HYPERBOLICITY, IRRATIONALITY EXPONENTS AND THE ETA INVARIANT

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Abstract. We consider the remainder term in the semiclassical limit formula of [25] for the eta invariant on a metric contact manifold, proving in general that it is controlled by volumes of recurrence sets of the Reeb flow. This particularly gives a logarithmic improvement of the remainder for Anosov Reeb flows, while for certain elliptic flows the improvement is in terms of irrationality measures of corresponding Floquet exponents.

1. Introduction

The eta invariant of Atiyah-Patodi-Singer [1] is formally the signature of a Dirac operator and appears as a correction term in the index theorem for manifolds with boundary. Much like the signature of a matrix, it is not in general continuous in the operator making it difficult to understand its behavior/asymptotics in parameters/limits. In previous work [22, 23, 24, 25] the author considered the asymptotics of the eta invariant of a coupled Dirac operator in the semiclassical limit. In particular [25, Thm 1.2] proved a semiclassical limit formula for the eta invariant on metric-contact manifold whose Reeb flow satisfies a non-resonance condition, i.e. rational independence of relevant Floquet exponents. In this article we consider the question of the remainder term in this formula, relating it to further dynamical properties such as volumes of recurrence sets. This particularly gives a logarithmic improvement of the remainder when the flow is Anosov. While for certain elliptic flows the remainder is related to finer properties such as irrationality measures and Diophantine approximation of the Floquet exponents.

Let us state our results precisely. Let $\left(X, g^{TX}\right)$ be a compact, spin Riemannian manifold of odd dimension $n = 2m+1$. Let $a \in \Omega^1(X)$ be a contact one form satisfying $a \wedge (da)^m \neq 0$. This gives rise to the contact hyperplane $H = \ker(a) \subset TX$ as well as the Reeb vector field R defined by $i_R da = 0$, $i_R a = 1$. The metric g^{TX} is assumed to be *strongly suitable* to the contact form a in the following sense: the contracted endomorphism $\mathfrak{J} : T_xX \to T_xX$, defined via $da(.,.) = g^{TX}(., \mathfrak{J}.)$, has spectrum $\mathrm{Spec}(\mathfrak{J}_x) = \{0\} \cup \{\pm i\mu_j\}_{j=1}^m$ independent of the point $x \in X$. This hypothesis includes all metric contact manifolds, in which case $\mathfrak{J}$ is a compatible almost complex structure on H and satisfies $\mathfrak{J}^2 = -1$.

Next let $\left(L, h^L\right) \to X$ be another complex Hermitian line bundle on X and A_0 a unitary connection on L. This gives rise to the family of unitary connections $A_h := A_0 + \frac{i}{h}a$, $h \in (0,1]$, and corresponding coupled Dirac operators

$$D_h := hD_{A_h} = hD_{A_0} + ic(a) : \; C^\infty(X; S \otimes L) \to C^\infty(X; S \otimes L) \tag{1.1}$$

with S denoting the corresponding spin bundle. The above being an elliptic operator has a discrete spectrum whose behavior as $h \to 0$ is of our interest. In particular, we shall be interested in the asymptotics of the eta invariant $\eta_h = \eta(D_h)$ of the Dirac operator (see 2.1 below). To state our result define the endomorphism define the endomorphisms $\left(\nabla^{TX}\mathfrak{J}\right)^0$:

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$R^{\perp} \to R^{\perp}$, $|\mathfrak{J}| : R^{\perp} \to R^{\perp}$, via

$$\begin{aligned}\left(\nabla^{TX}\mathfrak{J}\right)^{0} v &:= \left(\nabla_{v}^{TX}\mathfrak{J}\right) R, \quad \forall v \in R^{\perp},\\ |\mathfrak{J}| &:= \sqrt{-\mathfrak{J}^{2}}.\end{aligned} \tag{1.2}$$

Let T_0 be the shortest period of the Reeb vector field. For each $\varepsilon, T > 0$ we define the recurrence set

$$S_{T,\varepsilon} := \left\{ x \in X | \exists t \in \left[\frac{1}{2}T_0, T\right] \text{ s.t. } d^{g^{TX}}\left(e^{tR}x, x\right) \leq \varepsilon \right\} \tag{1.3}$$

as well as its extended neighborhood $S_{T,\varepsilon}^{e} := \left\{ x \in X | d^{g^{TX}}\left(x, S_{T,\varepsilon}\right) \leq \varepsilon \right\}$. Here $d^{g^{TX}}$ denotes the Riemannian distance on X and we also denote by $\mu_{g^{TX}}$ be the Riemannian volume below. Our first result is now the following.

Theorem 1. *Let a be a contact form and g^{TX} a strongly suitable metric. For each $\varepsilon = h^{\delta}$, $\delta \in \left[0, \frac{1}{2}\right)$, and possibly h-dependent $T = T(h)$ the rescaled eta invariant of the Dirac operator satisfies*

$$h^{m}\eta\left(D_h\right) = -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\frac{1}{m!}\int_X \left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)^{0}\right] a \wedge (da)^{m} + R(h) \tag{1.4}$$

$$\text{where} \qquad |R(h)| \leq \frac{\det|\mathfrak{J}|}{(4\pi)^{n/2}}T^{-1} + O\left(T^{-2} + \mu_{g^{TX}}\left(S_{T,\varepsilon}^{e}\right)\right). \tag{1.5}$$

As noted in [25], the leading term in (1.4) equals a multiple of the volume $-\frac{m}{2}\frac{1}{(2\pi)^{m+1}}\mathrm{vol}(X)$ in the case of a metric-contact manifold.

The formula (1.5) above can be made more explicit in specializations based on the estimates for the recurrence set (1.3). The first such corollary of the above is obtained by letting ε, T be $h-$independent. As $\varepsilon \to 0$ the volume of the recurrence set approaches the measure of the set of closed trajectories having period at most T. Letting $T \to \infty$ then gives the following.

Corollary 2. *Let a be a contact form and g^{TX} a strongly suitable metric. Assuming the set of closed Reeb trajectories is of measure zero, the rescaled eta invariant of the Dirac operator satisfies*

$$h^{m}\eta\left(D_h\right) = -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\frac{1}{m!}\int_X \left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)^{0}\right] a \wedge (da)^{m} + o(1). \tag{1.6}$$

The next specialization is when the Reeb flow is Anosov, whereby the union of closed Reeb trajectories automatically has null measure. The recurrence set in this case can be shown to satisfy an exponential estimate in time in terms of the topological entropy $\mathtt{h}_{\mathrm{top}}$ of the flow (see 5.1 below). Our main Theorem 1 then has the following corollary.

Corollary 3. *Let a be a contact form and g^{TX} a strongly suitable metric. Assuming the Reeb flow of a to be Anosov, the rescaled eta invariant of the Dirac operator satisfies*

$$h^{m}\eta\left(D_h\right) = -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\frac{1}{m!}\int_X \left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)^{0}\right] a \wedge (da)^{m} + O\left(|\ln h|^{-1}\right). \tag{1.7}$$

More precisely, the remainder above satisfies the estimate

$$|R(h)| \leq \mathtt{h}_{top}\left(\frac{8}{n^{2}}\frac{\det|\mathfrak{J}|}{(4\pi)^{n/2}}\right)|\ln h|^{-1} + o\left(|\ln h|^{-1}\right) \tag{1.8}$$

in terms of the topological entropy of the flow.

In cases where the flow is elliptic, the recurrence set can be harder to control. As a particular case we consider certain elliptic flows on Lens spaces. In dimension three it is known [17, Thm 1.2] that any contact manifold, all of whose Reeb orbits are non-degenerate and elliptic, is necessarily a Lens space. To define these, fix non-negative integers $q_0, q_1, \ldots, q_m$, $q_0 > 1$, as well as positive reals $a_0, \ldots, a_m$ such that $\left(a_0^{-1}a_1, \ldots, a_0^{-1}a_m\right) \notin \mathbb{Q}^m$. To such a tuple of reals is associated an irrationality exponent/measure $\nu\left(a_1, \ldots, a_{m+1}\right) > 1$ of Diophantine approximation (see 2.3). The Lens space is now defined as the quotient

$$X = L\left(q_0, q_1, \ldots, q_m; a_0, \ldots a_m\right) := E\left(a_0, \ldots, a_m\right)/\mathbb{Z}_{q_0} \quad \text{where} \tag{1.9}$$

$$E\left(a_0, \ldots, a_m\right) := \left\{\left(z_0, \ldots, z_m\right) \in \mathbb{C}^{m+1} | \sum_{j=0}^{m} a_j \left|z_j\right|^2 = 1\right\} \tag{1.10}$$

is the irrational ellipsoid. Above the $\mathbb{Z}_{q_0}$ action on the ellipsoid is given by $e^{\frac{2\pi i}{q_0}}\left(z_0, \ldots, z_m\right) = \left(e^{\frac{2\pi i}{q_0}}z_0, e^{\frac{2\pi i q_1}{q_0}}z_1, \ldots, e^{\frac{2\pi i q_m}{q_0}}z_m\right)$. The contact form is chosen to be

$$a = \sum_{j=0}^{m}\left(x_j dy_j - y_j dx_j\right)\bigg|_{E(a_0,\ldots,a_m)}, \quad z_j = x_j + iy_j, \tag{1.11}$$

the restriction of the tautological form on $\mathbb{C}^{m+1}$, which maybe seen to be $\mathbb{Z}_{q_0}$-invariant and hence descending to the Lens space quotient. We now have the following corollary of Theorem 1, below we denote by $O\left(h^{\alpha\pm}\right)$ a term which is $O\left(h^{\alpha\pm\varepsilon}\right)$, $\forall\varepsilon > 0$.

Corollary 4. *Let a be the tautological contact form and g^{TX} a strongly suitable metric on the Lens space $X = L\left(q_0, q_1, \ldots, q_m; a_0, \ldots, a_m\right)$ (1.9). The rescaled eta invariant of the Dirac operator satisfies*

$$h^m \eta\left(D_h\right) = -\frac{1}{2}\frac{1}{(2\pi)^{m+1}}\frac{1}{m!}\int_X \left[tr\ |\mathfrak{J}|^{-1}\left(\nabla^{TX}\mathfrak{J}\right)^0\right] a \wedge (da)^m + O\left(h^{\frac{1}{2\nu-1}-}\right) \tag{1.12}$$

where $\nu = \nu\left(a_0, a_1 - \frac{q_1}{q_0}a_0, \ldots, a_m - \frac{q_m}{q_0}a_0\right)$ denotes the irrationality exponent of the given tuple.

Our final result is quantum ergodicity for the Dirac operator 1.1, conditional on ergodicity of the Reeb flow. We refer to Theorem 11 in Section 6 for the precise statement. Although it is not the main interest here, it is included to show the direct connection of the authors work with the results of [7]. The final Theorem 11 is the semiclassical Dirac operator analogue of the main result therein.

The Corollary 2 already improves [25, Thm 1.2] proving the same limit formula (1.6) therein under the weaker assumption of null measure of the closed trajectories, thus akin to the remainder improvement in the Weyl law of Duistermaat-Guillemin [11]. However it should be noted that this weaker dynamical assumption in 2 is insufficient to obtain the Gutzwiller trace formula of [25, Thm 1.1]. The next corollary 3 is analogous to the remainder improvement in the Weyl law for the Laplacian by Berard [3]. For the Weyl law analogues of the remainder improvements Theorem 1 and 4 we refer to [18, Sec. 4.5] and [10, Ch. 11], although not explicitly written it is inherent in the methods therein.

However, we importantly note that the Fourier integral calculus that is used in deriving sharp Weyl laws and remainder estimates is largely unavailable in our context, the basic reason being that the Dirac operator (1.1) is non-scalar with its principal symbol being non-diagonalizable along its characteristic variety. The difficulties are in spirit close to those in the analysis of the wave equation of a hypoelliptic operator with double characteristics [19, 26]. The method here is based on the authors earlier work [22, 23, 24, 25], which in turn uses a combination of Birkhoff normal forms, almost analytic continuations, microhyperbolicity and local index theory arguments. New components here include a microlocal trace expansion, analysis of recurrence sets and a refined Tauberian argument. In 5.1 an exponential estimate for the recurrence set of an Anosov flow is proved based on a seemingly new bound on its topological entropy. A conjectured characterization of topological entropy is stated, based on which the estimate (1.8) in Corollary 3 can be improved by a factor (see remark 10 below). An analogous microlocal and second-microlocal trace expansion to the one here was also derived by the author recently [26] in the context of the hypoelliptic Laplacian of sub-Riemannian geometry. A related hypoelliptic, second-microlocal Weyl law has also recently appeared in [29] after [7].

The eta invariant asymptotics considered here further has applications to contact geometry. It was originally motivated by the proof of the three dimensional Weinstein conjecture using Seiberg-Witten theory by Taubes [28]. The semiclassical limit formula for the eta invariant of [25] was recently used by the author in [9] to improve the remainder term in the asymptotics of embedded contact homology (ECH) capacities [8]. Our main theorem here as well as its corollaries could have further applications in this direction.

The paper is organized as follows. In first section Section 2 we begin with background notions used in the paper including the requisites on Dirac operators 2.1, semiclassical analysis 2.2 and Diophantine approximation 2.3. In Section 3 we prove a microlocal trace expansion for the Dirac operator. This is later used in Section 4 to prove our main Theorem 1. In Section 5 we consider recurrence sets in particular Anosov 5.1 and elliptic 5.2 cases cases and prove the corollaries 2, 3 and 4. In the final Section 6 we prove quantum ergodicity for the Dirac operator.

2. Preliminaries

2.1. **Spectral invariants of the Dirac operator.** We begin by stating the requisites about Dirac operators used in the paper, [4] provides a standard reference. Let $\left(X, g^{TX}\right)$ be a compact, oriented, Riemannian manifold of odd dimension $n = 2m+1$. It shall be further equipped with a spin structure, i.e. a $\mathrm{Spin}\,(n)$ principal bundle $\mathrm{Spin}\,(TX) \to SO\,(TX)$ that is an equivariant double covering of the $SO\,(n)$ principal bundle $SO\,(TX)$ of orthonormal frames in TX. The unique irreducible representation of $\mathrm{Spin}\,(n)$ gives rise to the associated spin bundle $S = \mathrm{Spin}\,(TX) \times_{\mathrm{Spin}(n)} S_{2m}$. The Levi-Civita connection ∇^{TX} on the tangent bundle TX lifts to a connection on $\mathrm{Spin}\,(TX)$ thus in turn giving rise to the spin connection ∇^{S} on the spin bundle S. The Clifford multiplication endomorphism $c : T^*X \to S \otimes S^*$ arises from the standard representation of the Clifford algebra of T^*X and satisfies

$$c(a)^2 = -|a|^2, \quad \forall a \in T^*X.$$

Next choose $\left(L, h^L\right)$ a Hermitian line bundle on X along with A_0 be a unitary connection on it. Given any one-form $a \in \Omega^1(X;\mathbb{R})$ on X we may form the family $\nabla^h = A_0 + \frac{i}{h}a$, $h \in (0,1]$ of unitary connections on L. Denote the corresponding tensor product connection on $S \otimes L$ by $\nabla^{S\otimes L} := \nabla^S \otimes 1 + 1 \otimes \nabla^h$. Each such connection defines a coupled Dirac operator

$$D_h := hD_{A_0} + ic\,(a) = hc \circ \left(\nabla^{S\otimes L}\right) : C^\infty(X; S \otimes L) \to C^\infty(X; S \otimes L)$$

for $h \in (0,1]$. The Dirac operator D_h being elliptic and self-adjoint has a discrete spectrum of eigenvalues.

