Spectral Asymptotics for Coupled Dirac Operators

by

Nikhil Savale

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Author . . . .
Department of Mathematics
August 9, 2012

Certified by . . . .
Tomasz Mrowka
Singer Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Paul Seidel
Chairman, Department Committee on Graduate Students

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Abstract

In this thesis, we study the problem of asymptotic spectral flow for a family of coupled Dirac operators. We prove that the leading order term in the spectral flow on an n dimensional manifold is of order $r^{\frac{n+1}{2}}$ followed by a remainder of $O(r^{\frac{n}{2}})$. We perform computations of spectral flow on the sphere which show that $O(r^{\frac{n-1}{2}})$ is the best possible estimate on the remainder.

To obtain the sharp remainder we study a semiclassical Dirac operator and show that its odd functional trace exhibits cancellations in its first $\frac{n+3}{2}$ terms. A normal form result for this Dirac operator and a bound on its counting function are also obtained.

Thesis Supervisor: Tomasz Mrowka
Title: Singer Professor of Mathematics

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Chapter 1

Introduction

1.1 Motivation from the Weinstein conjecture

The motivation for the main problem of this thesis has come from the recent proof of the Weinstein conjecture in dimension three by Taubes [40]. A contact three manifold Y is one that is equipped with a one form a such that $a \wedge da$ is nowhere vanishing. The associated Reed vector field R to the one form a is defined by the equations $i_R da = 0$ and $a(R) = 1$. The Weinstein conjecture says that the Reeb vector field R always has a closed orbit. In [40] the following perturbed version of the three dimensional Seiberg Witten equations is considered

$$cl(*F_A) = r(\Phi\Phi^* - \frac{1}{2}|\Phi|^2 - a) \tag{1.1}$$

$$D_A\Phi = 0. \tag{1.2}$$

Here A denotes a connection on the determinant line bundle of a Spin^c structure with associated Dirac operator D_A , Φ is a spinor and $cl : T^*Y \to End(S)$ denotes Clifford multiplication map. A Reeb orbit arises from solutions to (1.1)-(1.2) with a uniform bound on their energy as $r \to \infty$. Solutions to (1.1)-(1.2) are given by a non-vanishing theorem in Monopole Floer homology of Kronheimer and Mrowka [26]. The bound on the energy follows if one considers solutions which represent a generator in Floer homology of a fixed grading. The relative grading in Floer homology is given by the spectral flow function for the Hessian of the Chern-Simons-Dirac functional. It is hence important to investigate the asmptotics of the spectral flow function for large r.

1.2 The problem of spectral flow

The Seiberg Witten equations are the variational equations associated with the Chern Simons Dirac functional. The Dirac operator appears as a component in the Hessian of the Chern Simons Dirac functional. One then has to give an estimate on the asymptotics of the spectral flow function $sf\{D_{A_0+sa}\}$, $0 \leq s \leq r$, as $r \to \infty$. Here A_0 is a fixed connection on the determinant line bundle and a is a purely imaginary one form. The spectral flow function is defined to be the number of eigenvalues of D_{A_0+sa} which go from being negative to positive as s goes from 0 to r. The following result appears as proposition 5.5 in [40]

Theorem 1.2.1. *The spectral flow function satisfies the asymptotics*

$$sf\{D_{A+sa}\} = -\frac{r^2}{32\pi^2}\int_Y a \wedge da + O(r^{\frac{15}{8}}(lnr)^{\frac{3}{2}}) \tag{1.3}$$

on a three manifold, as $r \to \infty$.

Another subsequent paper of Taubes [39] proves a similar result on higher dimensional manifolds with a leading term of order $r^{\frac{n+1}{2}}$ and a remainder of $O(r^p)$ with $p = \frac{n}{2} + \frac{n-1}{2(n+1)} + \epsilon$, $\forall \epsilon > 0$. The result above leads us to ask what the sharpest asymptotics are for the second order term in the spectral flow estimate (1.3). This is the main question of this thesis and we prove the following result in this regard.

Theorem 1.2.2. *On a manifold of odd dimension n the spectral flow function for the family of Dirac operators $D_{A_0+sa}, 0 \leq s \leq r$ coupled to the connections $A_0 + sa$ satisfies the asymptotics*

$$sf\{D_{A_0+sa}\} = r^{\frac{n+1}{2}}\left(\frac{i}{4\pi}\right)^{\frac{n+1}{2}}\frac{1}{(\frac{n+1}{2})!}\int_Y a \wedge (da)^{\frac{n-1}{2}} + O(r^{\frac{n}{2}}) \tag{1.4}$$

as $r \to \infty$.

To describe the main arguments in the proof of theorem (1.2.2), we first associate to the family of Dirac operators D_{A_0+sa} the operator on $Y \times [0,r]_s$ given by

$$D = \frac{\partial}{\partial s} + D_{A_0+sa}. \tag{1.5}$$

One has from [1] that the spectral flow for the family $D_{A_0+sa}, 0 \leq s \leq r$ is given by the index of the operator D subject to the Atiyah-Patodi-Singer boundary condition on the boundary $Y \times \{0, r\}$. The index of D is now given by Atiyah-Patodi-Singer index theorem as

$$\begin{aligned} sf\{D_{A_0+sa}\} &= ind(D) && (1.6)\\ &= r^{\frac{n+1}{2}} \left(\frac{i}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{\left(\frac{n+1}{2}\right)!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + \frac{1}{2}(\eta^r - \eta^0) + O(r^{\frac{n-1}{2}}). && (1.7) \end{aligned}$$

Here the integral term, and the $O(r^{\frac{n-1}{2}})$ term appear from the usual Atiyah-Singer integral. The terms η^0 and η^r denote the eta invariants of D_{A_0} and D_{A_0+ra} respectively, where the eta invariant η_A of an operator A is defined to be the value at zero of the meromorphic continuation of the function

$$\eta_A(z) = \dim \ker(A) + \sum_{\substack{\lambda \neq 0 \\ \lambda \in \mathrm{Spec}(A)}} \mathrm{sign}(\lambda)|\lambda|^{-z}. \tag{1.8}$$

The problem now reduces to finding the optimal asymtotics for the eta invariant η^r as $r \to \infty$. Letting $A = A_0 + ra$ denote an r dependent connection, we next express the eta invariant in terms of the traces

$$\eta^r = \left(\int_0^T \frac{1}{\sqrt{\pi t}} tr(D_A e^{-tD_A^2}) dt\right) + trf(\sqrt{T} D_A). \tag{1.9}$$

Here the second term denotes the functional trace corresponding to the function $f = \mathrm{sign}(x)\mathrm{erfc}(x)$ with $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy$ being the complementary error function.

The main work involved in proving theorem (1.2.2) is in deriving the heat trace estimates

$$|tr(e^{-tD_A^2})| \leq \frac{c_1}{t^{n/2}} e^{c_2 rt}, \quad |tr(D_A e^{-tD_A^2})| \leq c_3 r^{\frac{n+1}{2}} e^{c_4 rt}, \tag{1.10}$$

for uniform constants $c_i, 0 \leq i \leq 4$. These are proved using the maximum principle and small time expansions for the heat trace. Since $\text{erfc}(x) < e^{-x^2}, \forall x$, the desired estimate $\eta^r = O(r^{\frac{n}{2}})$ on the eta invariant follows using these trace estimates and substituting $T = \frac{1}{r}$ in (1.9), hence proving theorem (1.2.2).

The theorem (1.2.2) however does not say anything about the sharpness of the estimate (1.4), and we do not believe this to be the case. To study the question of sharpness we shall perform some computations for spectral flow. In particular we shall compute the spectral flow function for the odd dimensional sphere S^{2m+1} with its unique spin structure. The result we have is the following.

Theorem 1.2.3. *Let S be the unique spin bundle on S^{2m+1}. Consider the trivial Hermitian line bundle $\mathbb{C}$ with connection $d - ira$ where a is the standard contact form. The eigenvalues with multiplicities for the coupled Dirac operator D_{ra} acting on sections of $S \otimes \mathbb{C}$ are given by*

i. $\lambda = r - (a + m + \frac{1}{2})$, *for $a \in \mathbb{N}_0$ with multiplicity $\binom{m+a}{m}$*
ii. $\lambda = (-1)^m (r + a + m + \frac{1}{2})$, *for $a \in \mathbb{N}_0$ with multiplicity $\binom{m+a}{m}$*
iii.

$$\lambda = \frac{(-1)^{j+1}}{2} \pm \sqrt{(a_1 - a_2 + 2j - m + r + 1)^2 + 4(j + a_1 + 1)(m - j + a_2)}, \tag{1.11}$$

for $a_1, a_2 \in \mathbb{N}_0$, $j = 0, \ldots, m-1$, each with multiplicity

$$\frac{(m + a_1)!(m + a_2)!(a_1 + a_2 + 1 + m)}{m! j! (m - j - 1)! a_1! a_2! (a_1 + j + 1)(a_2 + m - j)}. \tag{1.12}$$

Hence its spectral flow function is given by

$$sf(D, D_{ra}) = \sum_{a=0}^{\left[r-m-\frac{1}{2}\right]} \binom{m+a}{m}. \tag{1.13}$$

This computation shows that the optimal possible asymptotic formula for the spectral flow function is as given by the following conjecture.

Conjecture 1.2.1. *On a manifold of odd dimension n the spectral flow function for the family of Dirac operators $D_{A+sa}, 0 \leq s \leq r$ coupled to the connections $A + sa$ satisfies the asymptotics*

$$sf\{D_{A_0+sa}\} = r^{\frac{n+1}{2}} \left(\frac{i}{4\pi}\right)^{\frac{n+1}{2}} \frac{1}{\left(\frac{n+1}{2}\right)!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + O(r^{\frac{n-1}{2}}) \tag{1.14}$$

as $r \to \infty$.

The above result has been proved by C. J. Tsai for certain specific three manifolds in [41].

The line of argument in the previous section can potentially be improved if one uses the substitution $T = \frac{1}{r^2}$ in (1.9) instead. With this substitution one is reduced to finding the sharpest asymptotics for the functional trace $trE(\frac{1}{r}D_{A_0} + cl(a))$, where the function $E = \text{sign}(x)\text{erfc}(x)$. If one thinks of $\frac{1}{r} = h$ as a semiclassical parameter, this problem appears to be one of semiclassical analysis as in [13], [21], [29] and [44]. These techniques provide a full trace expansion for $trf(D_h)$, where $D_h = hD_{A_0} + a$, in powers of h under the assumption that f is smooth. Although a general expansion begins with the power h^{-n}, in the case of the semiclassical Dirac operator we are able to show that this trace exhibits cancellations in its first $\frac{n+3}{2}$ terms when the function f is odd.

Theorem 1.2.4. *Let $f \in \mathcal{S}$ be an odd Schwartz function. There is a trace expansion*

$$trf(D_h) \sim h^{-\frac{n-3}{2}} c_{\frac{n+3}{2}} + h^{-\frac{n-5}{2}} c_{\frac{n+5}{2}} + \dots \tag{1.15}$$

for some constants c_i, $\frac{n+3}{2} \leq i$.

In the semiclassical terminology, conjecture (1.2.1) is reduced to the statement $trE(D_h) = O(h^{-\frac{n-1}{2}})$. Theorem (1.2.4) still does not prove this since the function E has a discontinuity at the origin and is not Schwartz.

Spectral asymptotics for counting functions of eigenvalues have been well studied in the literature. Namely, given a positive elliptic operator P of order m on a manifold X, consider $N(R)$ to be the number of eigenvalues of P less than R. The famous Weyl asymptotic formula gives the following asymptotics for the counting function $N(R)$

$$N(R) = R^{\frac{n}{m}} \mathrm{vol}(\{(x,\xi) \in T^*X | p(x,\xi) \leq 1\}) + O(R^{\frac{n-1}{m}}), \tag{1.16}$$

as $R \to \infty$. Here $p(x,\xi)$ denotes the symbol of the operator P. Weaker estimates on the remainder had earlier been obtained using heat trace methods in [4] and [28]. The optimal estimate of $O(R^{\frac{n-1}{m}})$ for the remainder was first proved by Hormander in [23] using wave trace methods and Fourier integral operators. The counting function $N(R)$ can be expressed as the spectral flow function of the family $P - s, 0 \leq s \leq R$. The problem of considering general asymptotics for the spectral flow function of a family appears to be new.

In the semiclassical context one is interested in the asymptotics for the counting function $N_h(a,b)$. This equals the number of eigenvalues of a semiclassical operator P_h in the interval $[a,b]$. Sharp asymptotics for these counting functions are known for scalar operators or non-scalar operators with a smoothly diagonalizable symbol [13], [25]. These formulas also require that a and b not be critical values of the symbol of P_h. In the case of the Dirac operator D_h^2 we are able to estimate such a counting function near the critical value 0 of its symbol.

Theorem 1.2.5. *For $c > 0$ be any positive real, the counting function*

$$N_h(-ch^{\frac{1}{2}}, ch^{\frac{1}{2}}) = O(h^{-\frac{n}{2}}) \tag{1.17}$$

near $h = 0$.

1.3 Outline

In Chapters 2 and 3 we provide some technical background required to prove the results of the thesis. In Chapter 2 we derive the asymttotic expansion for the heat kernel and its trace. A proof of Weyl's law for Dirac operators is included. In Chapter 3 we prove the Atiyah-Patodi-Singer index theorem and certain results on the eta invariants of Dirac operators. Here we also define spectral flow and give its relation to the APS index.

In Chapter 4 we prove theorem (1.2.2) following some bounds on the heat trace.

In Chapter 5 we consider the semiclassical Dirac operator. Here we prove the results (1.2.4) and (1.2.5).

In Chapter 6 we perform computations for spectral flow. We shall prove (1.2.3) giving the spectrum of the Dirac operator on S^n and showing that the result (1.2.1) is the best possible.

Finally, in appendices A and B we develop the necessary techniques from semiclassical analysis required in Chapter 5.

Chapter 2

The Heat kernel expansion

2.1 Dirac operators

We begin with the notion of a generalized Dirac operator. Such an operator exists on any Clifford bundle. A Clifford bundle S is a complex vector bundle with a connection ∇, a hermitian inner product $\langle , \rangle$ and a 'Clifford multiplication' map which is a morphism of vector bundles $cl : T^*M \otimes S \to S$. This morphism has the property that $cl(v)^2 s = -\langle v, v\rangle s$ for every cotangent vector v and $s \in S$. In addition there are compatibility conditions between any two of these structures given by:

1. (∇ and $\langle , \rangle$) $d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$ for any pair of sections s_1 and s_2.
2. (∇ and cl) $\nabla_X(\omega.s) = (\nabla_X^{L.C.}\omega).s + \omega.(\nabla_X s)$, for any vector field X. Here $\nabla^{L.C}$ denotes the Levi Civita connection on T^*M and Clifford multiplication is denoted by the shorthand $\omega.s = cl(\omega \otimes s)$.
3. (cl and $\langle , \rangle$) $\langle \omega.s_1, s_2\rangle + \langle s_1, \omega.s_2\rangle$ for any $\omega \in T^*M$ and $s_1, s_2 \in S$.

Given such a Clifford bundle, we can define a corresponding first order operator $D : C^\infty(S) \to C^\infty(S)$. This is defined via $D = cl \circ \nabla$ and called the Dirac operator.

2.2 Asymptotic expansion of the kernel

Next we shall be concerned with finding the asymptotics of the trace of the evolution operator e^{-tD^2}. It is well known that this operator is smoothing. This means that it has a smooth kernel $k_t(x,y) \in C^\infty(M \times M; \pi_1^* S \otimes \pi_2^* S^*)$ for all time $t > 0$ and that the trace is $Tr(e^{-tD^2}) = \int_M tr(k_t(x,x)) dvol$. This integral expression says that to find the asymptotics of the trace it will suffice to find the asymptotics of the kernel. In fact it is possible to get a complete asymptotic expansion for the kernel near $t = 0$. Before we give the expansion of the kernel we define what is meant by a full asymptotic expansion below.

Definition 2.2.1. *Let $f : \mathbb{R}_+ \to B$ be a function on the positive real line with values in a Banach space B. We say that f has the asymptotic expansion*

$$f(t) \sim \sum_{i=0}^{\infty} f_i(t) \tag{2.1}$$

near $t = 0$ if $f_i : \mathbb{R}_+ \to B$ is a set of functions such that the remainders of $R_N(t) = f(t) - \sum_{i=0}^{i=N} f_i(t)$ are eventually of an arbitrarily small order. That is, for every r there is an N_r such that $N \geq N_r \implies ||R_N(t)|| = o(t^r)$.

Knowing the heat kernel on Euclidean space to be $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$ we guess that the kernel $k_t(x,y)$ ought to be related to

$$h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\rho(x,y)^2}{4t}} \tag{2.2}$$

where $\rho(x,y)$ denotes the geodesic distance between points x and y. In practice we would like h to be smooth. Hence, we will let $\rho(x,y)$ be the geodesic distance when x is within the injectivity radius of y and continue ρ smoothly outside as long as it is bounded below $\rho(x,y) > \alpha > 0$ in this region. The asymptotic expansion that we look for will be of the type $k_t(x,y) \sim h_t(x,y)(s_0(x,y) + ts_1(x,y) + t^2 s_2(x,y) + \cdots)$, where s_i are smooth sections of $\pi_1^* S \otimes \pi_2^* S^*$. Before we prove this expansion and find the coefficients s_i we will need a lemma to help us with our computations.

Lemma 2.2.2. 1. *Let D be a Dirac operator on a Clifford bundle S. Then for* *every section s of S and every smooth function f on M*

$$D^2(fs) = fD^2 s - 2\nabla_{\nabla f} s + (\Delta f)s \tag{2.3}$$

2. *Let $h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\rho^2}{4t}}$ where $\rho = \rho(x,y)$ denotes the function defined earlier. Let i_g be the injectivity radius of (M,g) and let $g = det(g_{ij})$ be the determinant of the metric. Then the identity*

$$\partial_t h + \Delta_x h = \frac{\rho h}{4gt}\frac{\partial g}{\partial \rho} \tag{2.4}$$

holds in a neighbourhood U_{i_g} of distance i_g of the diagonal in $M \times M$.

Proof. 1. We compute in geodesic coordinates centered at a point. We use the compatibility rules for a Clifford bundle to get

$$D^2 fs = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} e_j e_i s + \sum_{i,j} e_j e_i \left(\frac{\partial f}{\partial x_j} \nabla_{e_i} s + \frac{\partial f}{\partial x_i} \nabla_{e_j} s \right) + f \sum_{i,j} e_j e_i \nabla_{e_j} \nabla_{e_i} s. \tag{2.5}$$

The first term only contributes when $i = j$ and gives $(\Delta f)s$, the second also has cancellations for $i \neq j$ to give $-2\nabla_{\nabla f} s$ and the last equals $fD^2 s$.

2. We fix the point y and compute in geodesic coordinates centered at y. The Laplacian in coordinates is given by $\Delta h = -\frac{1}{\sqrt{g}}\partial_i(\sqrt{g} g^{ij} \partial_j h) = -\frac{1}{2g} g^{ij}(\partial_i g)(\partial_j h) - (\partial_i g^{ij})(\partial_j h) - g^{ij}(\partial_i \partial_j h)$. Now we use $\rho^2 = \sum_i x_i^2$ in geodesic coordinates to get $\partial_i h = -\frac{x_i h}{2t}$. This gives

$$\Delta h = \frac{1}{4gt} g^{ij} x_j h(\partial_i g) + \frac{1}{2t} x_j h(\partial_i g^{ij}) + \frac{1}{2t} g^{ii} h - \frac{1}{4t^2} g^{ij} x_i x_j h \tag{2.6}$$

$$= \frac{1}{4gt} g^{ij} x_j h(\partial_i g) + \frac{1}{2t} h \partial_i (g^{ij} x_j) - \frac{1}{4t^2} g^{ij} x_i x_j h \tag{2.7}$$

where we have combined the middle two terms. Now it is a consequence of the Gauss's lemma that $g^{ij} x_j = x_i$ and $g^{ij} x_i x_j = \rho^2$ in geodesic coordinates. Using this we get $\Delta h = \frac{\rho h}{4gt}\frac{\partial g}{\partial \rho} + \frac{nh}{2t} - \frac{\rho^2 h}{4t^2}$. Here $\frac{\partial}{\partial \rho} = \frac{1}{\rho}x^i\partial_i$ is the radial vector field in geodesic coordinates. The time derivative is easily calculated to be $\partial_t h = -\frac{nh}{2t} + \frac{\rho^2 h}{4t^2}$. Adding the two gives us the result. □

Having this lemma in hand we are now ready to derive the full asymptotic expansion of the kernel.

Theorem 2.2.3. *There is an asymptotic expansion for the kernel $k_t(x, y)$ of the type*

$$k_t(x,y) \quad \sim \quad h_t(x,y)\Bigg(s_0(x,y) + ts_1(x,y) + t^2 s_2(x,y) + \cdots\Bigg) \tag{2.8}$$

which is valid in the Banach space $C^k(M \times M)$ for every k. Here s_i are smooth sections of $\pi_1^* S \otimes \pi_2^* S^*$.

Proof. We first show that it is possible to find s_i such that for each partial sum $k_t^N = h(\sum_{i=0}^N t^i s_i)$ we have $(\partial_t + D^2)k_t^N = e_t^N$, where e_t^N is a smooth section whose C^k norm satisfies the bound

$$\|e_t^N\|_{C^k} \leq C_N t^{N-k-\frac{n}{2}} \tag{2.9}$$

for $t < 1$. To this end we apply the heat operator to the expansion term by term while trying to get rid of lower order terms. Using the lemma we get $(\partial_t + D^2)ht^i s_i = ht^{i-1}(\rho\nabla_{\partial_\rho} s_i + is_i + \frac{\rho}{4g}\frac{\partial g}{\partial \rho} s_i) + ht^i(D^2 s_i)$. Now comparing coefficients of t^{i-1} gives us the equations

$$(\rho\nabla_{\partial_\rho} + i + \frac{\rho}{4g}\frac{\partial g}{\partial \rho})s_i = \begin{cases} 0 & \text{if } i = 0 \\ -D^2 s_{i-1} & \text{if } i \geq 1 \end{cases} \tag{2.10}$$

These are a set of linear first order equations which can be solved with the help the integrating factor $\rho^{i-1}g^{1/4}$ to give

$$\nabla_{\partial_\rho}(\rho^i g^{1/4} s_i) = \begin{cases} 0 & \text{if } i = 0 \\ -\rho^{i-1}g^{1/4}(D^2 s_{i-1}) & \text{if } i \geq 1. \end{cases} \tag{2.11}$$

We can first solve for s_0 uniquely given $s_0(y,y)$. We set $s_0(y,y) = 1$. The reason for this choice is because we will need the expansion to tend to $\delta(x-y)$ as $t \to 0$ for it to approximate the kernel. For $i \geq 1$ the equation (2.11) gives s_i in terms of s_{i-1} up to a constant multiple of term which is of order r^{-i} near $r = 0$. Smoothness near 0 requires this constant of integration to vanish and hence we have solved for all s_i's. Notice that since the formula (2.4) is only valid in U_{i_g}, the s_i's are only determined in this neighborhood. However since the heat kernel is concentrated near the diagonal for small time we may set the s_i's arbitrarily outside this neighbourhood. To prove the bound (2.9) on e_t^N, we first prove it inside U_{i_g}. Here $e_t^N = ht^N(D^2 s_N)$ and it's C^k norm will involve terms of order atmost $t^{N-k-\frac{n}{2}}$ near $t = 0$. Outside U_{i_g} we have that $\rho(x,y) > \alpha > 0$ and the fact that $e^{-\alpha/t}$ is of order t^∞ near $t = 0$ gives us the estimate in this region.

