right)\\ &\leq c_2\, r^{\frac{n}{2}}\left\|\langle x\rangle^{\frac{n}{2}+2}\varphi\right\|_{C^0}\left(\sum_{k=0}^{\infty}\frac{1}{\langle k+1\rangle^{2}}\right).\end{aligned}$$

Finally to prove the third part note that (4.8) is true for the family of odd Schwartz functions $\varphi_t=xe^{-tx^2}$ on account of (4.2). Since the convergence in (4.2) is uniform it may be differentiated to obtain (4.8) for the odd functions $\varphi=x^{2m+1}e^{-x^2}, m\in\mathbb{N}_0$. Now (4.7) along with the fact that the span of $\left\{x^{2m+1}e^{-x^2}\right\}$ is dense in the space of odd Schwartz functions gives (4.8). ∎

Now consider the rescaled spectral measure of D_r^2 given by

$$\mu_r = r^{-\frac{n}{2}} \sum_{\lambda \in Spec(D_r)} \delta_{\lambda^2}.$$

Let $C_0^0\left(\mathbb{R}_{\geq 0}\right)$ denote the space of bounded, continuous functions on $\mathbb{R}_{\geq 0}$ vanishing at ∞. Consider the Banach space $\mathcal{B} = \langle x \rangle^{-n-2} C_0^0\left(\mathbb{R}_{\geq 0}\right)$. By a measure here we shall mean an element of the dual space $\mathcal{B}'$. By (4.7) μ_r is a family of uniformly bounded measures in $\mathcal{B}'$. We now derive a formula for the limit of the measures μ_r as $r \to \infty$. Expecting the Laplace transform of the limit measure to be given by the integral in (4.1), we find the Laplace inverse of its integrand. Let $2n_y + 1$ be the dimension of the kernel of A_y at any point $y \in Y$ and $m_y = \frac{n-(2n_y+1)}{2}$. Let $u(s)$ be the Heaviside step function and $Z(k)$ denote the number of non-zero components of a multi-index $k \in \mathbb{N}_0^{m_y}$. We then compute

$$\begin{aligned}
\frac{1}{(4\pi t)^{\frac{n}{2}}} \det{}^{\frac{1}{2}} \left(\frac{tA}{\tanh tA} \right) &= \frac{t^{-n_y - \frac{1}{2}}}{(4\pi)^{\frac{n}{2}}} \prod_{\lambda_i^y > 0} \frac{\lambda_i^y}{\tanh(t\lambda_i^y)} \\
&= \frac{t^{-n_y - \frac{1}{2}}}{(4\pi)^{\frac{n}{2}}} \prod_{\lambda_i^y > 0} \lambda_i^y (1 + 2e^{-2t\lambda_i} + 2e^{-4t\lambda_i} + \ldots) \\
&= \frac{t^{-n_y - \frac{1}{2}}}{(4\pi)^{\frac{n}{2}}} \left(\prod_{\lambda_i^y > 0} \lambda_i^y \right) \left(\sum_{k \in \mathbb{N}_0^{m_y}} 2^{Z(k)} e^{-2tk\cdot\lambda} \right) \\
&= \frac{\left(\prod_{\lambda_i^y > 0} \lambda_i^y \right)}{(4\pi)^{\frac{n}{2}}} \mathcal{L}_{s \to t} \left(s^{n_y - \frac{1}{2}} * \left(\sum_{k \in \mathbb{N}_0^{m_y}} 2^{Z(k)} \delta(s - 2k \cdot \lambda) \right) \right) \\
&= \frac{\left(\prod_{\lambda_i^y > 0} \lambda_i^y \right)}{(4\pi)^{\frac{n}{2}}} \mathcal{L}_{s \to t} \left(\sum_{k \in \mathbb{N}_0^{m_y}} 2^{Z(k)} u(s - 2k \cdot \lambda)(s - 2k \cdot \lambda)^{n_y - \frac{1}{2}} \right).
\end{aligned}$$

Motivated by this pointwise calculation on Y we define the measure

$$\mu_\infty^y = \frac{\left(\prod_{\lambda_i^y > 0} \lambda_i^y \right)}{(4\pi)^{\frac{n}{2}}} \left(\sum_{k \in \mathbb{N}_0^{m_y}} 2^{Z(k)} u(s - 2k \cdot \lambda)(s - 2k \cdot \lambda)^{n_y - \frac{1}{2}} \right).$$

We now have the following proposition.

Proposition 4.3. *Each μ_∞^y is a measure in $\mathcal{B}'$ satisfying $\|\mu_\infty^y\|_{\mathcal{B}'} \leq C$ for some uniform constant C independent of y. Furthermore, the family of measures $\mu_\infty^y \in \mathcal{B}'$ is weakly continuous in y.*

Proof. For $\varphi\in\mathcal{B}$ we estimate

$$\begin{aligned}
\left|\mu_{\infty}^{y}(\varphi)\right| &= \left|\frac{\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)}{(4\pi)^{\frac{n}{2}}}\left(\sum_{k\in\mathbb{N}_0^{m_y}}2^{Z(k)}\left(\int_{2k\cdot\lambda}^{\infty}\varphi(s)(s-2k\cdot\lambda)^{n_y-\frac{1}{2}}ds\right)\right)\right|\\
&\leq \frac{\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)}{(2\pi)^{\frac{n}{2}}}\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\sum_{k\in\mathbb{N}_0^{m_y}}\left(\int_{2k\cdot\lambda}^{\infty}\left\langle s\right\rangle^{-\frac{n}{2}-2}(s-2k\cdot\lambda)^{n_y-\frac{1}{2}}ds\right)\right)\\
&= \frac{\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)}{(2\pi)^{\frac{n}{2}}}\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\sum_{k\in\mathbb{N}_0^{m_y}}\left(\int_{0}^{\infty}\frac{s^{n_y-\frac{1}{2}}}{\left\langle s+2k\cdot\lambda\right\rangle^{n_y+m_y+2+\frac{1}{2}}}ds\right)\right)\\
&\leq \frac{\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)}{(2\pi)^{\frac{n}{2}}}\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\sum_{k\in\mathbb{N}_0^{m_y}}\frac{1}{\left\langle 2k\cdot\lambda\right\rangle^{m_y+\frac{3}{2}}}\left(\int_{0}^{\infty}\frac{s^{-\frac{1}{2}}}{\left\langle s\right\rangle}ds\right)\right)\\
&\leq C\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\left(\sum_{k\in\mathbb{N}_0^{m_y}}\frac{1}{\left\langle 2k\cdot\lambda\right\rangle^{m_y+\frac{3}{2}}}\right). \qquad (4.9)
\end{aligned}$$

Now if $\sup_{y\in Y}\left\Vert A_y\right\Vert=\alpha$ then each $\lambda_i^y\leq\alpha$. If N_R denotes the cardinality of the set $S_R=\left\{k\in\mathbb{N}_0^{m_y}|2k\cdot\lambda\leq R\right\}$ we have the bound $N_R\leq C(R+\alpha)^{m_y}\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)^{-1}$ for C depending only n. This is obtained on observing that the union of the m_y-parallelotope's based at points of S_R can be covered by the appropriately large ball. Hence we may estimate (4.9) further by

$$\begin{aligned}
& C\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\left(\sum_{k\in\mathbb{N}_0^{m_y}}\frac{1}{\left\langle 2k\cdot\lambda\right\rangle^{m_y+\frac{3}{2}}}\right)\\
\leq\; & C\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\left(\sum_{l\in\mathbb{N}_0}\frac{N_{l+1}}{\left\langle l\right\rangle^{m_y+\frac{3}{2}}}\right)\\
\leq\; & C_1\left\Vert\varphi\right\Vert_{\mathcal{B}}\left(\sum_{l\in\mathbb{N}_0}\frac{(l+\alpha)^{m_y}}{\left\langle l\right\rangle^{m_y+\frac{3}{2}}}\right)\\
\leq\; & C_2\left\Vert\varphi\right\Vert_{\mathcal{B}}. \qquad (4.10)
\end{aligned}$$

Now we prove the second part of the proposition. By (4.10) each μ_{∞}^{y} lies in $\mathcal{B}'$ and we need to show that $\mu_{\infty}^{y}(\varphi)$ is a continuous function of y for every $\varphi\in\mathcal{B}$. First note that

$$\mu_\infty^y(e^{-ts}) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \det{}^{\frac{1}{2}} \left(\frac{tA_y}{\sinh tA_y} \right) \tag{4.11}$$

by construction. Hence $\mu_\infty^y(e^{-s})$ is a continuous function of y. By differentiating (4.11) further we obtain that $\mu_\infty^y(s^m e^{-s})$ is a continuous function of y for each $m \in \mathbb{N}_0$. The result now follows on knowing that $\|\mu_\infty^y\|_{\mathcal{B}'}$ is uniformly bounded in y and the span of $s^m e^{-s}$ is dense in $\mathcal{B}$. □

Next we define the measure μ_∞ via

$$\mu_\infty(\varphi) = \int_Y \mu_\infty^y(\varphi) dy.$$

By proposition 4.3, we have that μ_∞ is a well defined element of $\mathcal{B}'$.

Proposition 4.4. *We have the weak convergence $\mu_r \rightharpoonup \mu_\infty$ in $\mathcal{B}'$.*

Proof. By (4.1) we have that $\mu_r(e^{-ts}) \to \mu_\infty(e^{-ts})$ and since this limit is uniform on compact intervals of time it may be differentiated to obtain $\mu_r(s^m e^{-s}) \to \mu_\infty(s^m e^{-s}), \forall m \in \mathbb{N}_0$. Weak convergence again follows on knowing that $\|\mu_r\|_{\mathcal{B}'}$ is uniformly bounded in r and the span of $s^m e^{-s}$ is dense in $\mathcal{B}$. □

Finally, we shall need the following information about the limit measure.

Proposition 4.5. *Let $\varphi \in \mathcal{B}$, $0 \le \varphi \le 1$ be such that supp(φ) $\subseteq [0, \epsilon]$, $0 < \epsilon < 1$. Then we have*

$$\mu_\infty(\varphi) \le C\epsilon^{\frac{1}{2}} \tag{4.12}$$

for some constant C independent of φ, ϵ.

Proof. We estimate

$$\begin{aligned}
|\mu_{\infty}^{y}(\varphi)| &= \left|\frac{\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)}{(4\pi)^{\frac{n}{2}}}\left(\sum_{k\in\mathbb{N}_0^{m_y}}2^{Z(k)}\left(\int_{2k\cdot\lambda}^{\infty}\varphi(s)(s-2k\cdot\lambda)^{n_y-\frac{1}{2}}ds\right)\right)\right| \\
&\leq c_1\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\left(\sum_{k\in\mathbb{N}_0^{m_y}}\left(\int_0^{\infty}\varphi(u+2k\cdot\lambda)u^{n_y-\frac{1}{2}}du\right)\right) \\
&\leq c_2\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\left(\sum_{2k\cdot\lambda\leq\epsilon}\left(\int_0^{\epsilon}u^{n_y-\frac{1}{2}}du\right)\right) \\
&\leq c_2\left(\prod_{\lambda_i^y>0}\lambda_i^y\right)\frac{\epsilon^{n_y+\frac{1}{2}}}{n_y+\frac{1}{2}}N_{\epsilon} \\
&\leq c_3\epsilon^{n_y+\frac{1}{2}}(\epsilon+\alpha)^{m_y} \\
&\leq c_4\epsilon^{\frac{1}{2}}.
\end{aligned}$$

For a constant c_4 independent of y. The proposition now follows on integration over Y. □

We are now ready to give the proof of Theorem 1.1 below.