The eta function of D_h is defined by the formula

$$\eta\left(D_h, s\right) := \sum_{\substack{\lambda \neq 0 \\ \lambda \in \operatorname{Spec}(D_h)}} \operatorname{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \operatorname{tr}\left(D_h e^{-tD_h^2}\right) dt, \tag{2.1}$$

$\forall s \in \mathbb{C}$. We shall use the convention that $\operatorname{Spec}(D_h)$ is a multiset with each eigenvalue of D_h being counted with its multiplicity. From the Weyl law for elliptic operators, the above series is seen to converge for $\operatorname{Re}(s) > n$. Further in [1, 2] it was shown that the eta function above (2.1) has a meromorphic continuation to the entire complex s-plane and further has no pole at zero. The eta invariant of the Dirac operator D_h is then defined to be the value of (2.1) at zero

$$\eta_h := \eta\left(D_h, 0\right). \tag{2.2}$$

From (2.1) the above is seen formally to be the signature of the Dirac operator, i.e. the difference between the number of its positive/negative eigenvalues. A variant of the above, known as the reduced eta invariant, is defined by including the zero eigenvalue with an appropriate convention

$$\begin{aligned} \bar{\eta}_h :=& \frac{1}{2}\left\{k_h + \eta_h\right\} \\ k_h :=& \dim \ker \ \left(D_h\right). \end{aligned}$$

Much like the signature of a matrix, the eta invariant is left unchanged under positive scaling

$$\eta\left(D_h, 0\right) = \eta\left(cD_h, 0\right); \quad \forall c > 0. \tag{2.3}$$

With

$$K_{t,h}\left(x, x'\right) := D_h e^{-tD_h^2}\left(x, x'\right) \in C^{\infty}\left(X \times X; S \otimes S^*\right)$$

being the Schwartz kernel of the given heat operator, defined with respect to the Riemannian volume density, we denote by $\operatorname{tr}\left(K_{t,h}\left(x, x\right)\right)$ its point-wise trace along the diagonal. A generalization of the eta function (2.1) is then given by

$$\eta\left(D_h, s, x\right) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \operatorname{tr}\left(K_{t,h}\left(x, x\right)\right) dt, \tag{2.4}$$

as a function of $s \in \mathbb{C}$, $x \in X$. It was shown in [5, Thm 2.6] that the function $\eta\left(D_h, s, x\right)$ is holomorphic in s and smooth in x for $\operatorname{Re}(s) > -2$. This from the above (2.4) is clearly equivalent to

$$\operatorname{tr}\left(K_{t,h}\right) = O\left(t^{\frac{1}{2}}\right), \quad \text{as } t \to 0. \tag{2.5}$$

The integral

$$\eta_h = \int_0^{\infty} \frac{1}{\sqrt{\pi t}} \operatorname{tr}\left(D_h e^{-tD_h^2}\right) dt. \tag{2.6}$$

is convergent with its value being the eta invariant (2.2).

2.2. **The Semi-classical calculus.** Next we state some requisite facts from semi-classical analysis that shall be used in the paper, [14, 31] provide the standard references. For any $l \times l$ complex matrix $A = (a_{ij}) \in \mathfrak{gl}(l)$, we denote $|A| = \max_{ij}|a_{ij}|$. The symbol space $S^m\left(\mathbb{R}^{2n}; \mathbb{C}^l\right)$ is defined as the space of maps $a : (0, 1]_h \to C^{\infty}\left(\mathbb{R}^{2n}_{x,\xi}; \mathfrak{gl}(l)\right)$ for which each semi-norm

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} \langle\xi\rangle^{-m+|\beta|} \left|\partial_x^{\alpha} \partial_{\xi}^{\beta} a(x, \xi; h)\right|, \quad \alpha, \beta \in \mathbb{N}_0^n,$$

is finite. The more refined class $a \in S^m_{\mathrm{cl}}\left(\mathbb{R}^{2n};\mathbb{C}^l\right)$ of classical symbols consists of those for which there exists an h-independent sequence a_k, $k=0,1,\ldots$ of symbols satisfying

$$a-\left(\sum_{k=0}^{N}h^k a_k\right)\in h^{N+1}S^m\left(\mathbb{R}^{2n};\mathbb{C}^l\right),\ \forall N. \tag{2.7}$$

Any given $a \in S^m\left(\mathbb{R}^{2n};\mathbb{C}^l\right), S^m_{\mathrm{cl}}\left(\mathbb{R}^{2n};\mathbb{C}^l\right)$ in one of the symbol classes above defines a one-parameter family of operators $a^W \in \Psi^m\left(\mathbb{R}^{2n};\mathbb{C}^l\right), \Psi^m_{\mathrm{cl}}\left(\mathbb{R}^{2n};\mathbb{C}^l\right)$ via Weyl quantization whose Schwartz kernel is given by

$$a^W := \frac{1}{(2\pi h)^n}\int e^{i(x-y).\xi/h}a\left(\frac{x+y}{2},\xi;h\right)d\xi.$$

The above pseudodifferential classes of operators are closed under the usual operations of composition and formal-adjoint. Furthermore the classes are invariant under changes of coordinates and basis for $\mathbb{C}^l$. Thus one may invariantly the classes of operators $\Psi^m(X;E), \Psi^m_{\mathrm{cl}}(X;E)$ acting on $C^\infty(X;E)$ associated to any complex, Hermitian vector bundle $\left(E,h^E\right)$ on a smooth compact manifold X.

The principal symbol of a classical pseudodifferential operator $A \in \Psi^m_{\mathrm{cl}}(X;E)$ is defined as an element in $\sigma(A) \in S^m(X;\mathrm{End}(E)) \subset C^\infty(X;\mathrm{End}(E))$. It is given by $\sigma(A)=a_0$ the leading term in the symbolic expansion (2.7) of its full Weyl symbol. The principal symbol is multiplicative, commutes with adjoints and fits into a symbol exact sequence

$$\begin{aligned}
\sigma(AB) &= \sigma(A)\,\sigma(B)\\
\sigma(A^*) &= \sigma(A)^*
\end{aligned}$$

$$0\to h\Psi^m_{\mathrm{cl}}(X;E)\to\Psi^m_{\mathrm{cl}}(X;E)\xrightarrow{\sigma}S^m(X;\mathrm{End}(E)), \tag{2.8}$$

where the formal adjoints above are defined with respect to the same Hermitian metric h^E. The quantization map

$$\begin{aligned}
&\mathrm{Op}: S^m(X;\mathrm{End}(E))\to\Psi^m_{\mathrm{cl}}(X;E)\quad\text{satisfying}\\
&\sigma(\mathrm{Op}(a)) = a\in S^m(X;\mathrm{End}(E))
\end{aligned} \tag{2.9}$$

gives an inverse to the principal symbol map and we sometimes use the alternate notation $\mathrm{Op}(a)=a^W$. The quantization map above is however non-canonical and depends on the choice of a coordinate atlas, with local trivializations for E, as well as a subordinate partition of unity. From the multiplicative property of the symbol (2.8), it then follows that $\left[a^W,b^W\right]\in h\Psi^{m-1}_{\mathrm{cl}}(X;E)$ when $b\in S^0(X)$ is a scalar function. We shall then define $H_b(a):=\frac{i}{h}\sigma\left(\left[a^W,b^W\right]\right)\in S^{m-1}(X;\mathrm{End}(E))$ however noting again that its definition depends on the quantization scheme, and in particular the local trivializations used in defining Op. It is given however by the Poisson bracket $H_b(a)=\{a,b\}$ assuming that both sides are computed in the same defining trivialization.

Each $A\in\Psi^m_{\mathrm{cl}}(X;E)$ has a wavefront set defined invariantly as a subset $WF(A)\subset\overline{T^*X}$ of the fibrewise radial compactification of the cotangent bundle T^*X. It is locally defined as follows, $(x_0,\xi_0)\notin WF(A)$, $A=a^W$, if and only if there exists an open neighborhood $(x_0,\xi_0;0)\in U\subset\overline{T^*X}\times(0,1]_h$ such that $a\in h^\infty\langle\xi\rangle^{-\infty}C^k\left(U;\mathbb{C}^l\right)$ for all k. The wavefront set satisfies the basic properties under addition, multiplication and adjoints $WF(A+B)\subset WF(A)\cap WF(B)$, $WF(AB)\subset WF(A)\cap WF(B)$ and $WF(A^*)=WF(A)$. The wavefront set $WF(A)=\emptyset$ is empty if and only if $A\in h^\infty\Psi^{-\infty}(X;E)$ while we say that two operators $A=B$ microlocally on $U\subset\overline{T^*X}$ if $WF(A-B)\cap U=\emptyset$.

An operator $A \in \Psi_{\mathrm{cl}}^{m}(X; E)$ is said to be elliptic if $\langle\xi\rangle^{m} \sigma(A)^{-1}$ exists and is uniformly bounded on $T^{*}X$. If $A \in \Psi_{\mathrm{cl}}^{m}(X; E)$, $m > 0$, is formally self-adjoint such that $A + i$ is elliptic then it is essentially self-adjoint (with domain $C_{c}^{\infty}(X; E)$) as an unbounded operator on $L^{2}(X; E)$. Beals's lemma further implies that its resolvent $(A - z)^{-1} \in \Psi_{\mathrm{cl}}^{-m}(X; E)$, $z \in \mathbb{C}$, $\mathrm{Im} z \neq 0$, exists and is pseudo-differential. The Helffer-Sjöstrand formula now expresses the function $f(A)$, $f \in \mathcal{S}(\mathbb{R})$, of such an operator in terms of its resolvent

$$f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-1} dz d\bar{z},$$

with $\tilde{f}$ denoting an almost analytic continuation of f. One further has $WF(f(A)) \subset \Sigma_{\mathrm{spt}(f)}^{A} := \bigcup_{\lambda \in \mathrm{spt}(f)} \Sigma_{\lambda}^{A}$ where

$$\Sigma_{\lambda}^{A} = \left\{ (x, \xi) \in T^{*}X \middle| \det(\sigma(A)(x, \xi) - \lambda I) = 0 \right\}. \tag{2.10}$$

is classical λ-energy level of A.

2.2.1. *The class* $\Psi_{\delta}^{m}(X)$. We shall also need a more exotic class of scalar symbols $S_{\delta}^{m}(\mathbb{R}^{2n}; \mathbb{C})$ defined for each $0 \leq \delta < \frac{1}{2}$. A function $a : (0, 1]_{h} \rightarrow C^{\infty}\left(\mathbb{R}_{x,\xi}^{2n}; \mathbb{C}\right)$ is said to be in this class if and only if

$$\|a\|_{\alpha,\beta} := \sup_{x,\xi,h} h^{(|\alpha|+|\beta|)\delta} \left| \partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi; h) \right| \tag{2.11}$$

is finite $\forall \alpha, \beta \in \mathbb{N}_{0}^{n}$. This class of operators is also closed under the standard operations of composition, adjoint and changes of coordinates allowing for the definition of the same exotic pseudo-differential algebra $\Psi_{\delta}^{m}(X)$ on a compact manifold. The class $S_{\delta}^{m}(X)$ is a family of functions $a : (0, 1]_{h} \rightarrow C^{\infty}(T^{*}X; \mathbb{C})$ satisfying the estimates (2.11) in every coordinate chart and induced trivialization. Such a family can be quantized to $a^{W} \in \Psi_{\delta}^{m}(X)$ satisfying $a^{W} b^{W} = (ab)^{W} + h^{1-2\delta} \Psi_{\delta}^{m+m'-1}(X)$, $\frac{i}{h^{1-2\delta}} \sigma\left(\left[a^{W}, b^{W}\right]\right) = [\{a, b\}]$ for another $b \in S_{\delta}^{m'}(X)$. The operators in $\Psi_{\delta}^{0}(X)$ are uniformly bounded on $L^{2}(X)$. Finally, the wavefront an operator $A \in \Psi_{\delta}^{m}(X; E)$ is similarly defined and satisfies the same basic properties as before.

2.3. **Diophantine approximation.** We finally collect some requisite notions from Diophantine approximation. These shall be useful later in 5.2. We refer to the texts [6, 27] for the background and proofs of the statements below.

Let $a \in \mathbb{R}$ be a real number. Its irrationality exponent/measure is defined by

$$\mu(a) := \inf \left\{ \mu \middle| \left| a - \frac{p}{q} \right| < \frac{1}{q^{\mu}}, \text{ has finitely many rational solutions } \frac{p}{q} \in \mathbb{Q} \right\} \tag{2.12}$$

$$= \inf \left\{ \mu \middle| \exists C > 0 \text{ s.t. } \left| a - \frac{p}{q} \right| > \frac{C}{q^{\mu}}, \forall \frac{p}{q} \in \mathbb{Q} \setminus \{a\} \right\} \tag{2.13}$$

where we set $\mu(a) = \infty$ when the sets above are empty.

It is easy to check that $\mu(a) = 1$ for $a \in \mathbb{Q}$ rational. A theorem of Dirichlet shows that $\mu(a) \geq 2$ for a irrational. The map $\mu : \mathbb{R} \setminus \mathbb{Q} \rightarrow [2, \infty)$ is known to be surjective while $\mu(a) = 2$ for almost all reals with respect to the Lebesgue measure. A number a with $\mu(a) = 2$ and for which the infimum in (2.13) is attained is called badly approximable. Roth's theorem shows that $\mu(a) = 2$ for irrational algebraic integers, it is conjectured however that no such (of degree at least 3) is badly approximable. Furthermore conversely there are transcendental a with $\mu(a) = 2$, Euler's number e being such an example. The reals a for which $\mu(a) = \infty$ are called a Liouville numbers, these form a dense albeit Lebesgue measure zero subset of the reals.