Elementary estimates show that $k_t^N \to \delta(x-y)$ as $t \to 0$. Now if r_t^N is the unique solution to the equation

$$(\partial_t + D^2) r_t^N = -e_t^N \tag{2.12}$$

with the initial condition $r_0 = 0$, this initial condition clearly implies that

$$k_t^N + r_t^N \to \delta(x-y) \text{ as } t \to 0. \tag{2.13}$$

Also $k_t^N + r_t^N$ satisfies

$$(\partial_t + D^2)(k_t^N + r_t^N) = 0. \tag{2.14}$$

The heat kernel is the unique time-dependent section which satisfies (2.13) and (2.14). Hence we have that $k_t = k_t^N + r_t^N$ and thus r_t^N is the remainder to the expansion whose order we have to determine. To do this, apply Duhamel's formula to (2.12) to write $r_t^N = \int_0^t e^{-(t-t')D^2} e_t^N dt'$. The fact that e^{-tD^2} is bounded on every Sobolev space gives

$$\|r_t^N\|_k \leq t \sup_{0 \leq t' \leq t} \|e_{t'}^N\|_k \leq K_0 t \sup_{0 \leq t' \leq t} \|e_{t'}^N\|_{C^k} \leq K_1 t^{N-k-\frac{n}{2}+1}. \tag{2.15}$$

Here the second inequality is from the fact that, on a compact manifold, the kth Sobolev norm is bounded by a multiple of the C^k norm and the third inequality is the bound given by (2.9). Finally Sobolev's inequality gives $\|r_t^N\|_{C^l} \leq K_2\|r_t^N\|_k \leq K_3 t^{N-k-\frac{n}{2}+1}$ for $l + \frac{n}{2} < k$. Thus we have that the order of the remainder, in any Banach space C^l, becomes arbitrarily small as $N \to \infty$. □

2.3 Weyl asymptotics

As a consequence of the asmptotic expansion for the heat kernel we now derive the well known Weyl asmptotic formula. The operator D^2 is an elliptic, positive, formally self adjoint operator of second order. Standard elliptic theory for self adjoint elliptic operators tells us that such operators have a discrete spectrum of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$$

tending to infinity. The object of interest here is the counting function (with multiplicity) for the number of eigenvalues of D^2 less than a certain magnitude $N(R) = \max\{i|\lambda_i \leq R\}$. The following theorem gives the asymptotics for the function $N(R)$ for large R.

Theorem 2.3.1. *(Weyl's law) The counting function for the eigenvalues of D^2 satisfies*

$$N(R) = \frac{vol(M)}{(4\pi)^{n/2}\Gamma(n/2+1)} R^{n/2} + o(R^{n/2}) \tag{2.16}$$

near $R = \infty$.

What allows us then to go from the asymptotics of the heat kernel to the asymptotics of the counting function $N(R)$ is the so called 'Tauberian theorem' from real analysis. Since this is motivating for the rest of the proof, we give this part of the argument here.

Theorem 2.3.2. *(Karamata) Let μ be a positive measure on $\mathbb{R}_+$ such that*

$$\lim_{t\to 0} t^\alpha \int_0^\infty e^{-t\lambda} d\mu(\lambda) = C. \tag{2.17}$$

Then

$$\lim_{x\to\infty} x^{-\alpha} \int_0^x d\mu(\lambda) = \frac{C}{\Gamma(\alpha+1)}. \tag{2.18}$$

Proof. First we show that for any continuous function f on the interval $[0,1]$

$$\lim_{t\to 0} t^\alpha \int_0^\infty f(e^{-t\lambda}) e^{-t\lambda} d\mu(\lambda) = \frac{C}{\Gamma(\alpha)} \int_0^\infty f(e^{-t}) t^{\alpha-1} e^{-t} dt. \tag{2.19}$$

To see this we approximate f by a Weirstrass polynomial p such that $|f(x)-p(x)| < \epsilon$ for $x \in [0,1]$. The positivity of the measure is used here to see that the difference of the corresponding intergrals is small. Following this, it suffices to prove the claim for polynomials and hence for monomials $f(x) = x^k$. The claim is true for monomials since

$$\lim_{t\to 0} t^\alpha \int_0^\infty e^{-(k+1)t\lambda} d\mu(\lambda) = C(k+1)^{-\alpha} = \frac{C}{\Gamma(\alpha)} \int_0^\infty e^{-kt} t^{\alpha-1} e^{-t} dt, \tag{2.20}$$

where the first equality follows by (2.17) and the second defines the Gamma function.

Now we apply the lemma of the previous paragraph to the function g which equals 0 on $[0, 1/e)$ and x^{-1} on $[1/e, 1]$. This gives

$$\lim_{t\to 0} t^\alpha \int_0^{t^{-1}} d\mu(\alpha) = \frac{C}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} dt = \frac{C}{\Gamma(\alpha+1)} \tag{2.21}$$

as required. Although g isn't continous we can still apply the lemma to it since g can be approximated by continuous functions. □

The general idea behind Tauberian theorems is to relate the behaviour near infinity of the function to behaviour near zero of its (Laplace) transform. Deriving the behaviour of the transform from that of the function is usually easy and known as an 'Abelian Theorem'. Conversely, deriving the behavior of the function from that of its transform is more subtle and requires and additional Tauberian condition on the function. In this context the fact that the counting function $N(R)$ is non-decreasing (or its derivative $N'(R)$, the spectral measure, is positive) is the Tauberian condition.

Now we are ready to finish the proof of Weyl's law. Firstly, given a bounded linear operator between two Banach spaces $A : B \to B'$ we can compose an asymptotic expansion with values in B with A to get an asymptotic expansion with values in B'. The trace is a bounded linear operator from $C^k(M \times M)$ to $\mathbb{R}$. Thus applying the trace gives us the asymptotic expansion of the trace from that of the kernel and we get

$$Tr(e^{-tD^2}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots) \tag{2.22}$$

where

$$a_i = \int_M tr(s_i(y, y))dvol \tag{2.23}$$

Since we had $s_0(y, y) = 1$, we have $a_0 = rank(S)vol(M)$. Finally applying the Tauberian theorem to the spectral measure $\sum_i \delta_{\lambda_i}$ gives us

$$N(R) = \frac{rank(S)vol(M)}{(4\pi)^{n/2}\Gamma(n/2 + 1)}R^{n/2} + o(R^{n/2}). \tag{2.24}$$

Chapter 3

Spectral flow and the APS index

In this section we will recall the Atiyah-Patodi-Singer index theorem and its relation with spectral flow. The results of this chapter will be important to prove the estimate on spectral flow in chapter 4. We shall begin with the statement of the index theorem. Let X be a compact manifold with boundary $\partial X = Y$. Let E and F be vector bundles over X and let $D : C^\infty(X, E) \to C^\infty(X, F)$ be a first order elliptic operator. Now assume that there exists a collar neighbourhood of the boundary $Y \times I \overset{i}{\hookrightarrow} X$ and a vector bundle E_0 on Y such that there are identifications $i_E : E \overset{\sim}{\to} \pi^* E_0$ and $i_F : F \overset{\sim}{\to} \pi^* E_0$. Further assume that there exists a self-adjoint (with respect to density dy), elliptic operator $A : C^\infty(Y, E_0) \to C^\infty(Y, E_0)$ such that in a neighbourhood of the boundary D corresponds via the identifications to

$$D = i_F^{-1} \circ \left(\frac{\partial}{\partial u} + A \right) \circ i_E. \tag{3.1}$$

Here u denotes the coordinate on the interval and the operator $\frac{\partial}{\partial u} + A$ needs to be defined in these special coordinates on the product. We also assume that the Hermitian inner products on E_0, E and F agree under the identifications and the density dx agrees with the density $dydu$ on the collar. Consider now the operator $P \circ r : H^s(X, E) \to H^{s-\frac{1}{2}}(Y, E_0)$, $(s > \frac{1}{2})$ which is the composition of the restriction map $r : H^s(X, E) \to H^{s-\frac{1}{2}}(Y, E_0)$ with the projection P onto the nonnegative part of the spectrum of A. Let $H^s(X, E, P)$ denote the kernel of this composition, which is itself a Hilbert space. The aim is to prove the following theorem

Theorem 3.0.3. *(Atiyah-Patodi-Singer) Consider the operator* $D : H^s(X, E, P) \to H^{s-1}(X, F)$ *for* $s \geq 1$*.*

a. *D is Fredholm.*

b. *The index of D is given by*

$$ind(D) = \int_X ch(\sigma(D))Td(X) - \left(\frac{h + \eta_A(0)}{2}\right) \tag{3.2}$$

where

(a) *$\int_X ch(\sigma(D))Td(X)$ is the usual Atiyah-Singer integral.*

(b) *$h = dim\ ker(A)$.*

(c) *the eta invariant is formally defined via*

$$\eta_A(s) = \sum_{\lambda \neq 0} sign\lambda |\lambda|^{-s}, s \in \mathbb{C} \tag{3.3}$$

where the sum runs over the eigenvalues of A. This formal series converges for $Re(s)$ large and has an analytic continuation to the whole s-plane with a finite value at 0 which appears in (3.2)*.*

The proof of the above theorem requires some preparation. First we do some computations on an analogous situation on the cylinder in the next section.

3.1 Computations on the cylinder

Let Y be a compact manifold with a vector bundle $E_0 \to Y$. Let $A : C^\infty(Y, E) \to C^\infty(Y, E)$ be a first order, self-adjoint, elliptic differential operator. Consider the product $Y \times \mathbb{R}_+$ of Y with the nonnegative real line and let $E = \pi^* E_0$ be the pullback of E under the projection onto Y. Consider the differential operator

$$D = \frac{\partial}{\partial u} + A : C^\infty(Y \times \mathbb{R}_+; E) \to C^\infty(Y \times \mathbb{R}_+; E) \tag{3.4}$$

The Sobolev space $\bar{H}^s(Y \times \mathbb{R}_+, E)$ denotes the space of restrictions to $Y \times \mathbb{R}_{>0}$ of elements in $H^s(Y \times \mathbb{R}, E)$. Let $r : C^\infty(X, E) \to C^\infty(Y, E_0)$ denote the restriction map and let $P : C^\infty(Y, E_0) \to C^\infty(Y, E_0)$ be the projection onto the nonnegative eigenspace of A. Define

$$C^\infty(X, E, P) = \{u \in C^\infty(X, E) | P \circ r(u) = 0\}. \tag{3.5}$$

One can analogously define $\bar{H}^s(X, E, P)$ for $s > \frac{1}{2}$. We prove the following lemma

Lemma 3.1.1. *There exists a linear operator $Q : C_c^\infty(X, E) \to C^\infty(X, E, P)$ such that*

i. *$DQg = g, \forall g \in C_c^\infty(X, E)$.*

ii. *$QDf = f, \forall f \in C_c^\infty(X, E, P)$.*

iii. *The kernel of Q, $K_Q(y, u; z, v)$ is C^∞ for $u \neq v$.*

iv. *Q extends to a map $Q : \bar{H}^{s-1}(X, E) \to \bar{H}^s_{loc}(X, E, P)$ for $s \geq 1$.*

Proof. First we prove iv. We need the following descriptions for the relevant Sobolev spaces involved.

$$\begin{aligned} \bar{H}^s(X, E) &= \{f = \sum f_\lambda(u)\phi_\lambda | f_\lambda \in \bar{H}^s(\mathbb{R}_+), \sum_{i+j \leq s} \langle\lambda\rangle^{2i} \|f_\lambda\|_j^2 < \infty\} &(3.6)\\ \bar{H}^s_{loc}(X, E) &= \{f = \sum f_\lambda(u)\phi_\lambda | \forall \phi \in C_c^\infty(X), \phi f \in \bar{H}^s\} &(3.7)\end{aligned}$$

Now to define Qg we must solve $Df = g$ with boundary conditions. If $f = \sum f_\lambda \phi_\lambda$ this amounts to solving $(\partial_u + \lambda) f_\lambda = g_\lambda$ with $f_\lambda(0) = 0$ for $\lambda \geq 0$. We do this in three cases.

(a) First let $\lambda > 0$. Let g^e_λ be an extension of g_λ such that

$$\int \frac{\hat{g^e_\lambda}}{\lambda + i\xi} = 0 \tag{3.8}$$

and $g^e_\lambda \in \bar{H}^s$. To show that such an extension exists first start with an arbitrary extension g^1_λ with $\int \frac{\hat{g^1_\lambda}}{\lambda + i\xi} = c$. Now consider $g^e_\lambda = g^1_\lambda + c\alpha$ where $\alpha \in C_c^\infty(\mathbb{R}_{<0})$

such that $\int e^{\lambda u}\alpha = -1$. Clearly g^e_λ is still an extension. Now $\alpha = (\partial_u + \lambda)\beta$ where $\beta = e^{-\lambda u}\int_{-\infty}^u e^{\lambda u'}\alpha(u')du'$. Hence $\hat{\alpha} = (i\xi + \lambda)\hat{\beta}$ and we have $c\int \frac{\hat{\alpha}}{i\xi+\lambda} = c\int \hat{\beta} = c\beta(0) = -c$ which implies (3.8). Now define

$$f_\lambda = r\mathcal{F}^{-1}\left(\frac{\hat{g^e_\lambda}}{\lambda + i\xi}\right) \quad \text{for} \quad \lambda > 0. \tag{3.9}$$

This restriction can be shown to be independent of the extension g^e_λ. This is because for $\hat{\phi} \in C_c^\infty(\mathbb{R}_{>0})$ we have $f_\lambda(\hat{\phi}) = \mathcal{F}^{-1}\left(\frac{\hat{g^e_\lambda}}{\lambda+i\psi}\right)(\hat{\phi}) = \frac{\hat{g^e_\lambda}}{\lambda+i\psi}(\phi) = \hat{g^e_\lambda}\left(\frac{\phi}{\lambda+i\psi}\right) = g^e_\lambda(\hat{\psi})$ where $(\lambda - \partial_x)\hat{\psi} = \hat{\phi}$. Hence $\hat{\psi} = -e^{\lambda u}\int_0^u e^{-\lambda u'}\hat{\phi}du' + (\int e^{-\lambda u'}\hat{\phi}du')e^{\lambda u}$ and we may further compute $g^e_\lambda(\hat{\psi}) = g(\chi_+\hat{\psi}) + g^e_\lambda(\chi_-\hat{\psi}) = g(\chi_+\hat{\psi}) + g^e_\lambda(e^{\lambda u}) = g(\chi_+\hat{\psi}) + \hat{g^e_\lambda}(\frac{1}{\lambda+i\psi}) = g(\chi_+\hat{\psi})$ which proves the independence of the extension. Now

$$\begin{aligned} \|f_\lambda\|_{s+1} &\leq \|\mathcal{F}^{-1}\left(\frac{\hat{g^e_\lambda}}{\lambda + i\xi}\right)\|_{s+1} && (3.10)\\ &= \left(\int \frac{\langle\xi\rangle^{2s+2}}{|\lambda + i\xi|^2}|\hat{g^e_\lambda}|^2 d\xi\right) && (3.11)\\ &\leq C\left(\int \langle\xi\rangle^{2s}|g^e_\lambda|^2\right) && (3.12)\\ &= C\|g^e_\lambda\|_s. && (3.13)\end{aligned}$$

We may also estimate

$$\begin{aligned} \|g^e_\lambda\|_s &\leq \|g^1_\lambda\|_s + c\|\alpha\|_s && (3.14)\\ &\leq C\|g^1_\lambda\|_s && (3.15)\end{aligned}$$

which finally gives

$$\|f_\lambda\|_{s+1} \leq C\|g_\lambda\|_s \tag{3.16}$$

for some uniform constant C in λ. A similar argument gives

$$\|\lambda f_\lambda\|_{s+1} \leq C\|g_\lambda\|_s \tag{3.17}$$

and $f_\lambda(0) = 0$ clearly follows from (3.8).

(b) Now consider $\lambda < 0$. Here let g_λ^e be an arbitrary extension of g_λ and define

$$f_\lambda = r\mathcal{F}^{-1}\left(\frac{\hat{g_\lambda^e}}{\lambda + i\xi}\right). \tag{3.18}$$

Similar arguments to case (a) show that this restriction is independent of g_λ^e and that we have the estimates

$$\begin{aligned} \|f_\lambda\|_{s+1} &\leq C\|g_\lambda\|_s & (3.19) \\ \|\lambda f_\lambda\|_{s+1} &\leq C\|g_\lambda\|_s & (3.20) \end{aligned}$$

(c) Finally consider $\lambda = 0$. Let $g_0 \in \bar{H}^s$ for $s \geq 0$. Let g_0^e be a extension of g_0 and define

$$f_0 = r\mathcal{F}^{-1}\left(\frac{\hat{g_0^e}}{i\xi}\right). \tag{3.21}$$

Here $\frac{\hat{g_0^e}}{i\xi}$ is the distribution defined via

$$\int \frac{\hat{g_0^e}}{i\xi}\phi d\xi = \int \hat{g_0^e}\psi d\xi \tag{3.22}$$

where ψ is the unique test function satisfying $\phi = \phi(0) + i\xi\psi$. Here the convergence of (3.22) follows as $\psi \sim \frac{\phi(0)}{i\xi}$ as $\xi \to \infty$. Again the restriction is independent of g_0^e as if $\hat{\phi} \in C_c^\infty(\mathbb{R}_{>0})$ then we may compute

$$\mathcal{F}^{-1}\left(\frac{\hat{g}_0^e}{i\xi}\right) = \frac{\hat{g}_0^e}{i\xi}(\phi) \tag{3.23}$$
$$= \int \hat{g}_0^e\psi \tag{3.24}$$
$$= \int g_0^e\hat{\psi} \tag{3.25}$$

where $\hat{\psi} = \int_{-\infty}^{x}\hat{\phi} - (\int\hat{\phi})H$. Now we see $\sup(\hat{\psi}) \subset \mathbb{R}_{\geq 0}$ and hence (3.25) is indpendent of the extension g_0^e. Now clearly $\partial_u f_0 = g_0$ and hence $f_0 \in \bar{H}^{s+1}_{\text{loc}}$ by local elliptic regularity. Also since $s \geq 0$ we have $f_0 \in C^0$ and $\sup(f_0) \subset \mathbb{R}_{\geq 0}$ can be shown by an argument similar to that showing the indepence of extension. Hence we have that $f_0(0) = 0$.

Now to show $\sum f_\lambda\phi_\lambda \in \bar{H}^{s+1}_{\text{loc}}$ it suffices to show $\sum_{\lambda\neq 0}\sum_{i+j\leq s+1}\langle\lambda\rangle^{2i}\|f_\lambda\|_j^2 < \infty$. But this follows from (3.16), (3.17), (3.19) and (3.20).

Parts (i) and (ii) of the proposition are easily checked. For part (iii) let $|A| = AP - A(1-P)$ and define

$$K_t = \chi_{\geq 0}e^{-t|A|}P - \chi_{\leq 0}e^{t|A|}(1-P). \tag{3.26}$$

It is clear that K_t is smoothing for $t \neq 0$ with smooth kernel $K(y,z,t)$. The kernel of Q is seen to be $Q(y,u;z,v) = K(y,z,u-v)$ and is hence smooth for $u \neq v$. □

Our next task is to construct kernels for the operators e^{-tD^*D} and e^{-tDD^*} on the cylinder $Y \times \mathbb{R}_+$. Let ϕ_λ be the eigenfunctions for A on Y with eigenvalue λ. Let $\pi : (Y\times\mathbb{R}_+)\times(Y\times\mathbb{R}_+) \to Y\times Y$ denote the projection given by $\pi(y,u;z,v) = (y,z)$. Define $s_\lambda = \pi^*(\pi_1^*\phi_\lambda \otimes \pi_2^*\phi_\lambda)$ which is a smooth section of $E\boxtimes E^* = \pi^*(\pi_1^*E_0 \otimes \pi_2^*E_0)$. Now we define functions on $\mathbb{R}_+\times\mathbb{R}_+$ for each λ via

$$f_\lambda = \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left(\exp\left(\frac{-(u-v)^2}{4t}\right) - \exp\left(\frac{-(u+v)^2}{4t}\right)\right), \tag{3.27}$$

$$g_\lambda = \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left(\exp\left(\frac{-(u-v)^2}{4t}\right) + \exp\left(\frac{-(u+v)^2}{4t}\right)\right) + \lambda e^{-\lambda(u+v)}\mathrm{erfc}\left(\frac{u+v}{2\sqrt{t}} - \lambda\sqrt{t}\right), \tag{3.28}$$

where erfc is the complementary error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-r^2}dr$. Now we define the kernels K_1, K_2 of e^{-tD^*D}, e^{-tDD^*} respectively via

$$K_1 = \sum_{\lambda \geq 0} f_\lambda s_\lambda + \sum_{\lambda<0} g_\lambda s_\lambda \tag{3.29}$$

$$K_2 = \sum_{\lambda \geq 0} g_{-\lambda} s_\lambda + \sum_{\lambda<0} f_\lambda s_\lambda. \tag{3.30}$$

Here the series (3.29) and (3.30) converge in the C^∞ topology on $[\delta,\infty]_t \times (Y \times \mathbb{R}_+) \times (Y \times \mathbb{R}_+)$ for any $\delta > 0$. This can be seen from the fact the e^{-tA^2} is smoothing on Y and the inequalities $\mathrm{erfc}(x) < e^{-x^2} < 1$. The next proposition shows that our construction does infact give the kernels of the fundamental solutions of the relevant boundary value problems.

Proposition 3.1.2. *The kernels K_1, K_2 defined in* (3.29) *and* (3.30) *satisfy*

i. $\partial_t K_1 + (D^*D)_p K_1 = 0$ *and* $\partial_t K_2 + (DD^*)_p K_2 = 0$

ii. $P \circ rK_1(.,q) = 0$, $(1-P)\circ r(DK_1(.,q)) = 0$ *and* $(1-P)\circ rK_2(.,q) = 0$, $P \circ r(D^*K_2(.,q)) = 0$

iii. *If $K = K_1 - K_2$ and $K(t) = \int_0^\infty \int_Y trK(y,u;y,u)dydu$ then we have an asymptotic expansion*

$$K(t) \sim \sum_{k\geq -n} a_k t^{\frac{1}{2}k} as \ : \ t \to 0. \tag{3.31}$$

Moreover $a_0 = -\left(\frac{\eta_A(0)+h}{2}\right)$ where $h = dimkerA$ and $\eta_A(0)$ is a finite value at zero for the analytic continuation of the eta function (3.3).

Proof. (i) Since the convergence in (3.29) and (3.30) is uniform one may check this by differentiating each term. This now follows from the fact that f_λ and g_λ are in the kernel of $\partial_t - \partial_u^2 + \lambda^2$ for each λ.

(ii) This follows by checking $f_\lambda(0,v) = 0$ if $\lambda \geq 0$ and $(\partial_u g_\lambda + \lambda g_\lambda)(0,v) = 0$ if $\lambda < 0$ for all v.

