Asymptotics of the Eta invariant

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The Dirac Operator

M^n oriented, Riemannian manifold.

A Clifford module $\left(S, h^S, \nabla^S, c\right)$ is a Hermitian vector bundle equipped with unitary connection ∇^S and a compatible morphism of algebra bundles $c : Cl\left(T^*X\right) \to \mathrm{End}\left(S\right)$.

We can then form the **Dirac operator**

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1. M^n spin manifold and $S = S^{TX} \otimes L$ is the spin bundle twisted by a Hermitian line bundle L. A unitary connection A on L now gives the corresponding Dirac operator
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$D_A : C^\infty\left(S^{TX} \otimes L\right) \to C^\infty\left(S^{TX} \otimes L\right)$.
2. X complex manifold with $\mathcal{L} \to X$ Hermitian holomorphic line bundle with holomorphic derivative $\bar{\partial}_{\mathcal{L}} : \mathcal{A}^{0,*}\left(\mathcal{L}\right) \to \mathcal{A}^{0,*}\left(\mathcal{L}\right)$. If X additionally Kahler, then

$$D = \sqrt{2}\left(\bar{\partial}_{\mathcal{L}} + \bar{\partial}_{\mathcal{L}}^*\right)$$

is a Dirac operator.

Spectral Invariants

The Dirac operator is self-adjoint and elliptic. Hence it has a discrete spectrum of real eigenvalues

$$\dots \lambda_{-1} \leq 0 \leq \lambda_0 \leq \lambda_1 \leq \dots$$

One can form the spectral invariants $k_D = \dim \ker(D)$, the determinant of D^2 and the signature of D.

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Signature of D

$$\begin{aligned} \eta_D \quad &" = " \quad \sum_j \operatorname{sign}(\lambda_j) \\ &" = " \quad \sum_j \left(\int_0^\infty \frac{1}{\sqrt{\pi t}} \lambda_j e^{-t\lambda_j^2} \, dt \right) \end{aligned}$$

$$\boxed{\eta_D = \int_0^\infty \frac{1}{\sqrt{\pi t}} \operatorname{tr}\left(D e^{-tD^2} \right) dt}$$

Spectral Invariants

Determinant of D^2

$$\det\left(D^2\right) \quad = \quad e^{-\zeta_D'(0)}$$

$$\boxed{\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left\{ \mathrm{tr}\left(e^{-tD^2} \right) - k_D \right\} t^{s-1} dt, \quad \mathrm{Re}(s) \gg 0.}$$

Asymptotics of Spectral Invariants

Consider Y^n oriented, Riemannian spin of odd dimension with Hermitian line bundle L. Fix base unitary connection A_0, and imaginary one form $a \in \Omega^1(Y; i\mathbb{R})$. This gives family of connections $A_r = A_0 + ra$ and associated Dirac operators

$$D_r = D_{A_r} : C^\infty\left(S^{TY} \otimes L\right) \to C^\infty\left(S^{TY} \otimes L\right).$$

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Consider the asscociated spectral invariants

$$\begin{aligned} k_r &= k_{D_r} \\ \eta_r &= \eta_{D_r} \\ d_r &= \det\left(D_r^2\right). \end{aligned}$$

What happens to these as $r \to \infty$?

Analogous problem

X complex (not nec. Kahler), $\mathcal{L} \to X$ Hermitian holomorphic line bundle

$$D_p = \sqrt{2}\left(\bar{\partial}_{\mathcal{L}^{\otimes p}} + \bar{\partial}^*_{\mathcal{L}^{\otimes p}}\right) : \mathcal{A}^{0,*}\left(\mathcal{L}^{\otimes p}\right) \to \mathcal{A}^{0,*}\left(\mathcal{L}^{\otimes p}\right).$$

Similar spectral quantities k_p, η_p, d_p. What happens to these as $p \to \infty$. Here p is analogous to r.

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Application to the Grauert- Riemenschneider conjecture.

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(Bismut-Vasserot '89) $\mathcal{L}$ positive.

$$\tau_p = \log\left[\frac{\det\left(D_p^2|_{\mathcal{A}^{0,even}}\right)}{\det\left(D_p^2|_{\mathcal{A}^{0,odd}}\right)}\right] = O\left(p^{\frac{n}{2}}\right).$$

Arithmetic applications by Faltings ('84), Vojta ('89).