A generalization of the above, the irrationality exponent of simultaneous Diophantine approximation, can be defined for a tuple of real numbers $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$, $n \geq 2$, via

$$\nu(a_1, \ldots, a_n) \coloneqq \inf\left\{\nu | \exists C > 0 \text{ s.t. } d\left((ta_1, \ldots, ta_n); \mathbb{Z}^n\right) > Ct^{1-\nu}, \forall (ta_1, \ldots, ta_n) \notin \mathbb{Z}^n\right\}, \tag{2.14}$$

where d above denotes the distance from the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. The above (2.14) is seen to be related to the exponent (2.13) via $\mu(a) = \nu(1, a)$ and is scale invariant $\nu(ca_1, ca_2) = \nu(a_1, a_2)$, $c \neq 0$. Further it is easy to check that $\nu(a_1, \ldots, a_n) \leq \min_{i \neq j} \nu(a_i, a_j)$ and that $\nu(1, a_2, \ldots, a_n) = 1$ for $(a_2, \ldots, a_n) \in \mathbb{Q}^{n-1}$. The higher dimensional analogue of Dirichlet's theorem says $1 + \frac{1}{n-1} \leq \nu(1, a_2, \ldots, a_n)$ for $(a_2, \ldots, a_n) \notin \mathbb{Q}^{n-1}$ with again the equality holding for almost all tuples. The higher dimensional analogue of Roth's theorem is due to Schmidt: if $a_2, \ldots, a_n$ are algebraic integers such that $\{1, a_2, \ldots, a_n\}$ are rationally independent then $\nu(1, a_2, \ldots, a_n) = 1 + \frac{1}{n-1}$.

3. Microlocal trace expansion

The Dirac operator D_h (1.1) has principal symbol and characteristic variety

$$\sigma(D)(x, \xi) = c(\xi + a) \in C^\infty(T^*X; \mathrm{End}(S)) \tag{3.1}$$

$$\begin{aligned} \Sigma &\coloneqq \{(x, \xi) | \sigma(D)(x, \xi) = 0\} \\ &= \{(x, \xi) | \xi = -a(x)\} \end{aligned} \tag{3.2}$$

given by Clifford multiplication and the graph of the one form a respectively.

In [24, Sec. 7] an on diagonal expansion for functions $\phi\left(\frac{D}{\sqrt{h}}\right)$, $\phi \in \mathcal{S}(\mathbb{R})$, of the Dirac operator was proved. Namely we showed the existence of tempered distributions

$$U_{j,x}(s) \in C^\infty(X; S \otimes L \otimes \mathcal{S}'(\mathbb{R}_s)),$$

$j \in \mathbb{N}_0$, $x \in X$, such that

$$\phi\left(\frac{D}{\sqrt{h}}\right)(x, x) = h^{-n/2}\left(\sum_{j=0}^{N} U_{j,x}(\phi) h^{j/2}\right) + h^{(N+1-n)/2} O\left(\sum_{k=0}^{n+1} \left\| \langle\xi\rangle^N \hat{\phi}^{(k)} \right\|_{L^1}\right) \tag{3.3}$$

$\forall \phi \in \mathcal{S}(\mathbb{R}_s), x \in X$, $N \in \mathbb{N}$. Here we show a further microlocal version of this result, considering functions of $B\phi\left(\frac{D}{\sqrt{h}}\right)$, $\phi \in \mathcal{S}(\mathbb{R})$, $B \in \Psi^0_{\mathrm{cl}}(X; S \otimes L)$. The leading part of this expansion shall be shown to concentrate on the characteristic variety Σ (3.2).

We first fix some terminology. Fixing a point $p \in X$ there is an orthonormal basis $e_{0,p} = \frac{R}{|R|}, \{e_{j,p}, e_{j+m,p}\}_{j=1}^m \in R^\perp$, of the tangent space at p consisting of eigenvectors of $\mathfrak{J}_p$ with eigenvalues $0, \pm i\mu_j$, $j = 1, \ldots, m$, such that

$$da(p) = \sum_{j=1}^{m} \mu_j e^*_{j,p} \wedge e^*_{j+m,p}. \tag{3.4}$$

Using the parallel transport from this basis, fix a geodesic coordinate system $(x_0, \ldots, x_{2m})$ on an open neighborhood of $p \in \Omega$. Let $e_j = w^k_j \partial_{x_k}$, $0 \leq j \leq 2m$, be the local orthonormal frame of TX obtained by parallel transport of $e_{j,p} = \partial_{x_j}\big|_p$, $0 \leq j \leq 2m$, along geodesics. We then have

$$w_j^k g_{kl} w_r^l = \delta_{jr},$$
$$w_j^k\big|_p = \delta_j^k,$$

with g_{kl} being the components of the metric in these coordinates. Choose an orthonormal basis $\{s_{j,p}\}_{j=1}^{2m}$for S_p in which Clifford multiplication

$$c\left(e_j\right)\big|_p = \gamma_j \tag{3.5}$$

is standard. Choose an orthonormal basis $\mathtt{1}_p$ for L_p. Parallel transport the bases $\{s_{j,p}\}_{j=1}^{2m}$, $\mathtt{1}_p$ along geodesics using the spin connection ∇^S and unitary family of connections $\nabla^h = A_0 + \frac{i}{h}a$ to obtain trivializations $\{s_j\}_{j=1}^{2m}$, $\mathtt{1}$ of S, L on Ω. Since Clifford multiplication is parallel, the relation (3.5) now holds on Ω. The connection $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ can be expressed in this frame and these coordinates as

$$\nabla^{S\otimes L} = d + A_j^h dx^j + \Gamma_j dx^j, \tag{3.6}$$

where each A_j^h is a Christoffel symbol of ∇^h and each Γ_j is a Christoffel symbol of the spin connection ∇^S. Since the section $\mathtt{1}$ is obtained via parallel transport along geodesics, the connection coefficient A_j^h maybe written in terms of the curvature $F_{jk}^h dx^j \wedge dx^k$ of ∇^h

$$A_j^h(x) = \int_0^1 d\rho \left(\rho x^k F_{jk}^h\left(\rho x\right)\right). \tag{3.7}$$

The dependence of the curvature coefficients F_{jk}^h on the parameter h is seen to be linear in $\frac{1}{h}$ via

$$F_{jk}^h = F_{jk}^0 + \frac{i}{h}\left(da\right)_{jk} \tag{3.8}$$

despite the fact that they are expressed in the h dependent frame $\mathtt{1}$. This is because a gauge transformation from an h independent frame $\mathtt{1}_0$ into $\mathtt{1}$ changes the curvature coefficient by conjugation. Since L is a line bundle this is conjugation by a function and hence does not change the coefficient. Furthermore, the coefficients in the Taylor expansion of (3.8) at 0 maybe expressed in terms of the covariant derivatives $\left(\nabla^{A_0}\right)^l F_{jk}^0$, $\left(\nabla^{A_0}\right)^l (da)_{jk}$ evaluated at p.

The gauge transformation relating the h-dependent frame $\mathtt{1}$ with the h-independent frame $\mathtt{1}_0$ is given by

$$\mathtt{1} = c\exp\left\{\underbrace{-\int_0^1 \rho\, d\rho x^j A_{0,j}\left(\rho x\right) - \frac{i}{h}\underbrace{\int_0^1 \rho\, d\rho x^j a_j\left(\rho x\right)}_{:=\varphi}}_{\varphi_h}\right\}\mathtt{1}_0$$

$$\varphi = x^j a_j\left(0\right) + O\left(x^2\right) \tag{3.9}$$

where $A_0 + \frac{i}{h}a$ denotes the connection form for ∇^h in the $\mathtt{1}_0$ trivialization while $c > 0$ is a constant which may be taken to be 1 by an appropriate choice of $\mathtt{1}_0$.

Next, using the Taylor expansion

$$\left(da\right)_{jk} = \left(da\right)_{jk}\left(0\right) + x^l a_{jkl}, \tag{3.10}$$

we see that the connection $\nabla^{S\otimes L}$ has the form

$$\nabla^{S\otimes L}=d+\left[\frac{i}{h}\left(\frac{x^k}{2}\left(da\right)_{jk}(0)+x^kx^lA_{jkl}\right)+x^kA^0_{jk}+\Gamma_j\right]dx^j \tag{3.11}$$

where

$$\begin{aligned}A^0_{jk} &= \int_0^1 d\rho\left(\rho F^0_{jk}\left(\rho x\right)\right)\\ A_{jkl} &= \int_0^1 d\rho\left(\rho a_{jkl}\left(\rho x\right)\right)\end{aligned}$$

and Γ_j are all independent of h. Finally from (3.5) and (3.11) we may write down the expression for the Dirac operator (1.1), given as $D=hc\circ\left(\nabla^{S\otimes L}\right)$, in terms of the chosen frame and coordinates to be

$$D=\gamma^r w^j_r\left[h\partial_{x_j}+i\frac{x^k}{2}\left(da\right)_{jk}(0)+ix^kx^lA_{jkl}+h\left(x^kA^0_{jk}+\Gamma_j\right)\right] \tag{3.12}$$

$$\begin{aligned}&=\gamma^r\left[w^j_rh\partial_{x_j}+iw^j_r\frac{x^k}{2}\left(da\right)_{jk}(0)+\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]+ \\ &\quad\gamma^r\left[iw^j_rx^kx^lA_{jkl}+hw^j_r\left(x^kA^0_{jk}+\Gamma_j\right)-\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]\in\Psi^1_{\mathrm{cl}}\left(\Omega^0_s;\mathbb{C}^{2^m}\right)\end{aligned} \tag{3.13}$$

In the second expression above both square brackets are self-adjoint with respect to the Riemannian density $e^1\wedge\ldots\wedge e^n=\sqrt{g}dx\coloneqq\sqrt{g}dx^1\wedge\ldots\wedge dx^n$ with $g=\det\left(g_{ij}\right)$. Again one may obtain an expression self-adjoint with respect to the Euclidean density dx in the framing $g^{\frac{1}{4}}u_j\otimes 1, 1\leq j\leq 2^m$, with the result being an addition of the term $h\gamma^jw^k_jg^{-\frac{1}{4}}\left(\partial_{x_k}g^{\frac{1}{4}}\right)$.

Let i_g be the injectivity radius of g^{TX} . Define the cutoff $\chi\in C^\infty_c\left(-1,1\right)$ such that $\chi=1$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$. We now modify the functions w^k_j, outside the ball $B_{i_g/2}\left(p\right)$, such that $w^k_j=\delta^k_j$ (and hence $g_{jk}=\delta_{jk1_0}$) are standard outside the ball $B_{i_g}\left(p\right)$ of radius i_g centered at p. This again gives

$$\begin{aligned}\mathbb{D}&=\gamma^r\left[w^j_rh\partial_{x_j}+iw^j_r\frac{x^k}{2}\left(da\right)_{jk}(0)+\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right]+ \\ &\quad\chi\left(\left|x\right|/i_g\right)\gamma^r\left[iw^j_rx^kx^lA_{jkl}+hw^j_r\left(x^kA^0_{jk}+\Gamma_j\right)-\frac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}\left(g^{\frac{1}{2}}w^j_r\right)\right] \\ &\quad\in\Psi^1_{\mathrm{cl}}\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)\end{aligned} \tag{3.14}$$

as a well defined operator on $\mathbb{R}^n$ formally self adjoint with respect to $\sqrt{g}dx$. Again $\mathbb{D}+i$ being elliptic in the class $S^0\left(m\right)$ for the order function

$$m=\sqrt{1+g^{jl}\left(\xi_j+\frac{x^k}{2}\left(da\right)_{jk}(0)\right)\left(\xi_l+\frac{x^r}{2}\left(da\right)_{lr}(0)\right)},$$

the operator $\mathbb{D}$ is essentially self adjoint.

Letting $H\left(s\right)\in\mathcal{S}'\left(\mathbb{R}_s\right)$ denote the Heaviside distribution, below we define the following elementary tempered distributions

$$v_{a;p}\left(s\right)\coloneqq s^a,\ a\in\mathbb{N}_0 \tag{3.15}$$

$$\begin{aligned}v_{a,b,c,\varLambda;p}\left(s\right)&\coloneqq\partial^a_s\left[\left|s\right|s^b\left(s^2-2\varLambda\right)^{c-\frac{1}{2}}H\left(s^2-2\varLambda\right)\right], \\ &\qquad\left(a,b,c;\varLambda\right)\in\mathbb{N}_0\times\mathbb{Z}\times\mathbb{N}_0\times\mu.\left(\mathbb{N}^m_0\setminus 0\right)\end{aligned} \tag{3.16}$$

as in [24, Sec. 7].

We now have the following.

Theorem 5. *Let $B \in \Psi^0_{\mathrm{cl}}(X; S \otimes L)$ be a classical pseudodifferential operator. There exist tempered distributions $U_{B,j,x} \in C^\infty(X; End(S \otimes L) \otimes \mathcal{S}'(\mathbb{R}_s))$, $j = 0, 1, 2, \ldots$, such that one has an expansion*

$$B\phi\left(\frac{D}{\sqrt{h}}\right)(x,x) = h^{-n/2}\left(\sum_{j=0}^{N} U_{B,j,x}(\phi)\, h^{j/2}\right) + h^{(N+1-n)/2} O\left(\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^{2N}\hat{\phi}^{(k)}\right\|_{L^1}\right) \tag{3.17}$$

for each $N \in \mathbb{N}$, $x \in X$ and $\phi \in \mathcal{S}(\mathbb{R}_s)$.

Each coefficient of the expansion above can be written in terms of (3.15), (3.16)

$$U_{B,j,x}(s) = \sum_{a \le 2j+2} c_{B,j;a}(x)\, s^a + \sum_{\substack{\Lambda \in \mu.\left(\mathbb{N}_0^m \setminus 0\right). \\ a, |b|, c \le 4j+4}} c_{B,j;a,b,c,\Lambda}(x)\, v_{a,b,c,\Lambda;p}(s). \tag{3.18}$$

for some sections $c_{j;a}, c_{j;a,b,c,\Lambda} \in C^\infty(X; End(S \otimes L))$. Moreover, the leading coefficient is given by

$$U_{B,0,x} = \left(b_0|_\Sigma\right).U_{0,x} \tag{3.19}$$

in terms of the leading coefficient of (3.3) *and the principal symbol $b_0 = \sigma(B)$.*

Proof. We begin by writing $\phi = \phi_0 + \phi_1$, with

$$\begin{aligned}
\phi_0(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi s}\hat{\phi}(\xi)\,\chi\left(\frac{2\xi\sqrt{h}}{i_g}\right) d\xi \\
\phi_1(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi s}\hat{\phi}(\xi)\left[1 - \chi\left(\frac{2\xi\sqrt{h}}{i_g}\right)\right] d\xi
\end{aligned}$$

via Fourier inversion.

First considering ϕ_1, integration by parts gives the estimate

$$\left|s^{n+1}\phi_1(s)\right| \le C_N h^{\frac{N-1}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^{2N}\hat{\phi}^{(k)}\right\|_{L^1}\right),$$

$\forall N \in \mathbb{N}$. Hence,

$$\left\|D^{n+1-a} B\phi_1\left(\frac{D}{\sqrt{h}}\right) D^a\right\|_{L^2 \to L^2} \le C_N h^{\frac{n+N}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^{2N}\hat{\phi}^{(k)}\right\|_{L^1}\right),$$

$\forall N \in \mathbb{N}$, $\forall a = 0, \ldots, n+1$. Semi-classical elliptic estimate and Sobolev's inequality now give the estimate

$$\left|B\phi_1\left(\frac{D}{\sqrt{h}}\right)\right|_{C^0(X \times X)} \le C_N h^{\frac{n+N}{2}}\left(\sum_{k=0}^{n+1}\left\|\xi^{2N}\hat{\phi}^{(k)}\right\|_{L^1}\right) \tag{3.20}$$

$\forall N \in \mathbb{N}$, on the Schwartz kernel.