(iii) Absolute convergence of the integral $K(t) = \int_0^\infty \int_Y \mathrm{tr}K(y,u;y,u)dydu$ again follows from the fact that e^{-tA^2} is smoothing on Y, $\mathrm{erfc}(x) < e^{-x^2}$ and that e^{-x^2} is absolutely integrable on $[0,\infty)$. We may then compute

$$\begin{aligned}
K(t) &= \int_0^\infty \int_Y \mathrm{tr}K(y,u;y,u)dydu && (3.32)\\
&= \int_0^\infty \sum_\lambda \mathrm{sign}(\lambda)\left(-\frac{e^{-\lambda^2 t}e^{-u^2 t}}{\sqrt{\pi t}} + |\lambda| e^{2|\lambda|u}\mathrm{erfc}\left(\frac{u}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right) du && (3.33)\\
&= \int_0^\infty \sum_\lambda \mathrm{sign}(\lambda)\frac{\partial}{\partial u}\left(\frac{1}{2}e^{2|\lambda|u}\mathrm{erfc}\left(\frac{u}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right) du && (3.34)\\
&= -\sum_\lambda \frac{\mathrm{sign}(\lambda)}{2}\mathrm{erfc}(|\lambda|\sqrt{t}) && (3.35)
\end{aligned}$$

where we have adopted the convention that $\mathrm{sign}(0) = 1$. Uniform convergence of (3.35) in the C^∞ topology on $[\delta,\infty)_t$ allows us to differentiate

$$K'(t) = \frac{1}{\sqrt{4\pi t}} \sum_\lambda \lambda e^{-\lambda^2 t}. \qquad (3.36)$$

Now $K(t)$ may be identified with the trace of the trace class operator $B = -\frac{|A|}{2A}\mathrm{erfc}(|A|\sqrt{t})$, given by $B\phi_\lambda = -\frac{\mathrm{sign}(\lambda)}{2}\mathrm{erfc}(|\lambda|\sqrt{t})\phi_\lambda$. We *assume* that such an operator has a trace expansion

$$K(t) = \mathrm{tr}B \sim \sum_{k \geq -n} a_k t^{\frac{1}{2}k} \text{ as } t \to 0. \qquad (3.37)$$

Now $K(t) + \frac{1}{2}h \to 0$ exponentially as $t \to 0$ and $|K(t)| \leq Ct^{-\frac{1}{2}n}$ as $t \to 0$. Hence we see that $\int_0^\infty (K(t) + \frac{1}{2}h)t^{s-1}dt$ converges for $\mathrm{Re}(s) > \frac{n}{2}$. Integrating by parts and using (3.36) gives

$$\int_0^\infty (K(t)+\frac{1}{2}h)t^{s-1}dt = \frac{1}{s}\int_0^\infty K'(t)t^s dt \tag{3.38}$$

$$= -\frac{\Gamma(s+\frac{1}{2})}{2s\sqrt{\pi}}\sum_{\lambda\neq 0}\frac{\text{sign}(\lambda)}{|\lambda|^{2s}} \tag{3.39}$$

where the convergence of the series (3.39) follows from the uniform convergence of (3.36) and absolute convergence of the respective integrals. Hence we see that the $\eta(2s)$ function is a well defined holomorphic function for $\text{Re}(s) > \frac{n}{2}$ given by

$$\eta(2s) = -\frac{2s\sqrt{\pi}}{\Gamma(s+\frac{1}{2})}\int_0^\infty (K(t)+\frac{1}{2}h)t^{s-1}dt. \tag{3.40}$$

Now we may use the trace expansion (3.37) to define an analytic continuation for this eta function via

$$\eta(2s) = -\frac{2s\sqrt{\pi}}{\Gamma(s+\frac{1}{2})}\left(\sum_{k=-n}^{N}\frac{\bar{a}_k}{\frac{1}{2}k+s}+\theta_N(s)\right) \tag{3.41}$$

where $\bar{a}_k = a_k, \forall k$ except $\bar{a}_0 = a_0 + \frac{1}{2}h$. Also the function $\theta_N(s)$ is defined via

$$\theta_N(s) = \int_0^\infty \left(K(t)+\frac{1}{2}h-\chi(\sum_{k=-n}^{N}\bar{a}_k t^{-\frac{k}{2}})\right)t^{s-1}dt \tag{3.42}$$

where χ is the characteristic function on the interval $[0,1]$. This integral (3.42) now converges and defines a holomorphic function for $\text{Re}(s) > -\frac{N+1}{2}$ by the asymptotic expansion (3.37). The fact that (3.41) analytically continues (3.40) follows simply from $\int_0^1 \bar{a}_k t^{\frac{k}{2}+s-1} = \frac{\bar{a}_k}{\frac{1}{2}k+s}$. Finally substituting $s=0$ in (3.41) gives

$$\eta(0) = -2\bar{a}_0 = -(2a_0+h). \tag{3.43}$$

□

3.2 The index formula

Before we proceed to prove the index theorem it will be useful to prove a uniqueness theorem for the heat kernels of the relevant boundary value problems. We assume below that we are in the situation described in the introduction, which means that X is now a compact manifold with boundary. The following proposition establishes some unique properties satisfied by the relevant heat kernels.

Proposition 3.2.1. *There exists a unique time dependent section* $K_t \in C^1(\mathbb{R}_{>0}; C^2(X \times X; \pi_1^* E \otimes \pi_2^* E^*))$ *which satisfies*

i. $\partial_t K_t + (D^*D)_p K_t = 0$

ii. $K_t s \underset{C^2}{\to} s$ *as* $t \to 0$ *for every* $s \in C^\infty(X, E)$

iii. $P \circ r(K_t(.,q)) = 0$ *and* $(1-P) \circ r(DK_t(.,q)) = 0$ *for every* $q \in X$.

Proof. Let K_t^1 and K_t^2 be two such time dependent sections. Their difference $H_t = K_t^1 - K_t^2$ then satisfies hypotheses i and iii as well as $H_t s \underset{C^2}{\to} 0$ as $t \to 0$. Hence $s_t = H_t s$ satisfies $\partial_t s_t + (D^*D)s_t = 0$, $s_t \underset{L^2}{\to} 0$ and $P \circ rs = 0$ and $(1-P) \circ rDs = 0$. We can then compute

$$\begin{aligned} \partial_t \|s_t\|^2 &= \int_X \langle -D^*Ds_t, s_t\rangle dx && (3.44)\\ &= -\int_X \langle Ds_t, Ds_t\rangle dx + \int_Y \langle rDs_t, rs_t dy\rangle && (3.45)\\ &= -\|Ds_t\|^2 \le 0. && (3.46) \end{aligned}$$

Here the boundary term $\int_Y \langle rDs_t, rs_t\rangle$ in (3.45) comes from Stokes theorem assuming D has the special form (3.1) in the collar neighbourhood of Y and the fact that the density dx on X agrees with the density $dudy$ on the collar. This boundary term vanishes as the restrictions rDs_t and rs_t are orthogonal under the assumptions $P \circ rs = 0$ and $(1-P) \circ rDs = 0$. Hence (3.46) gives $\|s_t\| \le \|s_\epsilon\|$ for $\epsilon < t$. Taking the limit $\epsilon \to 0$ gives $\|s_\epsilon\| \to 0$ and we see that $s_t = 0$ for $t > 0$. This means that $H_t s = 0$ for each s and we must have $H_t = 0$ and $K_t^1 = K_t^2$. □

We are now ready to prove the index theorem. We first prove part (a) of theorem

(3.0.3) which claims that D is Fredholm with the relevent boundary condition.

Proof of theorem (3.0.3) *(a).* The proof that D is Fredholm employs the construction of a parametrix for D as follows. Let $\rho(a,b)$ denote a smooth function of the real variable u which satisfies

$$\rho = 0 \quad \text{for} \quad u \leq a \quad \text{and} \quad \rho = 1 \quad \text{for} \quad u \geq b. \tag{3.47}$$

Now define the four functions

$$\phi_2 = \rho\left(\frac{1}{4}, \frac{1}{2}\right) \quad , \quad \psi_2 = \rho\left(\frac{1}{2}, \frac{3}{4}\right) \tag{3.48}$$

$$\phi_1 = 1 - \rho\left(\frac{3}{4}, 1\right) \quad , \quad \psi_1 = 1 - \psi_2. \tag{3.49}$$

Now we define a paramatrix R for D via

$$R = \phi_1 Q_1 \psi_1 + \phi_2 Q_2 \psi_2. \tag{3.50}$$

Here $Q_1 = i_E^{-1} \circ Q \circ i_F$ is the operator defined in lemma (3.1.1) after identifications. And Q_2 is a parametrix for D on the double $\tilde{X}$ of X. We claim that R is a parametrix for D in the sense that $RD - I$ and $DR - I$ are both compact operators. First consider $RD - I : \bar{H}^s(X,E,P) \to \bar{H}^s(X,E,P)$ for $s \geq 1$ and compute

$$RD - I = \phi_1 Q_1 \psi_1 D + \phi_2 Q_2 \psi_2 D - I \tag{3.51}$$

$$= (\phi_1 Q_1 D \psi_1 + \phi_1 Q_1 [\psi_1, D]) + (\phi_2 Q_2 D \psi_2 + \phi_2 Q_2 [\psi_2, D]) - I \tag{3.52}$$

$$= (\phi_1 \psi_1 + \phi_1 Q_1 [\psi_1, D]) + (\phi_2 \psi_2 + \phi_2 S \psi_2 + \phi_2 Q_2 [\psi_2, D]) - I \tag{3.53}$$

$$= \phi_1 Q_1 [\psi_1, D] + \phi_2 Q_2 [\psi_2, D] + \phi_2 S \psi_2. \tag{3.54}$$

Here (3.53) follows from the construction of Q_1 and the fact that Q_2 is a left parametrix in the interior for D with smooth error S. Line (3.54) follows from $\phi_i \psi_i = \psi_i$ and $\psi_1 + \psi_2 = 1$. Finally observe that the commutators $[\psi_1, D]$, $[\psi_2, D]$ are of zeroth order and $\phi_2 S\psi_2$ is smoothing. Hence $RD - I : \bar{H}^s(X,E,P) \to \bar{H}^{s+1}(X,E,P)$ increases regularity by 1 and hence by Rellich lemma is a compact operator on $\bar{H}^s(X,E,P)$. A similar argument shows that $DR - I$ is compact and proves that D is Fredholm. □

Before we prove the index formula we will need a proposition identifying the cokernel of D with the kernel of its adjoint D^*. This is a non trivial matter on a manifold with boundary and is proved next.

Proposition 3.2.2. *Let D^* be the formal L^2 adjoint of D. The orthogonal complement of the range of $D : H^1(X,E,P) \to L^2(X,E)$ is isomorphic to the kernel of $D^* : H^1(X,E,1-P) \to L^2(X,E)$*

Proof. Under the assumptions we must have that $D^* = -\frac{\partial}{\partial u} + A$ on the collar $Y \times I$. Let $v \in L^2(X,E)$ be such that

$$\langle Du, v\rangle \quad \forall u \in H^1(X,E,P). \tag{3.55}$$

First we show that the restriction of v to $\dot{X} = X\backslash(Y \times [0,\frac{1}{2}])$ is smooth. To this end consider $\phi_2 v$ and observe that we must have $\langle Du, \phi_2 v\rangle = 0 \quad \forall \mathrm{sup}(u) \subset \dot{X}$. Hence $\langle u, D^*(\phi_2 v)\rangle_{(1,-1)} \quad \forall \mathrm{sup}(u) \subset \dot{X}$ where $\langle\rangle_{(1,-1)}$ denotes the L^2 pairing between H^1 and H^{-1}. Hence $D^*(\phi_2 v)|_{\dot{X}} = 0$ and we must have that $v|_{\dot{X}}$ is smooth by elliptic regularity for D^* in the interior. Moreover $(D^* v)|_{\dot{X}} = 0$ as $\phi_2 = 1$ on $\dot{X}$. Now we show v is smooth on $Y \times [0,\frac{3}{4}]$ and $(1-P)\circ r(u) = 0$. By (3.55) we have that

$$\langle Du, \phi_1 v\rangle = 0 \quad \forall u \in H^1(X,E,P) \quad \text{with} \quad \mathrm{sup}(u) \subset [0,\frac{3}{4}]. \tag{3.56}$$

Let $\phi_1 v = \sum f_\lambda \phi_\lambda$. Consider $u = Q(\phi f_\lambda \phi_\lambda)$ where $\mathrm{sup}(\phi) \subset [0,\frac{3}{4}]$. Now in the case where $\lambda < 0$ we have $sup(u) \subset [0,\frac{3}{4}]$ and we may apply (3.56) to get

$$\begin{aligned} 0 = \langle Du, \phi_1 v\rangle &= \langle DQ(\phi f_\lambda \phi_\lambda), \phi_1 v\rangle && (3.57)\\ &= \phi\|f_\lambda \phi_\lambda\|^2. && (3.58)\end{aligned}$$

This implies that

$$f_\lambda|_{(0,\frac{3}{4})} = 0 \quad \text{for} \lambda < 0. \tag{3.59}$$

Now consider $v_+ = \sum_{\lambda \geq 0} f_\lambda \phi_\lambda$. By (3.55) and (3.59) we now have that

$$\langle Du, v_+ \rangle = 0 \quad \forall u \in H^1(X, E, P) \quad \text{with} \quad \sup(u) \subset [0, \frac{3}{4}]. \tag{3.60}$$

Hence $\langle u, D^* v_+ \rangle_{(1,-1)} = 0$ where there is no boundary term due to the fact that $P \circ r(u) = 0$. Hence $D^* v_+|_{[0,\frac{3}{4}]} = 0$ and combining this with (3.59) gives $D^* v|_{(0,\frac{3}{4})} = 0$. We may hence solve $f_\lambda(u) = e^{\lambda - \lambda c} f_\lambda(c)$ for $c \in (\frac{1}{2}, \frac{3}{4})$. This formula along with the fact that v is smooth on $Y \times (\frac{1}{2}, \frac{3}{4})$ gives that v is smooth on $Y \times [0, \frac{3}{4}]$. Hence v is smooth on Y and in the kernel of $D^* : H^1(X, E, 1-P) \to L^2(X, E)$. Conversely elements in this kernel are easily seen to be in the orthogonal complement of the range of D.

□

Having established the isomorphism of the cokernel of D and the kernel of D^* we can now finish the proof of the index formula.

Proof of theorem (3.0.3) *(b).* By proposition (3.2.2) we may write $\text{ind}(D) = \text{dimker}(D) - \text{dimker}(D^*)$. By an argument similar to part (a) we see that the operator $D^*D : H^2(X, E; 1-P, P) \to L^2(X, E)$ is Fredholm where

$$H^2(X, E; 1-P, P) = \{u \in H^2(X, E) | P \circ ru = 0, (1-P) \circ rDu = 0\} \tag{3.61}$$

Its generalized inverse $(D^*D)^{-1}$ is a self-adjoint compact operator on $L^2(X, E)$ and hence has a complete orthonormal basis of eigenvectors $\{s_\lambda\}$. These are also eigenvectors of D^*D with a discrete set of eigenvalues $\lambda \to \infty$. A similar parametrix for D^*D also shows that $s_\lambda \in C^\infty(X, E)$ for each λ. Now we define an operator e^{-tD^*D} on $L^2(X, E)$ via $e^{-tD^*D} s_\lambda = e^{-t\lambda} s_\lambda$. This operator maps $L^2(X, E)$ into

$$H^{2n}(X,E;1-P,P,n) = \{u \in H^{2n}(X,E) | P \circ (D^*D)^i u = 0,$$
$$(1-P) \circ rD(D^*D)^i u = 0 \quad \forall i < n\} \tag{3.62}$$

Hence the operator e^{-tD^*D} is a smoothing operator with kernel $K_t(p,q) \in C^\infty(X \times X, \pi_1^*E \otimes \pi_2^*E^*)$. It is hence trace class with trace $\mathrm{Tr}(e^{-tD^*D}) = \sum_\lambda e^{-t\lambda}$. Similar statements hold for e^{-tDD^*}. The nonzero eigenvalues of D^*D and DD^* coincide as $s_\lambda \mapsto Ds_\lambda$ defines an isomorphism between the λ-eigenspaces of D^*D and DD^* with inverse $t_\lambda \mapsto \frac{1}{\lambda}D^*t_\lambda$. Also the nullspace of D^*D coincides with the nullspace of D while the nullspace of DD^* coincides with that of D^*. Hence we have that

$$\mathrm{ind}(D) = \mathrm{Tr}e^{-tD^*D} - \mathrm{Tr}e^{-tDD^*}. \tag{3.63}$$

Now we define a time evolution operator to approximate e^{-tD^*D} via

$$e_t = \phi_1 e_1 \psi_1 + \phi_2 e_2 \psi_2 \tag{3.64}$$

where e_1 is the corresponding evolution operator on the cylinder whose kernel is (3.29) and e_2 is the evolution operator for $(\partial_t + D^*D)$ on the double of X. If E_t is the kernel of e_t then elementary estimates show that $R_t = (\partial_t + D^*D)E_t$ is exponentially small, as $t \to 0$, in C^k norm for any k. Now Duhamel's principle shows that

$$H_t^1 = E_t - \int_0^t e^{-(t-t')D^*D} R_{t'} dt' \tag{3.65}$$

satisfies $(\partial_t + (D^*D)_p)H_t = 0$. Also proposition (3.1.2) shows that $P \circ rE_t(.,q) = 0$ and $(1-P) \circ rDE_t(.,q) = 0$ for each q. Further $E_t s \underset{C^2}{\to} s$ as $t \to 0$ is clear from the definitions of e_1 and e_2. Hence by proposition (3.2.1) H_t^1 is the unique heat kernel for e^{-tD^*D}. A similar construction relates the heat kernel H_t^2 for e^{-tDD^*} to the approximate kernel F_t constructed from evolution operators f_1 and f_2 for $(\partial_t + DD^*)$ on the cylinder and the double respectively. Now since R_t is exponentially small as $t \to 0$ and e^{-tD^*D} is bounded on any Sobolev space, (3.65) implies that H_t^1 and E_t

have the same asymptotics as $t \to 0$. Hence

$$\begin{aligned} \mathrm{ind}(D) &= \mathrm{Tr} e^{-tD^*D} - \mathrm{Tr} e^{-tDD^*} && (3.66) \\ &= \lim_{t\to 0}(\mathrm{Tr} H_t^1 - \mathrm{Tr} H_t^2) && (3.67) \\ &= \lim_{t\to 0}(\mathrm{Tr} E_t - \mathrm{Tr} F_t) && (3.68) \\ &= \lim_{t\to 0}\left(\int_0^1 \int_Y \psi_1(y) tr K_t(y,u;y,u)dydu + \int_X \psi_2(x) tr \tilde{K}_t(x)dx\right) && (3.69) \end{aligned}$$

where K_t is defined as in proposition (3.1.2) and $\tilde{K}_t$ denotes the kernel of $e^{-tD^*D} - e^{-tDD^*}$ on the double of X. The last equality follows from the definitions of E_t and F_t and the fact that $\phi_i \psi_i = \psi_i$. Now proposition (3.1.2) gives the asymptotic expansion

$$\left(\int_0^1 \int_Y \psi_1(y) tr K_t(y,u;y,u)dydu\right) \sim \sum_{k\geq -n} a_k t^{\frac{1}{2}k} \tag{3.70}$$

while we have an asymptotic expansion

$$\int_{\tilde{X}} tr\tilde{K}_t(x)dx \sim \sum_{k\geq -n}\left(\int_{\tilde{X}} \alpha_k(x)dx\right) t^{\frac{1}{2}k} \tag{3.71}$$

for the trace $\mathrm{Tr}(e^{-tD^*D} - e^{-tDD^*})$ on the double $\tilde{X}$ of X. Here $\alpha_k(x)$ are local functions of the operators D^*D and DD^*. Under the assumptions these two operators are isomorphic on the collar $Y \times I$ and since $\psi_2 = 1$ outside the collar we may replace the expansion (3.71) with

$$\int_X \psi_2(x) tr\tilde{K}_t(x)dx \sim \sum_{k\geq -n}\left(\int_X \alpha_k(x)dx\right) t^{\frac{1}{2}k}. \tag{3.72}$$

Now substituting (3.70) and (3.72) into (3.69) gives

$$\mathrm{ind}(D) = \lim_{t\to 0}\left(\sum_{k\geq -n} a_k t^{\frac{1}{2}k} + \sum_{k\geq -n}\left(\int_X \alpha_k(x)dx\right) t^{\frac{1}{2}k}\right). \tag{3.73}$$

Since the limit exists we must have $a_k = -\int_X \alpha_k(x)dx$ for $k < 0$ and

$$\begin{aligned}
\operatorname{ind}(D) &= \int_X \alpha_0(x)dx + a_0 && (3.74)\\
&= \int_X ch(\sigma(D))Td(X) - \left(\frac{h+\eta_A(0)}{2}\right) && (3.75)
\end{aligned}$$

where the last line follows from proposition (3.1.2) and the local index theorem on the double of X. □

3.3 Eta invariants of Dirac operators

In this section we consider the eta invariants of Dirac operators. The main result which appears below says that the eta function of a generalized Dirac operator is holomorphic in the part of the complex plane where $\mathrm{Re}(s) > -\frac{1}{2}$. Following equations (3.31), (3.36) and (3.41) this is equivalent to the fact that the trace $\mathrm{Tr}(De^{-tD^2}) \in t^{\frac{1}{2}}C^\infty([0,\infty))$ and exhibits cancellations. This fact will be used in deriving estimates on the heat trace and spectral flow. The result appearing below was originally proved in [1], [2] and using a different technique in [9]. The proof we give below follows proposition 8.35 in [32].

Theorem 3.3.1. *Let $D : C^\infty(Y,S) \to C^\infty(Y,S)$ be a generalized Dirac operator on an n dimensional manifold Y. Then $\eta_D(s)$ is a holomorphic function for $Re(s) > -\frac{1}{2}$.*

Proof. Define $\tilde{Y}^0 = Y \times S^1$ with the product metric $\tilde{g}^0 = d\theta^2 + g$. Let (S,ρ,∇,h) be the Clifford bundle on Y corresponding to D, and define a Clifford bundle $(\tilde{S}^0,\tilde{\rho}^0,\tilde{\nabla}^0,\tilde{h}^0)$ on $\tilde{Y}^0$ via $\tilde{S}^0 = \pi_1^*S \oplus \pi_1^*S$. Clifford multiplication $\tilde{\rho}^0$ on $\tilde{S}^0$ is defined by

$$\tilde{\rho}^0(d\theta) = \begin{bmatrix} & -I \\ I & \end{bmatrix}, \quad \tilde{\rho}^0(\alpha_y) = \begin{bmatrix} & \rho(\alpha_y) \\ \rho(\alpha_y) & \end{bmatrix}. \tag{3.76}$$

and the connection $\tilde{\nabla}^0$ and metric $\tilde{h}^0$ on $\tilde{S}^0$ are simply pulled back from S. The Dirac operator $\tilde{D}^0$ corresponding to this Clifford bundle on $\tilde{Y}^0$ can be computed to be

$$\tilde{D}^0 = \begin{bmatrix} & -I \\ I & \end{bmatrix} \frac{\partial}{\partial\theta} + \begin{bmatrix} & I \\ I & \end{bmatrix} D. \tag{3.77}$$

Now we wish to consider the manifold $\tilde{Y} = Y \times S^1$ with the warped metric

$$\tilde{g} = d\theta^2 + e^{2\phi(\theta)} g, \tag{3.78}$$

where $\phi \in C^\infty(S^1)$ is a smooth real valued function on S^1. One can first calculate how the Levi-Civita connection on $Y \times S^1$ changes under the warping. If $\tilde{\nabla}^{0\,L.C.}$ denotes the Levi-Civita connection for the product metric and $\tilde{\nabla}^{L.C.}$ denotes the Levi-Civita connection for the warped metric (3.78) then we have that the two differ by a one form $\tilde{\nabla}^{L.C.} - \tilde{\nabla}^{0\,L.C.} = \omega$ where

$$\omega(\partial_\theta) d\theta = 0, \qquad \omega(\partial_\theta)\alpha = -\frac{\partial\phi}{\partial\theta}\alpha, \tag{3.79}$$

$$\omega(X) d\theta = \frac{\partial\phi}{\partial\theta} e^{2\phi} i_X g, \qquad \omega(X)\alpha = -\frac{\partial\phi}{\partial\theta}\alpha(X) d\theta \tag{3.80}$$

for all α, X denote a one form and a vector field on Y respectively. Now we define a Clifford bundle for the warped metric with $\tilde{S} = \tilde{S}^0$. One must change Clifford multiplication appropriately so that unit elements square to -1. Hence we set

$$\tilde{\rho}(d\theta) = \begin{bmatrix} & -I \\ I & \end{bmatrix}, \quad \tilde{\rho}(\alpha_y) = e^{-\phi} \begin{bmatrix} & \rho(\alpha_y) \\ \rho(\alpha_y) & \end{bmatrix}. \tag{3.81}$$

The connection must also be changed appropriately to keep it compatible with Clifford multiplication. Hence we set $\tilde{\nabla} = \tilde{\nabla}^0 + \Omega$ where

$$\Omega(\partial_\theta)s = 0, \quad \Omega(X)s = -\frac{1}{2}\frac{\partial\phi}{\partial\theta} e^{\phi} \tilde{\rho}^0(i_X g)\tilde{\rho}^0(d\theta). \tag{3.82}$$

The metric $\tilde{h} = \tilde{h}^0$ on the Clifford bundle is left unchanged. It is now an excercise to show that $(\tilde{S}, \tilde{\rho}, \tilde{\nabla}, \tilde{h})$ is a Clifford bundle on $\tilde{Y}$ for the warped metric $\tilde{g}$. The corresponding Dirac operator $\tilde{D}$ can be computed to be

$$\tilde{D} = \begin{bmatrix} & -I \\ I & \end{bmatrix} \left(\frac{\partial}{\partial\theta} + F(\theta) \right) + e^{-\phi} \begin{bmatrix} & I \\ I & \end{bmatrix} D, \tag{3.83}$$

where $F(\theta)$ is the function of θ given by

$$F(\theta) = \frac{n}{2} \frac{\partial\phi}{\partial\theta} e^{\phi}. \tag{3.84}$$

The motivation for the above construction comes from the corresponding formulas in the case where Y is an odd spin manifold. The spin structure on Y and the trivial spin structure on S^1 combine to give a spin structure on $Y \times S^1$. Choosing the warped metric on the product one can compute and verify the spin Dirac operator on the product to be given by (3.83).