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Kahler-Einstein program: Asymptotics of the projector $\Pi_p : C^\infty\left(\mathcal{L}^{\otimes p}\right) \to \ker\left(D_p\right)$.

New Results

Coming back to

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(S.) $k_r = o\left(r^{\frac{n}{2}}\right)$.
(Taubes '07) $\eta_r = O\left(r^p\right), \ p = \frac{n}{2} + \frac{n-1}{2(n+1)} + \epsilon, \ \forall \epsilon > 0.$
(S.) $\eta_r = o\left(r^{\frac{n}{2}}\right)$.
Application to proving the Weinstein conjecture in dimension 3.
(S.) $d_r = o\left(r^{\frac{n}{2}}\right)$.

The Weinstein Conjecture

Given (Y^3, a) contact manifold (i.e. $a \wedge da \neq 0$). Its Reeb vector field R defined via $i_R da = 0$, $a(R) = 1$ has a closed orbit.

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\begin{aligned}
c(*F_A) &= r\left(\Phi\Phi^* - \frac{1}{2}|\Phi|^2 - a\right) \\
D_A\Phi &= 0,
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here A is a unitary connection on L and $\Phi \in C^\infty\left(S^{TY} \otimes L\right)$.

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These are variational equations for the functional

$$\begin{aligned} \mathrm{CSD}(A, \Phi) &= \underbrace{\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0})}_{cs(A)} + \frac{1}{2}\int_Y \langle D_A\Phi, \Phi\rangle\, dy \\ &\quad - \frac{r}{2}\int_Y a \wedge F_A. \end{aligned}$$

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Infinite dimensional Morse Theory for this functional gives Monopole Floer group $\widehat{HM}(Y, L)$. The grading of a generator (A_r, Φ_r) is

$$\mathrm{gr}\,(A_r, \Phi_r) = \eta\,(\mathrm{Hess\ CSD}\,(A_r, \Phi_r)) - cs\,(A_r)\,.$$

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The set $\Phi^{-1}(0)$ converges to a union of Reeb orbits as $r \to \infty$ assuming we have a bound on $cs(A_r)$.
Fixing the grading, this reduces to a bound on η_r.

Bound on the Eta invariant

To bound the eta invariant we use the integral formula

$$\begin{aligned}\eta_r &= \int_0^\infty \frac{1}{\sqrt{\pi t}} \mathrm{tr}\left(D_r e^{-tD_r{}^2}\right) dt \\ &= \int_0^1 \frac{1}{\sqrt{\pi t}} \mathrm{tr}\left(D_r e^{-tD_r{}^2}\right) dt + \int_1^\infty \frac{1}{\sqrt{\pi t}} \mathrm{tr}\left(D_r e^{-tD_r{}^2}\right) dt \\ &= \int_0^1 \frac{1}{\sqrt{\pi t}} \mathrm{tr}\left(D_r e^{-tD_r{}^2}\right) dt + \mathrm{tr}\ E\left(D_r\right)\end{aligned}$$

Where $E(x) = \mathrm{sign}(x)\left[\frac{2}{\sqrt{\pi}} \int_{|x|}^\infty e^{-s^2} ds\right] = \mathrm{sign}(x)\,\mathrm{erfc}(|x|)$. This is discontinuous and has a non-local trace

Local Index theory expansions

Maximum principle or otherwise gives the bound

$$\begin{aligned} r^{-\frac{n}{2}} \left| e^{-tD_r^2}(x,y) \right| &\leq Ce^{-r\frac{\rho(x,y)^2}{4t}} e^t. \\ \text{where } \rho(x,y) &= \text{geodesic distance function.} \end{aligned}$$

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Local index theory (or semiclassical analysis) now gives expansions

$$
\begin{aligned}
\mathrm{tr}\left(e^{-tD_r{}^2}\right) &\sim r^{\frac{n}{2}}\left(c_0(t) + c_1(t)\, r^{-1} + \ldots\right) \\
\mathrm{tr}\left(D_r e^{-tD_r{}^2}\right) &\sim r^{\frac{n-1}{2}}\left(d_0(t) + d_1(t)\, r^{-1} + \ldots\right)
\end{aligned}
$$