Next, considering ϕ_0, we first use the change of variables $\alpha = \xi\sqrt{h}$ to write

$$\phi_0\left(\frac{D}{\sqrt{h}}\right) = \frac{1}{2\pi\sqrt{h}}\int_{\mathbb{R}} e^{i\alpha\left(D_{A_0} + ih^{-1}c(a)\right)}\hat{\phi}\left(\frac{\alpha}{\sqrt{h}}\right)\chi\left(\frac{2\alpha}{i_g}\right) d\alpha.$$

Now since $D = \mathbb{D}$ on $B_{i_g/2}(p)$, we may use the finite propagation speed of the wave operators $e^{i\alpha h^{-1}D}$, $e^{i\alpha h^{-1}\mathbb{D}}$ and microlocality of $A \in \Psi^0_{\mathrm{cl}}(X)$ to conclude

$$B\phi_0\left(\frac{D}{\sqrt{h}}\right)(p,\cdot) = B\phi_0\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(0,\cdot)\,. \tag{3.21}$$

The right hand side above is defined using functional calculus of self-adjoint operators, with standard local elliptic regularity arguments implying the smoothness of its Schwartz kernel. By virtue of (3.20), a similar estimate for $B\phi_1\left(\frac{\mathbb{D}}{\sqrt{h}}\right)$, and (3.21) it now suffices to consider $B\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)$.

We now introduce the rescaling operator $\mathscr{R} : C^\infty\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right) \to C^\infty\left(\mathbb{R}^n;\mathbb{C}^{2^m}\right)$, $(\mathscr{R}s)(x) := s\left(\frac{x}{\sqrt{h}}\right)$. Conjugation by $\mathscr{R}$ amounts to the rescaling of coordinates $x \to x\sqrt{h}$. A Taylor expansion in (3.14) now gives the existence of classical (h-independent) self-adjoint, first-order differential operators $\mathtt{D}_j = a^k_j(x)\,\partial_{x_k} + b_j(x)$, $j = 0,1\ldots$, with polynomial coefficients (of degree at most $j+1$) as well as h-dependent self-adjoint, first-order differential operators $\mathrm{E}_j = \sum_{|\alpha|=N+1} x^\alpha\left[c^k_{j,\alpha}(x;h)\,\partial_{x_k} + d_{j,\alpha}(x;h)\right]$, $j = 0,1\ldots$, with uniformly C^∞ bounded coefficients $c^k_{j,\alpha}$, $d_{j,\alpha}$ such that

$$\mathscr{R}\mathbb{D}\mathscr{R}^{-1} = \sqrt{h}\mathtt{D} \quad \text{with} \tag{3.22}$$

$$\mathtt{D} = \left(\sum_{j=0}^{N} h^{j/2}\mathtt{D}_j\right) + h^{(N+1)/2}\mathrm{E}_{N+1},\ \forall N. \tag{3.23}$$

The coefficients of the polynomials $a^k_j(x)$, $b_j(x)$ again involve the covariant derivatives of the curvatures F^{TX}, F^{A_0} and da evaluated at p. Furthermore, the leading term in (3.23) is easily computed

$$\mathtt{D}_0 = \gamma^j\left[\partial_{x_j} + i\frac{x^k}{2}(da)_{jk}(0)\right] \tag{3.24}$$

$$= \gamma^0\partial_{x_0} + \underbrace{\gamma^j\left[\partial_{x_j} + \frac{i\lambda_j(p)}{2}x_{j+m}\right] + \gamma^{j+m}\left[\partial_{x_{j+m}} - \frac{i\lambda_j(p)}{2}x_j\right]}_{:=\mathtt{D}_{00}} \tag{3.25}$$

using (3.4), (3.10). It is now clear from (3.22) that

$$\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(x,x') = h^{-n/2}\phi(\mathtt{D})\left(\frac{x}{\sqrt{h}},\frac{x'}{\sqrt{h}}\right). \tag{3.26}$$

Next, let $I_j = \{k = (k_0,k_1,\ldots)\,|k_\alpha \in \mathbb{N},\ \sum k_\alpha = j\}$ denote the set of partitions of the integer j and set

$$\mathtt{C}^z_j = \sum_{k\in I_j}(z-\mathtt{D}_0)^{-1}\left[\Pi_\alpha\left[\mathtt{D}_{k_\alpha}(z-\mathtt{D}_0)^{-1}\right]\right]. \tag{3.27}$$

Local elliptic regularity estimates again give

$$(z-\mathtt{D})^{-1} = O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{loc}}}\left(|\mathrm{Im}z|^{-1}\right) \quad \text{and}$$
$$\mathtt{C}^z_j = O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{loc}}}\left(|\mathrm{Im}z|^{-2j-2}\right),$$

$j=0,1,\ldots$. A straightforward computation using (3.23) then yields

$$(z-\mathtt{D})^{-1}-\left(\sum_{j=0}^{N}h^{j/2}\mathtt{C}_{j}^{z}\right)=O_{L_{\mathrm{loc}}^{2}\to L_{\mathrm{loc}}^{2}}\left(\left(\left|\mathrm{Im}z\right|^{-2}h^{\frac{1}{2}}\right)^{N+1}\right). \tag{3.28}$$

A similar expansion as (3.23) for the operator $\left(1+\mathtt{D}^{2}\right)^{(n+1)/2}(z-\mathtt{D})$ also gives the bounds

$$\left(1+\mathtt{D}^{2}\right)^{-(n+1)/2}(z-\mathtt{D})^{-1}-\left(\sum_{j=0}^{N}h^{j/2}\mathtt{C}_{j,n+1}^{z}\right)=O_{H_{\mathrm{loc}}^{s}\to H_{\mathrm{loc}}^{s+n+1}}\left(\left(\left|\mathrm{Im}z\right|^{-2}h^{\frac{1}{2}}\right)^{N+1}\right) \tag{3.29}$$

$\forall s\in\mathbb{R}$, for classical (h-independent) Sobolev spaces H_{loc}^{s}. Here each $\mathtt{C}_{j,n+1}^{z}=O_{H_{\mathrm{loc}}^{s}\to H_{\mathrm{loc}}^{s+n+1}}\left(\left|\mathrm{Im}z\right|^{-2j-2}\right)$ with the leading term

$$\mathtt{C}_{0,n+1}^{z}=\left(1+\mathtt{D}_{0}^{2}\right)^{-(n+1)/2}(z-\mathtt{D}_{0})^{-1}.$$

Finally, plugging the expansion (3.29) into the Helffer-Sjostrand formula

$$\phi(\mathtt{D})=-\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{\varrho}(z)\left(1+\mathtt{D}^{2}\right)^{-(n+1)/2}(z-\mathtt{D})^{-1}dzd\bar{z},$$

with $\varrho(x)\coloneqq\langle x\rangle^{n+1}\phi(x)$, gives

$$\phi(\mathtt{D})(y,0)=\left(\sum_{j=0}^{N}h^{j/2}U_{j,x}(\phi)(y,0)\right)+h^{(N+1)/2}O\left(\sum_{k=0}^{n+1}\left\Vert \langle\xi\rangle^{N}\hat{\phi}^{(k)}\right\Vert_{L^{1}}\right), \tag{3.30}$$

$\forall y\in\mathbb{R}^{n}$, using Sobolev's inequality. Here each

$$U_{j,x}(\phi)(y,0)=-\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{\varrho}(z)\,\mathtt{C}_{j,n+1}^{z}(y,0)\,dzd\bar{z}\in\mathrm{End}S_{x}^{TX} \tag{3.31}$$

defines a smooth family (in $p\in X$) of distributions U_{j} and the remainder term in (3.30) comes from the estimate $\bar{\partial}\tilde{\varrho}=O\left(\left|\mathrm{Im}z\right|^{2N+2}\sum_{k=0}^{n+1}\left\Vert \langle\xi\rangle^{2N}\hat{\phi}^{(k)}\right\Vert_{L^{1}}\right)$ on the almost analytic continuation (cf. [31] Sec. 3.1). Substituting (3.30) with $y=0$ into (3.26) gives the diagonal expansion for the Schwartz kernel of $\phi\left(\frac{D}{\sqrt{h}}\right)$.

To describe the diagonal expansion for the Schwartz kernel of the composition $B\phi\left(\frac{D}{\sqrt{h}}\right)$ one has to first express the kernel expansion for $\phi\left(\frac{D}{\sqrt{h}}\right)$ in an $h-$independent frame $\mathtt{l}_{0}$, used in expressing the Schwartz kernel of $B=\frac{1}{(2\pi h)^{n}}\int e^{i(x-y)\frac{\xi}{h}}b(x,\xi;h)\,d\xi$. Using (3.9) we may write the composition

$$\begin{aligned}
B\phi\left(\frac{D}{\sqrt{h}}\right)(0,0) &= \frac{1}{(2\pi h)^{n}}\int d\xi dy\,e^{-\frac{iy.\xi}{h}}b(0,\xi;h)\,e^{\varphi_{h}(y)}h^{-n/2}\phi(\mathtt{D})\left(\frac{y}{\sqrt{h}},0\right)\\
&= \frac{1}{(2\pi h)^{n}}\int d\xi dy'\,e^{-\frac{iy'.\xi}{\sqrt{h}}}b(0,\xi;h)\,e^{\varphi_{h}\left(y'\sqrt{h}\right)}\phi(\mathtt{D})(y',0)\\
&= \frac{1}{(2\pi h)^{n}}\int d\xi dy'\,e^{-\frac{iy'.(\xi+a(0))}{\sqrt{h}}}b(0,\xi;h)\underbrace{e^{\varphi_{h}\left(y'\sqrt{h}\right)+\frac{iy.a(0)}{\sqrt{h}}}}_{=:e\left(y';\sqrt{h}\right)}\phi(\mathtt{D})(y',0)\\
&= \frac{1}{\left(2\pi\sqrt{h}\right)^{n}}\int d\xi'dy'\,e^{-iy'.\xi'}b\left(0,\xi'\sqrt{h}-a;h\right)\underbrace{e^{\varphi_{h}\left(y'\sqrt{h}\right)+\frac{iy'.a(0)}{\sqrt{h}}}}_{=:e\left(y';\sqrt{h}\right)}\phi(\mathtt{D})(y',0) \qquad (3.32)
\end{aligned}$$

having used to the two changes of variables $y' = \frac{y}{\sqrt{h}}$ and $\xi' = \frac{\xi + a(0)}{\sqrt{h}}$.

Now

$$e\left(y;\sqrt{h}\right) := e^{\varphi_h\left(y\sqrt{h}\right)+\frac{iy.a(0)}{\sqrt{h}}} \sim 1 + \sum_{j=1}^{\infty} h^{j/2} e_j\left(y\right) \tag{3.33}$$

has an asymptotic expansion in powers of $h^{1/2}$ on account of (3.9). Plugging the above (3.33), the classical symbolic expansion for $b\left(x,\xi;h\right)$ and (3.30) gives the expansion (3.17) as well as the calculation of the leading term (3.19).

To elucidate the structure (3.18) of the coefficients, note that the symbolic/Taylor expansion of the total symbol for b in (3.32) yields

$$b\left(0,\xi'\sqrt{h}-a\left(0\right);h\right) \sim \sum_{j=0}^{\infty} h^{j/2} b_j\left(\xi'\right) \quad \text{with each}$$

$$b_j\left(\xi\right) = \sum_{|\alpha|\leq j} \left(\xi'\right)^{\alpha} b_{j,\alpha}$$

being polynomial in ξ' of degree at most j. Plugging the last equation above into (3.32) then gives that each coefficient in (3.17) is a sum of the form

$$U_{B,j,x}\left(\phi\right) = \sum_{|\alpha|+j_1+j_2\leq j} b_{j_1,j_2,\alpha}\left[\partial_y^{\alpha}\left(e_{j_1}\left(y\right)U_{j_2,x}\left(y,0\right)\left(\phi\right)\right)\right]_{y=0}.$$

From here it follows that the distributions $U_{B,j,x}$ have the same type of structure as was shown for $U_{j,x}$ in [24, Prop. 7.2], cf. eqn 7.33 and following ones therein. $\square$

By integrating the pointwise traces of the the distributions in (3.17) we may further define

$$\begin{aligned} u_{B,j} &= \int_X u_{B,j,x} dx \quad \text{with} \\ u_{B,j,x} &:= \text{tr } U_{B,j,x} \in C^{\infty}\left(X;\mathcal{S}'\left(\mathbb{R}_s\right)\right) \end{aligned} \tag{3.34}$$

for $j = 0, 1, \ldots$. As with $U_{B,j,x}$, the distributions $u_{B,j}$ also have the same structure (3.18). In particular we have

$$\text{sing spt}\left(u_{B,j}\right) \subset \mathbb{R}\setminus\left(-\sqrt{2\mu_1},\sqrt{2\mu_1}\right) \tag{3.35}$$

as with [24, Cor. 7.3].

The above now gives a corresponding generalization of [24, Thm 1.3]. Choose $f \in C_c^{\infty}\left(-\sqrt{2\mu_1},\sqrt{2\mu_1}\right)$. With $0 < T' < T_0$ and let $\theta \in C_c^{\infty}\left(\left(-T_0,T_0\right);\left[0,1\right]\right)$ such that $\theta\left(x\right) = 1$ on $\left(-T',T'\right)$. Let

$$\begin{aligned} \mathcal{F}^{-1}\theta\left(x\right) &:= \check{\theta}\left(x\right) = \frac{1}{2\pi}\int e^{ix\xi}\theta\left(\xi\right)d\xi \\ \mathcal{F}_h^{-1}\theta\left(x\right) &:= \frac{1}{h}\check{\theta}\left(\frac{x}{h}\right) = \frac{1}{2\pi h}\int e^{\frac{i}{h}x\xi}\theta\left(\xi\right)d\xi. \end{aligned}$$

Theorem 6. *There exist smooth functions $u_{B,j} \in C^{\infty}\left(\mathbb{R}\right)$ such that there is a trace expansion*

$$\begin{aligned} &tr\left[Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right] = \\ &tr\left[Bf\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right] = h^{-m-1}\left(\sum_{j=0}^{N-1} f\left(\lambda\right)u_{B,j}\left(\lambda\right)h^{j/2} + O\left(h^{N/2}\right)\right) \end{aligned} \tag{3.36}$$

for each $N \in \mathbb{N}$,$\lambda \in \mathbb{R}$.