Now to prove that $\eta_D(s)$ is holomorphic for $\mathrm{Re}(s) > -\frac{1}{2}$ it is enough by (3.41) to prove that $a_k = 0$ for $k < 0$. Hence via (3.31) and (3.36) it is enough to prove that the asymptotic expansion for the trace satisfies $\mathrm{Tr}(De^{-tD^2}) \in t^{\frac{1}{2}} C^\infty([0, \infty))$. Now let k_t denote the kernel of De^{-tD^2} on Y and let $\tilde{k}_t^0$ denote the kernel of $De^{-t(\tilde{D}^0)^2}$ on $\tilde{Y}$. We will infact prove that $tr(k_t) \in t^{\frac{1}{2}} C^\infty([0, \infty) \times Y)$ where tr is the trace taken pointwise on the diagonal in Y. First let $i_{\theta_1,\theta_2} : Y \times Y \to \tilde{Y} \times \tilde{Y}$ denote the inclusion given by $i_{\theta_1,\theta_2}(y_1, y_2) = (y_1, \theta_1; y_2, \theta_2)$ for any pair $(\theta_1, \theta_2) \in S^1 \times S^1$. Equation (3.77) may be squared to give $(\tilde{D}^0)^2 = D_\theta^2 + D^2$ where $D_\theta = \frac{1}{i} \frac{\partial}{\partial\theta}$. Since D_θ and D commute it now follows that

$$i^*_{\theta_1,\theta_2} \tilde{k}_t^0 = \Theta(t; \theta_1 - \theta_2)\{k_t \oplus k_t\} \tag{3.85}$$

where $\Theta(t; \theta_1 - \theta_2)$ denotes the heat kernel for $e^{-tD_\theta^2}$ written in terms of the Jacobi theta function

$$\Theta(t; \theta) = \frac{1}{2\pi} \sum_n e^{-n^2 t} e^{in\theta}. \tag{3.86}$$

Since $d\theta^2$ is a flat metric on S^1 with no curvature by (3.85) one has the following asymptotic expansion in a neighbourhood of the diagonal

$$i^*_{\theta_1,\theta_2}\tilde{k}^0_t \sim \frac{1}{\sqrt{4\pi t}}e^{-\frac{|\theta_1-\theta_2|^2}{4t}}\{k_t \oplus k_t\}. \tag{3.87}$$

Hence we must have that

$$tr(k_t) \sim \sqrt{\pi t}\, tr(i^*_{\theta,\theta}\tilde{k}^0_t) \tag{3.88}$$

for any $\theta \in S^1$.

Consider now the square of the warped Dirac operator (3.83) which is

$$\tilde{D}^2 = -(\partial_\theta + F)^2 + e^{-2\phi}D^2 + \begin{bmatrix} I & \\ & -I \end{bmatrix} e^{-\phi}\frac{\partial\phi}{\partial\theta}D. \tag{3.89}$$

We now let the function ϕ depend on a smooth parameter s. The variation of (3.89) with respect to s can then be computed to be

$$\frac{\partial}{\partial s}\tilde{D}^2 = -\dot{F}(\partial_\theta + F) - (\partial_\theta + F)\dot{F} - 2\dot{\phi}e^{-2\phi}D^2 + e^{-\phi}(\dot{\phi}' - \dot{\phi}\phi')\begin{bmatrix} I & \\ & -I \end{bmatrix} D, \tag{3.90}$$

where we have used $\dot{\phi}$ and ϕ' to denote derivatives with respect to s and θ respectively. We further choose the family $\phi_s(\theta)$ such that $\phi_0(\theta) = 0$ for all θ. Using (3.84) this reduces (3.90) to

$$\frac{\partial}{\partial s}\tilde{D}^2(0) = -\frac{n}{2}(\dot{\phi}'\partial_\theta + \partial_\theta\dot{\phi}') - 2\dot{\phi}D^2 + \dot{\phi}'\begin{bmatrix} I & \\ & -I \end{bmatrix} D. \tag{3.91}$$

Now Duhamel's principle says that the derivative of the heat kernel may be written as

$$\frac{\partial}{\partial s}e^{-t\tilde{D}^2} = -\int_0^t e^{-(t-r)\tilde{D}^2}\,\frac{\partial\tilde{D}^2}{\partial s}\,e^{-r\tilde{D}^2}dr \tag{3.92}$$

which on setting $s = 0$ gives

$$\frac{\partial}{\partial s} e^{-t\tilde{D}^2}(0) = -\int_0^t e^{-(t-r)(\tilde{D}^0)^2} \frac{\partial \tilde{D}^2}{\partial s}(0)\, e^{-r(\tilde{D}^0)^2} dr. \tag{3.93}$$

Following (3.90) and (3.93) we may relate the trace $trDe^{-tD^2}$ with the variation of the supertrace $\frac{\partial}{\partial s} e^{-t\tilde{D}^2}(0)$. The theorem now follows from the local index theorem.

□

3.4 Relation with spectral flow

Here we breifly recall the notion of spectral flow and its relation with the index. We refer to [2] for the proofs of several assertions made here with [10] giving a more detailed account. Consider a continuous one parameter family of elliptic self-adjoint operators A_t, for $0 \leq t \leq 1$, of order m acting on sections of a vector bundle E on a manifold Y. Spectral flow counts the net number of eigenvalues which which change sign from negative to positive as t varies. To elaborate, first replace the family by $F_t = (1 + A_t^2)^{-\frac{1}{2}} A_t$ to obtain a continuous family of self-adjoint Fredholm operators on $L^2(Y, E)$. Let $\hat{\mathcal{F}}$ denote the space of Fredholm self-adjoint operators on a Hilbert space. It consists of three connected components $\hat{\mathcal{F}} = \hat{\mathcal{F}}_+ \cup \hat{\mathcal{F}}_* \cup \hat{\mathcal{F}}_-$, with $\hat{\mathcal{F}}_\pm$ consisting of the operators with essential spectrum contained in $\mathbb{R}_\pm$. Since the Dirac operator has spectrum going to $\pm\infty$ we have $\{1, -1\} \subset \sigma_{ess}(F_t)$ for all t. Hence it suffices to define spectral flow for a continuous path of operators in $\hat{\mathcal{F}}_*$. To this end we note that the space $\hat{\mathcal{F}}_*$ weakly retracts onto the smaller space

$$\hat{F}^\infty = \{B \in \hat{\mathcal{F}}_* |\ \|B\| = 1,\ \sigma_{ess}(B) \subset \{1, -1\},\ \sigma(B) \text{ is finite}\}. \tag{3.94}$$

Now, given a continuous path of operators B_t in $\hat{F}^\infty$ we have that the spectrum of the family is given by a finite sequence of continuous functions $\sigma(B_t) = \{\lambda_0(t), \lambda_1(t), \ldots, \lambda_m(t)\}$. Set the number of positive and negative crossings crossings to be

$$n_+ = \{i | \lambda_i(0) < 0 \leq \lambda_i(1)\}, \quad n_- = \{i | \lambda_i(1) < 0 \leq \lambda_i(0)\} \tag{3.95}$$

repectively. The spectral flow of the family is defined to be the integer $\mathrm{sf}\{A_t\} = n_+ - n_-$.

In the case that we have a family of first order differential operators A_t acting on the space of sections $C^\infty(Y;E)$ the spectral flow function can be related to the index of a Fredholm operator. First consider the product $X = Y \times [0,1]$ with the bundle E pulled back from Y (still denoted by E). Define a subspace of $H^1(X;E)$ via

$$H^1(X;E,\pi_{\geq}) = \{u \in H^1(X;E) |\ \pi_{\geq 0}u(.,0) = 0,\ \pi_{<0}u(.,1) = 0\} \tag{3.96}$$

where $\pi_{\geq 0}$ and $\pi_{<0}$ denote the projections onto the eigenspaces of A_0 and A_1 spanned by the nonnegative and negative eigenvalues repectively. Consider the operator $D = \frac{\partial}{\partial t} + A_t : H^1(X;E,\pi_{\geq}) \to L^2(X;E)$. If one perturbs the family to assume that it is constant near the ends of the cylinder then it is proved in [1] that this operator is Fredholm. Furthermore the index of this operator is the spectral flow of this family

$$\mathrm{sf}\{A_t\} = \mathrm{ind}(D). \tag{3.97}$$

The index of D is in turn given by the Atiyah-Patodi-Singer index theorem [1] as

$$\mathrm{ind}(D) = \int_X ch(\sigma(D))Td(X) + \frac{1}{2}(\bar{\eta}^{A_1} - \bar{\eta}^{A_0}). \tag{3.98}$$

Here the first term is the usual Atiyah-Singer integral. The term $\bar{\eta}^A = \bar{\eta}^A(0)$ is the reduced eta invariant which is the value at zero of the reduced eta function $\bar{\eta}^A(s)$ formally defined via

$$\bar{\eta}^A(s) = \dim \ker A + \sum_\lambda \mathrm{sign}\lambda|\lambda|^{-s}, s \in \mathbb{C}. \tag{3.99}$$

The sum above runs over the eigenvalues of A. This formal series converges and defines a holomorphic function for $\mathrm{Re}(s) > n$. It has a meromorphic continuation to the whole s-plane with a finite value at 0 which appears in (3.98).

In order to prove (3.97) one first proves it to be true in the case where A_t is a periodic family. In this case the family gives a continuous path $F_t : S^1 \to \hat{\mathcal{F}}_*$. In [3] it is shown that the the space $\hat{\mathcal{F}}_*$ is a classifying space for K^1. Hence the homotopy class $[F_t] \in K^1(S^1) = \mathbb{Z}$ gives an element in K theory which is the index of the family. We claim that this index equals the spectral flow of the family. This follows on showing that both the index and spectral flow are invariant under homotopy and checking them to be equal on the generator of $\pi_1(\hat{\mathcal{F}}_*)$ (the family with spectrum $n + t, n \in \mathbb{Z}$). Finally it remains to show that the index of the family $\mathrm{ind}(A_t)$ coincides with $\mathrm{ind}(D)$. Here $\mathrm{ind}(D)$ can be written using (3.98) where the two boundary contributions from A_0 and A_1 now cancel. The integral term now equals $\mathrm{ind}(A_t)$ by an application of the index theorem.

Finally having proved (3.97) for a periodic family it suffices to prove it for a single path of operators connecting A_0 and A_1.

Chapter 4

Spectral flow for the Dirac operator

We now come to one of our main results. This is an estimate on spectral flow for a family of coupled Dirac operators. To state the result, consider an oriented Riemannian spin manifold Y of odd dimension n. Let S be the spin bundle on Y corresponding to a given spin structure. Let L be another Hermitian line bundle on Y. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(Y; i\mathbb{R})$ be an imaginary one form on Y. Then we have a family $A_0 + sa$ of unitary connections on L. Each such connection gives rise to a coupled Dirac operator $D_{A_0+sa} : C^\infty(Y; S \otimes L) \to C^\infty(Y; S \otimes L)$. The Dirac operator being elliptic and self-adjoint has a discrete set of eigenvalues. The object of interest in these notes is the spectral flow function $\mathrm{sf}\{D_{A_0+sa}\}$, for $0 \le s \le r$, and its asymptotics for large r. In particular we shall prove the following

Theorem 4.0.1. *The spectral flow function for the family of Dirac operators D_{A_0+sa} satisfies*

$$sf\{D_{A_0+sa}\} = r^{\frac{n+1}{2}} \left(\frac{i}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{(\frac{n+1}{2})!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + O(r^{\frac{n}{2}}) \tag{4.1}$$

as $r \to \infty$.

4.1 Estimating spectral flow

Now we return to the problem of estimating the spectral flow of the family D_{A_0+sa}, for $0 \leq s \leq r$. As noted in the previous section the spectral flow function $\mathrm{sf}\{D_{A_0+ra}\}$ equals the index of the operator $D = \frac{\partial}{\partial t} + D_{A_0+ta} : H^1(X; E, \pi_{\geq}) \to L^2(X; E)$ where $X = Y \times [0, r]$. The index is again given by index formula (3.2). The integral term can now be simplified according to 4.3 in [1] to give

$$\begin{aligned} \mathrm{sf}\{D_{A_0+sa}\} &= \mathrm{ind}(D) &(4.2)\\ &= \int_X ch(L)\hat{A}(X) + \frac{1}{2}(\bar{\eta}^r - \bar{\eta}^0). &(4.3) \end{aligned}$$

Here the terms appearing in the integral are the Chern character form of L, computed using the connection $A_0 + sa$, and $\hat{A}$ genus of X. The terms $\bar{\eta}^0$ and $\bar{\eta}^r$ denote the reduced eta invariants of D_{A_0} and D_{A_0+ra} respectively. The leading order term in s in the integrand can now be computed from the definitions to be

$$ch(L)\hat{A}(X) = \left(\frac{i}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{\left(\frac{n+1}{2}\right)!} \left(\frac{n+1}{2}\right) s^{\frac{n-1}{2}} ds \wedge a \wedge (da)^{\frac{n-1}{2}} + O(s^{\frac{n-1}{2}}). \quad (4.4)$$

Which on integration simplifies (4.3) to

$$\mathrm{sf}\{D_{A_0+sa}\} = r^{\frac{n+1}{2}} \left(\frac{i}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{\left(\frac{n+1}{2}\right)!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + \frac{1}{2}(\bar{\eta}^r - \bar{\eta}^0) + O(r^{\frac{n-1}{2}}). \quad (4.5)$$

Hence to prove (4.1) it remains to prove that

$$\bar{\eta}^r = O(r^{\frac{n}{2}}) \quad (4.6)$$

as $r \to \infty$. In order to prove (4.6) we first note, following section 2 of [1], that the eta invariant appears as the zeroth order term in a trace expansion

$$\frac{1}{2}\bar{\eta}^r = -a_0^r \tag{4.7}$$

where

$$K_t^r := \mathrm{tr} B_t^r \sim \sum_{k \geq -n} a_k^r t^{\frac{1}{2}k}. \tag{4.8}$$

The operator B_t^r is given by functional calculus

$$B_t^r \phi_\lambda = -\frac{\mathrm{sign}\lambda}{2} \mathrm{erfc}(|\lambda|\sqrt{t})\phi_\lambda \tag{4.9}$$

acting on the eigenvectors ϕ_λ of D_{A_0+ra} with eigenvalue λ and erfc denotes the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} d\xi. \tag{4.10}$$

Next we use a theorem, see [9], asserting that on odd manifolds the reduced eta function for Dirac operators is holomorphic in the region of the complex plane given by $\mathrm{Re}(s) > -\frac{1}{2}$. The poles of the eta function are given by $-\frac{1}{2}k$ with corresponding residues being the coefficients a_k in the trace expansion (4.8). Hence this theorem is equivalent to the fact that the trace K_t is a smooth function of t near zero. The time derivative of this trace is the trace $K_t' = \frac{1}{\sqrt{4\pi t}} \mathrm{Tr}(D_A e^{-tD_A^2})$, where $A = A_0 + ra$. Hence this theorem also follows from theorem 8.35 in [32]. In fact [32] proves that the same is true for the pointwise trace $tr(D_A e^{-tD_A^2})$ along the diagonal.

Using the smoothness of K_t near zero we may now rewrite (4.7) and (4.8) as

$$\frac{1}{2}\bar{\eta}^r = -a_0^r = -K_0^r. \tag{4.11}$$

The fundamental theorem of calculus gives

$$-K_0^r = -K_t^r + \int_0^t {K'}_{t'}^r dt'. \tag{4.12}$$

Hence to bound $\bar{\eta}^r$ it suffices to bound the right hand side of (4.12) and the traces

K_t^r and K'^r_t uniformly in r. Here the trace K_t may be estimated by

$$|K_t| \leq \frac{1}{\sqrt{\pi}} \mathrm{Tr}(e^{-tD_A^2}), \tag{4.13}$$

which simply follows from the inequality $\mathrm{erfc}(x) < e^{-x^2}$. Hence it now suffices to get uniform bounds in r on the the heat trace $\mathrm{tr}(e^{-tD_A^2})$ and the trace $\mathrm{tr}(D_A e^{-tD_A^2})$ which we do next.

4.1.1 Bound on the heat trace

Let $H_t(x,y)$ denote the kernel of the evolution operator $e^{-tD_A^2}$. In this section we derive a bound on this heat kernel following [40]. First consider the function $h_t(x,y)$ defined by

$$h_t(x,y) = \frac{e^{-\frac{\rho(x,y)^2}{4t}}}{(4\pi t)^{n/2}} \tag{4.14}$$

where $\rho(x,y)$ is the distance function on Y when x is within the injectivity radius of y. Outside of this region $\rho(x,y)$ is set arbitrarily as long as is uniformly bounded from below there $\rho(x,y) > \epsilon > 0$. The heat kernel bound is now given by the following proposition

Proposition 4.1.1. *The heat kernel $H_t(x,y)$ satisfies*

$$|H_t(x,y)| \leq C_1 h_t(x,y) e^{C_2 rt}, \tag{4.15}$$

where C_1, C_2 are some positive constants independent of r.

Proof. First observe that for fixed y the section $s_t(.) = H_t(.,y)$ satisfies the heat equation $\partial_t s_t = -D_A^2 s_t$. The Weitzenbock formula

$$D_A^2 = \nabla_A^* \nabla_A + \frac{cl(F_A)}{2} + \frac{\kappa}{4} \tag{4.16}$$

now gives that the function $f_t = |s_t|$ obeys the inequality

$$\partial_t f_t \leq -d^* df_t + c_1(r+1) f_t \tag{4.17}$$

for some constant c_1 independent of r. Hence the function $f'_t = e^{-c_1(r+1)t} f_t$ satisfies the inequality

$$\partial_t f'_t \leq -d^* df'_t. \tag{4.18}$$

Now standard asymptotics for the heat kernel H_t as in chapter 7 of [37] give

$$f'_t \sim |H_t(x,y)| \sim h_t(x,y) \sim \Phi_t(x,y) \quad \text{as} \quad t \to 0 \tag{4.19}$$

where $\Phi_t(x,y)$ denotes the heat kernel e^{-td^*d} for the Laplace operator acting on functions on Y. Now using (4.18) and (4.19) an application of the maximum principle for the heat equation gives that

$$f'_t \leq c_2 \Phi_t(x,y) \leq c_3 h_t(x,y) \tag{4.20}$$

holds for $t \leq 1$ and some constants c_2, c_3. Hence

$$f_t \leq c_3 h_t(x,y) e^{-c_1(r+1)t} \tag{4.21}$$

for $t \leq 1$ and the proposition follows. □

We note that the above proposition immedietly gives the bound

$$\mathrm{Tr}(e^{-tD_A^2}) \leq \frac{c_1}{t^{n/2}} e^{c_2 rt} \tag{4.22}$$

on the heat trace for constants c_1, c_2 uniform in r.

In order to obtain a better estimate on spectral flow, we will need another bound comparing the heat kernel to Mehler's kernel. To recall the definition of Mehler's kernel first define the function

$$m_t(exp_v(y), y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} det^{\frac{1}{2}} \left(\frac{rtda}{\sinh rtda} \right) exp \left\{ -\frac{1}{4t} g(v, rtda \coth(rtda) v) \right\} \tag{4.23}$$

in a geodesic neighborhood of the diagonal in $Y \times Y$. Here $v \in T_yY$ and the two from da is considered as an element of $C^\infty(End(T_yY))$ using the metric. Next let π_1 and π_2 denote the projections onto the two factors of $Y \times Y$ and define the section $e^{-\frac{tF}{2}}$ of $\pi_1^* S \otimes \pi_2^* S^*$ in a geodesic neighborhood of the diagonal. This restricts to $e^{-\frac{tF}{2}}|_\Delta = e^{-\frac{tcl(F_A)}{2}}$ at the diagonal Δ and is parallel along geodesics $(exp_{tv}(y), y)$. If i_g denotes the injectivity radius of Y consider the cutoff

$$\chi(x) = \begin{cases} 1 & \text{if } x \leq \frac{i_g}{2}; \\ 0 & \text{if } x > i_g, \end{cases}$$

and define Mehler's kernel as

$$M_t(x,y) = \chi(\rho(x,y)) m_t(x,y) e^{-\frac{tF}{2}}. \tag{4.24}$$

Theorem 4.1.2. *There exist positive constants C_1 and C_2 independent of r such that*

$$|H_t(x,y) - M_t(x,y)| \leq C_1 h_{2t}(x,y) t^{\frac{1}{2}} e^{C_2 rt}. \tag{4.25}$$

Proof. First fix a set of geodesic coordinates centered at y. Now choose a basis s_α for S_y and a basis l for L_y. Parallel transport this basis along geodesics to obtain trivializations $s_\alpha(x)$ and $l(x)$ of S and L respectively near y. Now define local orthonormal sections of $(S \otimes L) \otimes (S \otimes L)^*_y$ via

$$t_{\alpha\beta} = s_\alpha(x) \otimes l(x) \otimes s^*_\beta \otimes l^*. \tag{4.26}$$

The connection ∇_A can be expressed in this frame and these coordinates via

$$\nabla_i^A = \partial_i + A_i + \Gamma_i \tag{4.27}$$

where each A_i is a Christoffel symbol of A (or $\dim(S \otimes L)_y$ copies of it) and each Γ_i is a Christoffel symbol of the Spin connection on S. Since the section $l(x)$ is obtained via parallel transport along geodesics the connection coefficient A_i maybe written in terms of the curvature F_{ij} of A via

$$A_i(x) = \int_0^1 d\rho(\rho x^j F^A_{ij}(\rho x)). \tag{4.28}$$

The dependence of the curvature coefficients F_{ij} on the parameter r is seen to be linear $F_{ij} = F^{A_0}_{ij} + r(da)^0_{ij}$ despite the fact that they are expressed in the r dependent frame l. This is because a change of frame into l is conjugation by a function which leaves the coefficient unchanged. Further using the Taylor expansion $(da)_{ij} = (da)_{ij}(0) + x^k a_{ij,k}$, we see that the connection ∇^A has the form

$$\nabla^A_i = \partial_i + \frac{1}{2} r x^j (da)_{ij}(0) + x^j A^0_{ij} + r x^j x^k A_{ij,k} + \Gamma_i, \tag{4.29}$$

where $A^0_{ij} = \int_0^1 d\rho(\rho F^{A_0}_{ij}(\rho x))$, $A_{ij,k} = \int_0^1 d\rho(\rho a_{ij,k}(\rho x))$ and Γ_i are all independent of r. Now it follows from the Weitzenbock formula that the operator D^2_A may be written as

$$D^2_A = A + E, \quad \text{with} \tag{4.30}$$

$$A = -\partial_i^2 - r(da)_{ij}(0) x^j \partial_i - \frac{r^2}{4} x^i x^j \left(\sum_k (da)_{ik}(0)(da)_{jk}(0) \right) + cl\left(\frac{F_A}{2}\right) \quad \text{and} \tag{4.31}$$

$$E = P_{ijkl} x^k x^l \partial_i \partial_j + Q_{ijk} r x^j x^k \partial_i + R_i \partial_i + S_{ijk} r^2 x^i x^j x^k + T_i r x^i + U, \tag{4.32}$$

and where P, Q, R, S, T and U are each smooth endomorphisms of $S \otimes L$ independent of r. Since $(\partial_t + D^2)H_t = 0$ we now have

$$(\partial_t + D^2_A)(H_t - M_t) = -(\partial_t + A)M_t - EM_t. \tag{4.33}$$

By Mehler's formula, see section 4.2 in [8], we have $(\partial_t + A)M_t = 0$ for $\rho(x, y) < \frac{i_g}{2}$. Hence using (4.24) and (4.32) we may write the right hand side of (4.33) as a sum

$$-(\partial_t + A)M_t - EM_t = \sum_{(k,d,I)} t^k r^d x^I f_{k,d,I}(x) f(rt) M_t, \tag{4.34}$$

where each $(k, d, I) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0^n$ and satisfies the inequality

$$d \leq k + \frac{|I|}{2} + \frac{1}{2}. \tag{4.35}$$

Also the function f appearing in (4.34) satisfies an exponential bound $|f(x)| < c_1 e^{c_2 x}$. Now since the kernels H_t and M_t both have the same asymptotics as $t \to 0$ Duhamel's principle gives via (4.33) that

$$H_t - M_t = \int_0^t e^{-(t-s)D_A^2} \{-(\partial_s + A)M_s - EM_s\} \, ds. \tag{4.36}$$

Now we substitute (4.34) into (4.36). Following this we use the bound (4.15) for the heat kernel, the bound

$$|M_t(x, y)| \leq c_3 e^{c_4 r t} h_t(x, y) \tag{4.37}$$

for uniform constants c_3 and c_4 and the bound $|f(x)| \leq c_1 e^{c_2 x}$. These bounds along with the inequalities

$$x^I h_t(x, 0) \leq C t^{\frac{1}{2}|I|} h_{2t}(x, 0), \tag{4.38}$$

$$\int_0^t ds \left(\int_Y dy h_{t-s}(x, y) h_{2s}(y, 0) \right) \leq C t h_{2t}(x, 0) \tag{4.39}$$

and (4.35) can be used to estimate the right hand side of (4.36) to give (4.25).