Proof of Theorem 1.1. Since the eta invariant is unchanged under rescaling it suffices to consider $\bar{\eta}(D_r)$. We then write

$$\begin{aligned}
\bar{\eta}(D_r) &= \frac{1}{2}\left\{\dim\,\ker(D_r)+\int_0^{\infty}\frac{1}{\sqrt{\pi t}}\mathrm{tr}\left(D_re^{-tD_r^2}\right)dt\right\} \\
&= \frac{1}{2}\left\{\int_0^{1}\frac{1}{\sqrt{\pi t}}\mathrm{tr}\left(D_re^{-tD_r^2}\right)dt+\mathrm{tr}\,E(D_r)\right\}. \qquad (4.13)
\end{aligned}$$

Here $E(x)=\mathrm{sign}(x)\mathrm{erfc}(|x|)=\mathrm{sign}(x)\cdot\frac{2}{\sqrt{\pi}}\int_{|x|}^{\infty}e^{-s^2}ds$ with the convention $\mathrm{sign}(0)=1$. The first summand of (4.13) is $o(r^{\frac{n}{2}})$ on account of the uniform convergence in (4.2). To bound the second term we fix $0<\epsilon<1$ and define Schwartz functions $\varphi_1,\varphi_2\in\mathcal{S}$ satisfying.

(1) φ_1 odd, φ_2 even
(2) $-1\leq\varphi_1\leq 1,\ 0\leq\varphi_2\leq 1$
(3) $\varphi_1(x)=E(x)$ for $x\notin\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]$
(4) $supp(\varphi_2)\subset[-\epsilon,\epsilon]$,$\varphi_2(x)=1$ for $x\in\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]$.

Since φ_2 is even we may also assume $\varphi_2(x) = \bar{\varphi}_2(x^2)$. We then have

$$\begin{aligned} r^{-\frac{n}{2}}\operatorname{tr} E(D_r) &\leq r^{-\frac{n}{2}}\left(\left|\operatorname{tr} E(D_r) - \operatorname{tr}\varphi_1(D_r)\right| + \left|\operatorname{tr}\varphi_1(D_r)\right|\right) \\ &\leq r^{-\frac{n}{2}}\left(2N_r\left(\frac{\epsilon}{2}\right) + \left|\operatorname{tr}\varphi_1(D_r)\right|\right) \qquad \text{(by 3)} \\ &\leq r^{-\frac{n}{2}}\left(2\operatorname{tr}\varphi_2(D_r) + \left|\operatorname{tr}\varphi_1(D_r)\right|\right) \qquad \text{(by 4)} \\ &= 2\mu_r(\bar{\varphi}_2) + r^{-\frac{n}{2}}\left|\operatorname{tr}\varphi_1(D_r)\right| \\ &\leq 2\left|\mu_r(\bar{\varphi}_2) - \mu_\infty(\bar{\varphi}_2)\right| + 2\left|\mu_\infty(\bar{\varphi}_2)\right| + r^{-\frac{n}{2}}\left|\operatorname{tr}\varphi_1(D_r)\right| \\ &\leq c\epsilon^{\frac{1}{4}}, \end{aligned}$$

for r sufficiently large by (4.8), (4.12) and the weak convergence $\mu_r \rightharpoonup \mu_\infty$.

□

5. Eta invariant of a circle bundle

In this section we consider the eta invariant in a specialized case. In particular, we let Y to be the total space of principal circle bundle $S^1 \to Y^{2m+1} \xrightarrow{\pi} X^{2m}$ over a base of even dimension $2m = n-1$. We assume that X is oriented and equipped with a metric g^{TX} and a spin structure. Let $TS^1 = T^V Y \subset TY$ denote the subbundle of TY consisting of the vertical tangent vectors. Let $T^H Y \subset TY$ be another subbundle corresponding to a connection on the fibration and hence giving an invariant splitting

$$TY = T^V Y \oplus T^H Y \tag{5.1}$$

of the tangent bundle. The projection π gives an identification $T^H Y = \pi^* TX$. Consider the trivializing section of TS^1 given by $e_y = \frac{\partial}{\partial t}(e^{it}.y)|_{t=0} \in T_y S^1$, the infinitesimal generator of the S^1 action. Let g^{TS^1} be the metric on TS^1 such that $\|e\|_{g^{TS^1}} = 1$. We now consider the adiabatic family of metrics

$$g^{TY}_{\varepsilon} = g^{TS^1} \oplus \varepsilon^{-1}\pi^* g^{TX} \tag{5.2}$$

on Y as in [5].

Let $\nabla^{TY,\varepsilon}, \nabla^{TX}$ denote the Levi-Civita connections of $g^{TY}_{\varepsilon}, g^{TX}$ respectively. The connection $\nabla^{TY,\varepsilon}$ need not preserve the splitting (5.1). Let p^{TS^1}, p^H denote the projections of TY onto $TS^1, T^H Y$ respectively. Define a connection on TS^1 via $\nabla^{TS^1} = p^{TS^1}\nabla^{TY,\varepsilon}$. As shown in section 4 of [5], the connection ∇^{TS^1} is independent of ε. In the case of circle bundles this is easily checked by showing that e is ∇^{TS^1} -parallel via

$$\begin{aligned} \left\langle \nabla^{TS^1}_U e, e \right\rangle &= \left\langle \nabla^{TY,\varepsilon}_U e, e \right\rangle \\ &= \frac{1}{2} U\left\langle e, e\right\rangle = 0, \quad \forall U \in TY. \end{aligned} \tag{5.3}$$

Define the second connection ∇ on TY to be $\nabla = \nabla^{TS^1} \oplus \pi^* \nabla^{TX}$. The connection ∇ does preserve the splitting of TX but need not be torsion free. Let T denote the torsion tensor of ∇ and define the difference tensor

$$S^\varepsilon = \nabla^{TY,\varepsilon} - \nabla.$$

Since $\nabla^{TY,\varepsilon}$ is torsion free, for tangent vectors $U, V, W \in TY$ we have

$$S^\varepsilon(U)V - S^\varepsilon(V)U = -T(U,V).$$

Since both $\nabla^{TY,\varepsilon}, \nabla$ are compatible with g_ε^{TY}, we also have

$$\langle S^\varepsilon(U)V, W\rangle + \langle V, S^\varepsilon(U)W\rangle = 0$$

where $\langle\rangle = g_\varepsilon^{TY}$. The last two equations give

$$\langle S^\varepsilon(U)V, W\rangle = \frac{1}{2}\left[\langle T(V,W),U\rangle - \langle T(W,U),V\rangle - \langle T(U,V),W\rangle\right]. \tag{5.4}$$

Next we let $g^{TY} = g_1^{TY}$, $\nabla^{TY} = \nabla^{TY,1}$ and $S = \nabla^{TY} - \nabla$ be the metric, Levi-Civita connection and the difference tensor respectively when the adiabatic parameter $\varepsilon = 1$ is set to one. It is clear from equations (5.2) and (5.4) that

$$p^H S^\varepsilon = \varepsilon p^H S, \tag{5.5}$$

$$p^{TS^1} S^\varepsilon = p^{TS^1} S. \tag{5.6}$$

Let f_i be a locally defined orthonormal frame of vector fields on the base X and $\tilde{f}_i$ their lifts to Y. The torsion tensor T can be computed in the following cases to be

1. $T(e,e) = 0$ as T is antisymmetric,
2. $T(e,\tilde{f}) = \nabla_e \tilde{f} - \nabla_{\tilde{f}} e - [e,\tilde{f}] = -[e,\tilde{f}] = 0$ for $\tilde{f} \in T^H Y$, as $\tilde{f}$ is S^1 invariant,
3. $T(\tilde{f}_1,\tilde{f}_2) = R(f_1,f_2)$ the curvature of the S^1 connection, as ∇^{TX} is torsion free.

Following the above computation of the torsion tensor, (5.4) now clearly implies

$$S(e)e = 0. \tag{5.7}$$

Define e^* to be the one form which annihilates $T^H M$ and $e^*(e) = 1$. We then compute $de^*\left(\tilde{f}_1,\tilde{f}_2\right) = -\left\langle \left[\tilde{f}_1,\tilde{f}_2\right], e\right\rangle = R\left(f_1,f_2\right)$ is the curvature of the S^1 connection.

5.1. **Splitting of the Dirac operator.** The spin structure on TX can be pulled back to one on $T^H Y$. Combined with the trivial spin structure on TS^1 this gives a spin structure on TY. If $S_\pm^{TX}$denote the bundles of positive and negative spinors on X, we have the identification $S^{TY} = \pi^*(S_+^{TX} \oplus S_-^{TX})$. This in turn gives

$$C^\infty(Y, S^{TY}) = C^\infty(X; (S_+^{TX} \oplus S_-^{TX}) \otimes C^\infty(Y_x)).$$

We now decompose

$$C^\infty(Y_x) = \bigoplus_{k\in\mathbb{Z}} E_k \tag{5.8}$$

according to the eigenspaces of e. Each E_k corresponds to the eigenvalue $-ik$ and gives a line bundle over X. Let $\mathcal{L} \to X$ be the Hermitian line bundle over X corresponding to the standard representation of S^1. Note that we may reconstruct Y as the space of unit elements in $\mathcal{L}$. We now also have the identification $E_k = \mathcal{L}^{\otimes k}$. For any point $y_x \in Y_x \subset \mathcal{L}_x$, this identification maps $y_x^{\otimes k}$ to $\{f(y_x e^{i\theta}) = e^{-ik\theta}\}$ and we extend it by linearity. Hence we have the decomposition of the space of spinors on Y into

$$C^\infty(Y, S^{TY}) = \bigoplus_{k\in\mathbb{Z}} C^\infty(X; (S_+^{TX} \oplus S_-^{TX}) \otimes \mathcal{L}^{\otimes k}). \tag{5.9}$$

Finally, we twist the spin bundle $S^{TY} \otimes \mathbb{C} = S^{TY}$ by the trivial Hermitian line bundle but equipped with the family of unitary connections $A_r = d + ire^*$.