Proof. We break up the trace using

$$\theta(x)=\theta_{\epsilon}(x)+\underbrace{\left[\theta(x)-\theta_{\epsilon}(x)\right]}_{\vartheta(x)}$$

where $\theta_{\epsilon}(x):=\theta\left(\frac{x}{h^{\epsilon}}\right)$, $\epsilon\in\left(\frac{1}{4},\frac{1}{2}\right)$. The second function in the break up satisfies $\vartheta\in C_{c}^{\infty}\left(\left(T'h^{\epsilon},T\right);[-1,1]\right)$ and one has

$$\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_{h}^{-1}\vartheta\right)\left(\lambda\sqrt{h}-D\right)\right]=O\left(h^{\infty}\right). \tag{3.37}$$

The proof of the above is the same as [24, Lem 3.1] which already uses a microlocal partition of the trace cf. [24, Eq. 3.1 and 3.2].

Next we come to the trace involving $\theta_{\epsilon}(x)$ and write

$$\begin{aligned}\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_{h}^{-1}\theta_{\epsilon}\right)\left(\lambda\sqrt{h}-D\right)\right]&=\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h^{1-\epsilon}}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h^{1-\epsilon}}\right)\right]\\&=\frac{h^{-\frac{1}{2}}}{2\pi}\int dt\,\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right]\theta\left(th^{\frac{1}{2}-\epsilon}\right).\end{aligned} \tag{3.38}$$

Next, the expansion Theorem 5, with $\phi(x)=f(x)e^{it(\lambda-x)}$, combined with the smoothness of u_j on $\operatorname{spt}(f)\subset\left(-\sqrt{2\mu_1},\sqrt{2\mu_1}\right)$ (3.35) gives

$$\begin{aligned}\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right]&=e^{it\lambda}h^{-n/2}\left(\sum_{j=0}^{N}h^{j/2}\widehat{fu_{B,j}}(t)\right)\\&\quad+h^{(N+1-n)/2}\underbrace{O\left(\sum_{k=0}^{n+1}\left\|\langle\xi\rangle^{2N}\hat{\phi}^{(k)}(\xi-t)\right\|_{L^{1}}\right)}_{=O\left(\langle t\rangle^{2N}\right)}.\end{aligned} \tag{3.39}$$

Finally, plugging (3.39) into (3.38) and using $\theta\left(th^{\frac{1}{2}-\epsilon}\right)=1+O\left(h^{\infty}\right)$ gives via Fourier inversion

$$\begin{aligned}&\frac{h^{-\frac{1}{2}}}{2\pi}\int dt\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)e^{it\left(\lambda-\frac{D}{\sqrt{h}}\right)}\right]\theta\left(th^{\frac{1}{2}-\epsilon}\right)\\&=h^{-m-1}\left(\sum_{j=0}^{N}h^{j/2}f(\lambda)u_{B,j}(\lambda)\right)+O\left(h^{2N\left(\epsilon-\frac{1}{4}\right)-m-1}\right)\end{aligned} \tag{3.40}$$

as required. □

3.1. **Estimates in Ψ_{δ}^{0}.** In the next section we shall also need estimates on the microlocal trace (3.17) in the more general class from 2.2.1. These follow from arguments similar to the ones in the proofs of Theorem 5, Theorem 6. Firstly, the formula (3.19) then shows that the microlocal Weyl measure of $\frac{D}{\sqrt{h}}$ concentrates on Σ (cf. [7, Thm 4.1], [26, Thm. 25]).

More generally, denote by $\pi_{\Sigma}:T^{*}X\rightarrow\Sigma$, $\pi_{\Sigma}(x,\xi)=(x,-a(x))$ the projection onto the characteristic variety. Then for $B\in\Psi_{\delta}^{0}(X)$ the equations (3.26), (3.30) and (3.32) imply

$$\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)\right]\leq Ch^{-n/2}\mu\left(\pi_{\Sigma}\left(WF(B)\right)\right)$$

with μ denoting the pullback Riemannian measure on Σ and the constant $C = C\left(\|b\|_{0,0}\right)$ depending only on the sup norm (2.11) of the symbol of B.

For the estimate generalizing Theorem 6, the equations corresponding to (3.37), (3.38) and (3.40) give

$$\operatorname{tr}\left[Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right] \leq Ch^{-m-1}\mu\left(\pi_\Sigma\left(WF\left(B\right)\right)\right),$$

with again the constant $C = C\left(\|b\|_{0,0}\right)$ depending only on the sup norm of the symbol of B.

4. Eta remainder asymptotics

In this section we shall prove the main theorem Theorem 1.

4.1. **Partitions adapted to recurrence.** We shall first choose a microlocal partition of unity adapted to the recurrence sets $S_{T,\varepsilon}$ and $S^e_{T,\varepsilon}$ (1.3). We recall that $\varepsilon = h^\delta$ for $\delta \in \left[0, \frac{1}{2}\right)$. With $\chi \in C_c^\infty(-1,1)$ satisfying $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ we choose a pseudodifferential operator $B = b^W \in \Psi^0_\delta(X)$ of the form

$$b = \chi\left(\frac{|\xi + a|}{h^\delta}\right) b_0(x) \qquad \text{with}$$

$$b_0 = \begin{cases} 1 & \text{on } S_{T,\varepsilon} \\ 0 & \text{on } \left(S^e_{T,\varepsilon}\right)^c \end{cases}.$$

The existence of b_0 satisfying the correct symbolic estimates follows by an application of the Whitney extension theorem [16, Sec. 2.3]. In particular this gives

$$\mu\left(\pi_\Sigma\left(WF\left(B\right)\right)\right) \leq \mu^{g^{TX}}\left(S^e_{T,\varepsilon}\right). \tag{4.1}$$

Next following [25, Lem. 3.3] we note that near each $x \in X \setminus S_{T,\varepsilon}$ there is a local Darboux chart $\varphi_x : N_x \xrightarrow{\sim} C_{\varepsilon_0 h^\delta, T} \subset \mathbb{R}^n$ into a cylinder $C_{\varepsilon_0 h^\delta, T} := B_{\mathbb{R}^{2m}}\left(\varepsilon_0 h^\delta\right) \times (0,T)_{x_0} \subset \mathbb{R}^n_x$ of radius $\varepsilon_0 h^\delta$ and height T in Euclidean space. For each such Darboux chart $\varphi_x : N_x \xrightarrow{\sim} C_{\varepsilon h^\delta, T} \subset \mathbb{R}^n$ we set $N^0_x := \varphi_x^{-1}\left(C_{\frac{\varepsilon h^\delta}{8}, \frac{T}{8}}\right)$. By compactness we may choose a finite set of these such that $\bigcup_{u=1}^N N^0_{x_u}$, $N = N_h = O\left(h^{-\delta}\right)$, cover $X \setminus S_{T,\varepsilon}$. Denote by $\tilde{S} \subset T^*X$ the inverse image of any subset $S \subset X$ under the projection $\pi : T^*X \to X$ and by $c_\delta := ch^\delta$ the h-dependent constant for each h-independent constant c.

For $\delta \in \left[0, \frac{1}{2}\right)$, $\tau > 0$, a (Ω, τ, δ)-microlocal partition of unity is defined to be a collection of zeroth-order self -adjoint pseudo-differential operators

$$\mathcal{P} = \left\{A_u \in \Psi^0_\delta(X) \,|\, 0 \leq u \leq N_h\right\} \cup \left\{B \in \Psi^0_\delta(X)\right\}$$

satisfying

$$
\begin{aligned}
\sum_{u=0}^{N_h} A_u + B &= 1\\
N_h &= O\left(h^{-\delta}\right)\\
WF\left(A_0\right) \subset\ U_0 \subset &\ \overline{T^*X} \setminus \Sigma^D_{\left[-\frac{\tau_\delta}{64},\frac{\tau_\delta}{64}\right]}\\
WF\left(A_u\right) \Subset\ U_u \subset &\ \Sigma^D_{\left[-\tau_\delta,\tau_\delta\right]} \cap \tilde{N}^0_{x_u},\ 1 \leq u \leq N\\
WF\left(B\right) \Subset\ V \subset &\ \Sigma^D_{\left[-\tau_\delta,\tau_\delta\right]} \cap \tilde{S}^e_{T,\varepsilon},
\end{aligned}
\tag{4.2}
$$

for some open cover $\left\{U_u\right\}_{u=0}^{N} \cup V$ of T^*X . For such a partition $\mathcal{P}$ define the pairs of indices

$$
\begin{aligned}
I_{\mathcal{P}} &= \left\{\left(u,u'\right)|u \leq u',\ WF\left(A_u\right) \cap WF\left(A_{u'}\right) \neq \emptyset\right\}\\
J_{\mathcal{P}} &= \left\{u|WF\left(A_u\right) \cap WF\left(B\right) \neq \emptyset\right\}.
\end{aligned}
\tag{4.3}
$$

An augmentation $(\mathcal{P};\mathcal{V},\mathcal{W})$ of this partition consists of an additional collection of open sets

$$
\begin{aligned}
\mathcal{V} &= \left\{V^1_{uu'}\right\}_{(u,u')\in I_{\mathcal{P}}} \cup \left\{V^2_u\right\}_{u\in J_{\mathcal{P}}}\\
\mathcal{W} &= \left\{W^1_{uu'}\right\}_{(u,u')\in I_{\mathcal{P}}} \cup \left\{W^2_u\right\}_{u\in J_{\mathcal{P}}}
\end{aligned}
$$

satisfying

$$
\begin{aligned}
WF\left(A_u\right) \cap WF\left(A_{u'}\right) &\subset W^1_{uu'}\\
& \cap\\
WF\left(A_u\right) \cup WF\left(A_{u'}\right) &\subset V^1_{uu'} \Subset \Sigma^D_{\left[-2\tau_\delta,2\tau_\delta\right]} \cap \tilde{N}_{x_u},\\
WF\left(A_u\right) \cap WF\left(B\right) &\subset W^2_u\\
& \cap\\
WF\left(A_u\right) \cup WF\left(B\right) &\subset V^2_u \Subset \Sigma^D_{\left[-2\tau_\delta,2\tau_\delta\right]} \cap \tilde{N}_{x_u}.
\end{aligned}
\tag{4.4}
$$

Next with $d = \sigma\left(D\right)$, for each pair of indices in (4.3) we set

$$
T_{uu'} := \frac{1}{\inf_{(g,\mathtt{v})\in\mathcal{G}_{uu'}\times S^0_\delta(X;U(S))} \left|H_{g,\mathtt{v}}d\right|},
\tag{4.5}
$$

$$
S_u := \frac{1}{\inf_{(g,\mathtt{v})\in\mathcal{H}_{u}\times S^0_\delta(X;U(S))} \left|H_{g,\mathtt{v}}d\right|},\qquad \text{with}
\tag{4.6}
$$

$$
\mathcal{G}_{uu'} := \left\{g \in S^0_\delta\left(T^*X;[0,1]\right) |\ g|_{W^1_{uu'}} = 1,\ g|_{\left(V^1_{uu'}\right)^c} = 0\right\}
\tag{4.7}
$$

$$
\mathcal{H}_{u} := \left\{g \in S^0_\delta\left(T^*X;[0,1]\right) |\ g|_{W^2_{u}} = 1,\ g|_{\left(V^2_{u}\right)^c} = 0\right\}
\tag{4.8}
$$

and $\left|H_{g,\mathtt{v}}d\right| := \sup\left\|\left\{\mathtt{v}^*d\mathtt{v},g\right\}\right\|$ with the bracket being computed in terms of the chosen and induced trivialization/coordinates on $N_{x_u}, \tilde{N}_{x_u}$. A function in $\mathcal{G}_{uu'}$ or $\mathcal{H}_u$ shall be referred to as a trapping/microlocal weight function. Finally, the *extension/trapping time* of an augmented (Ω,τ,δ)-partition $(\mathcal{P};\mathcal{V},\mathcal{W})$ is set to be

$$
T_{(\mathcal{P};\mathcal{V},\mathcal{W})} := \min\left\{\min\left\{T_{uu'}\right\}_{(u,u')\in I_{\mathcal{P}}}, \min\left\{S_u\right\}_{u\in J_{\mathcal{P}}}\right\}.
\tag{4.9}
$$

It was shown in [25, Prop. 3.4] that for each $T>0, \delta\in\left(0,\frac{1}{2}\right)$ and τ sufficiently small there exists an augmented (Ω,τ,δ)-partition of unity $(\mathcal{P};\mathcal{V},\mathcal{W})$ with

$$T_{(\mathcal{P};\mathcal{V},\mathcal{W})}>T. \tag{4.10}$$

Next for each $\theta\in C_c^\infty\left(\left(T_0,T\right);[-1,1]\right)$, $f\in\mathcal{S}\left(\mathbb{R}\right)$ and $A,B\in\Psi_\delta^0\left(X\right)$ set

$$\mathcal{T}_{A,B}^{\theta}\left(D\right):=\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}\left(z\right)\check{\theta}\left(\frac{\lambda-z}{\sqrt{h}}\right)\text{tr }\left[A\left(\frac{1}{\sqrt{h}}D-z\right)^{-1}B\right]dzd\bar{z}$$

with $\tilde{f}$ being an almost analytic continuation of f. Then for $T<T_{(\mathcal{P};\mathcal{V},\mathcal{W})}$ it was shown in [25, Lemma 3.6] that

$$\mathcal{T}_{A_u,A_v}^{\theta}\left(D\right),\ \mathcal{T}_{A_u,B}^{\theta}\left(D\right),\ \mathcal{T}_{B,A_u}^{\theta}\left(D\right)=O\left(h^\infty\right). \tag{4.11}$$

We remark that an albeit arduous inspection of the arguments in [24, 25], which we do not repeat here, shows that the equations above (4.10), (4.11) hold for h-dependent times $T=T\left(h\right)$.