□

4.1.2 Bound on the trace of $D_A e^{-tD_A^2}$

We now turn to bound the pointwise $tr(D_A e^{-tD_A^2})$. To this end first consider the expansion for the heat kernel $H_t(x, y)$ given by

$$H_t(x, y) \sim h_t(x, y) \left(b_0(x, y) + b_1(x, y) t + b_2(x, y) t^2 + \ldots \right). \tag{4.40}$$

Such a kernel expansion is not unique and only the restriction to the diagonal of the coefficients b_k are defined uniquely. In (4.40) we refer to the coefficients b_k generated by solving a recursive system of transport equations along geodesics as in chapter 7 of [37]. The kernel $L_t(x,y)$ of $D_A e^{-tD_A^2}$ is simply $L_t = D_A H_t$ and hence has an expansion given by

$$\begin{aligned} L_t(x,y) \sim & h_t(x,y) cl\left(-\frac{\rho d\rho}{2t}\right)\left(b_0(x,y) + b_1(x,y)t + b_2(x,y)t^2 + \ldots\right) \\ & + h_t(x,y)\left(D_A b_0(x,y) + D_A b_1(x,y)t + D_A b_2(x,y)t^2 + \ldots\right). \end{aligned} \tag{4.41}$$

As noted earlier the pointwise trace $tr(D_A e^{-tD_A^2})$ along the diagonal has an expansion starting with a leading term of order $t^{\frac{1}{2}}$. Since the restriction to the diagonal of the term $cl\left(-\frac{\rho d\rho}{2t}\right)$ in (4.41) is zero this implies that $D_A b_k(x,x) = 0$ for $k < \frac{n+1}{2}$ at each point on the diagonal. To bound the trace of L_t we will need the following lemma giving a schematic expression for the coefficients $b_k(x,y)$.

Lemma 4.1.3. *For each $k \geq 0$ and each $y \in Y$ consider $i_y^* b_k \in C^\infty(Y; (S \otimes L) \otimes (S \otimes L)_y^*)$, the pullback of the heat kernel coefficient under the inclusion $i_y(x) = (x,y)$. There exists a local orthonormal basis of sections $t_{\alpha\beta} \in C^\infty(Y; (S \otimes L) \otimes (S \otimes L)_y^*)$ in which the heat kernel coefficient maybe be written as $i_y^* b_k = \sum f_{\alpha\beta} t_{\alpha\beta}$. Moreover, in geodesic coordinates centered at y, the functions $f_{\alpha\beta}$ have the form*

$$f_{\alpha\beta} = \sum_{(d,I)} r^d x^I f_{d,I} \tag{4.42}$$

where each $(d,I) \in \mathbb{N}_0 \times \mathbb{N}_0^n$ and satisfies the inequality $d \leq k + \frac{1}{2}|I|$. Moreover, the functions $f_{d,I}$ appearing in (4.42) are indpendent of r.

Proof. We again work in the geodesic coordinate system centered at y and the trivializations of S and L used in the proof of (4.1.2). The heat kernel coefficients $b_k(x,y)$ are given, see chapter 7 of [37], by the recursion

$$
\begin{aligned}
b_0(x,y) &= \sum_\alpha g^{-1/4}(x) t_{\alpha\alpha}, && (4.43)\\
b_k(x,y) &= -\frac{1}{g^{1/4}(x)} \int_0^1 \rho^{k-1} g^{1/4}(\rho x) D_A^2 b_{k-1}(\rho x) d\rho, \quad \text{for} \quad k \geq 1, && (4.44)
\end{aligned}
$$

where g denotes the determinant of the metric on Y. Hence b_0 is clearly seen to be of the form (4.42). Equations (4.30) and (4.44) imply that b_k has the form (4.42) assuming it to be true for b_{k-1} and hence the lemma follows by induction on k. □

Following this we are ready to bound the pointwise trace $tr(D_A e^{-tD_A^2})$. The above lemma will play an important role in the proposition below.

Proposition 4.1.4. *The pointwise trace $tr(D_A e^{-tD_A^2})$ satisfies the estimate*

$$
\|tr(D_A e^{-tD_A^2})\|_{C^0} \leq C_1 r^{\frac{n}{2}} e^{C_2 rt} \tag{4.45}
$$

for constants C_1 and C_2 independent of r.

Proof. Consider the remainder in the kernel expansion (4.41) given by

$$
L_t^{\frac{n-1}{2}} = L_t - D_A(h_t(b_0(x,y) + \ldots + t^{\frac{n-1}{2}} b_{\frac{n-1}{2}})). \tag{4.46}
$$

This is seen to equal $L_t^{\frac{n-1}{2}} = D_A H_t^{\frac{n-1}{2}}$ with $H_t^{\frac{n-1}{2}}$ being the analogous remainder in the kernel expansion for the heat trace

$$
H_t^{\frac{n-1}{2}} = H_t - h_t(b_0(x,y) + \ldots + t^{\frac{n-1}{2}} b_{\frac{n-1}{2}}). \tag{4.47}
$$

Hence applying the heat operator we see that

$$
\begin{aligned}
(\partial_t + D_A^2)(L_t^{\frac{n-1}{2}}) &= (\partial_t + D_A^2)(D_A H_t^{\frac{n-1}{2}}) && (4.48)\\
&= h t^{\frac{n-1}{2}} \left\{ -D_A^3 b_{\frac{n-1}{2}} + cl\left(\frac{\rho d\rho}{2t}\right) D_A^2 b_{\frac{n-1}{2}} \right\}. && (4.49)
\end{aligned}
$$

Now since $L_t^{\frac{n-1}{2}} \to 0$ as $t \to 0$ we have by Duhamel's principle that

$$
\begin{aligned}
L_t^{\frac{n-1}{2}}(y,y) \;&= \int_0^t ds \Big(\int_Y dx\, H_{t-s}(y,x) h_s(x,y) s^{\frac{n-1}{2}} \Big\{ -D_A^3 b_{\frac{n-1}{2}}(x,y) \\
&\qquad + cl\left(\tfrac{\rho d\rho}{2s}\right) D_A^2 b_{\frac{n-1}{2}}(x,y) \Big\} \Big).
\end{aligned} \tag{4.50}
$$

We denote by $U_t^{\frac{n-1}{2}}$ and $V_t^{\frac{n-1}{2}}$ the kernels obtained by replacing H_{t-s} in (4.50) by $(H_{t-s} - M_{t-s})$ and M_{t-s} respectively. It is clear that $L_t^{\frac{n-1}{2}} = U_t^{\frac{n-1}{2}} + V_t^{\frac{n-1}{2}}$. To bound $U_t^{\frac{n-1}{2}}$, we work in geodesic coordinates and the frame introduced in theorem (4.1.2). The Dirac operator, by (4.29), is seen to be of the form

$$D_A = A_i \partial_i + r x_i B_i + C \tag{4.51}$$

where A_i, B_i and C are endomorphisms of $S \otimes L$ independent of r. Using (4.42) and (4.51) we may write

$$D_A^3 b_{\frac{n-1}{2}} = \sum f_{\alpha\beta}^1 t_{\alpha\beta}, \quad D_A^2 b_{\frac{n-1}{2}} = \sum f_{\alpha\beta}^2 t_{\alpha\beta}, \tag{4.52}$$

where

$$f_{\alpha\beta}^1 = \sum_{(d,I)\in S_1} x^I r^d f_{d,I}^1, \qquad \text{with} \quad d \le \tfrac{n-1}{2} + \tfrac{1}{2}|I| + \tfrac{3}{2} \quad \forall (d,I) \in S_1 \qquad \text{and} \tag{4.53}$$

$$f_{\alpha\beta}^2 = \sum_{(d,I)\in S_2} x^I r^d f_{d,I}^2, \qquad \text{with} \quad d \le \tfrac{n-1}{2} + \tfrac{1}{2}|I| + 1 \quad \forall (d,I) \in S_2. \tag{4.54}$$

Now a combination of (4.25), (4.38), (4.53) and (4.54) gives the estimate

$$|U_t^{\frac{n-1}{2}}(y,y)| \le c_1 r^{\frac{n}{2}} e^{c_2 r t}. \tag{4.55}$$

Next to estimate $V_t^{\frac{n-1}{2}}$ we first use a Taylor expansion to write

$$f_{d,I}^1 = g_{d,I}^1 + x_i h_{d,I,i}^1 \quad \text{and} \quad f_{d,I}^2 = g_{d,I}^2 + x_i h_{d,I,i}^2 \tag{4.56}$$

where each of $g^1_{d,I}$ and $g^2_{d,I}$ is an even function in these coordinates. Also we let

$$\bar{f}^1_{\alpha\beta} = \sum_{(d,I)\in S_1} x^I r^d g^1_{d,I}, \qquad \text{with} \quad d = \tfrac{n-1}{2} + \tfrac{1}{2}|I| + \tfrac{3}{2} \qquad \text{and} \tag{4.57}$$

$$\bar{f}^2_{\alpha\beta} = \sum_{(d,I)\in S_2} x^I r^d g^2_{d,I}, \qquad \text{with} \quad d = \tfrac{n-1}{2} + \tfrac{1}{2}|I| + 1. \tag{4.58}$$

Now the terms which correspond to $\bar{f}^1_{\alpha\beta}$ and $\bar{f}^2_{\alpha\beta}$ under (4.52) are seen to contribute 0 to $V_t^{\frac{n-1}{2}}(y,y)$. This is because their contribution corresponds to the integral of an odd function in these coordinates. The rest of the terms contributing to $V_t^{\frac{n-1}{2}}(y,y)$ can again be estimated using (4.37), (4.38), (4.53) and (4.54) to give

$$|V_t^{\frac{n-1}{2}}(y,y)| \leq c_3 r^{\frac{n}{2}} e^{c_4 rt}. \tag{4.59}$$

Following (4.55) and (4.59) we obtain the estimate

$$|L_t^{\frac{n-1}{2}}(y,y)| \leq c_5 r^{\frac{n}{2}} e^{c_6 rt}, \tag{4.60}$$

for constants c_5 and c_6 independent of r. Finally theorem 8.35 in [32] is equivalent to the fact that the pointwise trace $tr(D_A e^{-tD_A^2}) = tr(L_t) = tr(L_t^{\frac{n-1}{2}})$ and hence the proposition follows from (4.60).

□

The above proposition is similar to lemma 2.6 in [39] although we have arrived at it a little differently. We can now finish the proof of theorem (4.0.1). The relevant observations were made in the beginning of this section and we summarize them below.

Proof of theorem (4.0.1). The spectral flow function is given by (4.5) to be

$$\mathrm{sf}\{D_{A_0+sa}\} = r^{\frac{n+1}{2}} \left(\frac{i}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{(\frac{n+1}{2})!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + \frac{1}{2}(\bar{\eta}^r - \bar{\eta}^0) + O(r^{\frac{n-1}{2}}). \tag{4.61}$$

The reduced eta invariant

$$\bar{\eta}^r = -2a_0^r \tag{4.62}$$

is the zeroth order term in the trace expansion

$$K_t^r := \mathrm{tr} B_t^r \sim \sum_{k \geq -n} a_k^r t^{\frac{1}{2}k}. \tag{4.63}$$

Here B_t^r is given by functional calculus

$$B_t^r \phi_\lambda = -\frac{\mathrm{sign}\lambda}{2} \mathrm{erfc}(|\lambda|\sqrt{t})\phi_\lambda \tag{4.64}$$

acting on the eigenvectors ϕ_λ of D_{A_0+ra} with eigenvalue λ. By theorem 8.35 in [32] K_t^r is a smooth function of t near 0 and we have

$$\frac{1}{2}\bar{\eta}^r = -a_0^r = -K_0^r = -K_t^r + \int_0^t K'^r_{t'} dt' \tag{4.65}$$

Next we bound

$$|K_t^r| \leq \frac{1}{\sqrt{\pi}} \mathrm{Tr}(e^{-tD_A^2}) \leq \frac{c_1}{t^{n/2}} e^{c_2 rt} \tag{4.66}$$

which follows from the inequality $\mathrm{erfc}(x) < e^{-x^2}$ and (4.22). Also (4.45) implies $|K'^r_{t'}| \leq c_3 r^{\frac{n}{2}} t'^{-\frac{1}{2}} e^{c_4 rt'}$ and hence

$$\left| \int_0^t K'^r_{t'} dt' \right| \leq c_3 r^{\frac{n}{2}} t^{\frac{1}{2}} e^{c_4 rt}. \tag{4.67}$$

Now (4.65), (4.66) and (4.67) give

$$|\bar{\eta}^r| \leq c_5 e^{c_6 rt} \left(\frac{1}{t^{n/2}} + r^{\frac{n}{2}} t^{\frac{1}{2}} \right). \tag{4.68}$$

Substituting $t = \frac{1}{r}$ gives $\bar{\eta}^r = O(r^{\frac{n}{2}})$ and hence the theorem follows from (4.61).

□

The main theorem (4.0.1) of this chapter does not say anything about the sharpness of the estimate (4.1), and it is not believed to be to be the optimal result. The conjectured sharp result is as stated by the proposition below.

Conjecture 4.1.1. *On a manifold of odd dimension n the spectral flow function for the family of Dirac operators $D_{A+sa}, 0 \leq s \leq r$ coupled to the connections $A + sa$ satisfies the asymptotics*

$$sf\{D_{A_0+sa}\} = r^{\frac{n+1}{2}} \left(\frac{i}{4\pi}\right)^{\frac{n+1}{2}} \frac{1}{\left(\frac{n+1}{2}\right)!} \int_Y a \wedge (da)^{\frac{n-1}{2}} + O(r^{\frac{n-1}{2}}) \tag{4.69}$$

as $r \to \infty$.

Hence the conjectured optimal result is $O(r^{\frac{1}{2}})$ sharper than theorem (4.0.1). In the next chapter we give some partial results towards proving (4.1.1). In chapter 6 we shall perform some explicit computations of spectral flow which show that this is the best possible result.

Chapter 5

A semiclassical Dirac operator

In this chapter we give some partial results towards proving (4.1.1) stated at the end of the last chapter. To begin with, by (4.61) and (4.62) the estimate (4.69) is equivalent to

$$\bar{\eta}^r = O(r^{\frac{n-1}{2}}). \tag{5.1}$$

Following this equations (4.64) and (4.65) give

$$\frac{1}{2}\bar{\eta}^r = \int_0^t K'^r_{t'} dt' + trE(\sqrt{t}D_A) \tag{5.2}$$

with $E(x) = -\frac{\text{sign}(x)}{2}\text{erfc}(|x|\sqrt{t})$. Following the bound (4.66) and the substitution $t = \frac{1}{r^2}$, it now suffices to prove $trE(\frac{1}{r}D_{A_0} + cl(a)) = O(r^{\frac{n-1}{2}})$. Substituting $h = \frac{1}{r}$ to be a semiclassical parameter conjecture (4.1.1) reduces to proving

$$trE(hD_{A_0} + cl(a)) = O(h^{-\frac{n-1}{2}}). \tag{5.3}$$

5.1 The odd functional trace

In an attempt to prove (5.3) we shall consider the traces $trf(D_h)$ where D_h is the semiclassical Dirac operator

$$D_h = hD_{A_0} + cl(a). \tag{5.4}$$

The semiclassical symbol of this operator is

$$\sigma_{sl}(D_h) = cl(\xi + a) \tag{5.5}$$

and it is hence elliptic and self adjoint. On fixing a nowhere vanishing $\frac{1}{2}$-density on X we may also think of D_h as an operator on $S \otimes L$ valued $\frac{1}{2}$-densities. The methods of [13] and [44], reviewed in the appendix, give a trace expansion for $trf(D_h)$ when f is a Schwartz function. Our main result is that this expansion shows cancellations in its first $\frac{n+3}{2}$ terms when the function f is odd. This is the proposition below.

Proposition 5.1.1. *Let $f \in \mathcal{S}$ be an odd Schwartz function. There is a trace expansion*

$$trf(D_h) \sim h^{-\frac{n-3}{2}} c_{\frac{n+3}{2}} + h^{-\frac{n-5}{2}} c_{\frac{n+5}{2}} + \dots \tag{5.6}$$

for some constants c_i, $\frac{n+3}{2} \leq i$.

Proof. By proposition (A.5.5) we have a trace expansion

$$trf(D_h) \sim c_0(f)h^{-n} + c_1(f)h^{-n+1} + \dots \tag{5.7}$$

for each function $f \in \mathcal{S}$. Also setting $r = \frac{1}{h}$ and $t = \tau h^2$ in proposition (4.1.4) gives the trace bound

$$|trD_h e^{-\tau D_h^2}| \leq C_1 h^{-\frac{n-2}{2}} e^{C_2 \tau h} \tag{5.8}$$

for some constants C_1, C_2 independent of h and τ. This implies that the coefficients in (5.7) must satisfy $c_i(f) = 0$, $i \leq \frac{n+1}{2}$ for $f = f_\tau = xe^{-\tau x^2}$. This is a smooth family of Schwartz functions $f_\tau : \mathbb{R}^{>0}_\tau \to \mathcal{S}$. Hence using proposition (A.5.6) we may differentiate the trace expansion (5.7) for f_τ to obtain the expansion for $\frac{\partial^k f_\tau}{\partial \tau^k} = (-1)^k x^{2k+1} e^{-\tau x^2}$. This gives that the coefficients in (5.7) must satisfy $c_i(f) = 0$,

$i \leq \frac{n+1}{2}$ for $f = x^{2k+1}e^{-\tau x^2}$. Now set $\tau = 1$ and note that the span of the functions $x^{2k+1}e^{-x^2}$ is dense in the space of odd Schwartz functions. Hence we must have $c_i(f) = 0$, $i \leq \frac{n+1}{2}$ for any odd Schwartz function f. □

5.2 A normal form result for D_h^2

Proposition (5.1.1) still does not prove the estimate (5.3) since the function $E(x)$ has a discontinuity at the origin. In order to get an understanding of the functional trace $trE(D_h)$ an analysis of the kernel of the wave operator $f(D_h)e^{\frac{itD_h}{h}}$ appears necessary. The wave kernel has been analyzed in [18] for operators whose symbols are smoothly diagonalizable over the cotangent bundle. The symbol $\sigma(D_h)$ however is not smoothly diagonalizable on T^*X since its eigevalues $\pm|\xi + a| \in C^\infty(T^*X)$ are not smooth along the locus $\Sigma = \{(x,\xi)|\xi = -a\}$. The kernel for $f(D_h)e^{\frac{itD_h}{h}}$ being related to the solution operator of $(h^2\partial_t^2 + D_h^2)$, we attempt to find a normal form for $\sigma(D_h^2)$ along Σ.

In order to obtain the normal form result of this section we first review some facts about Hamiltonian linear transformations following [20]. Given a symplectic vector space (V, ω) its space of Hamiltonian transformations is simply the Lie algebra $\mathfrak{sp}(V, \omega)$ of its symplectic group

$$\mathfrak{sp}(V, \omega) = \{A : V \to V | \omega(v_1, Av_2) + \omega(Av_1, v_2) = 0\}. \tag{5.9}$$

We now have the following lemma.

Lemma 5.2.1. *Let $A \in \mathfrak{sp}(V, \omega)$ be a Hamiltonian transformation. If λ is an eigenvalue of A then so are $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$.*

Proof. First we extend A and ω to $V_\mathbb{C} = V \otimes \mathbb{C}$ by complex linearity and bilinearity respectively. Now we show that $-\lambda$ is an eigenvalue. Consider the map $\rho : V_\mathbb{C} \to V_\mathbb{C}^*$ given by $\rho(v) = i_v\omega$. Since $A \in \mathfrak{sp}(V, \omega)$ we have $\rho \circ A = -A^* \circ \rho$. Thus the λ eigenspace of A is mapped to the $-\lambda$ eigenspace of A^*. However if A has the Jordan blocks

$$\begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix} \tag{5.10}$$

with respect to some basis, then A^* has the blocks

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \tag{5.11}$$

with respect to the dual basis. Hence the eigenvalues of A and A^* are the same and hence $-\lambda$ is an eigenvalue of A. The fact that $\pm\bar{\lambda}$ are eigenvalues follows from the fact that A is a real linear trasformation and its eigenvalues come in complex conjugate pairs. □

An easy consequence of the above proposition is that the generalized nullspace of a Hamiltonian transformation is of even dimension. Next we show that on a 4 dimesional symplectic vector space a symmetric, positive semi-definite inner product and the symplectic form maybe simultaneously put in standard form. The result follows from the general canonical form result for Hamiltonian transformations appearing in [27]. However we shall be content with the 4 dimensional case below.

Lemma 5.2.2. *Let H be a symmetric, positive semi-definite inner product of rank 3 on a symplectic vector space (V, ω) of dimension 4. There exists a basis for V in which we simultaneously have*

$$H = \begin{bmatrix} \mu & & & \\ & \mu & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \quad \textit{and} \quad \omega = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \tag{5.12}$$

for some $\mu > 0$.

Proof. Consider the linear transformation $A : V \to V$ defined via

$$H(v_1, v_2) = \omega(v_1, Av_2), \quad \forall v_1, v_2 \in V. \tag{5.13}$$

Since H is symmetric and ω antisymmetric we have $A \in \mathfrak{sp}(V, \omega)$. Being of rank 3, H and hence A have one dimensional kernels. The generalized nullspace of A is hence of even dimension 2 or 4. In the latter case we have a basis for V in which A has the Jordan block form

$$A = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix}. \tag{5.14}$$

The condition $A \in \mathfrak{sp}(V, \omega)$ gives that ω is of the form

$$\omega = \begin{bmatrix} 0 & -a & 0 & c \\ a & 0 & c & 0 \\ 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix} \quad \text{and hence} \quad H = \omega A = \begin{bmatrix} -a & 0 & -c & 0 \\ 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.15}$$

in this basis. If $c = 0$, H has rank less than 3 and if $c \neq 0$ H is seen not to be positive semi-definite. Hence the generalized nullspace of A is of dimension 2. Following Lemma (5.2.1) the remaining 2 dimensions are accounted for by a pair of eigenspaces $V_{\pm\lambda}$ with λ real or purely imaginary. Since $A \in \mathfrak{sp}(V, \omega)$ it follows that the generalized nullspace is ω-orthogonal to $V_{\pm\lambda}$. If λ is real we have a basis for V in which

$$A = \begin{bmatrix} \lambda & & & \\ & -\lambda & & \\ & & 0 & \\ & & 1 & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} & & 1 & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \quad \text{and hence} \quad H = \begin{bmatrix} & \lambda & & \\ \lambda & & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \tag{5.16}$$

which is not positive semi-definite. Hence the eigenvalues are purely imaginary $\lambda = i\mu, \mu \in \mathbb{R}$. This gives a basis for V, as a real vector space, in which

$$A = \begin{bmatrix} & -\mu & & \\ \mu & & & \\ & & 0 & \\ & & 1 & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} & & 1 & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \quad \text{and hence} \quad H = \begin{bmatrix} \mu & & & \\ & \mu & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \tag{5.17}$$

as required. □

We now come to the normal form result for $\sigma(D_h^2)$. In the proposition below we assume that the manifold X is of dimension 3.