We now consider how the family of coupled Dirac operators $D_{A_r,\varepsilon}$ decomposes with respect to (5.9). Let $\nabla^{S^{TY},\varepsilon}, \tilde{\nabla}$ denote the lifts of $\nabla^{TY,\varepsilon}, \nabla$ to the spin bundle. Let c^ε stand for the Clifford multiplication associated to the metric g_ε^{TY}. We let $e_i = \varepsilon^{1/2}\tilde{f}_i$, where $\tilde{f}_i$ denote the locally defined horizontal lifts introduced earlier, and also adopt the notation $e = e_0$. Using (5.4) and the computation of the torsion tensor done in the previous subsection, we now compute

$$\begin{aligned} D_{A_r,\varepsilon} &= \sum_{i=0}^{2m} c^\varepsilon(e_i)\nabla_{e_i}^{S^{TY},\varepsilon} + irc^\varepsilon(e) \\ &= c^\varepsilon(e)\tilde{\nabla}_e + \sum_{i=1}^{2m} c^\varepsilon(e_i)\tilde{\nabla}_{e_i} + \frac{1}{4}\sum_{ijk} \langle S^\varepsilon(e_i)e_j, e_k\rangle\, c^\varepsilon(e_i)c^\varepsilon(e_j)c^\varepsilon(e_k) + irc^\varepsilon(e) \\ &= \bigoplus_k \begin{bmatrix} k - \frac{i\varepsilon}{4}c(R) - r & \varepsilon^{1/2}D_-^{B,k} \\ \varepsilon^{1/2}D_+^{B,k} & -k + \frac{i\varepsilon}{4}c(R) + r \end{bmatrix}. \end{aligned} \tag{5.10}$$

Here $D_\pm^{X,k}$ denotes the coupled Dirac operators acting on sections of $S_\pm^{TX} \otimes \mathcal{L}^{\otimes k}$ and $c(R) = \sum_{i<j} R(f_i, f_j)c(f_i)c(f_j)$ denotes the Clifford multiplication by the curvature tensor R on the base X.

5.2. **The Kahler case.** We now specialize to the case when X is a complex manifold, with complex structure J. We further assume $\mathcal{L}$ to be a positive, holomorphic, Hermitian line bundle on X . The curvature R of the associated holomorphic connection is now a $(1,1)$ form. Positivity of $\mathcal{L}$ here means that $\frac{1}{2}R = \omega$ is a Kahler form on X (i.e. $\omega(.,J.) = g^{TX}(.,.)$ is a metric). A spin structure on X corresponds to a holomorphic, Hermitian square root $\mathcal{K}$ of the canonical line bundle $\mathcal{K}^{\otimes 2} = K_X$ (cf . [10]). The corresponding bundles of positive and negative spinors are $\Lambda^{even}T^{0,1*} \otimes \mathcal{K}$ and $\Lambda^{odd}T^{0,1*} \otimes \mathcal{K}$ respectively while the spin Dirac operator is $\sqrt{2}(\bar{\partial}_{\mathcal{K}} + \bar{\partial}^*_{\mathcal{K}})$ with $\bar{\partial}_{\mathcal{K}}$ being the holomorphic derivative on $\Lambda^* T^{0,1*} \otimes \mathcal{K}$. Similarly the twisted Dirac operator acting on sections of $\Lambda^* T^{0,1*} \otimes \mathcal{K} \otimes \mathcal{L}^{\otimes k}$is given by $\sqrt{2}(\bar{\partial}_{\mathcal{K}\otimes\mathcal{L}^{\otimes k}} + \bar{\partial}^*_{\mathcal{K}\otimes\mathcal{L}^{\otimes k}})$. Denote the holomorphic derivative $\bar{\partial}_{\mathcal{K}\otimes\mathcal{L}^{\otimes k}}$ on $\mathcal{K} \otimes \mathcal{L}^{\otimes k}$ by the shorthand $\bar{\partial}_k$ and let $\Delta_{\bar{\partial}_k} = \bar{\partial}_k \bar{\partial}^*_k + \bar{\partial}^*_k \bar{\partial}_k$ denote the Hodge Laplacian. Clifford multiplication by the Kahler form is $c(\omega) = i(2N - m)$ where N is the number operator which acts as multiplication by p on $\Lambda^p T^{0,1*}$. Hence the formula (5.10) for the Dirac operator is seen to specialize to

$$D_{A_r,\varepsilon} = \bigoplus_k \begin{bmatrix} k + \varepsilon(N - \frac{m}{2}) - r & (2\varepsilon)^{1/2}(\bar{\partial}_k + \bar{\partial}^*_k) \\ (2\varepsilon)^{1/2}(\bar{\partial}_k + \bar{\partial}^*_k) & -k - \varepsilon(N - \frac{m}{2}) + r \end{bmatrix}. \tag{5.11}$$

Denote by $\mathcal{A}^{0,p}\left(\mathcal{L}^{\otimes k}\right)$ the space of $\mathcal{L}^{\otimes k}$-valued $(0,p)$ forms. Let

$$\mathcal{A}^{0,p}\left(\mathcal{L}^{\otimes k}\right) = \bigoplus_{\mu \geq 0} E^{p,k}_\mu$$

be the spectral decomposition of $\Delta_{\bar{\partial}_k}$ where $E^{p,k}_\mu$ denotes the eigenspace with eigenvalue $\frac{1}{2}\mu^2$. From (5.11) it is clear that $\left[D_{A_r}, \Delta_{\bar{\partial}_k}\right] = 0$ and hence the Dirac operator preserves the eigenspaces $\bigoplus_p E^{p,k}_\mu$ of $\Delta_{\bar{\partial}_k}$ for each μ. Let $\dim E^{p,k}_\mu = e^{p,k}_\mu$ and define $d^{p,k}_\mu = e^{p,k}_\mu - e^{p-1,k}_\mu + \ldots + (-1)^p e^{0,k}_\mu$.

Lemma 5.1. *For each positive $\mu > 0$ we have $d^{p,k}_\mu \geq 0$. Furthermore there exists a collection of $\bar{\partial}^*_k$-closed p forms $\left\{\omega^p_j\right\}_{j=1}^{d^{p,k}_\mu}$, such that $\left\{\omega^p_j\right\}_{j=1}^{d^{p,k}_\mu} \cup \left\{\bar{\partial}_k \omega^{p-1}_j\right\}_{j=1}^{d^{p-1,k}_\mu}$ is a basis of $E^{p,k}_\mu$.*

Proof. We proceed by induction on p. Clearly $d^{0,k}_\mu = e^{0,k}_\mu \geq 0$. We take $\left\{\omega^0_j\right\}_{j=1}^{d^{0,k}_\mu}$to be any basis of $E^{0,k}_\mu$. Now assume that ω^p_j have been defined. Since they are $\bar{\partial}^*_k$-closed we have

$$\begin{aligned} \bar{\partial}^*_k \bar{\partial}_k \omega^p_j &= \Delta_{\bar{\partial}_k} \omega^p_j &= \frac{1}{2}\mu^2 \omega^p_j \\ \Delta_{\bar{\partial}_k} \bar{\partial}_k \omega^p_j &= \bar{\partial}_k \bar{\partial}^*_k \bar{\partial}_k \omega^p_j &= \frac{1}{2}\mu^2 \bar{\partial}_k \omega^p_j \end{aligned}$$

Hence the collection of forms $\left\{\bar{\partial}_k \omega^p_j\right\}_{j=1}^{d^{p,k}_\mu}$ is linearly independent inside $E^{p+1,k}_\mu$. This implies $d^{p+1,k}_\mu = e^{p,k}_\mu - d^{p,k}_\mu \geq 0$. We choose $\left\{\omega^{p+1}_j\right\}_{j=1}^{d^{p+1,k}_\mu}$ to be any basis for the orthogonal complement of the span the space of $\left\{\bar{\partial}_k \omega^p_j\right\}_{j=1}^{d^{p,k}_\mu}$ inside $E^{p+1,k}_\mu$. That each

ω_j^{p+1} is $\bar{\partial}_k^*$-closed follows from

$$\begin{aligned} \left\langle \bar{\partial}_k^* \omega_j^{p+1}, \omega_{j'}^p \right\rangle &= \left\langle \omega_j^{p+1}, \bar{\partial}_k \omega_{j'}^p \right\rangle &= 0 \\ \left\langle \bar{\partial}_k^* \omega_j^{p+1}, \bar{\partial}_k \omega_{j'}^{p-1} \right\rangle &= \left\langle \omega_j^{p+1}, \bar{\partial}_k^2 \omega_{j'}^{p-1} \right\rangle &= 0. \end{aligned}$$

Finally, $\left\{ \bar{\partial}_k \omega_j^{2m-1} \right\}_{j=1}^{d_{2m-1}}$ span $E_\mu^{2m,k}$ since the Dirac operator $\sqrt{2}(\bar{\partial}_k + \bar{\partial}_k^*)$ is an isomorphism between $E_\mu^{even,k}$ and $E_\mu^{odd,k}$ for $\mu > 0$. □

Following this lemma, we see using (5.11) that the Dirac operator preserves the two dimensional subspaces of spanned by $\left\{ \omega_j^p, \frac{\sqrt{2}}{\mu} \bar{\partial}_k \omega_j^p \right\}$, for each $0 \leq j \leq d_\mu^{p,k}$. Furthermore its restriction to this two dimensional subspace is given by the matrix

$$\begin{bmatrix} (-1)^p (k + \varepsilon(p - \frac{m}{2}) - r) & \mu \varepsilon^{1/2} \\ \mu \varepsilon^{1/2} & (-1)^{p+1} (k + \varepsilon(p + 1 - \frac{m}{2}) - r) \end{bmatrix}.$$

The eigenvalues of the above matrix are computed to be

$$\lambda = \frac{(-1)^{p+1} \varepsilon \pm \sqrt{(2k + \varepsilon(2p - m) - 2r + 1)^2 + 4\mu^2 \varepsilon}}{2}.$$

From (5.11) we also see that each of $E_0^{p,k}$ is an eigenspace of the $D_{A_r,\varepsilon}$ with eigenvalue $\lambda = (-1)^p (k + \varepsilon(p - \frac{m}{2}) - r)$. From Hodge theory, we have $E_0^{p,k} = H^p(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k})$ and we denote

$$h^{p,k} := e_0^{p,k} = \dim H^p(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k}).$$

To sum up we have the following computation.

Proposition 5.2. *The eigenvalues of the Dirac operator are given by the two types*

1. *Type 1:*
$$\lambda = (-1)^p (k + \varepsilon(p - \frac{m}{2}) - r), \; 0 \leq p \leq m, k \in \mathbb{Z}$$
with multiplicity $h^{p,k} = dim\, H^p(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k})$.
2. *Type 2:*
$$\lambda = \frac{(-1)^{p+1} \varepsilon \pm \sqrt{(2k + \varepsilon(2p - m) - 2r + 1)^2 + 4\mu^2 \varepsilon}}{2}, \; 0 \leq p \leq m, k \in \mathbb{Z}$$
and $\frac{1}{2}\mu^2$ *is a positive eigenvalue of* $\Delta_{\bar{\partial}_k}^p$. *The multiplicity of* λ *is* $d_\mu^{p,k} = e_\mu^{p,k} - e_\mu^{p-1,k} + \ldots + (-1)^p e_\mu^{0,k}$ *where* $e_\mu^{p,k}$ *is the multiplicity of* $\frac{1}{2}\mu^2$.