Integrating the second expansion shows $o\left(r^{\frac{n}{2}}\right)$ bound on the first summand of

$$
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Local Index theory expansions

In general for an smooth odd trace

$$\mathrm{tr}\ \varphi^{odd}\left(D_r\right) \sim r^{\frac{n-1}{2}}\left(C_0\left(\varphi\right) + C_1\left(\varphi\right) r^{-1} + \ldots\right).$$

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$$\lim_{r\to\infty} r^{-\frac{n}{2}} k_r \leq \lim_{r\to\infty} r^{-\frac{n}{2}} \mathrm{tr}\left(e^{-tD_r{}^2}\right) \leq c_0\left(t\right) \to 0, \quad \text{as} \quad t \to \infty.$$

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Start with $\mathcal{L} \to X^{2m}$ positive line bundle over complex manifold.

$$\begin{array}{ccl} S^1 & \to & Y^{2m+1} = \text{unit circle bundle of } \mathcal{L} \\ & & \downarrow \pi \\ & & X^{2m}, \qquad n = 2m+1. \end{array}$$

Chern connection on $\mathcal{L}$ gives splitting $TY = TS^1 \oplus \pi^* TX$. Choose adiabatic family of metrics

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A spin structure on X corresponds to a square root bundle $\mathcal{K}^{\otimes 2} = K_X = \Lambda^{top} T^{0,1} X$. Asscociated twisted Dirac operator

$$\sqrt{2}\left(\bar{\partial}_k + \bar{\partial}_k^*\right) = \sqrt{2}\left(\bar{\partial}_{\mathcal{K}\otimes\mathcal{L}^{\otimes k}} + \bar{\partial}^*_{\mathcal{K}\otimes\mathcal{L}^{\otimes k}}\right).$$

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The spin structures on TX, TS^1 combine to give one on TY. We twist the spin bundle trivially $S^{TY} \otimes \mathbb{C}$ but with the connections $d + ira$, a dual to generator of S^1 action. This gives our D_r^ε.

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By using Fourier modes along the S^1 it is possible to write the spectrum of D_r^ε in terms of the spectrum of $\Delta_k^p = \left(\bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k\right)|_{\Lambda^{0,p}}$.

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There are two types of eigenvalues.
Type 1.

$$\lambda = (-1)^p \left(k + \varepsilon \left(p - \frac{m}{2}\right) - r\right), \quad 0 \le p \le m, k \in \mathbb{Z}$$

multiplicity $= h^{p,k} = \dim\ H^p\left(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right)$.
Type 2.

$$\lambda = \frac{(-1)^{p+1}\varepsilon \pm \sqrt{(2k + \varepsilon(2p - m) - 2r + 1)^2 + 4\mu^2\varepsilon}}{2}, \ 0 \le p \le m, k \in$$

where $\frac{1}{2}\mu^2$ is a positive eigenvalue of Δ_k^p.

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where $\frac{1}{2}\mu^2$ is a positive eigenvalue of Δ_k^p.
Gives large kernel conjecture for the Dirac operator in this case.

Eta invariant of a circle bundle

This computation now gives an asymptotic formula for η_r^ε in terms of characteristic classes on the base.

$$\bar{\eta}^{r,\varepsilon} = \sum_{a=0}^{m} \left\{ \left(\frac{r^{a+1}}{(a+1)!} - \sum_{k=1}^{[r+\frac{\varepsilon m}{2}]} \frac{k^a}{a!} \right) \int_X c_1(\mathcal{L})^a \left[\mathrm{ch}(\mathcal{K})\mathrm{td}(X)\right]^{m-a} \right\} + O(1).$$

The above formula shows $\bar{\eta}^{r,\varepsilon}$ is discontinous of $O\left(r^{\frac{n-1}{2}}\right)$. Hence this would be the optimal estimate on the eta invariant.

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The above formula shows $\bar{\eta}^{r,\varepsilon}$ is discontinous of $O\left(r^{\frac{n-1}{2}}\right)$. Hence this would be the optimal estimate on the eta invariant. However this isn't an exact formula as there is an $O(1)$.