Proposition 5.2.3. *Let $\sigma = \sigma_{sl}(D_h^2)$ be the semiclassical symbol of the square of $D_h = hD_{A_0} + cl(a)$ with a being a contact one form. For every point $p \in \Sigma = \{(x,\xi) | \xi = -a(x)\}$, one has a germ of a symplectomorphism $\chi : (T^*\mathbb{R}^3, 0) \to (T^*X, p)$ near p such that*

$$\chi^*\sigma = \mu(x_2, x_3, \xi_3)(x_1^2 + \xi_1^2) + \xi_2^2 + (x_1^2 + \xi_1^2)^2 f(x_1^2 + \xi_1^2, x_2, x_3, \xi_3) + r_\infty. \tag{5.18}$$

Here μ is positive function on $\Sigma_0 = \{x_1 = \xi_1 = \xi_2 = 0\}$ and r_∞ is a function on $T^*\mathbb{R}^3$ vanishing to infinite order along Σ_0.

Proof. If $\pi : \Sigma \to X$ denotes the projection onto the base one clearly has $-\pi^* a = \alpha|_\Sigma$ with α being the tautological one form. Hence $\alpha|_\Sigma$ is contact and Darboux's theorem gives coordinates (y_1, y_2, y_3) on Σ such that

$$\alpha|_\Sigma = dy_1 + y_2 dy_3. \tag{5.19}$$

Consider the map $\chi : \Sigma_0 \to \Sigma$ given by $\chi(x_2, x_3, \xi_3) = (y_1, y_2, y_3)$. One clearly has $\chi^*(\omega|_\Sigma) = \omega_0|_{\Sigma_0}$ where ω_0 and ω are the symplectic forms on $T^*\mathbb{R}^3$ and T^*X respectively. An application of Moser's trick, Lemma 3.14 in [30], gives an extension to a symplectomorphism germ $\chi : (T^*\mathbb{R}^3, 0) \to (T^*X, p)$. Since σ vanishes to second order along Σ, the pullback $\sigma_0 = \chi^*\sigma$ vanishes to second order along Σ_0. The second order term in the Taylor expansion of σ_0 at p maybe thought of as a quadratic form in the variables x_1, x_2, ξ_1, ξ_2 of rank 3. By lemma (5.2.2) we have a linear symplectic change of these coordinates which diagonalizes this quadratic form to $\mu(x_1^2 + \xi_1^2) + \xi_2^2$ for some positive constant $\mu > 0$. Since Σ_0 is still the critical locus of the new quadratic form it is mapped to itself under this change of coordinates. Hence we may now assume that σ_0 has a Taylor expansion in the x_1, ξ_1, ξ_2 variables of the form

$$\sigma_0 = \mu_1(x_1^2 + \xi_1^2) + \mu_2\xi_2^2 + \mu_3(\xi_1^2 - x_1^2) + \mu_4 x_1\xi_1 + \mu_5 x_1\xi_2 + \mu_6\xi_1\xi_2 \pmod{O_3}. \tag{5.20}$$

Here μ_i are functions on Σ_0 such that $\mu_1(p) = \mu > 0, \mu_2(p) = 1$ and $\mu_i(p) = 0$ for $3 \le i \le 6$. Also O_N denotes the space of functions vanishing to order N on Σ_0. Next we note

$$\{O_M, O_N\} \subset O_{M+N-2} \tag{5.21}$$

and the commutation relations

$$\{x_1\xi_1, x_1^2 + \xi_1^2\} = 2(\xi_1^2 - x_1^2), \qquad \{x_1\xi_1, x_1\xi_2\} = x_1\xi_2 \tag{5.22}$$

$$\{x_1\xi_1, \xi_1^2 - x_1^2\} = 2(x_1^2 + \xi_1^2), \qquad \{x_1\xi_1, \xi_1\xi_2\} = \xi_1\xi_2. \tag{5.23}$$

Letting α denote a function on Σ_0, let $X_{\alpha x_1\xi_1}$ be the Hamiltonian vector field of $\alpha x_1\xi_1$.

We also define

$$
\begin{aligned}
\tilde{\sigma}_t &= (\mu_1 \cosh(2\alpha t) + \mu_3 \sinh(2\alpha t))(x_1^2 + \xi_1^2) + \mu_2 \xi_2^2 \\
&\quad + (\mu_1 \sinh(2\alpha t) + \mu_3 \cosh(2\alpha t))(\xi_1^2 - x_1^2) + \mu_4 x_1 \xi_1 \\
&\quad + \mu_5 e^{-\alpha t} x_1 \xi_2 + \mu_6 e^{\alpha t} \xi_1 \xi_2. \qquad (5.24)
\end{aligned}
$$

The commutation relations (5.21), (5.22) and (5.23) now allow us to compute $\partial_t \tilde{\sigma}_t = \{\alpha x_1 \xi_1, \tilde{\sigma}_t\} \pmod{O_3}$. Since $\tilde{\sigma}_0 = \sigma_0 \pmod{O_3}$, Duhamel's principle gives

$$
\tilde{\sigma}_t = \left(e^{tX_{\alpha x_1 \xi_1}}\right)^* \sigma_0 \pmod{O_3}, \qquad (5.25)
$$

with the right hand side being defined by Hamiltonian flow. Hence the symplectomorphism $e^{X_{\alpha x_1 \xi_1}}$ with α chosen such that $\tanh(2\alpha) = -\frac{\mu_3}{\mu_1}$ is seen to cancel the $(\xi_1^2 - x_1^2)$ coefficient of the Taylor expansion. Hence we now assume $\mu_3 = 0$ in (5.20). Similarly considering the symplectomorphism $e^{X_{\alpha(x_1^2 - \xi_1^2)}}$ with $\tanh(4\alpha) = -\frac{\mu_4}{2\mu_1}$ gets rid of the $x_1 \xi_1$ coefficient. Hence we may also set $\mu_4 = 0$ in (5.20). The symplectomorphism $e^{X_{\alpha \xi_1 \xi_2}}$ with $\alpha = \frac{\mu_5}{2\mu_1}$ cancels the $x_1 \xi_2$ coefficient and we may set $\mu_5 = 0$ in (5.20). And the symplectomorphism $e^{X_{\alpha x_1 \xi_2}}$ with $\alpha = -\frac{\mu_6}{2\mu_1}$ cancels the $\xi_1 \xi_2$ coefficient and we may also set $\mu_6 = 0$ in (5.20). Finally a change of coordinates in the variables x_2, x_3, ξ_2, ξ_3 sending $\sqrt{\mu_2}\xi_2$ to a coordinate function allows us to set $\mu_2 = 1$. It is important to carry out the computations in the order mentioned to ensure that terms once eliminated do not reappear later. Hence we are now reduced to a Taylor expansion for σ_0 of the form

$$
\sigma_0 = \mu(x_1^2 + \xi_1^2) + \xi_2^2 \pmod{O_3}. \qquad (5.26)
$$

Next we wish to improve the above equation to mod O_4. To this end, denoting $H_0 = \mu(x_1^2 + \xi_1^2) + \xi_2^2$, we further Taylor expand

$$\sigma_0 = H_0 + g_3 \pmod{O_4}, \quad \text{where} \tag{5.27}$$

$$g_3 = \sum_{a+b+c=3} r_{abc} x_1^a \xi_1^b \xi_2^c \tag{5.28}$$

for some functions r_{abc} on Σ_0. We claim that there exists a function $f \in O_3$ such that

$$\{f, H_0\} + g_3 = r_1(x_1^2 + \xi_1^2)\xi_2 + r_2\xi_2^3 \pmod{O_4}. \tag{5.29}$$

Introducing the complex coordinates $\zeta_1 = x_1 + i\xi_1$ and $\bar{\zeta}_1 = x_1 - i\xi_1$ we rewrite

$$g_3 = \sum_{a+b+c=3} \bar{r}_{abc} \zeta_1^a \bar{\zeta}_1^b \xi_2^c \tag{5.30}$$

$$H_0 = \mu \zeta_1 \bar{\zeta}_1 + \xi_2^2. \tag{5.31}$$

Observing the commutation relations $\{\zeta_1\bar{\zeta}_1, \zeta_1^a \bar{\zeta}_1{}^b\} = 2i(a-b)\zeta_1^a\bar{\zeta}_1{}^b$ and setting

$$\bar{f} = \sum_{a \neq b} \frac{\bar{r}_{abc}}{2i(a-b)\mu} \zeta_1^a \bar{\zeta}_1^b \xi_2^c \quad \text{gives} \tag{5.32}$$

$$\{\bar{f}, H_0\} + g_3 = \bar{r}_{111}(x_1^2 + \xi_1^2)\xi_2 + \bar{r}_{003}\xi_2^3 \pmod{O_4}. \tag{5.33}$$

Hence $f = Re(\bar{f})$ solves (5.29). Now considering the symplectomorphism $\chi_1 = e^{X_f}$ gives

$$\chi_1^* \sigma_0 = e^{ad_f}(H_0 + g_3) \pmod{O_4} \tag{5.34}$$

$$= H_0 + g_3 + \{f, H\} \pmod{O_4} \tag{5.35}$$

$$= H_0 + r_1(x_1^2 + \xi_1^2)\xi_2 + r_2\xi_2^3 \pmod{O_4}. \tag{5.36}$$

In order to get rid of the remaining terms first consider the symplectomorphism $e^{X_{\alpha\xi_2^2}}$

where α is a function on Σ_0 satisfying $2\partial_{x_2}\alpha + r_2 = 0$. The pullback of σ_0 under this symplectomorphism cancels the $r_2\xi_2^3$ term allowing us to set $r_2 = 0$. Finally the term $r_1(x_1^2 + x_1^2)$ is cancelled by the symplectomorphism $e^{X_{\alpha(x_1^2+\xi_1^2)}}$ with α satisfying $2\partial_{x_2}\alpha + r_1 = 0$. Hence we are now reduced to a Taylor expansion for σ_0 of the form

$$\sigma_0 = \mu(x_1^2 + \xi_1^2) + \xi_2^2 \quad (\text{mod } O_4). \tag{5.37}$$

Following this we inductively prove that for each N there exists a symplectomorphism χ_N such that

$$\chi_N^* \sigma_0 = H_0 + f_N(x_1^2 + \xi_1^2, x_2, x_3, \xi_3) \quad (\text{mod } O_N), \tag{5.38}$$

for some function $f_N \in O_4$. The case $N = 4$ is equation (5.37) and we now construct χ_{N+1}, assuming the existence of χ_N, for each $N \geq 4$. Hence we assume that σ_0 Taylor expansion

$$\sigma_0 = H_0 + f_N(x_1^2 + \xi_1^2, x_2, x_3, \xi_3) + g_N \quad (\text{mod } O_{N+1}) \tag{5.39}$$

with $g_N \in O_N$. Again we claim that we have a function $h_N \in O_N$ such that

$$\{h_N, H_0\} + g_N = \sum_{2a+b=N} r_{ab}(x_1^2 + \xi_1^2)^a \xi_2^b \quad (\text{mod } O_{N+1}). \tag{5.40}$$

As before if g_N has the Taylor expansion

$$g_N = \sum_{a+b+c=N} \bar{r}_{abc}\zeta_1^a \bar{\zeta}_1^b \xi_2^c \tag{5.41}$$

in complex coordinates, then $h_N = Re(\bar{h}_N)$ with

$$\bar{h}_N = \sum_{a \neq b} \frac{\bar{r}_{abc}}{2i(a-b)\mu} \zeta_1^a \bar{\zeta}_1^b \xi_2^c \tag{5.42}$$

is seen to solve (5.40). The symplectomorphism $e^{X_{h_N}}$ now reduces the Taylor expansion of σ_0 to the form

$$\sigma_0 = H_0 + f_N + \sum_{2a+b=N} r_{ab}(x_1^2+\xi_1^2)^a \xi_2^b \pmod{O_{N+1}}. \tag{5.43}$$

Finally we get rid of the terms $r_{ab}(x_1^2+\xi_1^2)^a\xi_2^b$ with $b \geq 1$. This is done using the symplectomorphism $e^{X_{\alpha_{ab}(x_1^2+\xi_1^2)^a\xi_2^{b-1}}}$ where $2\partial_{x_2}\alpha_{ab} + r_{ab} = 0$. This completes the induction step giving (5.38) for all N. The proposition now follows from an application of Borel's lemma.

□

The above normal form maybe extended to a slightly more general setting. Namely let $p(x,\xi) \in C^\infty(T^*X)$ be a symbol, on a 3 manifold X, with a Morse Bott critical locus of dimension 3. If additionally the symplectic form is maximally non-degenerate of rank 2 along the critical locus then the normal form result (5.2.3) holds, with the same proof, for such a symbol. Following this normal form result one hopes to be able to apply a Hermite transform, as appearing in [31], in the x_1, ξ_1 variables to (5.18). However we have not completed this line of argument at present.

5.3 A bound on the counting function

In this section we attempt to control the dimension of the nullspace dim $\ker(D_h)$ of the Dirac operator as $h \to 0$. To do this we analyze the number of eigenvalues of D_h in an $O(h^{\frac{1}{2}})$ interval around 0. This is the same as finding the number of eigenvalues of D_h^2 in an $O(h)$ interval around 0. The usual semiclassical method gives $O(h)$ information for eigenvalues around c assuming c is not a critical value of the symbol. However this is violated for the operator D_h^2 whose symbol does have 0 as a critical value. Counting functions near critical values were analyzed in [11] for scalar semiclassical operators. Here we modify their arguments to the non-scalar D_h^2.

The estimate on the counting function follows from a trace expansion. This expansion is derived by applying a stationary phase expansion to an oscillatory integral representation for the wave kernel. Below we give the required stationary phase formula.

Proposition 5.3.1. *For $a \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$ there exists an asymptotic expansion for the oscillatory integral*

$$I(a) = \int a e^{\frac{itx^2}{h}} dt dx \tag{5.44}$$

$$\sim h^{n/2}(\sum_{j\geq 0} a_j h^j) \tag{5.45}$$

in powers of h.

Proof. Let $\hat{a}$ denote the partial Fourier transform of a in t

$$\hat{a}(\tau, x) = \int e^{it\tau} a(t,x) dt. \tag{5.46}$$

The oscillatory integral (5.44) can be written as

$$I(a) = \int \hat{a}\left(\frac{x^2}{h}, x\right) dx \tag{5.47}$$

$$= h^{n/2} \int \hat{a}(y^2, y\sqrt{h}) dy. \tag{5.48}$$

The Taylor expansion for $\hat{a}(\tau, x)$ in x now gives the expansion $I(a) \sim h^{n/2}(\sum b_j h^{j/2})$ with

$$b_j = \sum_{|\alpha|=j} \int \frac{y^\alpha}{\alpha!} (\partial_x^\alpha \hat{a})(y^2, 0) dy. \tag{5.49}$$

The coefficients b_j, for j odd, correspond to odd integrals in (5.49) and must vanish, giving (5.45) with $a_j = b_{2j}$. □

The above proposition differs from the usual stationary phase formula since the phase function tx^2 is not Morse-Bott nondegenerate. Next we derive the required trace expansion.

Proposition 5.3.2. *There exists $T > 0$ sufficiently small such that for every Schwartz function ψ with $\hat{\psi} \in C_c^\infty(-T,T)$ one has the trace expansion*

$$tr\ \psi\left(\frac{D_h^2}{h}\right) f(D_h^2) \sim h^{-n/2}(a_0 + ha_1 + \cdots). \tag{5.50}$$

Proof. Let Λ be the Lagrangian given by proposition (B.4.1) with

$$p = \sigma_{sl}(D_h^2) = -|\xi + a|^2. \tag{5.51}$$

Let $x \in X$ and $K \subset \mathbb{R}^n_\xi$ be compact. Following proposition (B.4.3), one can find a coordinate neighbourhood U_x of x, $T_x > 0$ sufficiently small and a function $S \in C^\infty([-T_x, T_x] \times U_x \times K)$ satisfying

$$\begin{aligned} \partial_t S + p(x, \partial_x S) &= 0 & (5.52)\\ S|_{t=0} &= x.\xi. & (5.53) \end{aligned}$$

Proposition (B.4.3) further implies that the corresponding phase functions $\varphi \in C^\infty([-T_x, T_x] \times U_x \times U_y \times K)$

$$\varphi(x, y, t, \xi) = S(t, x, \xi) - y.\xi \tag{5.54}$$

give a collection of generating functions for Λ near $t = 0$. Combining this with proposition (B.4.2) we get that for $T > 0$ sufficiently small such that for every $\psi \in \mathcal{S}$ with $\hat{\psi} \in C_c^\infty(-T, T)$

$$\hat{\psi(t)} k_t(x, y) = h^{-n} \sum_{j=1}^{N} \int a_j e^{\frac{i\varphi_j}{h}} d\xi \tag{5.55}$$

mod $O(h^\infty)$. Here k_t denotes the kernel of $f(D_h^2) e^{-\frac{itD_h^2}{h}}$ and each φ_j is of the form (5.54). Now for any phase function of the form (5.54) the initial condition (5.53) gives that

$$S(t, x, \xi) - x.\xi = tF(t, x, \xi) \tag{5.56}$$

for some smooth function F. The Hamilton-Jacobi equation (5.52) also gives

$$F|_{t=0} = S_t|_{t=0} = -p(x, \xi) = |\xi + a|^2. \tag{5.57}$$

Introduce the new coordinates $(t, x, \eta) = (t, x, \xi + a)$. For $\eta = 0$, $\xi = -a$ which gives $H_p = 0$ and hence $e^{tH_p}(x, \xi) = (x, \xi)$ by (5.51). Thus (B.54) and (5.56) give

$$dF|_{\eta=0} = 0. \tag{5.58}$$

Moreover the Hessian of F

$$d_\eta^2 F|_{(\eta,t)=(0,0)} = g^{ij}(x) \tag{5.59}$$

equals the metric and is hence nondegenerate. Following (5.58) and (5.59), we may apply the Morse lemma with parameters (Lemma 1.2.2 in [14]) to get a further change of variables $(t, x, \zeta(t, x, \eta))$ such that $\zeta(t, x, 0) = 0$ and

$$F(t, x, \eta) = F(t, x, 0) + \zeta^2. \tag{5.60}$$

Also the formula (B.56) for S implies $F(t, x, 0) = 0$. Now the trace

$$tr\ \psi\left(\frac{D_h^2}{h}\right) f(D_h^2) = (2\pi)^{-1} \int \hat{\psi}(t)\ tr\left(f(D_h^2)e^{-\frac{itD_h^2}{h}}\right) dt \tag{5.61}$$

$$= (2\pi)^{-1} \int \hat{\psi}(t) k_t(x, x) dt dx. \tag{5.62}$$

Following (5.54), (5.55) and (5.60) this integral is a finite sum of integrals of the kind

$$h^{-n} \int \hat{\psi}(t) a_j(t, x, \zeta) e^{\frac{it\zeta^2}{h}} dt d\zeta dx \tag{5.63}$$

modulo $O(h^\infty)$. Finally the stationary phase lemma in proposition (5.3.1) gives the trace expansion (5.50). □

The above trace expansion now allows us to estimate the counting function of the Dirac operator in a small interval around 0. Namely for any $R > 0$ define $N_h(R)$ to be the maximum number of linearly independent eigenvectors of D_h with eigenvalues in $[-R, R]$. The following proposition gives a bound on this number.

Proposition 5.3.3. *For $c > 0$ be any positive real, the counting function*

$$N_h(ch^{\frac{1}{2}}) = O(h^{-\frac{n}{2}}) \tag{5.64}$$

near $h = 0$.

Proof. Let $T > 0$ be suffiently small as given by proposition (5.3.2). Choose $\psi \in \mathcal{S}$ such that $\hat{\psi} \in C_c^{\infty}(-T, T)$ and $\psi > 1$ on $[-c^2, c^2]$. Let f be any Schwartz function such that $f > 1$ near 0. Then one can estimate

$$\begin{aligned} N_h(ch^{\frac{1}{2}}) &\leq \sum_{\lambda \in Spec(D_h)} \psi\left(\frac{\lambda^2}{h}\right) f(\lambda^2) && (5.65) \\ &= O(h^{-\frac{n}{2}}) && (5.66) \end{aligned}$$

by the trace expansion (5.50). □

Chapter 6

Computations of Spectral flow

In this chapter we perform some computations of spectral flow for coupled Dirac operators. The spectrum of spin Dirac operators has been computed in several cases, a survery of known computations can be found in chapter 2 of [17]. We show how to modify some of these computations in the presence of a coupling. In particular we compute spectral flow for certain coupled Dirac operators on spheres and homogeneous Lens spaces. A consequence of these computations is the proof that conjecture (1.2.1) is the best possible estimate on spectral flow.

6.1 Spectral flow on S^3

Here we consider the spectral flow for a family of Dirac operators on S^3. Since $S^3 = SU(2) = \left\{ \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \middle| |z_1|^2 + |z_2|^2 = 1 \right\}$ is a Lie group it is parallelized by elements of the Lie algebra $e_i = \sigma_i \in su(2)$ which we take to be the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{6.1}$$

We think of e_1, e_2 and e_3 as three left invariant vector fields and hence first order differential operators on functions. They satisfy the standard commutation relations

$$[e_i, e_j] = 2\epsilon_{ijk}e_k, \tag{6.2}$$

where ϵ_{ijk} is the Levi-Civita symbol which equals ± 1 if (i, j, k) is an even/odd permutation of $(1, 2, 3)$ and zero otherwise. Since $H_2(S^3) = 0$, there exists only the trivial Spin^c structure on S^3 (which comes from the only Spin structure). The corresponding Spin bundle S in trivial with Clifford multiplication being given by the Pauli matrices $\rho(e_i) = \sigma_i$ in some basis for S. The Christoffel symbols for the Levi-Civita connection of the standard metric can be computed in the frame e_1, e_2, e_3 to be $\Gamma^k_{ij} = \epsilon_{ijk}$. The corresponding Dirac operator can be computed to be $D = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 - \frac{3}{2}$.

We now consider the Spin^c Dirac operator coupled to a unitary connection on $det(S)$. Since the determinant line bundle is trivial such a connection is given by an imaginary one form $a \in \Omega^1(S^3, i\mathbb{R})$. We shall choose a to be the *contact* one form $a = -ie_3^*$ and consider the one parameter family of Dirac operators D_{ra} for $r \in \mathbb{R}$. The object of interest here is the spectrum of this family and its corresponding spectral flow function $sf(D_0, D_{ra})$. The coupled Dirac operator can be written as

$$D_{ra} = D - \frac{ir}{2}\sigma_3 = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 - \frac{3}{2} - \frac{ir}{2}\sigma_3. \tag{6.3}$$

We can also compute $de_1^*(e_2, e_3) = e_2(e_1^*(e_3)) - e_3(e_1^*(e_2)) - e_1^*([e_2, e_3]) = -2$. And hence we have $de_1^* = -2e_2^* \wedge e_3^*$, $de_2^* = -2e_3^* \wedge e_1^*$ and $de_3^* = -2e_1^* \wedge e_2^*$. Using these we also have the following expression for the Laplacian on functions

$$\Delta = d^*d = -e_1^2 - e_2^2 - e_3^2. \tag{6.4}$$

The Laplacian can be considered to be acting componentwise on the sections of $S = \mathbb{C}^2$ as if $s = \begin{bmatrix} f \\ g \end{bmatrix}$ then $\Delta s = \begin{bmatrix} \Delta f \\ \Delta g \end{bmatrix}$. Using the expressions (6.3), (6.4) and the commutation relations (6.2) we have that $[D_{ra}, \Delta] = 0$ and hence the Dirac operator preserves the eigenspaces of the Laplacian acting on spinors. The eigenfunctions of Δ are the spherical harmonics and next we review their description in terms of harmonic polynomials.