The above proposition will allow us to compute an asymptotic formula for the eta invariant $\bar{\eta}^{r,\varepsilon} = \bar{\eta}(D_{A_r,\varepsilon})$ as $r \to \infty$, for each value of the adiabatic parameter ε. To this end we first compute the spectral flow function $\mathrm{sf}\{D_{A_s,\varepsilon}\}_{0 \leq s \leq r}$. Note that the eigenvalues of type 2 do not change sign when the corresponding positive eigenvalue of the Hodge Laplacian satisfies

$$\frac{1}{2}\mu^2 > \frac{\varepsilon}{8}. \tag{5.12}$$

Let K_X^* be the anticanonical bundle of X and define $R_{ij}^{\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*}$ to be the curvature of the connection on $\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*$. Let $d\bar{z}_i$ be an orthonormal basis of $T^{0,1^*}X$, with dz_i the dual basis of $T^{1,0^*}X$, and define $\lambda(R^{\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*}) = \sum_{ij} R_{ij}^{\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*} d\bar{z}_i \wedge i_{dz_j}$. Then the Bochner-Kodaira-Nakano formula (cf. [3] Proposition 3.71) asserts the existence of a positive operator $\Delta_{\bar{\partial}_k}^{p,0}$ such that

$$\Delta_{\bar{\partial}_k}^{p} = \Delta_{\bar{\partial}_k}^{p,0} + \lambda(R^{\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*}).$$

Now let α be a normalized eigenvector of $\Delta_{\bar{\partial}_k}^{p}$ with eigenvalue $\frac{1}{2}\mu^2$. We then compute

$$\begin{aligned}\frac{1}{2}\mu^2 = \left\langle \Delta_{\bar{\partial}_k}^{p}\alpha, \alpha \right\rangle &= \left\langle \Delta_{\bar{\partial}_k}^{p,0}\alpha + \lambda(R^{\mathcal{K}\otimes\mathcal{L}^{\otimes k}\otimes K_X^*})\alpha, \alpha \right\rangle \\ &\geq k\left\langle \lambda(R^{\mathcal{L}})\alpha, \alpha\right\rangle + \left\langle \lambda(R^{\mathcal{K}\otimes K_X^*})\alpha, \alpha\right\rangle > \frac{\varepsilon}{8},\end{aligned} \tag{5.13}$$

for $p > 0$ and $k \gg 0$ sufficiently large, via the positivity of $\mathcal{L}$. In the case where $p = 0$, since α is an eigenvector of $\Delta_{\bar{\partial}_k}^{0}$ with positive eigenvalue and $[\bar{\partial}_k, \Delta_{\bar{\partial}_k}] = 0$, we have that $\bar{\partial}_k s$ is nonzero eigenvector of $\Delta_{\bar{\partial}_k}^{1}$ with the same eigenvalue. Hence (5.12) holds for each positive eigenvalue of $\Delta_{\bar{\partial}_k}^{p}$ for all p and $k \gg 0$ sufficiently positive. Finally using duality we have that any positive eigenvalue of $\Delta_{\bar{\partial}_k}^{p}$ also obeys (5.12) for all p and $k \ll 0$ sufficiently negative. Hence we see that there are at most finitely many eigenvalues of $D_{A_s,\varepsilon}$ of type 2 that change sign as s varies from 0 to r. The contribution of the eigenvalues of type 1 to spectral flow is computed easily and we have

$$\mathrm{sf}\left\{D_{A_s,\varepsilon}\right\}_{0\leq s\leq r} = \sum_{0\leq k+\varepsilon(p-\frac{m}{2})\leq r} (-1)^{p+1}\, h^{p.k} + O(1).$$

By the Kodaira vanishing theorem we have $h^{p,k} = 0$ for $p > 0$ and $k \gg 0$ sufficiently large. We hence have

$$h^{0,k} = \chi(X, \mathcal{K}\otimes\mathcal{L}^{\otimes k}) = \int_X \mathrm{ch}(\mathcal{K}\otimes\mathcal{L}^{\otimes k})\mathrm{td}(X)$$

by the Hirzebruch-Riemann-Roch theorem. On using $\mathrm{ch}(\mathcal{K}\otimes\mathcal{L}^{\otimes k}) = \exp\{kc_1(\mathcal{L})\}\exp\{c_1(\mathcal{K})\}$ we have

$$\mathrm{sf}\left\{D_{A_s,\varepsilon}\right\}_{0\leq s\leq r} = -\sum_{k=0}^{\left[r+\frac{\varepsilon m}{2}\right]}\int_X \exp\{kc_1(\mathcal{L})\}\exp\{c_1(\mathcal{K})\}\,\mathrm{td}(X) + O(1).$$

Finally using the Atiyah-Patodi-Singer index theorem we have

$$\bar{\eta}^{r,\varepsilon} = \bar{\eta}^{0,\varepsilon} + 2\left\{\mathrm{sf}\left\{D_{A_s,\varepsilon}\right\}_{0\leq s\leq r} + \int_0^r ds\int_X \exp\{sc_1(\mathcal{L})\}\exp\{c_1(\mathcal{K})\}\,td(X)\right\}.$$

Hence we have proved

Theorem 5.3. *The eta invariant $\bar{\eta}^{r,\varepsilon}$ satisfies the asymptotics*

$$\bar{\eta}^{r,\varepsilon} = \sum_{a=0}^{m}\left\{\left(\frac{r^{a+1}}{(a+1)!} - \sum_{k=1}^{\left[r+\frac{\varepsilon m}{2}\right]}\frac{k^a}{a!}\right)\int_X c_1(\mathcal{L})^a\left[ch(\mathcal{K})td(X)\right]^{m-a}\right\} + O(1) \tag{5.14}$$

as $r\to\infty$.

The above result shows that the eta invariant is discontinuous of $O(r^{\frac{n-1}{2}})$ in this example.

5.3. **Computation of the eta invariant.** Although theorem Theorem 5.3 establishes an asymptotic formula for the eta invariant it does not provide an explicit computation for the eta invariant because of the $O(1)$ term in (5.14). In this subsection we give an explicit computation of the eta invariant $\bar{\eta}^{r,\varepsilon}$, assuming the value of the adiabatic parameter ε to be sufficiently small, using the adiabatic limit technique. We shall first compute the $\hat{\eta}$-form of Bismut and Cheeger [5] for circle bundles. This computation is similar to the one done by Zhang in [16] with the only difference being the presence of an extra coupling.

5.3.1. *The $\hat{\eta}$-form.* Let S^{TS^1} denote the spin bundle of TS^1 and $\nabla^{S^{TS^1}}$ be the lift of ∇^{TS^1} to S^{TS^1}. Let $\nabla^{S^{TS^1},r} = \nabla^{S^{TS^1}}\otimes 1 + 1\otimes(d+ire^*)$ denote the tensor product connection on $S^{TS^1}\otimes\mathbb{C}$. Consider the infinite dimensional bundles over X given by $H_x = C^\infty(Y_x, S^{TS^1}\otimes\mathbb{C})$ and $G_x = C^\infty(Y_x, TY)$. The connection $\nabla^{S^{TS^1},r}$ naturally lifts to a connection $\tilde{\nabla}^{S^{TS^1},r}$ on H. The torsion tensor T may be considered as an element of $T\in\Omega^2(X,G)$ and we may define Clifford multiplication by the torsion tensor as an element of $c(T)\in\Omega^2(\mathrm{End}(H))$. The fibrewise Dirac operator can also be defined as an element of $D^{S^1,r} = c(e)\nabla_e^{S^{TS^1},r}\in\mathrm{End}(H)$. The Bismut superconnection on H is defined via

$$A_u = \tilde{\nabla}^{S^{TS^1},r} + u^{1/2}D^{S^1,r} - (4u)^{-1/2}c(T).$$

The $\hat{\eta}$-form is the even form on X defined by

$$\hat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{tr}^{\mathrm{even}} \left[\left(D^{S^1,r} + (4u)^{-1} c(T) \right) e^{-A_u^2} \right] \frac{du}{2u^{1/2}}. \tag{5.15}$$

Let z be an auxiliary Grassman variable. Since the scalar curvature of the circle is zero equations (5.3) and (5.7) simplify (4.68)-(4.70) of [5] to give

$$-u \left(\nabla_e^{S^{TS^1},r} + \frac{R}{4u} + z\frac{c(e)}{2u^{1/2}} \right)^2 + irR = A_u^2 - z \left(u^{1/2} D^{S^1,r} + (4u)^{-1/2} c(T) \right), \tag{5.16}$$

where both sides are considered as operators on $\Omega^*(X,H)$. If $\mathrm{tr}^z(a+zb) = \mathrm{tr}(b)$ then the above curvature identity gives

$$\begin{aligned} &\mathrm{tr}^{\mathrm{even}} \left[\left(D^{S^1,r} + (4u)^{-1} c(T) \right) e^{-A_u^2} \right] \\ &\qquad = u^{-1/2} \mathrm{tr}^z \left[\exp \left\{ u \left(\nabla_e^{S^{TS^1},r} + \frac{R}{4u} + z\frac{c(e)}{2u^{1/2}} \right)^2 \right\} \right] \exp\left\{ -irR \right\}. \end{aligned}$$

Next, the trivialization given by e for TS^1 induces once for S^{TS^1}. This allows us to identify each fiber

$$\begin{aligned} H_x &= C^\infty(Y_x, S^{TS^1} \otimes \mathbb{C}) \\ &= C^\infty(Y_x) = \bigoplus_k E_k \end{aligned}$$

by (5.8). Using $c(e) = -i$, each E_k is seen to be an eigenspace of $D^{S^1,r}$ with eigenvalue $-k+r$. Hence $D^{S^1,r}$ is invertible for $r \notin \mathbb{Z}$. We then have

$$u^{-1/2} \mathrm{tr}^z \left[\exp \left\{ u \left(\nabla_e^{S^{TS^1},r} + \frac{R}{4u} + z\frac{c(e)}{2u^{1/2}} \right)^2 \right\} \right] \tag{5.17}$$

$$= u^{-1/2} \mathrm{tr}^z \left[\sum_{k=-\infty}^{\infty} \exp \left\{ u \left(ik + ir + \frac{R}{4u} - \frac{iz}{2u^{1/2}} \right)^2 \right\} \right] \tag{5.18}$$

$$= -i \sum_{k=-\infty}^{\infty} \left(\frac{\pi}{u} \right)^{3/2} e^{2\pi k i (\frac{R}{4ui} + r)} \cdot k \cdot e^{-\frac{k^2 \pi^2}{u}} \tag{5.19}$$

where the last equality follows from a Poisson summation formula. Hence

$$\frac{1}{\sqrt{\pi}}\int_0^\infty u^{-1/2}\mathrm{tr}^z\left[\exp\left\{u\left(\nabla_e^{S^{TS^1},r}+\frac{R}{4u}+z\frac{c(e)}{2u^{1/2}}\right)^2\right\}\right]\frac{du}{2u^{1/2}} \tag{5.20}$$

$$= \pi\int_0^\infty\sum_{k=1}^\infty k\cdot\sin\left(\frac{2\pi k}{u}\cdot\frac{R}{4i}+2\pi kr\right)e^{-\frac{k^2\pi^2}{u}}\frac{du}{u^2} \tag{5.21}$$

$$= \sum_{k=1}^\infty\frac{k\pi\sin(2\pi kr)+\cos(2\pi kr)\cdot\frac{R}{2i}}{k^2\pi^2+\left(\frac{R}{2i}\right)^2} \tag{5.22}$$

$$= \begin{cases}\frac{1}{2}\left[\frac{\exp\left((1-2\{r\})\frac{R}{2i}\right)}{\sinh\left(\frac{R}{2i}\right)}-\frac{1}{R/2i}\right] & \text{if } r\notin\mathbb{Z}\\ \frac{1}{2}\left[\frac{\frac{R}{2i}-\tanh\left(\frac{R}{2i}\right)}{\frac{R}{2i}\tanh\left(\frac{R}{2i}\right)}\right] & \text{if } r\in\mathbb{Z}.\end{cases} \tag{5.23}$$