Explicit computation of the eta invariant

To get an explicit computation of the eta invariant, we let $\varepsilon \to 0$.

$$\bar{\eta}^{r,\varepsilon} = \lim_{\varepsilon \to 0} \bar{\eta}^{r,\varepsilon} + \mathrm{sf}\left\{D_{A_r,\delta}\right\}_{0 \leq \delta \leq \varepsilon} + \frac{1}{(2\pi i)^{m+1}} \int_Z \hat{A}(R^{TZ})$$

Then $\lim_{\varepsilon \to 0} \bar{\eta}^{r,\varepsilon}$ exists by work of Bismut-Cheeger ('89), Dai ('91), Zhang ('94). The spectral flow function is found, for ε small, from our description of the spectrum. The third term is a characteristic class $Z = Y \times [0, \varepsilon]$.

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The result is a computation in terms of $h^{p,k} = \dim H^p\left(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k}\right)$ and characteristic classes of $TX^{1,0}$ and $\mathcal{L}$. This generalizes a computation of Nicolaescu ('97) in dimension 3.

Explicit computation of the eta invariant

$$\lim_{\varepsilon \to 0} \bar{\eta}^{r,\varepsilon} = \begin{cases} \frac{1}{2} \int_X \hat{A}(X) \left[\frac{\exp\left((1-2\{r\})\frac{c}{2}\right)}{\sinh\left(\frac{c}{2}\right)} - \frac{1}{c/2} \right] \exp\{rc\}, & \text{if } r \notin \mathbb{Z}, \\ \frac{1}{2} \left\{ \int_X \hat{A}(X) \left[\frac{\frac{c}{2} - \tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)} \right] \exp\{kc\} + h^{\frac{m}{2},k} \right. & \text{if } r = k, \\ \qquad \left. + \sum_{p > \frac{m}{2}} (-1)^p \, h^{p,k} - \sum_{p < \frac{m}{2}} (-1)^p \, h^{p,k} \right\}, & m \text{ even} \\ \frac{1}{2} \left\{ \int_X \hat{A}(X) \left[\frac{\frac{c}{2} - \tanh\left(\frac{c}{2}\right)}{\frac{c}{2}\tanh\left(\frac{c}{2}\right)} \right] \exp\{kc\} \right. & \text{if } r = k, \\ \qquad \left. + \sum_{p > \frac{m}{2}} (-1)^p \, h^{p,k} - \sum_{p < \frac{m}{2}} (-1)^p \, h^{p,k} \right\}, & m \text{ odd.} \end{cases}$$

Explicit computation of the eta invariant

$$
\begin{aligned}
\mathrm{sf}\,\{D_{A_r,\delta}\}_{0\leq\delta\leq\varepsilon} &= \sum_{p>\frac{m}{2},\mathrm{even}} \; \sum_{k=\lceil r-\varepsilon(p-\frac{m}{2})\rceil}^{\lceil r\rceil-1} h^{p,k} \\
&- \sum_{p>\frac{m}{2},\mathrm{odd}} \; \sum_{k=\lfloor r-\varepsilon(p-\frac{m}{2})\rfloor+1}^{\lfloor r\rfloor} h^{p,k} \\
&- \sum_{p<\frac{m}{2},\mathrm{even}} \; \sum_{k=\lceil r\rceil}^{\lceil r-\varepsilon(p-\frac{m}{2})\rceil-1} h^{p,k} \\
&+ \sum_{p<\frac{m}{2},\mathrm{odd}} \; \sum_{k=\lfloor r\rfloor+1}^{\lfloor r-\varepsilon(p-\frac{m}{2})\rfloor} h^{p,k}.
\end{aligned}
$$

Explicit computation of the eta invariant

$$\int_Z \hat{A}(R^{TZ}) = (2\pi)\int_0^{\varepsilon} d\delta \int_X \Omega_2 \exp\left\{\Omega_0\right\}, \quad \text{where}$$

$$\Omega_0 = 2\mathrm{tr}\left[p\left(R^{TX^{1,0}} + 2i\delta\omega\right)\right] + 2p\left(2i\delta\omega\right),$$

$$\Omega_2 = 2\mathrm{tr}\left[ip'\left(R^{TX^{1,0}} + i2\delta\omega\right)\right] + i2p'\left(2i\delta\omega\right)$$

and $p(z) = \frac{1}{2}\log\left(\frac{z/2}{\sinh(z/2)}\right)$.

Thank you.