6.1.1 Spherical harmonics

Here we describe the spectrum and the eigenfunctions of the Laplacian on the sphere S^{n-1} with the standard metric. First consider the fomula for the Laplacian on $\mathbb{R}^n$ in polar coordinates given by

$$\Delta_{\mathbb{R}^n} = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}} \tag{6.5}$$

where $\Delta_{S^{n-1}}$ is the Laplacian on the sphere. This formula implies that if p is a homogeneous harmonic polynomial on $\mathbb{R}^n$ of degree k then its restriction to S^{n-1} is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $k(k+n-2)$. Now we show that all the eigenfunctions of $\Delta_{S^{n-1}}$ are obtained in this way.

Let $\tilde{H}_k$ denote the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^n$. Let $\tilde{P}_k$ denote the space of all homogeneous polynomials of degree k on $\mathbb{R}^n$. We prove the following

Theorem 6.1.1.

$$\tilde{P}_k = \tilde{H}_k \oplus \tilde{P}_{k-2} \tag{6.6}$$

Proof. Define a positive definite inner product on $\tilde{P}_k$ via $\langle x_1^{k_1}\dots x_n^{k_n}, x_1^{l_1}\dots x_n^{l_n}\rangle = k_1!\dots k_n!\delta_{k_1 l_1}\dots\delta_{k_n l_n}$. Define $M : \tilde{P}_{k-2} \to \tilde{P}_k$ via $M(p) = (x_1^2 + \cdots x_n^2)p$. Clearly M is injective and an easy computation shows that $M^* = \Delta$ with the defined inner products. Now (6.6) is simply the fact that $\tilde{P}_k = \mathrm{Im}M \oplus \mathrm{Ker}M^*$. □

Now let P_k and H_k denote the restrictions of $\tilde{P}_k$ and $\tilde{H}_k$ to the sphere S^{n-1}.

Theorem 6.1.2. *The Spectrum of the Laplacian on S^{n-1} is given by*

$$Spec(\Delta) = \{k(n+k-2) | k = 0, 1, \dots\} \tag{6.7}$$

with the eigenvalue $\lambda = k(n+k-1)$ ocurring with multiplicity $\binom{n-1+k}{n-1} - \binom{n-3+k}{n-1}$.

Proof. Let f be any eigenfunction of Δ with eigenvalue λ. The set of all polynomials is L^2 dense in $C^\infty(S^{n-1})$ so we have a sequence of polynomials $p_i \underset{L^2}{\to} f$. If λ is not of the form $k(k+n-2)$ then f is orthogonal to H_k for all k. Hence by (6.6) f

is orthogonal to P_k for all k and $p_i \perp f$ which is a contradiction. If on the other hand $\lambda = k(n+k-2)$ for some k and $q_1, \ldots, q_m$ is a basis for H_k then consider $f' = f - \sum_i \langle f, q_i \rangle q_i$. Now f' is orthogonal to each H_k and a similar argument applied to f' gives that we cannot have a sequence of polynomials converging to f'. Hence $f' = 0$ and $f \in H_k$. Hence we have (6.7) and the multiplicity of each eigenvalue is easily found using (6.6). □

6.1.2 The Spectrum of D_{ra}

Now we have the decompositions $L^2(S^3) = \oplus H_k$ and $L^2(S) = \oplus H_k^2$ into eigenspaces of the Laplacian. Now since $[D_{ra}, \Delta] = 0$ the Dirac operator preserves this decomposition and it suffices to find the eigenvalues of the finite dimensional operator $D_{ra} : H_k^2 \to H_k^2$.

Now H_k is an $\mathfrak{su}(2)$ module since each e_i commutes with Δ. We wish to find the decomposition of H_k into irreducible submodules. First we define

$$
\begin{aligned}
H &= -ie_3 && (6.8)\\
X &= -\frac{1}{2}(e_2 + ie_1)\text{and} && (6.9)\\
Y &= \frac{1}{2}(e_2 - ie_1). && (6.10)
\end{aligned}
$$

Now if we use $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then we have that

$$
\begin{aligned}
H &= -\frac{\partial}{\partial r} + 2(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}) && (6.11)\\
X &= z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2} && (6.12)\\
Y &= -\bar{z}_2 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2} && (6.13)\\
\Delta_{\mathbb{R}^4} &= -4(\frac{\partial}{\partial z_1}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial z_2}\frac{\partial}{\partial \bar{z}_2}). && (6.14)
\end{aligned}
$$

This means that (6.8), (6.9) and (6.10) are the restrictions of (6.11), (6.12) and (6.13) to S^3. Now consider the $k+1$ homogeneous polynomials

$$p_a = z_1^a z_2^{k-a}, a = 0, \dots, k. \tag{6.15}$$

Each p_a is clearly in H_k by (6.14). Also (6.11) and (6.12) can be used to compute $Hp_a = kp_a$ and $Xp_a = 0$. Hence we have that the $su(2)$ submodule generated by p_a is a copy of $\mathrm{Sym}^k\mathbb{C}^2$ and is irreducible (see [15] chapter 11). These $k+1$ polynomials give $k+1$ irreducible $\mathfrak{su}(2)$ submodules isomorphic to $\mathrm{Sym}^k\mathbb{C}^2$ in H_k and since this accounts for all the $(k+1)^2$ dimensions of H_k we have that the decomposition of H_k into irreducible $\mathfrak{su}(2)$ modules is given by

$$H_k = \oplus_{a=0}^k [p_a]. \tag{6.16}$$

Now the Dirac operator (6.3) can be written as

$$D_{ra} = \begin{bmatrix} -H & -2Y \\ -2X & H \end{bmatrix} - \frac{3}{2} + \begin{bmatrix} \frac{r}{2} & \\ & -\frac{r}{2} \end{bmatrix}. \tag{6.17}$$

Hence it is clear that it preserves the decomposition $H_k^2 = \oplus [p_a]^2$. Now since each H_k is a copy of $\mathrm{Sym}^k\mathbb{C}^2$ it suffices to find the eigenvalues of $D_{ra} : (\mathrm{Sym}^k\mathbb{C}^2)^2 \to (\mathrm{Sym}^k\mathbb{C}^2)^2$. Since $\mathrm{Sym}^k\mathbb{C}^2$ is identified with the set of homogeneous polynomials of degree k in two variables x, y it has an obvious basis $h_a = x^a y^{k-a}$ with the action

$$\begin{aligned} Hh_a &= (2a-k)h_a && (6.18) \\ Xh_a &= (k-a)h_{a+1} && (6.19) \\ Yh_a &= ah_{a-1}. && (6.20) \end{aligned}$$

The vectors

$$\begin{bmatrix} 0 \\ h_0 \end{bmatrix} \text{ and } \begin{bmatrix} h_k \\ 0 \end{bmatrix} \tag{6.21}$$

are eigenvectors of D_{ra} with eigenvalue $-k - \frac{3}{2} - \frac{r}{2}$ and $-k - \frac{3}{2} + \frac{r}{2}$ respectively. To find the remaining eigenvalues we note that D_{ra} leaves invariant the spaces

$$V_a = \mathbb{C}\begin{bmatrix} h_a \\ 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 \\ h_{a+1} \end{bmatrix}, a = 0, \ldots, k-1. \tag{6.22}$$

and its restriction to each V_a is the matrix

$$\begin{bmatrix} k - 2a - \frac{3}{2} + \frac{r}{2} & -2(a+1) \\ 2a - 2k & 2a - k + 2 - \frac{3}{2} - \frac{r}{2}. \end{bmatrix} \tag{6.23}$$

Hence it remains to find the eigenvalues of these 2×2 matrices which can be done easily. Finally noting that each H_k consists of $k+1$ copies of $\mathrm{Sym}^k\mathbb{C}^2$ we have the following conclusion.

Theorem 6.1.3. *The eigenvalues and multiplicites of the Dirac operator D_{ra} on S^3 are*

$$\lambda = \begin{cases} -k - \frac{3}{2} \pm \frac{r}{2} & ; \\ -\frac{1}{2} \pm \sqrt{(k - 2a - \frac{r}{2} - 1)^2 + 4(a+1)(k-a)} & \textit{for } a = 0, \ldots, k-1. \end{cases}$$

where each occurs with multiplicity $k+1$ and $k = 0, 1, \ldots$.

From (6.1.3) it is possible to find the spectral flow for the family D_{ra}. First note that for $a = 0, \ldots, k-1$,

$$-\frac{1}{2} + \sqrt{(k - 2a - \frac{r}{2} - 1)^2 + 4(a+1)(k-a)} \geq \frac{3}{2} \quad \text{and} \tag{6.24}$$

$$-\frac{1}{2} - \sqrt{(k - 2a - \frac{r}{2} - 1)^2 + 4(a+1)(k-a)} \leq -\frac{5}{2}. \tag{6.25}$$

Hence for $r > 0$ only the eigenvalues of the type $\lambda = -k - \frac{3}{2} + \frac{r}{2}$ contribute to the spectral flow. Since each of these occurs with multiplicity $k+1$ we see that the spectral flow function is given by

$$sf(D, D_{ra}) = \sum_{k=1}^{[\frac{r-1}{2}]} k, \tag{6.26}$$

where [] denotes the greatest integer function. The asymptotics of this function for large r are hence seen to be

$$sf(D, D_{ra}) = \frac{1}{8}r^2 + O(r). \tag{6.27}$$

The $O(r)$ remainder term is seen to be optimal in this case since that it the size of the jump discontinuity near r.

6.2 Spectral flow on S^{2m+1}

The computation for the spectrum on the three sphere of last section uses the group structure on S^3 and does readily extend to higher dimensions. In this section we compute the spectrum, and the corresponding spectral flow function, for a family of coupled Dirac operators on the odd dimensional sphere. The computation is similar to the computation in [6] for the spectrum of the spin Dirac operator on Berger spheres. The only difference here is in the presence of a coupling. Since the odd dimensional sphere will be written as a homogeneous space, we will first begin with studying the Dirac operator on homogeneous spaces.

6.2.1 The Dirac operator on a homogeneous space

Consider an oriented Riemannian homogeneous space (M, g). This is an oriented Riemannian manifold possessing a smooth transitive action of a Lie group G by isometries. We shall assume G to be connected. Pick a point $p \in M$ and let $H = \mathrm{Stab}(p)$ be its stabilizer. This a closed subgroup of G and we may identify the $M = G/H$ with the coset space of H. Let $\pi : G \to M$ denote the natural projection map given by $\pi(g) = gp$. Choose an Ad_H invariant complement $\mathfrak{p}$ to the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Hence we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and that $\pi_* : \mathfrak{p} \to T_pM$ is an isomorphism.

The Riemannian metric gives an Ad_H invariant inner product $\langle , \rangle$ on $\mathfrak{p}$. Since G is connected its action is orientation preserving. Hence the image of the isotropy representation

$$\alpha : H \to SO(T_pM) \tag{6.28}$$

is contained in the special orthogonal group. Choose a positively oriented basis $X_1, \ldots, X_n$ for $\mathfrak{p}$ and denote the left invariant extensions of these to G by the same letters X_i. Let $\bar{X}_i = \pi_* X_i$ be the corresponding basis for T_pM.

Proposition 6.2.1. *The principal bundle of special orthogonal frames $SO(TM)$ and the tangent bundle TM maybe identified with*

$$SO(TM) = G \times_\alpha SO(T_pM) \tag{6.29}$$

$$TM = G \times_\alpha T_pM. \tag{6.30}$$

Here (6.29) *and* (6.30) *are the quotients of the respective products by the equivalence relations $[g, A] \sim [gh, \alpha(h^{-1})A]$ and $[g, v] \sim [gh, \alpha(h^{-1})v]$ with $g \in G$, $h \in H$, $v \in T_pM$ and $A \in SO(T_pM)$.*

Proof. The first identification is induced by the map

$$m : G \times SO(T_pM) \to SO(TM), \quad m(g, A) = (L_g)_*(A\bar{X}_i) \tag{6.31}$$

while the second is induced by

$$m : G \times T_pM \to TM, \quad m(g, v) = (L_g)_* v. \tag{6.32}$$

□

Now let $\alpha' : H \to Spin(T_pM)$ be a lift of the isotropy representation so that the following diagram commutes

$$\begin{array}{ccc} & & Spin(T_pM) \\ & \overset{\alpha'}{\nearrow} & \big\downarrow \theta \\ H & \xrightarrow{\alpha} & SO(T_pM) \end{array}$$

where $\theta : Spin(T_pM) \to SO(T_pM)$ is the usual double covering map. Define a spin structure on M by $Spin_{\alpha'}(M) = G \times_{\alpha'} Spin(T_pM)$. The covering map $Spin_{\alpha'}(M) \to SO(TM)$ is defined using the identification (6.29) and the usual double covering θ. In the case where G is simply connected all spin structures on M arise via such a lift of the isotropy representation (cf Lemma 3 in [5]).

Let $S_{\alpha'}$ denote the spin bundle corresponding to the spin structure $Spin_{\alpha'}(M)$. Let $cl : Spin(T_pM) \to U(S)$ be the spin representation.

Proposition 6.2.2. *The spin bundle $S_{\alpha'}$ maybe identified with*

$$S_{\alpha'} = G \times_{cl\circ\alpha'} S. \tag{6.33}$$

Under the identifications (6.30) *and* (6.33)*, Clifford multiplication is given by* $[g, v] \cdot [g, s] = [g, v \cdot s]$ *with* $s \in S$.

Proof. The spin bundle is defined as $S_{\alpha'} = Spin_{\alpha'}(M) \times_{cl} S$. The identification (6.33) is now induced by the map

$$m : G \times S \to S_{\alpha'}, \quad m(g, s) = [[g, 1], s]. \tag{6.34}$$

The formula for Clifford multiplication now follows from the definition. □

Now for $X, Y \in \mathfrak{g}$, denote by $[X, Y]_{\mathfrak{p}}$ the $\mathfrak{p}$ component of the Lie bracket $[X, Y]$. Define the constants

$$\alpha_{ijk} = \frac{1}{4}\left(\langle [X_i, X_j]_{\mathfrak{p}}, X_k\rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i\rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j\rangle\right) \tag{6.35}$$

$$\beta_i = \frac{1}{2}\sum_{j=1}^{n}\langle [X_j, X_i]_{\mathfrak{p}}, X_j\rangle. \tag{6.36}$$

From the identification (6.33) it follows that sections of $S_{\alpha'}$ correspond to $cl \circ \alpha'$ equivariant maps $\Psi : G \to S$. The section $\psi : M \to S_{\alpha'}$ corresponding to Ψ is given via $\psi(gp) = [g, \Psi(g)]$. The proposition below gives a formula for the Dirac operator.

Proposition 6.2.3. *Let D be the spin Dirac operator on $S_{\alpha'}$ and $\Psi : G \to S$ be a $cl \circ \alpha'$ equivariant map corresponding to a section ψ of $S_{\alpha'}$. The equivariant map $D\Psi : G \to S$ corresponding to $D\psi$ is given by*

$$D\Psi(g_0) = \sum_{i=1}^{n} \bar{X}_i \cdot X_i(\Psi)|_{g_0} + \left(\sum_{i=1}^{n} \beta_i \bar{X}_i + \sum_{i<j<k} \alpha_{ijk} \bar{X}_i \bar{X}_j \bar{X}_k \right) \cdot \Psi(g_0). \tag{6.37}$$

Proof. Let $p_0 = g_0 p$. Let $\sigma : M \to G$ be a local section of π near p_0 such that $\pi \circ \sigma = Id$ with

$$\sigma(p_0) = g_0 \quad \text{and} \tag{6.38}$$

$$T_{g_0}\sigma = (L_{g_0})_* \mathfrak{p}. \tag{6.39}$$

This gives a local trivialization of the principal bundle $Spin_{\alpha'}(M)$ via $\lambda(m) = [\sigma(m), 1]$. The induced trivialization of the frame bundle $SO(TM)$ is given by the local orthonormal frame $e_i = [\sigma(m), \bar{X}_i]$. The section ψ of $S_{\alpha'}$ is now given in the induced local trivialization via $[\sigma(m), s(m)]$ where $s(m) = \Psi(\sigma(m))$. The Dirac operator in this trivialization is given by

$$D\psi(p_0) = [\sigma(p_0), \sum_i e_i \cdot \frac{ds}{de_i}(p_0) + \frac{1}{2} \sum_i \sum_{j<k} \Gamma_{ij}^k (e_i e_j e_k) \cdot s(p_0)] \tag{6.40}$$

(see page 41 in [34]). Here Γ_{ij}^k denote the Christoffel symbols for the Levi-Civita connection defined via $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$. These can be computed to be

$$\Gamma_{ij}^k = \frac{1}{2} (a_{ijk} + a_{kij} - a_{jki}) \tag{6.41}$$

in terms of the Lie brackets in the orthonormal frame

$$a_{ijk} = \langle [e_i, e_j], e_k \rangle. \tag{6.42}$$

It follows from $\pi_*(X_i|_\sigma) = e_i$ and (6.39) that

$$\pi_*([X_i, X_j]|_{g_0}) = [e_i, e_j]|_{p_0}. \tag{6.43}$$

Hence we have that

$$a_{ijk}(p_0) = \langle [X_i, X_j]_{\mathfrak{p}}, X_k \rangle. \tag{6.44}$$

The formula (6.37) now follows from (6.40), (6.41) and (6.44). □

The group G acts on the space of $cl \circ \alpha'$ equivariant maps $\Psi : G \to S$ via

$$(g\Psi)(g_0) = \Psi(g^{-1}g_0). \tag{6.45}$$

This defines a representation of the group G on the Hilbert space $L^2(M, S_{\alpha'})$ of square integrable sections of $S_{\alpha'}$. We let $\hat{G}$ denote the set of all equivalence classes of irreducible representations $\rho : G \to U(V_\rho)$. Given $V_\rho \in \hat{G}$, let $Hom_H(V_\rho, S)$ denote the space of all H-module homomorphisms from V_ρ to S. The space $V_\rho \otimes Hom_H(V_\rho, S)$ admits a representation of G via $g(v \otimes A) = gv \otimes A$ for $g \in G, A \in Hom_H(V_\rho, S)$. This representation can be embedded into $L^2(M, S_{\alpha'})$ where $v \otimes A$ corresponds to the $cl \circ \alpha'$ equivariant map sending $g \mapsto A(\rho(g^{-1})v)$. The next proposition gives the decomposition of $L^2(M, S_{\alpha'})$ into irreducible components. This is theorem 5.3.6 in [42].

Theorem 6.2.4. *(Frobenius Reciprocity) The unitary representation $L^2(M, S_{\alpha'})$ of G is the unitary direct sum*

$$L^2(M, S_{\alpha'}) = \bigoplus_{\rho \in \hat{G}} V_\rho \otimes Hom_H(V_\rho, S) \tag{6.46}$$

over all irreducible representations $\rho \in \hat{G}$.

It is straightforward to observe from proposition (6.2.3) the Dirac operator commutes with the action of G. Hence the ρ-isotypical parts $V_\rho \otimes Hom_H(V_\rho, S)$ of (6.46) are invariant under the action of G. Proposition (6.2.3) can now be used to determine the restriction of D to each isotypical part below.

Proposition 6.2.5. *The restriction of the Dirac operator D to each isotypical part $V_\rho \otimes Hom_H(V_\rho, S)$ is given by $id \otimes D_\rho$ where*

$$D_\rho(A) = -\sum_i \bar{X}_i \cdot A(\pi_\rho)_*(X_i) + \left(\sum_{i=1}^n \beta_i \bar{X}_i + \sum_{i<j<k} \alpha_{ijk} \bar{X}_i \bar{X}_j \bar{X}_k\right) \cdot A. \tag{6.47}$$

Here $(\pi_\rho)_*$ is the derived action of $\mathfrak{g}$ on V_ρ.

Proof. Let $\Psi(g) = A(\rho(g^{-1})v)$ denote the $cl \circ \alpha'$ equivariant map corresponding to $v \otimes A \in V_\rho \otimes Hom_H(V_\rho, S)$. Following proposition (6.2.3) we have

$$D\Psi(g_0) = \sum_{i=1}^n \bar{X}_i \cdot X_i(\Psi)|_{g_0} + \left(\sum_{i=1}^n \beta_i \bar{X}_i + \sum_{i<j<k} \alpha_{ijk} \bar{X}_i \bar{X}_j \bar{X}_k\right) \cdot \Psi(g_0). \tag{6.48}$$

We may also compute

$$\begin{aligned} X_i(\Psi)|_{g_0} &= \frac{d}{dt}\Psi(g_0 e^{tX_i})|_{t=0} && (6.49)\\ &= \frac{d}{dt}A(\rho(e^{-tX_i} g_0^{-1})v)|_{t=0} && (6.50)\\ &= -A(\pi_\rho)_*(X_i)\rho(g_0^{-1})v && (6.51) \end{aligned}$$

The proposition now follows from (6.48) and (6.51). □

6.2.2 The group $\tilde{U}(m)$ and its representations

Our eventual goal is to compute the spectrum of the coupled Dirac operator on the odd dimensional sphere

$$S^{2m+1} = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} | |z_0|^2 + \ldots + |z_m|^2 = 1\}. \tag{6.52}$$

In order to do this we shall need to understand the double cover of the unitary group and its representations in this section. The sphere maybe written as the homogeneous space $S^{2m+1} = U(m+1)/U(m)$, with $U(m)$ denoting the unitary group. However the corresponding isotropy representation of $U(m)$ does not admit any lift to the spin group. For this reason we write $S^{2m+1} = \tilde{U}(m+1)/\tilde{U}(m)$. Here $\tilde{U}(m)$ denotes the double cover of $U(m)$ defined as

$$\tilde{U}(m) = \{(A, \alpha) \in U(m) \times U(1) | det(A) = \alpha^2\}. \tag{6.53}$$

It is clear that the natural projection of $\tilde{U}(m)$ onto $U(m)$ is a double cover. This projection also defines an action of $\tilde{U}(m+1)$ on S^{2m+1} via the natural action of $U(m+1)$ on $\mathbb{C}^{m+1}$. Choosing $p = (1, 0, \ldots, 0)$ gives $Stab(p) = \tilde{U}(m)$. The tangent space T_pS^{2m+1} is the span of $\partial_{y_0}, \partial_{x_\mu}, \partial_{y_\mu}$ with $\mu \geq 1$. The isotropy representation acts trivially on ∂_{y_0} and corresponds to the natural map $\imath : \tilde{U}(m) \to SO(2m)$ in the basis $\partial_{x_\mu}, \partial_{y_\mu}$ with $\mu \geq 1$. We now construct a lift of the isotropy representation below.

Proposition 6.2.6. *There exists a unique group homomorphism $\jmath : \tilde{U}(m) \to Spin(2m)$ such that*

$$\begin{array}{ccc} & & Spin(2m) \\ & \overset{\jmath}{\nearrow} & \downarrow \theta \\ \tilde{U}(m) & \xrightarrow{\imath} & SO(2m) \end{array}$$

is a commutative diagram.