Here $\{r\}$ denotes the fractional part of $r\notin\mathbb{Z}$. Let us denote the expression on line (5.23) by $f\left(\frac{R}{2i},r\right)$. We note that this is a periodic function in r of period 1. Hence we finally have that the eta form is given by

$$\hat{\eta}=f\left(\frac{R}{2i},r\right)\exp\{-irR\}. \tag{5.24}$$

5.3.2. *Adiabatic limit of the eta invariant.* Following the computation of the $\hat{\eta}$-form from the previous section we now compute the adiabatic limit of the eta invariant. First assume that the fibrewise Dirac operator $D^{S^1,r}$ is invertible, or $r\notin\mathbb{Z}$. The adiabatic limit of the eta invariant is then given by proposition 4.95 of [5] to be

$$\begin{aligned}\lim_{\varepsilon\to0}\eta^{r,\varepsilon} &= \frac{1}{(2\pi i)^m}\int_X\hat{A}(iR^{TX})\hat{\eta}\\ &= \int_X\hat{A}(X)\,f\left(\frac{c_1(\mathcal{L})}{2},r\right)\exp\{rc_1(\mathcal{L})\}.\end{aligned}$$

In the case where $r=k\in\mathbb{Z}$ we have that $\ker\left(D^{S^1,r}\right)=E_k=\mathcal{L}^{\otimes k}$ forms a vector bundle over the base X. Furthermore, it is clear from 5.2 that the dimension of the kernel of $D_{A_r,\varepsilon}$, for ε small, is given by

$$\dim\ker\left(D_{A_r,\varepsilon}\right)=\begin{cases}h^{\frac{m}{2},k} & \text{if } m \quad \text{even}\\ 0 & \text{if } m \quad \text{odd.}\end{cases} \tag{5.25}$$

Hence by Theorem 0.1 of [9] the adiabatic limit of the eta invariant exists in this case and is given by

$$\lim_{\varepsilon\to 0}\eta^{r,\varepsilon} = \frac{1}{(2\pi i)^m}\int_X \hat{A}(iR^{TX})\hat{\eta} + \eta(\bar{\partial}_k+\bar{\partial}_k^*) + \lim_{\varepsilon\to 0}\sum_{\lambda_0,\lambda_1=0}\mathrm{sgn}(\lambda_\varepsilon). \tag{5.26}$$

Here the third term denotes a sum over the eigenvalues of $D_{A_r,\varepsilon}$, which vanish to $O(\varepsilon)$ as $\varepsilon\to 0$, with the convention $\mathrm{sgn}(0)=0$. To compute this term note that the eigenvalues of type 2 in 5.2 do not vanish as $\varepsilon\to 0$ for $r\in\mathbb{Z}$. The eigenvalues of type 1 on the other hand vanish for $k=r$ and the third is seen to be

$$\lim_{\varepsilon\to 0}\sum_{\lambda_0,\lambda_1=0}\mathrm{sgn}(\lambda_\varepsilon) = \sum_{p>\frac{m}{2}}(-1)^p\, h^{p,k} - \sum_{p<\frac{m}{2}}(-1)^p\, h^{p,k}$$

Since the spectrum of $\sqrt{2}(\bar{\partial}_k+\bar{\partial}_k^*)$ is symmetric, we have $\eta(\bar{\partial}_k+\bar{\partial}_k^*)=0$. Denoting by $c=c_1(\mathcal{L})$, we now sum up the calculation of the adiabatic limit of the reduced eta invariant in all cases to be

$$\lim_{\varepsilon\to 0}\bar{\eta}^{r,\varepsilon} = \begin{cases} \frac{1}{2}\int_X \hat{A}(X)\left[\frac{\exp\left((1-2\{r\})\frac{c}{2}\right)}{\sinh\left(\frac{c}{2}\right)} - \frac{1}{c/2}\right]\exp\{rc\}, & \text{if } r\notin\mathbb{Z},\\ \frac{1}{2}\left\{\int_X \hat{A}(X)\left[\frac{\frac{c}{2}-\tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)}\right]\exp\{kc\} + h^{\frac{m}{2},k}\right. & \\ \qquad \left. + \sum_{p>\frac{m}{2}}(-1)^p\, h^{p,k} - \sum_{p<\frac{m}{2}}(-1)^p\, h^{p,k}\right\}, & \text{if } r=k\in\mathbb{Z},\ m \text{ even},\\ \frac{1}{2}\left\{\int_X \hat{A}(X)\left[\frac{\frac{c}{2}-\tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)}\right]\exp\{kc\}\right. & \\ \qquad \left. + \sum_{p>\frac{m}{2}}(-1)^p\, h^{p,k} - \sum_{p<\frac{m}{2}}(-1)^p\, h^{p,k}\right\}, & \text{if } r=k\in\mathbb{Z},\ m \text{ odd}. \end{cases}$$

5.3.3. *Spectral flow function.* To proceed with the computation of the eta invariant $\bar{\eta}^{r,\varepsilon}$ we attempt to compute the spectral flow function $\mathrm{sf}\{D_{A_r,\delta}\}_{0\le\delta\le\varepsilon}$. Let $\mathrm{Spec}^+(A)$ denote the positive spectrum of an operator A and define

$$M = \inf_{k,p}\left\{\frac{1}{2}\mu^2 \in \mathrm{Spec}^+\left(\Delta^p_{\bar{\partial}_k}\right)\right\}.$$

By the arguments in subsection 5.2 we have $M>0$. Furthermore, if we choose the adiabatic parameter small enough so that $\frac{\varepsilon}{8}<M$, the eigenvalues of type 2 in 5.2 do not contribute to spectral flow. The spectral flow from the eigenvalues of type 1

is easily computed to give

$$\begin{aligned}\mathrm{sf}\left\{D_{A_r,\delta}\right\}_{0\leq\delta\leq\varepsilon} = & \sum_{p>\frac{m}{2},\mathrm{even}} \sum_{k=\left\lceil r-\varepsilon\left(p-\frac{m}{2}\right)\right\rceil}^{\lceil r\rceil-1} h^{p,k} - \sum_{p>\frac{m}{2},\mathrm{odd}} \sum_{k=\left\lfloor r-\varepsilon\left(p-\frac{m}{2}\right)\right\rfloor+1}^{\lfloor r\rfloor} h^{p,k} \\ & - \sum_{p<\frac{m}{2},\mathrm{even}} \sum_{k=\lceil r\rceil}^{\left\lceil r-\varepsilon\left(p-\frac{m}{2}\right)\right\rceil-1} h^{p,k} + \sum_{p<\frac{m}{2},\mathrm{odd}} \sum_{k=\lfloor r\rfloor+1}^{\left\lfloor r-\varepsilon\left(p-\frac{m}{2}\right)\right\rfloor} h^{p,k}.\end{aligned}$$

Here $\lfloor x\rfloor$, $\lceil x\rceil$ stand for the floor and ceiling functions of x respectively.

5.3.4. *The transgression form.* Next let $\left\{\nabla^{\delta}\right\}_{0\leq\delta\leq\varepsilon}$ be any family of connections on TY such that $\nabla^0 = \nabla^{TY,0}, \nabla^{\varepsilon} = \nabla^{TY,\varepsilon}$. This family determines a connection ∇^{TZ} on the tangent bundle TZ of $Z = Y\times[0,\varepsilon]_{\delta}$ via

$$\nabla^{TZ} = d\delta\wedge\frac{\partial}{\partial\delta} + \nabla^{\delta}.$$

Let R^{TZ} be the curvature of ∇^{TZ}. By the Atiyah-Patodi-Singer index theorem we have

$$\bar{\eta}^{r,\varepsilon} - \lim_{\varepsilon\to 0}\bar{\eta}^{r,\varepsilon} = 2\left\{\mathrm{sf}\left\{D_{A_r,\delta}\right\}_{0\leq\delta\leq\varepsilon} + \frac{1}{(2\pi i)^{m+1}}\int_Z \hat{A}(R^{TZ})\right\}. \tag{5.27}$$

Note that this in particular implies that the integral term above is independent of the chosen family of connections. Here we shall compute the form $\hat{A}(R^{TZ})$. We choose the natural family of connections

$$\nabla^{\delta} = \nabla^{TY,\delta} = \nabla + \delta p^H S + p^{TS^1}S$$

by (5.5),(5.6). Denoting $p^H S = S^H, p^{TS^1}S = S^V$ by shorthands we have

$$\begin{aligned}R^{TZ} &= \left(d\delta\wedge\frac{\partial}{\partial\delta} + \nabla + \delta S^H + S^V\right)^2 \\ &= d\delta\wedge S^H + R^{TX} + \nabla S^V + S^V\wedge S^V \\ &\quad +\delta\left(\nabla S^H + S^H\wedge S^V + S^V\wedge S^H\right) + \delta^2 S^H\wedge S^H.\end{aligned}$$

Next we compute using (5.4)

$$\begin{aligned} S^H(e)e &= 0, \qquad S^H(e)f = Jf, \\ S^H(f)e &= Jf, \qquad S^H(f_1)f_2 = 0,\end{aligned}$$

as well as

$$\begin{aligned} S^V(e)e &= 0, \qquad S^V(e)f = 0, \\ S^V(f)e &= 0, \qquad S^V(f_1)f_2 = -\omega(f_1,f_2)e.\end{aligned}$$

where $f_1, f_2 \in T^H Y = \pi^* TX$. We may hence write $S^H = e^* \otimes J + \alpha_1$ where $e^* \otimes J, \alpha_1 \in \Omega^1(Y; \mathrm{End}(TY))$ with the only nonzero combination of α_1 being

$$\alpha_1(f)e = Jf. \tag{5.28}$$

Following this we may compute in symplectic geodesic coordinates to obtain $\nabla S^V = 0$, while computing in holomorphic geodesic coordinates yields $\nabla S^H = de^* \otimes J = 2\omega \otimes J \in \Omega^2(X; \mathrm{End}(TX))$. Computing further we find $S^V \wedge S^V = 0$. Another computation gives $S^H \wedge S^V = \Omega \in \Omega^2(X; \mathrm{End}(TX)) \subset \Omega^2(Y; \mathrm{End}(TY))$ is given by

$$\Omega(f_1, f_2)f = \omega(f_1, f)Jf_2 - \omega(f_2, f)Jf_1. \tag{5.29}$$

Also $S^V \wedge S^H = e^* \wedge \alpha_2$, where $\alpha_2 \in \Omega^1(Y; \mathrm{End}(TY))$ whose only nonzero combination is

$$\alpha_2(f_1)f = g(f_1, f)e. \tag{5.30}$$

And $S^H \wedge S^H = e^* \wedge \alpha_3$ where $\alpha_3 \in \Omega^1(Y; \mathrm{End}(TY))$ whose only nonzero combination is

$$\alpha_3(f_1)e = -f_1. \tag{5.31}$$

We hence have

$$R^{TZ} = d\delta \wedge e^* \otimes J + d\delta \wedge \alpha_1 + R^{TX} + 2\delta\omega \otimes J + \delta\Omega + \delta e^* \wedge \alpha_2 + \delta^2 e^* \wedge \alpha_3 \tag{5.32}$$

and we wish to compute

$$\begin{aligned} \hat{A}(R^{TZ}) &= \exp\left\{\mathrm{tr} p(R^{TZ})\right\}, \qquad \text{where} \\ p(z) &= \frac{1}{2}\log\left(\frac{z/2}{\sinh(z/2)}\right). \end{aligned}$$

Since $p(z)$ is an even function in z vanishing at zero, it has a power series

$$p(z) = p_2 z^2 + p_4 z^4 + \dots. \tag{5.33}$$

We shall begin our computation of $\mathrm{tr} p(R^{TZ})$ with the following lemma.