Next we turn to examine the trace $\mathcal{T}_{B,B}^{\theta}\left(D\right)$. We shall choose $\theta^1\in C_c^\infty\left(\left(-T_0,T_0\right)\right)$ such that $\theta^1\left(x\right)=1$ on $\left(-T',T'\right)$, $T'<T_0$, and $\check{\theta}^1\left(\xi\right)\geq 0$. Further we let $f\left(x\right)\geq 0$ with $f\left(0\right)=1$. Since the trace and trace norm of a positive self-adjoint operator agree we have

$$\begin{aligned}
&\left\|Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta^1\right)\left(\lambda\sqrt{h}-D\right)B\right\|_{\text{tr}}\\
=&\text{tr }\left[B^2f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta^1\right)\left(\lambda\sqrt{h}-D\right)\right]\\
\leq&h^{-m-1}\left[f\left(\lambda\right)\int_X\left(b_0^2\big|_{\xi=-a(x)}\right)u_{0,x}\left(\lambda\right)+O\left(h^{1/2}\right)\right]
\end{aligned} \tag{4.12}$$

from (3.19), (3.34) and (3.36). For $\theta_c^1\left(t\right):=\theta^1\left(t-c\right)$ one has $\mathcal{F}_h^{-1}\theta_c^1\left(x\right)=e^{i\frac{xc}{h}}\mathcal{F}_h^{-1}\theta^1\left(x\right)$. Hence $e^{ic\left(\lambda\sqrt{h}-D\right)}$ being a unitary operator, the left hand side of (4.12) is unchanged under translation of θ^1. By writing an arbitrary $\theta\in C_c^\infty\left(T_0,T\right)$, of possibly h-dependent compact support $T=T\left(h\right)$, as a sum of translates of functions with compact support in $\left(-T_0,T_0\right)$ we obtain

$$\begin{aligned}
&\left\|Bf\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)B\right\|_{\text{tr}}\\
\leq&Th^{-m-1}\left[f\left(\lambda\right)\int_X\left(b_0^2\big|_{\xi=-a(x)}\right)u_{0,x}\left(\lambda\right)+O_{\theta^1}\left(h^{1/2}\right)\right]\\
\leq&Th^{-m-1}\left[f\left(\lambda\right)\mu\left(S_{T,\varepsilon}^e\right)+O\left(h^{1/2}\right)\right].
\end{aligned}$$

Combined with (4.11) and [24, Thm 1.3] we have

$$\begin{aligned}
&\left|\text{tr }\left[f\left(\frac{D}{\sqrt{h}}\right)\left(\mathcal{F}_h^{-1}\theta\right)\left(\lambda\sqrt{h}-D\right)\right]-h^{-m-1}f\left(\lambda\right)u_0\left(\lambda\right)\right|\\
\leq&Th^{-m-1}f\left(\lambda\right)\mu\left(S_{T,\varepsilon}^e\right)+O\left(h^{1/2}\right)
\end{aligned} \tag{4.13}$$

for arbitrary $\theta\in C_c^\infty\left(-T,T\right)$, of possibly h-dependent compact support $T=T\left(h\right)$.

4.2. **Tauberian argument.** Next we consider irregular functional traces of $\frac{D}{\sqrt{h}}$. Note from [24, Cor. 7.3] that the distributions $u_j \in \mathcal{S}'(\mathbb{R})$ (3.17) are smooth near 0. Hence

$$u_j^{\pm}(x) := 1_{[0,\infty)}(\pm x)\, u_j(x) \in \mathcal{S}'(\mathbb{R}) \tag{4.14}$$

are well defined tempered distributions and we similarly define $f^{\pm}$ for any $f \in \mathcal{S}(\mathbb{R})$. We then have the following.

Lemma 7. *For any $f \in \mathcal{S}(\mathbb{R})$,*

$$\begin{aligned} &\left| tr\, f^{\pm}\left(\frac{D}{\sqrt{h}}\right) - \left[h^{-m-\frac{1}{2}} u_0^{\pm}(f) + h^{-m} u_1^{\pm}(f)\right]\right| \\ \leq & h^{-m}\|f\|_{C^0}\left[u_0(0)\, T^{-1} + O\left(T^{-2} + \mu\left(S^e_{T,\varepsilon}\right)\right)\right]. \end{aligned} \tag{4.15}$$

Proof. First choose $\theta \in \mathcal{S}(\mathbb{R})$ such that $\check{\theta} \geq \frac{1}{1+\epsilon}$ on $[0,1]$ and $1 = \theta(0) = \int d\xi \check{\theta}(\xi)$. For $T = T(h)$, set $\theta_T(x) = \theta(T^{-1}x)$ and let $N(a,b)$ denote the number of eigenvalues of $\frac{D_h}{\sqrt{h}}$ in the interval (a,b). Choosing $f(x) \geq 0$, the trace expansion (4.13) with $\lambda = 0$ now gives

$$\begin{aligned} \frac{T}{(1+\epsilon)\,h} N\left(0, T^{-1}\sqrt{h}\right) \leq & \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{T}{h}\check{\theta}\left(\frac{-TD}{h}\right)\right] \\ = & h^{-m-1}\left[f(0)\, u_0(0) + O\left(T\mu\left(S^e_{T,\varepsilon}\right)\right) + O\left(h^{1/2}\right)\right] \end{aligned}$$

$\forall \epsilon > 0$. And hence

$$N\left(0, T^{-1}\sqrt{h}\right) \leq h^{-m}\left[T^{-1} f(0)\, u_0(0) + O\left(\mu\left(S^e_{T,\varepsilon}\right)\right) + O\left(h^{1/2}\right)\right]. \tag{4.16}$$

By virtue of (3.3), we may assume $f \in C_c^\infty\left(-\sqrt{2\mu_1}, \sqrt{2\mu_1}\right)$. The spectral measure for $\frac{D}{\sqrt{h}}$ is defined as $\mathfrak{M}_f(\lambda') := \sum_{\lambda \in \mathrm{Spec}\left(\frac{D}{\sqrt{h}}\right)} f(\lambda)\, \delta(\lambda - \lambda')$. Next we choose $\theta \in \mathcal{S}(\mathbb{R})$ even and its transform satisfies $\operatorname{spt}\left(\check{\theta}\right) \subset [-1,1]$, $1 \geq \check{\theta}(\xi) \geq 0$, $\int \check{\theta}(\xi)\, d\xi = 1$. Setting $\theta_{\frac{1}{2}}(x) = \theta\left(\frac{x}{\sqrt{h}}\right)$, [24, Thm 1.3] up to its first two terms may be written as

$$\mathfrak{M}_f * \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda) = h^{-m-\frac{1}{2}}\left(f(\lambda)\, u_0(\lambda) + h^{1/2} f(\lambda)\, u_1(\lambda) + O(h)\right).$$

Both sides above involving Schwartz functions in λ, the remainder maybe replaced by $O\left(\frac{h}{\langle\lambda\rangle^2}\right)$. Integrating further gives

$$\begin{aligned} &\int_{-\infty}^0 d\lambda \int d\lambda' \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda - \lambda')\, \mathfrak{M}_f(\lambda') \\ = & h^{-m-\frac{1}{2}}\left(\int_{-\infty}^0 d\lambda f(\lambda)\, u_0(\lambda) + h^{1/2}\int_{-\infty}^0 d\lambda f(\lambda)\, u_1(\lambda) + O(h)\right). \end{aligned} \tag{4.17}$$

Now note

$$\int_{-\infty}^0 d\lambda \left(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}}\right)(\lambda - \lambda') = 1_{(-\infty,0]}(\lambda') + \phi\left(\frac{\lambda'}{\sqrt{h}}\right) \tag{4.18}$$

where $\phi(x) := \int_{-\infty}^0 dt \check{\theta}(t - x) - 1_{(-\infty,0]}(x)$ is a function that is rapidly decaying with all derivatives, odd and smooth on $\mathbb{R}_x \setminus 0$ and satisfies $\phi'(x) = \check{\theta}(-x)$ for $x \neq 0$.

Next with $T = T(h)$ and $x \geq 0$ we compute

$$
\begin{aligned}
&\left|\phi(x) - \phi * \check{\theta}_T(x)\right| \\
&= \left|\int dy \left[\phi(x) - \phi\left(x - T^{-1}y\right)\right] \check{\theta}(y)\right| \\
&\leq \int_{y \leq xT} dy \left|\phi'(c(x,y))\right| T^{-1} |y| \check{\theta}(y) + 2\int_{y \geq xT} dy \check{\theta}(y) \\
&\leq T^{-1} \underbrace{\int_{-\infty}^{xT} dy\, |y|\, \check{\theta}(y)}_{=\theta_1(xT)} + 2 \underbrace{\int_{y \geq xT} dy \check{\theta}(y)}_{=\theta_2(xT)}
\end{aligned}
\tag{4.19}
$$

where $c(x,y) \in [x - T^{-1}y, x]$, with a similar estimate for $x \leq 0$.

Next pairing the second term of (4.19) with $\mathfrak{M}_f(\lambda')$ gives

$$
\int d\lambda' \theta_2\left(\frac{\lambda' T}{\sqrt{h}}\right) \mathfrak{M}_f(\lambda') \leq h^{-m}\left[T^{-1}\|f\|_{C^0} u_0(0) + O\left(\mu\left(S^e_{T,\varepsilon}\right)\right) + O\left(h^{1/2}\right)\right]
\tag{4.20}
$$

on covering $\mathbb{R}_{\lambda'}$ intervals of size $O(T^{-1}h)$ and using the Weyl estimate (4.16). A similar estimate

$$
\int d\lambda' T^{-1}\theta_1\left(\frac{\lambda' T}{\sqrt{h}}\right) \mathfrak{M}_f(\lambda') = O\left(h^{-m}T^{-1}\left[T^{-1} + \mu\left(S^e_{T,\varepsilon}\right) + O\left(h^{1/2}\right)\right]\right)
\tag{4.21}
$$

then gives

$$
\begin{aligned}
&\int d\lambda' \left[\phi_R\left(\frac{\lambda'}{\sqrt{h}}\right) - \phi_R * \check{\theta}_T\left(\frac{\lambda'}{\sqrt{h}}\right)\right] \mathfrak{M}_f(\lambda') \\
&\leq h^{-m}\left[\|f\|_{C^0} u_0(0) T^{-1} + O\left(T^{-2} + \mu\left(S^e_{T,\varepsilon}\right) + h^{1/2}\right)\right].
\end{aligned}
\tag{4.22}
$$

on combining(4.19), (4.20) and (4.21).

The second term above is estimated on integrating (4.13) against ϕ

$$
\begin{aligned}
\int d\lambda' \phi * \check{\theta}_T\left(\frac{\lambda'}{\sqrt{h}}\right) \mathfrak{M}_f(\lambda') &= h^{-m}\left[\int d\lambda \phi(\lambda) f(0) u_0(0) + O\left(\mu\left(S^e_{T,\varepsilon}\right) + \frac{h^{1/2}}{T}\right)\right] \\
&= O\left(h^{-m}\left[\mu\left(S^e_{T,\varepsilon}\right) + h^{1/2}\right]\right)
\end{aligned}
\tag{4.23}
$$

since ϕ is an odd function. Finally combining (4.17), (4.18), (4.22) and (4.23) gives

$$
\begin{aligned}
\operatorname{tr} f^-\left(\frac{D}{\sqrt{h}}\right) &= \int d\lambda' 1_{(-\infty,0]}(\lambda') \mathfrak{M}_f(\lambda') \\
&= h^{-m-\frac{1}{2}}\left(\int_{-\infty}^0 d\lambda f(\lambda) u_0(\lambda) + h^{1/2}\int_{-\infty}^0 d\lambda f(\lambda) u_1(\lambda)\right) \\
&\quad + \int d\lambda' \phi\left(\frac{\lambda'}{\sqrt{h}}\right) \mathfrak{M}_f(\lambda') \\
&= h^{-m-\frac{1}{2}}\left(\int_{-\infty}^0 d\lambda f(\lambda) u_0(\lambda) + h^{1/2}\int_{-\infty}^0 d\lambda f(\lambda) u_1(\lambda)\right) + R(h)
\end{aligned}
$$

$$
\text{with} \quad R(h) \leq h^{-m}\left[\|f\|_{C^0} u_0(0) T^{-1} + O\left(T^{-2} + \mu\left(S^e_{T,\varepsilon}\right) + h^{1/2}\right)\right].
$$

It is finally an easy exercise to show that $T^{-2} + \mu\left(S^e_{T,\varepsilon}\right) \geq O\left(h^{1/2}\right)$ for $\varepsilon = h^\delta$, $\delta \in \left[0, \frac{1}{2}\right)$, completing the proof. □

We now prove our main result Theorem 1.

Proof of Theorem 1. The eta invariant being invariant under scaling we have

$$\eta\left(D_h\right)=\eta\left(\frac{D}{\sqrt{h}}\right)=\int_0^\infty dt\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]$$

$$=\int_0^1 dt\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]+\int_1^\infty dt\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]. \tag{4.24}$$

Using [23, Prop. 3.4 and Eq. 4.5] the first integral is seen to give

$$\int_0^1 dt\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]=h^{-m}\left[\int_0^1 dt\frac{1}{\sqrt{\pi t}}u_1\left(se^{-ts^2}\right)\right]+O\left(h^{-m+1}\right). \tag{4.25}$$

While the second integral is evaluated to be $\operatorname{tr} E\left(\frac{D}{\sqrt{h}}\right)$ where

$$E\left(x\right)\coloneqq\operatorname{sign}(x)\operatorname{erfc}(|x|)=\operatorname{sign}(x)\cdot\frac{2}{\sqrt{\pi}}\int_{|x|}^\infty e^{-s^2}ds$$

where we use the convention $\operatorname{sign}(0)=0$. The above function E being odd and the difference of two functions of the form (4.14) we obtain

$$\left|\operatorname{tr} E\left(\frac{D}{\sqrt{h}}\right)-h^{-m-\frac{1}{2}}\left[u_0\left(E\right)\right]+h^{-m}\left[u_1\left(E\right)\right]\right|\leq h^{-m}\left[u_0\left(0\right)T^{-1}+O\left(\mu\left(S^e_{T,\varepsilon}\right)+T^{-2}\right)\right]. \tag{4.26}$$

From [24, Prop. 7.4] $u_0\left(\lambda\right)$ is an even function of λ, hence the first evaluation on the left hand side above is 0. The second by definition is

$$u_1\left(E\right)=\int_1^\infty dt\frac{1}{\sqrt{\pi t}}u_1\left(se^{-ts^2}\right). \tag{4.27}$$

Finally combining (4.24), (4.25), (4.26), (4.27) with the computation in [25, Cor. 7.3] gives

$$\begin{aligned}&\left|\eta_h-h^{-m}\left(-\frac{1}{2}\frac{1}{\left(2\pi\right)^{m+1}}\frac{1}{m!}\int_X\left[\operatorname{tr}\frac{1}{|\mathfrak{J}|}\left(\nabla^{TX}\mathfrak{J}\right)^0\right]a\wedge\left(da\right)^m\right)\right|\\ \leq&h^{-m}\left[u_0\left(0\right)T^{-1}+O\left(\mu\left(S^e_{T,\varepsilon}\right)+T^{-2}\right)\right]\end{aligned}$$

The theorem now follows from the computation $u_0\left(0\right)=\frac{\det|\mathfrak{J}|}{\left(4\pi\right)^{n/2}}$ from [24, Prop. 7.4]. $\square$

5. Examples of recurrence

In this section we prove the corollaries of the main theorem Theorem 1. These shall be based on asymptotic volume estimates for the recurrence set in particular cases.

5.1. **Anosov flows.**

The Reeb flow is Anosov if there exist constants $c_1>0$ and an invariant continuous splitting

$$TX=\mathbb{R}\left[R\right]\oplus E^u\oplus E^s\quad\text{such that}$$

$$\begin{aligned}\left\|e^{tR}\big|_{E^s}\right\|&\leq e^{-c_1t},\\ \left\|e^{-tR}\big|_{E^u}\right\|&\leq e^{-c_1t},\end{aligned} \tag{5.1}$$

$\forall t>0$, where the norm is taken with respect to some restricted Riemann metric g. Below it shall be further useful to choose a metric g for which the Reeb orbits are geodesics, any metric satisfying $g\left(R,.\right)=\lambda$ has this property. The bounds on the recurrence set in this case are exponential and proved in 9 below. We shall first need some related concepts about Anosov flows.