Lemma 5.4. *We have the tensor identities*

(1) .

$$\Omega \wedge \Omega = 0 \tag{5.34}$$

(2) .

$$\Omega \wedge R^{TX} = R^{TX} \wedge \Omega = 0, \tag{5.35}$$

(3) .

$$R^{TX} \wedge \alpha_1 = 0, \tag{5.36}$$

(4) .

$$tr\left[(\Omega J)^{\wedge k}\right] = -\, 2^k \omega^{\wedge k}. \tag{5.37}$$

Proof. Let f_i denote an orthonormal basis of TX at a point. We also denote by S_k the group of permutations of $\{1, 2, \ldots, k\}$.

(1) We compute

$$\begin{aligned}
&\Omega \wedge \Omega(f_1, f_2, f_3, f_4)f \\
=&\frac{1}{4}\sum_{\sigma\in S_4} \operatorname{sgn}(\sigma)\, \Omega\left(f_{\sigma(1)}, f_{\sigma(2)}\right) \Omega\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f \\
=&\frac{1}{4}\sum_{\sigma\in S_4} \operatorname{sgn}(\sigma) \Big\{ \omega\left(f_{\sigma(1)}, Jf_{\sigma(4)}\right) \omega\left(f_{\sigma(3)}, f\right) Jf_{\sigma(2)} - \omega\left(f_{\sigma(1)}, Jf_{\sigma(4)}\right) \omega\left(f_{\sigma(3)}, f\right) Jf_{\sigma(2)} \\
&\qquad - \omega\left(f_{\sigma(1)}, Jf_{\sigma(4)}\right) \omega\left(f_{\sigma(3)}, f\right) Jf_{\sigma(2)} + \omega\left(f_{\sigma(1)}, Jf_{\sigma(4)}\right) \omega\left(f_{\sigma(3)}, f\right) Jf_{\sigma(2)} \Big\} \\
=&0,
\end{aligned} \tag{5.38}$$

since each of the four terms in (5.38) contains an expression of the type $\omega(f_i, Jf_j) = g^{TX}(f_i, f_j) = 0$.

(2) We have $R^{TX} \in \Omega^2(\mathfrak{so}(TX))$ and $[R^{TX}, J] = 0$ since the complex structure J is parallel. This gives the identity $\omega\left(f_1, R^{TX}(f_3, f_4) f_4\right) = g^{TX}\left(R^{TX}(f_3, f_4) f_1, Jf_4\right)$.

We then compute

$$
\begin{aligned}
&\Omega \wedge R^{TX}(f_1, f_2, f_3, f_4) f \\
=&\frac{1}{4} \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma)\, \Omega\left(f_{\sigma(1)}, f_{\sigma(2)}\right) R^{TX}\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f \\
=&\frac{1}{4} \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \Big\{ \omega\left(f_{\sigma(1)}, R^{TX}\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f\right) J f_{\sigma(2)} \\
&\qquad\qquad - \omega\left(f_{\sigma(2)}, R^{TX}\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f\right) J f_{\sigma(1)} \Big\} \\
=&\frac{1}{4} \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \Big\{ g^{TX}\left(R^{TX}\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f_{\sigma(1)}, Jf\right) J f_{\sigma(2)} \\
&\qquad\qquad - g^{TX}\left(R^{TX}\left(f_{\sigma(3)}, f_{\sigma(4)}\right) f_{\sigma(2)}, Jf\right) J f_{\sigma(1)} \Big\} \\
=&0,
\end{aligned}
$$

by Bianchi's identity. The computation $R^{TX} \wedge \Omega = 0$ is similar.

(3) We compute

$$
\begin{aligned}
&R^{TX} \wedge \alpha_1 (f_1, f_2, f_3)\, e \\
=&\frac{1}{2} \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, R^{TX}\left(f_{\sigma(1)}, f_{\sigma(2)}\right) \alpha_1\left(f_{\sigma(3)}\right) e \\
=&\frac{1}{2} \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, R^{TX}\left(f_{\sigma(1)}, f_{\sigma(2)}\right) J f_{\sigma(3)} \\
=&\frac{J}{2} \left\{ \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, R^{TX}\left(f_{\sigma(1)}, f_{\sigma(2)}\right) f_{\sigma(3)} \right\} \\
=&0,
\end{aligned}
$$

by Bianchi's identity.

(4) For each $1 \leq l \leq k$, define the transposition $\tau_l = (2l-1 \quad 2l) \in S_{2k}$. Given a subset $S \subset \{1, \ldots, k\}$, define the permutation

$$\tau_S = \prod_{l \in S} \tau_l$$

in S_{2k}. Using the definition (5.29) of Ω, we now compute

$$\begin{aligned}
&\operatorname{tr}\left[(\Omega J)^{\wedge k}(f_1,\dots,f_{2k})\right]\\
=&\frac{1}{2^k}\sum_{\sigma\in S_{2k}}\operatorname{sgn}(\sigma)\operatorname{tr}\left[\prod_{l=1}^{k}\Omega\left(f_{\sigma(2l-1)},f_{\sigma(2l)}\right)J\right]\\
=&\frac{1}{2^k}\sum_{\sigma\in S_{2k}}\operatorname{sgn}(\sigma)\left\{\sum_{i=1}^{2k}g^{TX}\left(\left[\prod_{l=1}^{k}\Omega\left(f_{\sigma(2l-1)},f_{\sigma(2l)}\right)J\right]f_i,f_i\right)\right\}\\
=&\frac{1}{2^k}\sum_{\sigma\in S_{2k}}\operatorname{sgn}(\sigma)\left\{\sum_{S\subset\{1,\dots,k\}}\operatorname{sgn}(\tau_S)\left[\prod_{l=1}^{k-1}\omega\left(f_{\tau_S\circ\sigma(2l)},f_{\tau_S\circ\sigma(2l+1)}\right)\right]\omega\left(f_{\tau_S\circ\sigma(2l)},f_{\tau_S\circ\sigma(1)}\right)\right\}\\
=&\sum_{S\subset\{1,\dots,k\}}\left\{\frac{1}{2^k}\sum_{\sigma\in S_{2k}}\operatorname{sgn}(\tau_S\circ\sigma)\left[\prod_{l=1}^{k-1}\omega\left(f_{\tau_S\circ\sigma(2l)},f_{\tau_S\circ\sigma(2l+1)}\right)\right]\omega\left(f_{\tau_S\circ\sigma(2l)},f_{\tau_S\circ\sigma(1)}\right)\right\}\\
=&-\sum_{S\subset\{1,\dots,k\}}\omega^{\wedge k}(f_1,\dots,f_{2k})\\
=&-2^k\omega^{\wedge k}(f_1,\dots,f_{2k}).
\end{aligned}$$

□

We now perform further computations. Note that since the complex structure is parallel, the complexification of the Levi-Civita connection ∇^{TX} preserves the holomorphic and anti-holomorphic tangent spaces $TX^{1,0},TX^{0,1}$. Let $\nabla^{TX^{1,0}},\nabla^{TX^{0,1}}$ be the restrictions of ∇^{TX} to $TX^{1,0},TX^{0,1}$ and let $R^{TX^{1,0}},R^{TX^{0,1}}$ denote their respective curvatures. One then has

$$\frac{1}{2}\operatorname{tr}\left[\left(JR^{TX}\right)^N\right]=\operatorname{tr}\left[\left(iR^{TX^{1,0}}\right)^N\right], \tag{5.39}$$

where the right hand side is now the trace of a complex linear endomorphism.

Before stating the next computation, define the sequence $\{\epsilon_i\}_{i=2}^{\infty}$ of integers via

$$\epsilon_N=\begin{cases}1 & \text{if } N=2,\\ 0 & \text{if } N>2.\end{cases}$$

Proposition 5.5. *Let $N\geq 2$ be an even integer. The following identities hold*

(1) .

$$tr\left[\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^N\right]=2tr\left[\left(R^{TX^{1,0}}+2i\delta\omega\right)^N\right]+2\left(2i\delta\omega\right)^N. \tag{5.40}$$

(2) .

(5.41)
$$tr\left[J\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N-1}\right]=2tr\left[i\left(R^{TX^{1,0}}+2i\delta\omega\right)^{N-1}\right]+2i\left(2i\delta\omega\right)^{N-1}+2\epsilon_{N}\omega.$$

(3) .

$$tr\left[\Omega J\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N-2}\right]=-2\epsilon_{N}\omega \tag{5.42}$$

Proof. (1). The expansion of $\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N}$consists of monomials in the three tensors $R^{TX},2\delta\omega\otimes J$ and $\delta\Omega$. Using $\left[R^{TX},\omega\otimes J\right]=0$ and (5.35) we see that a monomial containing both R^{TX} as well as Ω is necessarily zero. Hence

$$\begin{aligned}\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N}=&\left(R^{TX}+2\delta\omega\otimes J\right)^{N}\\&+\left(2\delta\omega\otimes J+\delta\Omega\right)^{N}-\left(2\delta\omega\otimes J\right)^{N}.\end{aligned} \tag{5.43}$$

The trace of the first summand on the right hand side of (5.46) is easily computed using (5.39) to be

$$\mathrm{tr}\left[\left(R^{TX}+2\delta\omega\otimes J\right)^{N}\right]=2\mathrm{tr}\left[\left(R^{TX^{1,0}}+2i\delta\omega\right)^{N}\right].$$

Next, for each $a=(a_1,\ldots,a_{k+1})\in\mathbb{N}_0^{k+1}$ we denote $|a|=\sum_{i=1}^{k+1}a_i$. Then the sum of the last two terms in (5.46) is

$$\begin{aligned}&\left(2\delta\omega\otimes J+\delta\Omega\right)^{N}-\left(2\delta\omega\otimes J\right)^{N}=\\&\qquad\sum_{k>0}\left\{\sum_{\substack{a\in\mathbb{N}_0^{k+1}\\|a|=N-k}}\left(2\delta\omega J\right)^{a_1}\delta\Omega\ldots\delta\Omega\left(2\delta\omega J\right)^{a_{k+1}}\right\}.\end{aligned} \tag{5.44}$$

Using identity (5.34), we see that the only non-zero terms in the sum (5.44) are ones satisfying the parity constraint

$$a_1+a_{k+1},a_2,\ldots,a_k\text{ odd}. \tag{5.45}$$

Furthermore, using (5.37), we may compute the trace of each summand in (5.44) satisfying (5.45) to be

$$\mathrm{tr}\left[\left(2\delta\omega J\right)^{a_1}\delta\Omega\ldots\delta\Omega\left(2\delta\omega J\right)^{a_{k+1}}\right]=-\left(-1\right)^{k}\left(2i\delta\omega\right)^{N}.$$

The number of $a \in \mathbb{N}_0^{k+1}$ with $|a| = N - k$ and satisfying (5.45) is easily computed to be $2\binom{\frac{N}{2}}{k}$. We hence have

$$\begin{aligned}
&\operatorname{tr}\left[(2\delta\omega \otimes J + \delta\Omega)^N - (2\delta\omega \otimes J)^N\right] \\
&= - (2i\delta\omega)^N \left\{ \sum_{k>0} (-1)^k \, 2\binom{\frac{N}{2}}{k} \right\} \\
&= 2\,(2i\delta\omega)^N .
\end{aligned}$$

(2). The proof is almost identical to part 1. Again we see that a monomial in the expansion of $J\left(R^{TX} + 2\delta\omega \otimes J + \delta\Omega\right)^{N-1}$ cannot contain both R^{TX} and Ω. Hence

$$\begin{aligned}
J\left(R^{TX} + 2\delta\omega \otimes J + \delta\Omega\right)^{N-1} = & J\Big[\left(R^{TX} + 2\delta\omega \otimes J\right)^{N-1} \\
& + (2\delta\omega \otimes J + \delta\Omega)^{N-1} - (2\delta\omega \otimes J)^{N-1}\Big].
\end{aligned} \tag{5.46}$$

The trace of the first term on the right hand side above is again easily computed using (5.39) to be

$$\operatorname{tr}\left[J\left(R^{TX} + 2\delta\omega \otimes J\right)^{N-1}\right] = 2\operatorname{tr}\left[i\left(R^{TX^{1,0}} + 2i\delta\omega\right)^{N-1}\right].$$

The sum of the last two terms in (5.46) is now

$$\begin{aligned}
& J\,(2\delta\omega \otimes J + \delta\Omega)^{N-1} - J\,(2\delta\omega \otimes J)^{N-1} = \\
& \sum_{k>0} \left\{ \sum_{\substack{a \in \mathbb{N}_0^{k+1} \\ |a| = N-k-1}} J\,(2\delta\omega J)^{a_1}\, \delta\Omega \ldots \delta\Omega\, (2\delta\omega J)^{a_{k+1}} \right\}.
\end{aligned} \tag{5.47}$$

Using identity (5.34), we see that the only non-zero terms in the sum (5.49) are ones satisfying the parity constraint

$$a_1 + a_{k+1} \text{ even}, \quad a_2, \ldots, a_k \text{ odd}. \tag{5.48}$$

Furthermore, using (5.37), we may compute the trace of each summand in (5.44) satisfying (5.48) to be

$$\operatorname{tr}\left[J\,(2\delta\omega J)^{a_1}\, \delta\Omega \ldots \delta\Omega\, (2\delta\omega J)^{a_{k+1}}\right] = -i\,(-1)^k\,(2i\delta\omega)^{N-1}.$$

The number of $a \in \mathbb{N}_0^{k+1}$ with $|a| = N-1-k$ and satisfying (5.48) is again computed to be $\binom{\frac{N}{2}}{k} + \binom{\frac{N}{2}-1}{k}$. We hence have

$$\begin{aligned}
&\operatorname{tr}\left[J\left(2\delta\omega\otimes J+\delta\Omega\right)^{N-1}-J\left(2\delta\omega\otimes J\right)^{N-1}\right]\\
=&-i\left(2i\delta\omega\right)^{N-1}\left\{\sum_{k>0}\left(-1\right)^{k}\left[\binom{\frac{N}{2}}{k}+\binom{\frac{N}{2}-1}{k}\right]\right\}\\
=&2i\left(2i\delta\omega\right)^{N-1}+2\epsilon_{N}\omega.
\end{aligned}$$

(3). Since $\Omega J\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N-2}$ already contains Ω, the identity (5.34) now implies

$$\begin{aligned}
&\Omega J\left(R^{TX}+2\delta\omega\otimes J+\delta\Omega\right)^{N-2}\\
=&\Omega J\left(2\delta\omega\otimes J+\delta\Omega\right)^{N-2}\\
=&\Omega J\left(2\delta\omega J\right)^{N-2}+\sum_{k>0}\left\{\sum_{\substack{a\in\mathbb{N}_0^{k+1}\\|a|=N-k-2}}\Omega J\left(2\delta\omega J\right)^{a_1}\delta\Omega\ldots\delta\Omega\left(2\delta\omega J\right)^{a_{k+1}}\right\}.
\end{aligned}\tag{5.49}$$

Using identity (5.34), we see that the only non-zero terms in the sum (5.49) are ones satisfying the parity constraint

$$a_1 \text{ even},\quad a_2,\ldots,a_{k+1} \text{ odd}. \tag{5.50}$$

Furthermore, using (5.37), we may compute the trace of each summand in (5.44) satisfying (5.50) to be

$$\operatorname{tr}\left[\Omega J\left(2\delta\omega J\right)^{a_1}\delta\Omega\ldots\delta\Omega\left(2\delta\omega J\right)^{a_{k+1}}\right]=-2\omega\left(-1\right)^{k}\left(2i\delta\omega\right)^{N-2}.$$

The number of $a \in \mathbb{N}_0^{k+1}$ with $|a| = N-2-k$ and satisfying (5.50) is again computed to be $\binom{\frac{N}{2}-1}{k}$. We hence have

$$\begin{aligned}
&\operatorname{tr}\left[\Omega J\left(2\delta\omega\otimes J+\delta\Omega\right)^{N-1}-J\left(2\delta\omega\otimes J\right)^{N-1}\right]\\
=&-2\omega\left(2i\delta\omega\right)^{N-2}-2\left(2i\delta\omega\right)^{N-2}\left\{\sum_{k>0}\left(-1\right)^{k}2\binom{\frac{N}{2}-1}{k}\right\}\\
=&-2\epsilon_{N}\omega.
\end{aligned}$$

□

Proposition 5.6. *Let $N \geq 2$ be an even integer. The following identity holds*

$$tr\left(R^{TZ}\right)^N = 2tr\left[\left(R^{TX^{1,0}} + 2i\delta\omega\right)^N\right] + 2\left(2i\delta\omega\right)^N + d\delta \wedge e^* \left\{2tr\left[iN\left(R^{TX^{1,0}} + i2\delta\omega\right)^{N-1}\right] + i2N\left(2i\delta\omega\right)^{N-1}\right\}. \tag{5.51}$$

Proof. Clearly $\left(R^{TZ}\right)^N$ is a sum of monomials in the seven tensors appearing on the right hand side of (5.32). Due to the $d\delta$ and e^* factors, a nonzero monomial appearing in $\left(R^{TZ}\right)^N$ is of atmost degree two in the four tensors $d\delta \wedge e^* \otimes J, d\delta \wedge \alpha_1, \delta e^* \wedge \alpha_2$ and $\delta^2 e^* \wedge \alpha_3$. Let P_i denote the sum of monomials of degree i in these four tensors appearing in the expansion of $\left(R^{TX}\right)^N$. Hence

$$\left(R^{TZ}\right)^N = P_0 + P_1 + P_2, \tag{5.52}$$

and we now compute the traces of P_0, P_1 and P_2.

Trace of P_0. It is clear that

$$\begin{aligned} \mathrm{tr} P_0 &= \mathrm{tr}\left[\left(R^{TX} + 2\delta\omega \otimes J + \delta\Omega\right)^N\right] \\ &= 2\mathrm{tr}\left[\left(R^{TX^{1,0}} + 2i\delta\omega\right)^N\right] + 2\left(2i\delta\omega\right)^N \end{aligned} \tag{5.53}$$

by (5.40).

Trace of P_1. A monomial in P_1 must contain exactly one occurrence of $d\delta \wedge \alpha_1, \delta e^* \wedge \alpha_2$ or $\delta^2 e^* \wedge \alpha_3$ and must not contain $d\delta \wedge e^* \otimes J$. From the formulas (5.28)-(5.31) for α_1, α_2 and α_3, it is clear that such a monomial switches the $T^H Y$ and $T^V Y$ summands. Hence we have

$$\mathrm{tr}\, P_1 = 0. \tag{5.54}$$

Trace of P_2. A nonzero monomial in P_2 must contain a single appearance of $d\delta$ and e^* each. It can hence be of following three types.

Type A. This type of monomial contains a single appearance of $d\delta \wedge e^* \otimes J$ and no appearances of $d\delta \wedge \alpha_1, \delta e^* \wedge \alpha_2$ or $\delta^2 e^* \wedge \alpha_3$. Let P_2^1 be the sum of all monomials of this type appearing in $\left(R^{TX}\right)^N$. Using the cyclicity of the trace we easily see that

$$\begin{aligned} \mathrm{tr} P_2^1 &= N\, d\delta \wedge e^*\, \mathrm{tr}\left[J\left(R^{TX} + 2\delta\omega \otimes J + \delta\Omega\right)^{N-1}\right] \\ &= d\delta \wedge e^* \left\{2\mathrm{tr}\left[iN\left(R^{TX^{1,0}} + 2i\delta\omega\right)^{N-1}\right] + i2N\left(2i\delta\omega\right)^{N-1} + 4\epsilon_N \omega\right\}, \end{aligned} \tag{5.55}$$

by (5.41).

Type B. This type of monomial contains a single appearance each of $d\delta \wedge \alpha_1, \delta e^* \wedge \alpha_2$ and no appearances of $d\delta \wedge e^* \otimes J$ or $\delta^2 e^* \wedge \alpha_3$. Let P_2^2 be the sum of all monomials of this type appearing in $\left(R^{TX}\right)^{N}$. From the formulas (5.28) and (5.30), we note that α_1 maps T^VY into T^HY while α_2 maps T^HY into T^VY. Hence in order to have a nonzero trace, a monomial of this type must be of the form

$$\begin{aligned} \delta e^* \wedge \alpha_2 \wedge A \wedge d\delta \wedge \alpha_1 \quad \text{or} \\ B \wedge d\delta \wedge \alpha_1 \wedge \delta e^* \wedge \alpha_2 \wedge C, \end{aligned}$$

where A, B and C are some monomials in the tensors $R^{TX}, 2\delta\omega \otimes J$ and $\delta\Omega$. Thus we see that in a monomial of this type $d\delta \wedge \alpha_1, \delta e^* \wedge \alpha_2$ appear consecutively after a cyclic permutation. Using the cyclicity of the trace we now have

$$\mathrm{tr} P_2^2 = N\,\mathrm{tr}\left[d\delta \wedge \alpha_1 \wedge \delta e^* \wedge \alpha_2 \wedge \left(R^{TX} + 2\delta\omega \otimes J + \delta\Omega\right)^{N-2}\right].$$

The identity

$$\alpha_1 \wedge \alpha_2 = -\,\Omega J$$

combined with (5.42) now gives

$$\mathrm{tr} P_2^2 = -\,4\epsilon_N \delta\omega. \tag{5.56}$$

Type C. The third type of monomial contains one appearance each of $d\delta \wedge \alpha_1$ and $\delta^2 e^* \wedge \alpha_3$ and no appearances of $d\delta \wedge e^* \otimes J$ or $\delta e^* \wedge \alpha_2$. However since $\alpha_1$and α_3 both annihilate T^HY and map T^VY into T^HY such a monomial must necessarily have trace zero.