Each $T>0$ defines the Bowen distance on X via

$$d_T^g\left(x_1,x_2\right):=\sup_{t\in[0,T]}d^g\left(e^{tR}x_1,e^{tR}x_2\right).$$

A (T,ϵ) separated subset $S\subset X$ is finite set in which any two distinct points are at least distance ϵ apart with respect to the above d_T. Denote by $N\left(T,\epsilon\right)$ the maximum cardinality of a (T,ϵ) separated set in X. The topological entropy of the flow is now defined

$$\mathtt{h}_{\text{top}}=\mathtt{h}_{\text{top}}\left(e^R\right):=\lim_{\epsilon\to 0}\left(\limsup_{T\to\infty}\frac{\ln N\left(T,\epsilon\right)}{T}\right)\tag{5.2}$$

and is known to be a topological invariant.

Next for each $s\in(0,1]$, let $\mathcal{D}_s\left(X\right)$, $\mathcal{D}_{s-}\left(X\right)$ respectively be the set of compatible distorted distance functions d on the manifold satisfying

$$\begin{aligned}&d^g\lesssim d\lesssim\left(d^g\right)^s\\&d^g\lesssim d\lesssim\left(d^g\right)^{s-\epsilon},\qquad\text{for some }\epsilon>0,\end{aligned}$$

respectively. It is clear that

$$\begin{aligned}\mathcal{D}_s\left(X\right)&\subset\mathcal{D}_{s'}\left(X\right),\\\mathcal{D}_{s-}\left(X\right)&\subset\mathcal{D}_{s'-}\left(X\right),\quad s'<s,\end{aligned}$$

while all distances in $\mathcal{D}_s\left(X\right)$, $\mathcal{D}_{s-}\left(X\right)$ define the same manifold topology. Furthermore $\mathcal{D}_1\left(X\right)$ is the set of all distances equivalent to the d^g and hence includes all Riemannian distances. To each distance $d\in\mathcal{D}_s\left(X\right),\mathcal{D}_{s-}\left(X\right)$ is associated the Lipschitz constant of the time one flow

$$L_d=L_d\left(e^R\right):=\sup_{x_1\neq x_2}\frac{d\left(e^Rx_1,e^Rx_2\right)}{d\left(x_1,x_2\right)}.\tag{5.3}$$

It shall also be useful to define the local skewness of the time one map

$$SL_d\left(e^R\right):=\sup_{\varepsilon>0}\inf_{0<d(x_1,x_2)<\varepsilon}\frac{d\left(e^Rx_1,e^Rx_2\right)}{d\left(x_1,x_2\right)}.$$

One now has the following inequalities for topological entropy of an Anosov flow.

Lemma 8. *The topological entropy* (5.2) *satisfies the inequalities*

$$\frac{n}{2}\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(X)}\ln L_d\right)\leq\mathtt{h}_{top}\leq n\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(X)}\ln L_d\right)$$

with respect to the infimum of the log Lipschitz constants (5.3) *in* $\mathcal{D}_{\frac{1}{2}-}\left(X\right)$.

Proof. With $\mathrm{HD}\left(d\right)$ denoting the Hausdorff dimension of the manifold with respect to a distance d, the inequalities

$$\mathrm{HD}\left(d\right)\ln SL_d\leq\mathtt{h}_{\text{top}}\leq\mathrm{HD}\left(d\right)\ln L_d\tag{5.4}$$

$\forall d\in\mathcal{D}_{\frac{1}{2}-}\left(X\right)$, are fairly well known ([13, 21], cf. [30, Thm. 7.15]). Furthermore, from the definition $\frac{n}{2}\leq\mathrm{HD}\left(d\right)\leq n$. This hence proves the lemma in one direction

$$\mathtt{h}_{\text{top}}\leq n\left(\inf_{d\in\mathcal{D}_{\frac{1}{2}-}(X)}\ln L_d\right).\tag{5.5}$$

It now remains to construct a sequence of distances $d_k\in\mathcal{D}_{\frac{1}{2}-}\left(X\right)$, $k=1,2,\ldots$ with $n\,\ln L_{d_k}$ approaching $2\mathtt{h}_{\text{top}}$ as $k\to\infty$. A sequence of distances d_k as desired can be constructed for expansive maps [13, 21], cf. also the related construction of the Hamenstädt distance [15]. The time one map e^R is unfortunately not expansive in the flow direction. The lack of expansiveness can however be replaced with the following instability property which is satisfied by e^R [20]: there exists a positive constant $c>0$ such that the following implication holds

$$y \neq e^{tR}x,\ \forall t\in\mathbb{R} \implies d^g\left(e^{jR}x, e^{jR}y\right) > c \text{ for some } j\in\mathbb{Z}. \tag{5.6}$$

In fact the proof therein gives the stronger statement: for any $\epsilon>0$ there exist positive constants $c>0$, $\alpha>1$, $\alpha_\epsilon := \alpha+\epsilon$ such that the stronger implication holds

$$\begin{aligned} & y \neq e^{tR}x,\ \forall t\in\mathbb{R},\ d^{g^{TX}}(x,y) < c, \\ \implies & \alpha d^{g^{TX}}(x,y) \le \max\left\{d^g\left(e^{R}x, e^{R}y\right), d^g\left(e^{-R}x, e^{-R}y\right)\right\} \le \alpha_\epsilon d^{g^{TX}}(x,y). \end{aligned}$$

The constant α is related to the exponent c_1 in the Anosov condition (5.1).

We then define

$$N(x,y) := \begin{cases} \infty & x=y \\ \inf\left\{N\in\mathbb{N}_0 \middle| \max_{j\in[-N,N]} d^{g^{TX}}\left(e^{jR}x, e^{jR}y\right) > c\alpha^{-|j|}\right\} & x\neq y. \end{cases}$$

The following bounds are easily derived

$$\max\left\{0, \frac{\ln\frac{c}{d(x,y)}}{\ln\alpha\alpha_\epsilon}\right\} \le N(x,y) \le \max\left\{0, \frac{\ln\frac{c}{d(x,y)}}{\ln\alpha}\right\}. \tag{5.7}$$

Next set

$$\rho(x,y) := \alpha^{-N(x,y)}, \quad \text{satisfying} \tag{5.8}$$

$$\frac{d(x,y)}{c} \le \rho(x,y) \le \left[\frac{d(x,y)}{c}\right]^{\ln\alpha/\ln(\alpha\alpha_\epsilon)} \qquad \text{for } d(x,y)\le c. \tag{5.9}$$

Thus it follows that ρ defines the same manifold topology as $d^{g^{TX}}$ although it does not quite define a distance. From (5.7) one further has $d^{g^{TX}}(x,y) \ge \frac{c}{2} \implies N(x,y) \le \frac{\ln 2}{\ln\alpha} \implies \alpha^N \le \alpha^{\frac{\ln 2}{\ln\alpha}} = 2$. An application of the triangle inequality for $d^{g^{TX}}$ gives

$$\begin{aligned} \min\{N(x,y), N(y,z)\} \le M + N(x,z) \qquad & \text{and} \\ & \rho(x,z) \le 2\max\{\rho(x,y), \rho(y,z)\} \quad \forall x,y,z\in X \end{aligned}$$

as a weak triangle inequality. An application of Frink's metrization theorem then gives the existence of a metric D on X satisfying

$$D(x,y) \le \rho(x,y) \le 4D(x,y) \tag{5.10}$$

and hence defining the same topology as $d^{g^{TX}}$. On account of (5.9) and (5.10) we have $D\in\mathcal{D}_{\frac{1}{2}-}(X)$.

Next it is an exercise to show that $\rho\left(e^{jR}x, e^{jR}y\right) \le \alpha^j\rho(x,y)$ with equality on some neighborhood $V_j\subset X\times X$ of the diagonal in the product. Using (5.10) this yields

$$\begin{aligned} D\left(e^{jR}x, e^{jR}y\right) &\le 4\alpha^j D(x,y) \quad \forall x,y\in X \\ D\left(e^{jR}x, e^{jR}y\right) &\ge \frac{1}{4}\alpha^j D(x,y) \quad \forall (x,y)\in V_j. \end{aligned} \tag{5.11}$$

And hence we

$$L_D\left(e^{jR}\right) \le 4\alpha^j \le 16 S L_D\left(e^{jR}\right). \tag{5.12}$$

Next we define the following distance equivalent to D via

$$d_k(x,y) := \max_{0\leq j\leq k-1} \frac{D\left(e^{jR}x, e^{jR}y\right)}{L_D^{j/n}}, \tag{5.13}$$

whose Lipschitz constant $L_{d_k}\left(e^R\right) = \left[L_D\left(e^{kR}\right)\right]^{1/k}$ is seen to be given in terms of the D-Lipschitz constant of the time k map. Using (5.4), (5.12) and $\mathtt{h}_{\text{top}}\left(e^{kR}\right) = k\mathtt{h}_{\text{top}}\left(e^R\right)$ this finally gives

$$\begin{aligned}
n\left(\frac{\ln\alpha}{\ln(\alpha\alpha_\epsilon)}\right)\ln L_{d_k}\left(e^R\right) &\leq \mathrm{HD}(d_k)\ln L_{d_k}\left(e^R\right) \\
&\leq \frac{\mathrm{HD}(d_k)}{k}\ln L_D\left(e^{kR}\right) \\
&\leq \frac{\mathrm{HD}(d_k)}{k}\left[\ln 16 + \ln SL_D\left(e^{kR}\right)\right] \\
&\leq \frac{\mathrm{HD}(d_k)}{k}\ln 16 + \frac{1}{k}\mathtt{h}_{\text{top}}\left(e^{kR}\right) \\
&= \frac{\mathrm{HD}(d_k)}{k}\ln 16 + \mathtt{h}_{\text{top}}\left(e^{R}\right) \\
&\leq \frac{n}{k}\ln 16 + \mathtt{h}_{\text{top}}\left(e^{R}\right).
\end{aligned}$$

Letting $k\to\infty$ and observing $\left(\frac{\ln\alpha}{\ln(\alpha\alpha_\epsilon)}\right)\to\frac{1}{2}$ as $\epsilon\to 0$ gives the result. $\square$

By the last Lemma Lemma 8, for each $\lambda > \frac{2}{n}\mathtt{h}_{\text{top}}$ we have

$$L_d \leq e^\lambda, \quad \text{for some } d\in\mathcal{D}_{\frac{1}{2}-}(X).$$

By the semi-group property of the flow $\exists c>0$ such that

$$d\left(e^{tR}x_1, e^{tR}x_2\right) \leq ce^{\lambda t}d(x_1,x_2).$$

From $d^g \lesssim d \lesssim (d^g)^{\frac{1}{2}-\epsilon}$ this further implies for each $\lambda > \frac{2}{n}\mathtt{h}_{\text{top}}$, $\exists c>0$ such that

$$d^g\left(e^{tR}x_1, e^{tR}x_2\right) \leq ce^{2\lambda t}d^g(x_1,x_2)$$

$$\left\|\left(e^{tR}\right)^* f\right\|_{C^2} \leq ce^{2\lambda t}\|f\|_{C^2} \tag{5.14}$$

$\forall x_1,x_2\in X,\ f\in C^2(X)$.

Using the above we shall now prove have the exponential bounds on the volume of the recurrence set

$$\mu\left(S_{T,\varepsilon}\right) = O\left(\varepsilon^n e^{2\lambda T}\right) \tag{5.15}$$

$$\mu\left(S^{\mathrm{e}}_{T,\varepsilon}\right) = O\left(\varepsilon^n e^{2\lambda T}\right), \tag{5.16}$$

following an argument in [12]. The above recurrence set has an obvious lift

$$\tilde{S}_{T,\varepsilon} := \left\{(x,t)\,|t\in\left[\frac{1}{2}T_0, T\right] \text{ s.t. } d^g\left(e^{tR}x, x\right)\leq\varepsilon\right\}\subset X\times\mathbb{R} \tag{5.17}$$

satisfying $\pi_X\left(\tilde{S}_{T,\varepsilon}\right) = S_{T,\varepsilon}$ under the projection onto the first X factor. The volume bounds (5.15), (5.16) shall then follow from (5.14) and the following proposition.

Proposition 9. *For each $\lambda > \frac{2}{n}\mathtt{h}_{top}$, the lift $\tilde{S}_{T,\varepsilon}$ (5.17) satisfies the volume estimate*

$$\mu_{X\times\mathbb{R}}\left(\tilde{S}_{T,\varepsilon}\right) = O\left(\varepsilon^{n}e^{2\lambda T}\right) \tag{5.18}$$

with respect to the product measure on $X\times\mathbb{R}$.

Proof. First we claim that there exist $C,\delta > 0$ of the following significance: for each pair $(x,t),(x',t')\in\tilde{S}_{T,\varepsilon}$ satisfying $|t-t'|\leq\delta$, $d^{g}(x,x')\leq\delta e^{-\lambda t}$ one has

$$|t-t'|\leq c\varepsilon,\quad d^{g}\left(x,\cup_{t\in[-1,1]}e^{tR}x'\right)\leq c\varepsilon. \tag{5.19}$$

By choosing δ sufficiently small and using (5.14) we work in a sufficiently small geodesic coordinate chart in Euclidean space. The Reeb direction and $E^{u}\oplus E^{s}$ being transverse, we may replace x' by $e^{tR}(x')$, $t\in[-1,1]$, to arrange $x-x'\in E^{u}(x)\oplus E^{s}(x)$. Using (5.14) and a Taylor expansion in x,t we obtain

$$\begin{aligned}\left|e^{tR}(x)-e^{t'R}(x')-de^{tR}(x)(x-x')-R\left(e^{tR}(x')\right)(t-t')\right| &\leq ce^{\lambda t}|x-x'|^{2}+c|t-t'|^{2}\\ &\leq c\delta|x-x'|+c\delta|t-t'|.\end{aligned}$$

Since $(x,t),(x',t')\in\tilde{S}_{T,\varepsilon}$the above gives

$$\begin{aligned}c\delta|x-x'|+c\delta|t-t'|+c\varepsilon &\geq\left|\left(I-de^{tR}(x)\right)(x-x')-R\left(e^{tR}(x')\right)(t-t')\right|\\ &\geq c'|x-x'|+c'|t-t'|\end{aligned}$$

with the second line above following from the Anosov property. It then remains to choose δ sufficiently small in relation to c,c' to obtain (5.19).