Adding (5.55) and (5.56) gives

$$\mathrm{tr} P_2 = d\delta \wedge e^* \left\{2\mathrm{tr}\left[iN\left(R^{TX^{1,0}} + 2i\delta\omega\right)^{N-1}\right] + i2N\left(2i\delta\omega\right)^{N-1}\right\}. \tag{5.57}$$

The proposition now follows from (5.52), (5.53), (5.54) and (5.57). □

Finally, substituting (5.51) into the power series (5.33), we now have

$$\begin{aligned} \mathrm{tr}\left\{p\left(R^{TZ}\right)\right\} &= \Omega_0 + d\delta \wedge e^* \wedge \Omega_2, \quad \text{where} \\ \Omega_0 &= 2\mathrm{tr}\left[p\left(R^{TX^{1,0}} + 2i\delta\omega\right)\right] + 2p\left(2i\delta\omega\right) \\ \Omega_2 &= 2\mathrm{tr}\left[ip'\left(R^{TX^{1,0}} + i2\delta\omega\right)\right] + i2p'\left(2i\delta\omega\right). \end{aligned}$$

We may now calculate

$$\begin{aligned}\hat{A}(R^{TZ}) &= \int_0^{\varepsilon}\int_Y \exp\{\Omega_0 + d\delta\wedge e^*\wedge\Omega_2\}\\ &= \int_0^{\varepsilon}\int_Y d\delta\wedge e^*\wedge\Omega_2\exp\{\Omega_0\}\\ &= (2\pi)\int_0^{\varepsilon}d\delta\int_X \Omega_2\exp\{\Omega_0\}.\end{aligned}$$

In view of equation (5.27), we now summarize the calculation of the eta invariant.

Theorem 5.7. *The eta invariant* $\bar{\eta}^{r,\varepsilon}$ *for* $\frac{\varepsilon}{8} < \inf_{k,p}\left\{\frac{1}{2}\mu^2 \in Spec^+\left(\Delta^p_{\bar{\partial}_k}\right)\right\}$ *is given by*

$$\bar{\eta}^{r,\varepsilon} = \lim_{\varepsilon\to 0}\bar{\eta}^{r,\varepsilon} + sf\{D_{A_r,\delta}\}_{0\le\delta\le\varepsilon} + \frac{1}{(2\pi i)^{m+1}}\int_Z \hat{A}(R^{TZ})$$

where the three terms above are given by

(1) the adiabatic limit:

$$\lim_{\varepsilon\to 0}\bar{\eta}^{r,\varepsilon} = \begin{cases}\frac{1}{2}\int_X \hat{A}(X)\left[\frac{\exp\left((1-2\{r\})\frac{c}{2}\right)}{\sinh\left(\frac{c}{2}\right)} - \frac{1}{c/2}\right]\exp\{rc\}, & \text{if } r\notin\mathbb{Z},\\ \frac{1}{2}\left\{\int_X \hat{A}(X)\left[\frac{\frac{c}{2}-\tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)}\right]\exp\{kc\} + h^{\frac{m}{2},k}\right. & \\ \qquad\left. + \sum_{p>\frac{m}{2}}(-1)^p h^{p,k} - \sum_{p<\frac{m}{2}}(-1)^p h^{p,k}\right\}, & \text{if } r=k\in\mathbb{Z},\ m \text{ even},\\ \frac{1}{2}\left\{\int_X \hat{A}(X)\left[\frac{\frac{c}{2}-\tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)}\right]\exp\{kc\}\right. & \\ \qquad\left. + \sum_{p>\frac{m}{2}}(-1)^p h^{p,k} - \sum_{p<\frac{m}{2}}(-1)^p h^{p,k}\right\}, & \text{if } r=k\in\mathbb{Z},\ m \text{ odd},\end{cases}$$

with $c = c_1(\mathcal{L})$.

(2) the spectral flow function:

$$\begin{aligned}sf\{D_{A_r,\delta}\}_{0\le\delta\le\varepsilon} = &\sum_{p>\frac{m}{2},even}\ \sum_{k=\lceil r-\varepsilon(p-\frac{m}{2})\rceil}^{\lceil r\rceil-1} h^{p,k} - \sum_{p>\frac{m}{2},odd}\ \sum_{k=\lfloor r-\varepsilon(p-\frac{m}{2})\rfloor+1}^{\lfloor r\rfloor} h^{p,k}\\ &- \sum_{p<\frac{m}{2},even}\ \sum_{k=\lceil r\rceil}^{\lceil r-\varepsilon(p-\frac{m}{2})\rceil-1} h^{p,k} + \sum_{p<\frac{m}{2},odd}\ \sum_{k=\lfloor r\rfloor+1}^{\lfloor r-\varepsilon(p-\frac{m}{2})\rfloor} h^{p,k}.\end{aligned}$$

(3) the transgression form:

$$\int_Z \hat{A}(R^{TZ}) = (2\pi)\int_0^{\varepsilon} d\delta \int_X \Omega_2 \exp\left\{\Omega_0\right\}, \quad \textit{where}$$

$$\begin{aligned}\Omega_0 =& 2tr\left[p\left(R^{TX^{1,0}} + 2i\delta\omega\right)\right] + 2p\left(2i\delta\omega\right),\\ \Omega_2 =& 2tr\left[ip'\left(R^{TX^{1,0}} + i2\delta\omega\right)\right] + i2p'\left(2i\delta\omega\right)\end{aligned}$$

and $p(z) = \frac{1}{2}\log\left(\frac{z/2}{\sinh(z/2)}\right)$.

We observe that the formula above expresses the eta invariant in purely topological terms on the base.

Finally, we show that our computation agrees with the one of Nicolaescu from [11] in dimension three. Consider the case when X is a oriented Riemann surface. We choose g^{TX} a metric of volume πl where l is a positive integer. Choose the complex structure $J = -\star$ on X, where $\star$ denotes the Hodge star. This gives a Kahler form ω satisfying $\int_X \omega = -\pi l$. Let $\mathcal{L} \to X$ be a Hermitian line bundle of degree $c_1(\mathcal{L}) = l$. This allows us to pick a connection on $\mathcal{L}$ with curvature $R = 2\omega$, which induces a holomorphic structure on $\mathcal{L}$. We may now choose Y to be the unit circle bundle in $\mathcal{L}$ over X equipped with the adiabatic family of metrics (5.2). We now specialize our formula for the eta invariant to compute $\bar{\eta}^{0,\varepsilon}$ in this case. Assuming the adiabatic parameter ε to be sufficiently small the spectral flow contribution in Theorem 5.7 is seen to vanish. Setting $r = 0$ the other terms in the formula are easily computed to give

$$\bar{\eta}^{0,\varepsilon} = \frac{c}{12} - \frac{1}{2}\left(h^{1,0} + h^{0,0}\right) + \frac{\varepsilon^2 l}{12} + \frac{\varepsilon}{12}\int_X \mathrm{tr}\left[\frac{iR^{TX^{1,0}}}{2\pi}\right].$$

Using Serre duality and Gauss-Bonnet we get

$$\bar{\eta}^{0,\varepsilon} = \frac{c}{12} - h^{0,0} + \frac{\varepsilon^2 l}{12} - \frac{\varepsilon\chi}{12}, \tag{5.58}$$

where χ is the Euler characteristic of the surface. However the adiabatic metrics in [11] were chosen to be of the form $r^2 g^{TS^1} \oplus \frac{1}{l}\pi^* g^{TX}$. This amounts to a rescaling and hence the substitution $\varepsilon = r^2 l$ in (5.58). Following this our formula is seen to agree in this case with Theorem 2.4 proved, by two different methods, in [11].

Appendix A. Estimates on Gaussian integrals

Here we prove some estimates on Gaussian integrals used in section 3

Lemma A.1. *There exist constants C_1, C_2 and C_3 depending only on the Riemannian manifold (Y, g), such that for any $x, z \in Y$ and $t, t' > 0$ we have the following inequalities*

(1) .

$$\int_Y h_t(x,y)dy \leq C_1, \tag{A.1}$$

(2) .

$$\int_Y h_t(x,y)h_{t'}(y,z)dy \leq C_2 h_{4(t+t')}(x,z) \tag{A.2}$$

(3) and

$$\int_0^t s^{-\frac{1}{2}}ds \left(\int_Y dy h_{2(t-s)}(x,y)h_{2s}(y,z)\right) \leq C_3 t^{\frac{1}{2}} h_{8t}(x,z). \tag{A.3}$$

Proof. (1) . Consider the ball $B = \{y|\rho(x,y) < i_g\}$ and split the integral A.1 into integrals over B and its complement B^c. Introducing geodesic coordinates, the integral over B can be bounded from above by the Euclidean integral $\int_{\mathbb{R}^n} \frac{e^{-\frac{|r|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}dr = 1$. For the integral over B^c, we use the inequality $\frac{e^{-\frac{\rho(x,y)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \leq \frac{(n/2)!}{\pi^{n/2} i_g^n}$ to get $\int_{B^c} h_t(x,y)dy \leq \frac{(n/2)!}{\pi^{n/2} i_g^n}\mathrm{vol}(Y)$.

(2) .Without loss of generality assume that $t \leq t'$. The triangle inequality gives the estimate $\frac{\rho(x,y)^2}{t} + \frac{\rho(y,z)^2}{t'} \geq \frac{\rho(x,z)^2}{2(t+t')}$. Using this we may bound

$$\begin{aligned}&\int_Y \frac{e^{-\frac{\rho(x,y)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \frac{e^{-\frac{\rho(y,z)^2}{4t'}}}{(4\pi t')^{\frac{n}{2}}} dy \\ \leq &\frac{e^{-\frac{\rho(x,z)^2}{16(t+t')}}}{(4\pi t')^{\frac{n}{2}}} \left(\int_Y \frac{e^{-\frac{\rho(x,y)^2}{8t}}}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\rho(y,z)^2}{8t'}} dy\right).\end{aligned}$$

Then via $\frac{1}{t'} \leq \frac{2}{t+t'}$ and $e^{-\frac{\rho(y,z)^2}{8t'}} \leq 1$ we may further bound this from above by $2^n h_{4(t+t')}(x,z) \left(\int_Y h_{2t}(x,y)dy\right)$. The estimate A.2 now follows from A.1.

(3). First use A.2 to estimate

$$\begin{aligned}&\int_0^t s^{-\frac{1}{2}} \left(\int_Y dy h_{2(t-s)}(x,y)h_{2s}(y,z)\right) ds \\ \leq &C_2 \int_0^t s^{-\frac{1}{2}} h_{8t}(x,z)ds = 2C_2 t^{\frac{1}{2}} h_{8t}(x,z).\end{aligned}$$

□

Acknowledgements. The author would like to thank Prof. J.-M. Bismut for several discussions and suggesting many key ideas presented here. The author is also grateful to his thesis advisor Tom Mrowka under whom this project was started. The author would like to thank the referees for constructive comments and suggestions to improve the paper.

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