Finally, let x_j, $j=1,\dots N$, be a maximal set of points such that $d(x_i,x_j)\geq\delta e^{-cT}$. As the balls $\left\{B_{\frac{\delta e^{-cT}}{2}}(x_j)\right\}_{j=1}^{N}$ centered at these points are disjoint, the bound $N\leq Ce^{ncT}$ follows by a computation of the total volume. Furthermore the sets

$$B_{j,k}\coloneqq B_{2\delta e^{-cT}}(x_j)\times\left[\frac{1}{2}T_0+k\delta,\frac{1}{2}T_0+(k+1)\delta\right],$$

and their intersections $S_{j,k}=\tilde{S}_{T,\varepsilon}\cap B_{j,k}$, $j=1,\dots N$, $k=0,\dots,1+[\delta^{-1}T]$, cover $X\times\mathbb{R}$ and $\tilde{S}_{T,\varepsilon}$ respectively. By (5.19) small $O(\varepsilon)$ size neighborhoods of the orbits

$$\left(\underbrace{\frac{1}{2}T_0+\left(k+\frac{1}{2}\right)\delta}_{=:t_k},\cup_{t\in[-1,1]}e^{(t_k+t)R}(x_j)\right)$$

of volume $O(\varepsilon^{n})$, then cover $\tilde{S}^{d}_{T,\varepsilon}$ proving (5.18). ☐

5.2. **Elliptic flows.** We now look at elliptic flows on Lens spaces. Given non-negative integers $q_0,q_1,\dots,q_m$, $q_0>1$, as well as positive reals $a_0,\dots,a_m$ such that $\left(a_0^{-1}a_1,\dots,a_0^{-1}a_m\right)\notin\mathbb{Q}^{m}$, the Lens space is defined as the quotient

$$X=L(q_0,q_1,\dots,q_m;a_0,\dots a_m)\coloneqq E(a_0,\dots,a_m)/\mathbb{Z}_{q_0}\quad\text{where} \tag{5.20}$$

$$E(a_0,\dots,a_m)\coloneqq\left\{(z_0,\dots,z_m)\in\mathbb{C}^{m+1}\middle|\sum_{j=0}^{m}a_j|z_j|^{2}=1\right\} \tag{5.21}$$

is the irrational ellipsoid. The $\mathbb{Z}_{q_0}$ action on the ellipsoid above is given by $e^{\frac{2\pi i}{q_0}}(z_0,\ldots,z_m) = \left(e^{\frac{2\pi i}{q_0}}z_0, e^{\frac{2\pi i q_1}{q_0}}z_1,\ldots,e^{\frac{2\pi i q_m}{q_0}}z_m\right)$.

The contact form is chosen to be

$$a = \sum_{j=0}^{m}(x_j dy_j - y_j dx_j)\Big|_{E(a_0,\ldots,a_m)}, \quad z_j = x_j + i y_j, \tag{5.22}$$

the restriction of the tautological form on $\mathbb{C}^{m+1}$, while its Reeb vector field is computed

$$R = \sum_{j=0}^{m} a_j\left(x_j\partial_{y_j} - y_j\partial_{x_j}\right).$$

Both of the above are seen to be $\mathbb{Z}_{q_0}$-invariant and hence descending to the Lens space quotient. The Reeb flow is then computed

$$e^{tR}\underbrace{\left[\left(z_0(0),\ldots,z_m(0)\right)\right]}_{=:x} = \left[\left(e^{ia_0t}z_0(0),\ldots,e^{ia_mt}z_m(0)\right)\right].$$

It is then an easy exercise to show that

$$e^{tR}x = x \iff \left(\underbrace{a_0q_0}_{=\tilde{a}_0}, \underbrace{a_1 - a_0q_1}_{=\tilde{a}_1},\ldots,\underbrace{a_m - a_0q_m}_{=\tilde{a}_m}\right) \in \mathbb{Z}^{m+1}$$

or more generally

$$\begin{aligned} d^{g^{TX}}\left(e^{tR}x, x\right) &\approx \sum_{j=1}^{n}\left|e^{i\tilde{a}_jt} - 1\right|\left|z_j(0)\right| \\ &\geq Ct^{1-\nu}\left(\min_j\left|z_j(0)\right|\right) \end{aligned}$$

from (2.14) where $\nu = \nu(\tilde{a}_0,\ldots,\tilde{a}_m) = \nu\left(a_0, a_1 - \frac{q_1}{q_0}a_0,\ldots,a_m - \frac{q_m}{q_0}a_0\right)$ denotes the irrationality index of the given tuple. Hence for $x \in S_{T,\varepsilon}$, $S^e_{T,\varepsilon}$, one has

$$\min_j\left|z_j(0)\right| < C\varepsilon T^{\nu-1}$$

from which we obtain

$$\mu\left(S_{T,\varepsilon}\right) = O\left(\varepsilon^2T^{2(\nu-1)}\right) \tag{5.23}$$

$$\mu\left(S^e_{T,\varepsilon}\right) = O\left(\varepsilon^2T^{2(\nu-1)}\right) \tag{5.24}$$

as the recurrence set bounds in this case.

5.3. **Proofs of corollaries.** We now prove the corollaries 2, 3, 4. These are seen to follow easily from the main theorem Theorem 1 along with the volume estimates on the recurrence sets from Section 5.

Proofs of Corollaries 2, 3 and 4. For Corollary 2 we choose ε, T to be h-independent. The formula (1.6) now follows easily from (1.4) by letting $\varepsilon \to 0$ and subsequently $T \to \infty$.

To prove Corollary 3 choose any $\lambda > \frac{2}{n}\mathtt{h}_{\mathrm{top}}$. Further set $\varepsilon = h^{\frac{1}{2}-}$ and $T = \frac{c\ln h^{-1}}{\lambda} = \frac{c|\ln h|}{\lambda}$ for $c < \frac{n}{4}$ in Theorem 1. The estimate (5.15) on the recurrence set then estimates the remainder (1.5) as

$$|R(h)| \leq \mathtt{h}_{\mathrm{top}}\left(\frac{8}{n^2}\frac{\det|\mathfrak{J}|}{(4\pi)^{n/2}}\right)|\ln h|^{-1} + o\left(|\ln h|^{-1}\right)$$

as required.

Finally for Corollary 4 we set $\varepsilon = h^{\frac{1}{2}-}$, $T = h^{-\frac{1}{2\nu-1}}$ in Theorem 1. The estimate (5.23) on the recurrence set then estimates the remainder (1.5)

$$R(h) = O\left(h^{\frac{1}{2\nu-1}-}\right)$$

as required. □

Remark 10. Conjecturally it is reasonable to expect

$$\inf_{d\in\mathcal{D}_1(X)} \ln L_d = \mathtt{h}_{\mathrm{top}} \tag{5.25}$$

as a characterization of topological entropy, and a strengthening of the entropy inequality in Lemma 8. If the above (5.25) were true, it would imply the sharper exponential bounds

$$\mu\left(S_{T,\varepsilon}\right) = O\left(\varepsilon^n e^{\lambda T}\right) \tag{5.26}$$

$$\mu\left(S^e_{T,\varepsilon}\right) = O\left(\varepsilon^n e^{\lambda T}\right), \tag{5.27}$$

$\forall\lambda > \frac{1}{n}\mathtt{h}_{\mathrm{top}}$, than (5.15) (5.16), on the recurrence set volumes of an Anosov flow. This in turn would improve the constant in the remainder estimate (1.8) of Corollary 3 by a factor of four

$$|R(h)| \leq \mathtt{h}_{\mathrm{top}}\left(\frac{2}{n^2}\frac{\det|\mathfrak{J}|}{(4\pi)^{n/2}}\right)|\ln h|^{-1} + o\left(|\ln h|^{-1}\right)$$

using the same argument as above.

6. Quantum ergodicity

In this section we shall prove quantum ergodicity for the magnetic Dirac operator D_h (1.1). The arguments of this section are valid under the weaker assumption of suitability (rather than strong suitability) for the metric g^{TX} from [24]. This means that the spectrum of the contracted endomorphism $\mathrm{Spec}(\mathfrak{J}_x) = \{0\}\cup\{\pm i\mu_j\nu(x)\}_{j=1}^m$ is equi-proportinal rather than constant, i.e. allowed to vary with a single smooth function $\nu \in C^\infty(X)$ on the manifold. As already noted in the introduction of [24] this weaker assumption is satisfied by all metrics in dimension 3.

We now state the result. For $a, b \in \mathbb{R}$ let

$$E_h(a,b) := \bigoplus_{\lambda\in\left[a\sqrt{h},b\sqrt{h}\right]} \ker\ [D_h - \lambda]$$

$$N_h(a,b) := \dim\ E_h(a,b) \tag{6.1}$$

denote the span of the eigenspaces for D_h with eigenvalues in, and the number of eigenvalues in, a given $\sqrt{h}$ size interval $\left[a\sqrt{h}, b\sqrt{h}\right]$. Further for each $h \in (0,1]$ we choose an orthonormal basis $\left\{\varphi_j^h\right\}_{j=1}^{N_h}$ for $E_h(a,b)$. A family of subsets $\Lambda_h \subset \{1,2,\ldots,N_h\}$, $h\in(0,1]$, is said to be of density one if it satisfies $\lim_{h\to 0}\frac{\#\Lambda_h}{N_h} = 1$.

The leading Weyl asymptotics for $N_h(a,b)$ is obtained from (3.3). The pointwise trace $u_{0,x}(s) := \mathrm{tr}\, U_{0,x}(s)$ of its leading term was computed in [24, Prop. 7.4] and is seen to be locally integrable in s. A Tauberian argument following [24, Prop. 7.1] then gives

$$N_h(a,b) \sim h^{-n/2} \int_X \mu_{a,b}$$
$$\text{for} \quad \mu_{a,b} := \left(\int_a^b ds\, u_{0,x}(s) \right) dx. \tag{6.2}$$

Our goal in this section will be to prove the following theorem.

Theorem 11. *Let a be a contact form and g^{TX} a suitable metric. Assume that the Reeb flow of a is ergodic.*

Then one has quantum ergodicity (QE): there exists a density one family of subsets $\Lambda_h \subset \{1,2,\ldots,N_h\}$, $h \in (0,1]$, such that

$$\langle B\varphi_j^h, \varphi_j^h \rangle \to \int_X (b_0|_\Sigma)\, \mu_{a,b},$$

$j \in \Lambda_h$, as $h \to 0$ for each $B \in \Psi^0_{\mathrm{cl}}(X)$, with homogeneous principal symbol $b_0 = \sigma(B) \in C^\infty(T^*X)$. In particular, the eigenfunctions get equidistributed $|\varphi_j^h|^2 dx \rightharpoonup \mu_{a,b}$, $j \in \Lambda_h$, as $h \to 0$ according to the measure (6.2).

Proof. Following a general outline for quantum ergodicity theorems (see for example [31, Sec. 15.4]) it suffices to prove a microlocal Weyl law

$$E(B) := \lim_{h \to 0} \frac{1}{N_h(a,b)} \sum_{j=1}^{N_h} \langle B\varphi_j^h, \varphi_j^h \rangle$$
$$= \int_X (b_0|_\Sigma)\, \mu_{a,b}, \tag{6.3}$$

and a variance estimate

$$V(B) := \lim_{h \to 0} \frac{1}{N_h(a,b)} \sum_{j=1}^{N_h} \left| \langle [B - E(B)]\, \varphi_j^h, \varphi_j^h \rangle \right|^2 = 0, \tag{6.4}$$

$\forall B \in \Psi^0_{\mathrm{cl}}(X)$, with $b = \sigma(B)$.

The microlocal Weyl law follows immediately by integrating the microlocal trace expansion Theorem 5 and the formula for its leading term (3.19) via a Tauberian argument.

For the variance estimate, firstly replacing B with $B - E(B)$, which has the same variance, one may assume $E(B) = 0$. An application of Cauchy-Schwartz (cf. [7, Lemma 4.1]) gives

$$V(B) \le E(B^* B)$$
$$V(B_2) = 0 \implies V(B_1) = V(B_1 + B_2). \tag{6.5}$$

The above along with (6.3) gives the variance estimate

$$b_0|_\Sigma = 0 \implies V(B) = 0 \tag{6.6}$$

for pseudodifferential operators with principal symbol vanishing on the characteristic variety.

Next, consider the lift of the Reeb vector field $\hat{R} \in C^\infty(T\Sigma)$ to the characteristic variety, satisfying $\pi_* \hat{R} = R$ under the natural projection $\pi : T^*X \to X$. We now prove the variance estimate for those pseudodifferential operators whose principal symbol satisfies

$$b_0|_\Sigma = \hat{R}(a|_\Sigma) \text{ for some } a \in S^0(T^*X) \implies V(B) = 0 \tag{6.7}$$

for some $a \in S^0(T^*X)$. To this end, we may use a partition of unity to suppose that the symbol a is supported in a small microlocal chart near the characteristic variety. Specifically we use the microlocal chart in which the normal form for the Dirac operator [24, Prop. 5.2] holds near Σ. In this chart one has $\Sigma = \{(x,\xi)\,|\xi_0 = \xi_1 = \ldots \xi_m = x_1 = \ldots x_m = 0\} \subset T^*\mathbb{R}^n_x$ in some phase space variables and one may Taylor expand $a = \underbrace{a_0(x_0, x_{m+1}\ldots x_{2m}, \xi_{m+1}, \ldots \xi_{2m})}_{=a|_\Sigma} + \underbrace{a_1}_{=O_\Sigma(1)}$ modulo a term $O_\Sigma(1)$ vanishing along Σ. Following [24, Prop. 5.2] or [24, eq. 5.8] we then compute the commutator

$$\begin{aligned}\left[a_0^W\sigma_0, D_h\right] &= c^W \\ \text{for} \quad c = &\underbrace{\partial_{x_0}a_0}_{=\hat{R}(a|_\Sigma)} + O_\Sigma(1) + O(h).\end{aligned}$$

Identifying the leading term above in terms of the lift of the Reeb vector field, and the variance of the above commutator being zero, we use (6.6) to obtain (6.6).

Finally, we choose a smooth family of symbols $b_t \in S^0(T^*X)$ such that $b_t|_\Sigma = \left(e^{t\hat{R}}\right)^*(b_0|_\Sigma)$ and set $B_t = b_t^W$. From (6.7) it then follows that $V\left(\frac{d}{dt}B_t\right) = 0$ and hence

$$\begin{aligned}V(B_0) = V(B_T) &= V\left(\underbrace{\frac{1}{T}\int_0^T B_t dt}_{=:\bar{B}_T}\right) \\ &\leq E\left(B_T^* B_T\right) = \int_X \left|\bar{b}_T|_\Sigma\right|^2 \mu_{a,b}\end{aligned}$$

for $\bar{b}_T := \frac{1}{T}\int_0^T b_t dt$, $\forall T$, from (6.5) and (6.3). Finally the ergodicity of the Reeb flow is equivalent to that of its lift $\hat{R}$. Hence by an application of the von Neumann mean ergodic theorem the last line above converges to zero as $T \to \infty$ completing the proof. ☐

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