Proof. The case $m = 1$ is easily verifiable. For $m \geq 2$, $\pi_1(\tilde{U}(m))$ is an order 2 subgroup of $\pi_1(U(m)) = \mathbb{Z}$. Its generator is mapped under $\imath_*$ to twice the generator of $\pi_1(SO(2m)) = \mathbb{Z}_2$ and is hence killed. Thus there must exist a unique lift of $\imath$ to a group homomorphism $\jmath : \tilde{U}(m) \to Spin(2m)$. To give an explicit formula for $\jmath$, let $(A, \alpha) \in \tilde{U}(m)$. Choose a unitary basis of eigenvectors $e_1, \ldots, e_m \in \mathbb{C}^m$ for A such that $Ae_k = e^{i\theta_k} e_k$. Considering $e_k, Je_k (= ie_k)$ as vectors in $\mathbb{R}^{2m}$ we claim that

$$\jmath(A, \alpha) = \prod_{k=1}^{m} \left(\cos(\tilde{\theta_k}) + \sin(\tilde{\theta_k}) e_k J e_k \right) \in Spin(2m). \tag{6.54}$$

Here $\tilde{\theta_k}$ are chosen such that $e^{i2\tilde{\theta_k}} = e^{i\theta_k}, \forall k$ and $\prod e^{i\tilde{\theta_k}} = \alpha$. It is easy to check that the right hand side of (6.54) is well-defined and gives a continuous map sending the identity in $\tilde{U}(m)$ to the identity in $Spin(2m)$. A direct computation also shows that this is a lift of the map $\imath$ and hence must equal to the unique group homomorphism $\jmath$. □

To find all the irreducible representations of $\tilde{U}(m)$ we first recall all the irreducible representations of the unitary group $U(m)$. In the case of $U(m)$, its irreducible representations are characterized by their highest weights which are m-tuples of integers $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ satisfyig

$$k_1 \geq k_2 \geq \ldots \geq k_m \tag{6.55}$$

(cf. Theorem 4 page 133 of [43]). We denote this representation by V_k. The standard representation Λ^1 of $U(m)$ corresponds to the weight $(1, 0, \ldots, 0)$, while its exterior powers Λ^j correspond to the the weight $(\underbrace{1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0)$. The dimension of the representation V_k is given by Weyl's dimension formula (cf. page 214 in [43])

$$\dim V_k = \frac{\prod_{1 \leq \mu < \nu \leq m} (k_\mu - k_\nu + \nu - \mu)}{1!2! \ldots (m-1)!}. \tag{6.56}$$

It is clear that each V_k induces an irreducible representation of $\tilde{U}(m)$. From the definition (6.53) we have another one dimensional representation of $\tilde{U}(m)$, which we denote by Σ_0, given by the projection onto $U(1)$. Since $-1 = (I, -1) \in \tilde{U}(m)$ acts by $-Id$ on Σ_0, it is not induced from a representation of $U(m)$. It is also easy to note that $\Sigma_0 \otimes \Sigma_0 = \Lambda^m$. Now given any irreducible representation V_ρ of $\tilde{U}(m)$, we have $\rho(-1)^2 = I$. Since -1 is in the center of $\tilde{U}(m)$, the ± 1 eigenspaces of $\rho(-1)$ are $\tilde{U}(m)$ invariant and we must have that $\rho(-1) = \pm I$ on V_ρ. In the case where $\rho(-1) = I$,

V_ρ is induced by a representation of $U(m)$. While in the case where $\rho(-1) = -I$ we have that $V_\rho \otimes \Sigma_0$ is induced from $U(m)$. Hence in conclusion, the irreducible representations of $\tilde{U}(m)$ are of two types. The first consists of the representations V_k induced from $U(m)$. While the second consists of those not induced from $U(m)$ and can be written in the form $V_k \otimes \Sigma_0$ or $V_k \otimes \Sigma_0^*$.

We shall also need to know how the irreducible representations V_k, $V_k \otimes \Sigma_0^*$ of $\tilde{U}(m+1)$ decompose when restricted to $\tilde{U}(m)$. The branching rules for $U(m)$ (cf. page 186 of [43]) state that we have the decomposition

$$V_k = \bigoplus_l V_l \tag{6.57}$$

where the direct sum is over the $l = (l_1, \ldots, l_m)$ satisfying

$$k_1 \geq l_1 \geq k_2 \geq l_2 \geq \ldots \geq l_m \geq k_{m+1}. \tag{6.58}$$

Tensoring with Σ_0^* now gives the analogous braching rules for $V_k \otimes \Sigma_0^*$.

Let $\delta_{\mu\nu}$ denote the matrix containing a 1 in the μth row and νth column and 0's otherwise. The Lie algebra $\mathfrak{u}(m)$ of $\tilde{U}(m)$ is spanned by $H_\mu = 2i\delta_{\mu\mu}$, $X_{\mu\nu} = 2(\delta_{\nu\mu} - \delta_{\mu\nu})$ for $\mu < \nu$ and $Y_{\mu\nu} = 2i(\delta_{\mu\nu} + \delta_{\nu\mu})$. The Casimir element

$$C = \sum_\mu H_\mu \circ H_\mu + \frac{1}{2} \sum_{\mu<\nu} (X_{\mu\nu} \circ X_{\mu\nu} + Y_{\mu\nu} \circ Y_{\mu\nu}) \tag{6.59}$$

commutes with the action of the lie algebra and hence acts by a constant c_k on any irreducible $\tilde{U}(m)$ module V_k. Now if

$$Z_{\mu\nu} = X_{\mu\nu} + iY_{\mu\nu} = -4\delta_{\mu\nu} \tag{6.60}$$

$$\bar{Z}_{\mu\nu} = X_{\mu\nu} - iY_{\mu\nu} = 4\delta_{\nu\mu} \tag{6.61}$$

then $[Z_{\mu\nu}, \bar{Z}_{\mu\nu}] = 8i(H_\mu - H_\nu)$ and we may write

$$C = \sum_{\mu} H_\mu \circ H_\mu + \frac{1}{4}\sum_{\mu<\nu}\left(Z_{\mu\nu} \circ \bar{Z}_{\mu\nu} + \bar{Z}_{\mu\nu} \circ Z_{\mu\nu}\right) \tag{6.62}$$

$$= \sum_{\mu} H_\mu \circ H_\mu + \sum_{\mu<\nu}\left(\frac{1}{2}\bar{Z}_{\mu\nu} \circ Z_{\mu\nu} + 2i(H_\mu - H_\nu)\right). \tag{6.63}$$

If v is the highest weight vector of V_k we have $H_\mu v = 2ik_\mu v$ while $Z_{\mu\nu}v = 0$. Hence the constant c_k is computed to be

$$c_k = -4\left(\sum_{a} k_\mu^2 + \sum_{\mu<\nu}(k_\mu - k_\nu)\right). \tag{6.64}$$

Similarly the action of the Casimir element on $(v \otimes 1) \in V_k \otimes \Sigma_0^*$ and hence this irreducible module is by the constant

$$\tilde{c}_k = -4\left(\sum_{a}\left(k_\mu - \frac{1}{2}\right)^2 + \sum_{\mu<u}(k_\mu - k_\nu)\right). \tag{6.65}$$

The highest weight vector v_l of the summand V_l in(6.57) is a weight vector of $U(m+1)$ with weight

$$\left(\sum k_\mu - \sum l_\mu, l_1, \ldots, l_m\right) \tag{6.66}$$

(cf. page 187 in [43]). Hence the action of $(\pi_k)_*(H_0)$ on $v_l \otimes 1$ is given by

$$(\pi_k)_*(H_0)(v_l \otimes 1) = 2i\left(\sum k_\mu - \sum l_\mu - \frac{1}{2}\right)(v_l \otimes 1). \tag{6.67}$$

The spin representation S is a representation of $\tilde{U}(m)$ via the composition with $\jmath$. To construct the spin representation S let V be the $2m$ dimensional subspace of T_pS^{2m+1} spanned by $\partial_{x_\mu}, \partial_{y_\mu}$ for $\mu \geq 1$. Let $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ with the two summands being the subspaces spanned by

$$\partial_{z_\mu} = \frac{1}{\sqrt{2}}(\partial_{x_\mu} - i\partial_{y_\mu}) \quad \text{and} \quad \partial_{\bar{z}_\mu} = \frac{1}{\sqrt{2}}(\partial_{x_\mu} + i\partial_{y_\mu}) \tag{6.68}$$

respectively. Define an inner product on $V \otimes \mathbb{C}$ which extends the metric on V by complex bilinearity. The spin module can be defined as

$$S = \bigoplus_{r=0}^{m} (\Lambda^r V^{1,0} \otimes \Sigma_0^*), \tag{6.69}$$

with

$$\begin{aligned} cl(\partial_{x_\mu})(\omega \otimes 1) &= (\partial_{z_\mu} \wedge \omega - \iota_{\partial_{\bar{z}_\mu}} \omega) \otimes 1, & (6.70)\\ cl(\partial_{y_\mu})(\omega \otimes 1) &= i(\partial_{z_\mu} \wedge \omega + \iota_{\partial_{\bar{z}_\mu}} \omega) \otimes 1 \quad \text{for } \mu \geq 1, \text{while} & (6.71)\\ cl(\partial_{y_0})(\omega \otimes 1) &= i(-1)^j (\omega \otimes 1) \quad \text{for } \omega \in \Lambda^j. & (6.72) \end{aligned}$$

There is a natural action of $U(m)$, and hence of $\tilde{U}(m)$, on $V^{1,0}$ in the complex basis ∂_{z_μ}. This induces a representation of $\tilde{U}(m)$ on $S = \Lambda^* V^{1,0} \otimes \Sigma_0^*$. It is a straightforward computation using (6.54) to check that this representation agrees with the representation $cl \circ \jmath : \tilde{U}(m) \to U(S)$. Hence (6.69) gives the decomposition of the spin representation into irreducible representations of $\tilde{U}(m)$.

6.2.3 The spectrum of the Dirac operator

Now we come to the computation of the Dirac spectrum on the sphere. Recall that sphere was written as the homogeneous space $S^{2m+1} = \tilde{U}(m+1)/\tilde{U}(m)$. The lift $\jmath$ of the isotropy representation gives rise to the unique spin structure on the sphere and a corresponding Dirac operator on the spin bundle S. Now let R denote the vector field which the infinitesimal generator for the diagonal S^1 action on S^{2m+1} via $e^{i\theta}(z_0, \ldots, z_m) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_m)$. The dual to R is a contact one form on S^{2m+1} which we denote by a. We now twist the spin bundle by the trivial Hermitian line bundle $\mathbb{C}$ equipped with the connection $d - ira$ for a parameter $r \geq 0$. We shall compute the spectrum of the corresponding coupled Dirac operator D_{ra}. The space of sections of the spin bundle S can again be decomposed

$$L^2(M, S) = \bigoplus_{\rho} V_\rho \otimes Hom_{\tilde{U}(m)}(V_\rho, S) \tag{6.73}$$

with the direct sum being taken over the irreducible representations ρ of $\tilde{U}(m+1)$. Following the decomposition (6.69) of S into irreducible representations we see that the ρ-isotypical part of (6.73) is nontrivial if and only if the restriction of V_ρ to $\tilde{U}(m)$ contains an irreducible representation of the form $\Lambda^j \otimes \Sigma_0$. Using the branching rule (6.57) we see that the V_ρ must be of one of the following types

I. $V_\rho = V_k \otimes \Sigma_0^*$ with $k = (0, \ldots, 0, b)$, $b \leq 0$ which contains $\Lambda^0 \otimes \Sigma_0^*$

II. $V_\rho = V_k \otimes \Sigma_0^*$ with $k = (a+1, 1, \ldots, 1)$, $a \geq 0$ which contains $\Lambda^m \otimes \Sigma_0^*$

III. $V_\rho = V_k \otimes \Sigma_0^*$ with $k = (a+1, \underbrace{1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0, b)$, $a \geq 0, b \leq 0, 0 \leq j \leq m-1$, which contains $\Lambda^j \otimes \Sigma_0^*$ and $\Lambda^{j+1} \otimes \Sigma_0^*$.

Under the projection $\pi : \tilde{U}(m) \to S^{2m+1}$, $\pi(A, \alpha) = Ap$ we have

$$\pi_*\left(\frac{1}{2}H_0\right) = \partial_{y_0}, \quad \pi_*\left(\frac{1}{2}X_{0\mu}\right) = \partial_{x_\mu}, \quad \pi_*\left(\frac{1}{2}Y_{0\mu}\right) = \partial_{y_\mu}. \tag{6.74}$$

Since the diagonal S^1 action commutes with the $\tilde{U}(m)$ action on the sphere, following proposition (6.2.5) the coupled Dirac operator preserves the decomposition (6.73) and its action on the ρ-isotypical part is by $id \otimes D_\rho$. In this case we may compute that each $\beta_i = 0$ while the only non-zero constants α_{ijk} are

$$\frac{1}{4}\left(\left\langle\left[\frac{1}{2}H_0, \frac{1}{2}X_{0\mu}\right]_{\mathfrak{p}}, \frac{1}{2}Y_{0\mu}\right\rangle + \left\langle\left[\frac{1}{2}X_{0\mu}, \frac{1}{2}Y_{0\mu}\right]_{\mathfrak{p}}, \frac{1}{2}H_0\right\rangle + \left\langle\left[\frac{1}{2}Y_{0\mu}, \frac{1}{2}H_0\right]_{\mathfrak{p}}, \frac{1}{2}X_{0\mu}\right\rangle\right) = -1. \tag{6.75}$$

Under the observation that the vector field R corresponds to $[g, \partial_{y_0}]$ in (6.30), D_ρ is given by the formula

$$D_\rho(A) = -\frac{1}{2}\partial_{y_0} \cdot A(\pi_\rho)_*(H_0) - \frac{1}{2}\sum_\mu \left\{\partial_{x_\mu} \cdot A(\pi_\rho)_*(X_{0\mu}) + \partial_{y_\mu} \cdot A(\pi_\rho)_*(Y_{0\mu})\right\}$$
$$- \sum_\mu \partial_{y_0}\partial_{x_\mu}\partial_{y_\mu} \cdot A - ir\partial_{y_0} \cdot A. \tag{6.76}$$

It now suffices to compute the spectrum of D_ρ under the three types I, II and III.

Type I

In this case D_ρ acts on the one dimensional space $End_{\tilde{U}(m)}(\Lambda^0 \otimes \Sigma_0^*)$. Since Clifford multiplication by ∂_{x_μ} and ∂_{y_μ} switch the $\Lambda^{odd/even} \otimes \Sigma_0^*$ parts we see that the middle term in (6.76) acts trivially. Following (6.67), (6.70) and (6.72) we see that the first term acts by $b-\frac{1}{2}$ the third acts by $-m$ and the last by r. Hence D_ρ has the eigenvalue

$$\lambda = b - m - \frac{1}{2} + r. \tag{6.77}$$

The multiplicity of this eigenvalue is the dimension of the representation V_k and is computed via (6.56) to be the binomial coefficient $\binom{m-b}{m}$.

Type II

In this case D_ρ acts on the one dimensional space $End_{\tilde{U}(m)}(\Lambda^m \otimes \Sigma_0^*)$. Again the middle term in (6.76) acts trivially. The first term now acts by $(-1)^m(a+\frac{1}{2})$ the third acts via $(-1)^m m$ and the last by $(-1)^m r$. Hence D_ρ has the eigenvalue

$$\lambda = (-1)^m(a + m + \frac{1}{2} + r). \tag{6.78}$$

whose multiplicity is again calculated via (6.56) to be $\binom{m+a}{m}$.

Type III

In this case D_ρ acts on the two dimensional space $End_{\tilde{U}(m)}(\Lambda^j \otimes \Sigma_0^*) \oplus End_{\tilde{U}(m)}(\Lambda^{j+1} \otimes \Sigma_0^*)$. Let A_1, A_2 denote the identity endomorphisms in the respective summands and let $D_\rho = \left(\begin{smallmatrix} x & u \\ v & y \end{smallmatrix}\right)$ in the basis given by the $A_i's$. Again since $cl(\partial_{x_\mu})$ and $cl(\partial_{y_\mu})$ switch the $\Lambda^{odd/even}$ parts we have that the off diagonal terms u and v come from the second term in (6.76). Similarly the diagonal terms x and y come from the first, third and last summands in (6.76). The terms x and y can be easily computed after noting that Clifford multiplication by $\omega = \sum \partial_{x_\mu}\partial_{y_\mu}$ acts via $i(2j-m)$ on $\Lambda^j \otimes \Sigma_0^*$. Hence using (6.67) and (6.72) we may compute

$$x = (-1)^j \left(a + b + 2j - m + r + \frac{1}{2}\right) \tag{6.79}$$

$$y = (-1)^{j+1} \left(a + b + 2j - m + r + \frac{3}{2}\right). \tag{6.80}$$

Next we compute

$$\left(D_\rho + \frac{1}{2}\partial_{y_0}\omega + ir\partial_{y_0}\right)^2 A = -\frac{1}{4}A \circ \sum_{i=1}^{m} \{(\pi_\rho)_*(X_{0\mu}) \circ (\pi_\rho)_*(X_{0\mu}) + (\pi_\rho)_*(Y_{0\mu}) \circ (\pi_\rho)_*(Y_{0\mu})\}$$

$$- \frac{1}{4}A \circ (\pi_\rho)_*(H_0) \circ (\pi_\rho)_*(H_0) + \frac{1}{2}\omega \cdot A \circ (\pi_\rho)_*(H_0)$$
$$+ \frac{3}{4}\omega^2 \cdot A + m(m+1)A. \tag{6.81}$$

Here we have used the commutation relations

$$[X_{0\mu}, X_{0\nu}] = [Y_{0\mu}, Y_{0\nu}] = 2X_{\mu\nu}, \quad [X_{0\mu}, Y_{0\nu}] = 2Y_{\mu\nu} \tag{6.82}$$

$$[H_0, X_{0\mu}] = -2Y_{0\mu}, \quad [H_0, Y_{0\mu}] = 2X_{0\mu}, \quad [X_{0\mu}, Y_{0\mu}] = 4(H_\mu - H_0) \tag{6.83}$$

as well as the formulas

$$(\pi_{cl\circ \mathrm{J}})_*(H_\mu) = \partial_{x_\mu}\partial_{y_\mu} \tag{6.84}$$

$$(\pi_{cl\circ \mathrm{J}})_*(X_{\mu\nu}) = \partial_{x_\mu} \cdot \partial_{x_\nu} + \partial_{y_\mu} \cdot \partial_{y_\nu} \tag{6.85}$$

$$(\pi_{cl\circ \mathrm{J}})_*(Y_{\mu\nu}) = \partial_{x_\mu} \cdot \partial_{y_\nu} - \partial_{y_\mu} \cdot \partial_{x_\nu} \tag{6.86}$$

for the derived action on the spin representation. Now we simplify (6.81) to give

$$\left(D_\rho + \frac{1}{2}\partial_{y_0}\omega + ir\partial_{y_0}\right)^2 A = \frac{1}{2}A \circ (\pi_\rho)_*(C_{\tilde{U}(m)} - C_{\tilde{U}(m+1)})$$
$$+ \frac{1}{4}A \circ (\pi_\rho)_*(H_0) \circ (\pi_\rho)_*(H_0) + \frac{1}{2}\omega \cdot A \circ (\pi_\rho)_*(H_0)$$
$$+ \frac{3}{4}\omega^2 \cdot A + m(m+1)A, \qquad (6.87)$$

where $C_{\tilde{U}(m)}$ and $C_{\tilde{U}(m+1)}$ denote the Casimir elements corresponding to $\tilde{U}(m+1)$ and $\tilde{U}(m)$ respectively. Now following (6.65) the action of the Casimir element $C_{\tilde{U}(m)}$ on $\Lambda^j \otimes \Sigma_0^*$ is given by the constant

$$\tilde{c}(1,\ldots,1,0,\ldots,0) = -4\left\{\frac{m}{4} + j(m-j)\right\}, \qquad (6.88)$$

while the action of $C_{\tilde{U}(m+1)}$ on V_ρ is given by

$$\tilde{c}(a+1,1,\ldots,1,0,\ldots,0,b) = -4\left\{\left(a+\frac{1}{2}\right)^2 + \left(b-\frac{1}{2}\right)^2 + \frac{m-1}{4} + m(a-b) + (m-j)(j+1)\right\}. \qquad (6.89)$$

Using (6.67) the action of $(\pi_\rho)_*(H_0)$ on the highest weight vectors v_j, v_{j+1} of the $\Lambda^j \otimes \Sigma_0^*$, $\Lambda^{j+1} \otimes \Sigma_0^*$ parts of V_ρ is given by

$$(\pi_\rho)_*(H_0)v_j = 2i(a+b+\frac{1}{2})v_j \quad \text{and} \quad (\pi_\rho)_*(H_0)v_{j+1} = 2i(a+b-\frac{1}{2})v_{j+1} \qquad (6.90)$$

respectively. Using these formulas we may compute

$$\left(D_\rho + \frac{1}{2}\partial_{y_0}\omega + ir\partial_{y_0}\right)^2 A_1 = \alpha A_1, \quad \left(D_\rho + \frac{1}{2}\partial_{y_0}\omega + ir\partial_{y_0}\right)^2 A_2 = \alpha A_2 \qquad (6.91)$$

with

$$\alpha = \alpha(a,b,j,m) = \left(a+b+j+\frac{1-m}{2}\right)^2 - 4(j+a+1)(b-m+j) \tag{6.92}$$

being the same constant for both A_1 and A_2. This now gives

$$\left(\begin{bmatrix} x & u \\ v & y \end{bmatrix} + \frac{(-1)^j}{2}\begin{bmatrix} m-2j-2r & 0 \\ 0 & 2j+2-m+2r \end{bmatrix}\right)^2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \tag{6.93}$$

which in turn is equivalent to the four equations

$$\left(x + \frac{(-1)^j}{2}(m-2j-2r)\right)^2 + uv = \alpha \tag{6.94}$$

$$\left(y + \frac{(-1)^j}{2}(2j+2-m+2r)\right)^2 + uv = \alpha \tag{6.95}$$

$$(x+y+(-1)^j)u = 0 \tag{6.96}$$

$$(x+y+(-1)^j)v = 0. \tag{6.97}$$

It hence gives

$$uv = -4(j+a+1)(b-m+j). \tag{6.98}$$

Now if λ_1, λ_2 denote the eigenvalues of D_ρ we have

$$\lambda_1 + \lambda_2 = trD_\rho = x + y = (-1)^{j+1}, \tag{6.99}$$

$$\lambda_1\lambda_2 = detD_\rho = xy - uv \tag{6.100}$$

$$= -\left(a+b+2j-m+r+\frac{1}{2}\right)\left(a+b+2j-m+r+\frac{3}{2}\right) + 4(j+a+1)(b-m+j). \tag{6.101}$$

Hence we may compute

$$\lambda_{1,2} = \frac{(-1)^{j+1}}{2} \pm \sqrt{(a+b+2j-m+r+1)^2 + 4(j+a+1)(m-j-b)}. \quad (6.102)$$

The multiplicity of each of these eigenvalues is the dimension of the representation V_k and is again computed via (6.56) to be

$$\frac{(m+a)!(m-b)!(a-b+1+m)}{m!j!(m-j-1)!a!(-b)!(a+j+1)(m-j-b)}. \quad (6.103)$$

We now summarize the computation of the spectrum in the theorem below.

Theorem 6.2.7. *The eigenvalues with multiplicities for the coupled Dirac operator D_{ra} on the odd sphere S^{2m+1} are given by*

i. $\lambda = r - (a + m + \frac{1}{2})$, *for $a \in \mathbb{N}_0$ with multiplicity $\binom{m+a}{m}$*

ii. $\lambda = (-1)^m(r + a + m + \frac{1}{2})$, *for $a \in \mathbb{N}_0$ with multiplicity $\binom{m+a}{m}$*

iii.

$$\lambda = \frac{(-1)^{j+1}}{2} \pm \sqrt{(a_1 - a_2 + 2j - m + r + 1)^2 + 4(j + a_1 + 1)(m - j + a_2)}, \quad (6.104)$$

for $a_1, a_2 \in \mathbb{N}_0$, $j = 0, \ldots, m-1$, each with multiplicity

$$\frac{(m+a_1)!(m+a_2)!(a_1+a_2+1+m)}{m!j!(m-j-1)!a_1!a_2!(a_1+j+1)(a_2+m-j)}. \quad (6.105)$$

We now compute the spectral flow function. It is easy to see that the eigenvalues of type *ii* are never zero. It is also easy to verify that the square root in (6.104) is atleast 2 and hence the eigenvalues of type *iii* are never zero. Hence the only eigenvalues which contribute to the spectral flow function are those of type *i*. The spectral flow function is now easily computed to be

$$sf(D, D_{ra}) = \sum_{a=0}^{[r-m-\frac{1}{2}]} \binom{m+a}{m}. \quad (6.106)$$

Using the binomial identity $\sum_{a=0}^{k} \binom{m+a}{m} = \binom{m+k+1}{m+1}$ the spectral flow function is seen to satisfy

$$sf(D, D_{ra}) = \frac{r^{m+1}}{(m+1)!} + O(r^m). \tag{6.107}$$

This is seen to be the sharp remainder since $O(r^m)$ is the size of the jump discontinuity in this example.

6.3 Spectral flow for $L(p, 1)$

In this section we compute the spectrum of coupled Dirac operators on homogeneous three dimensional Lens spaces. Let $SU(2) = S^3$ be the three sphere with the round metric. The three dimensional Lens space $L(p, 1)$ is the quotient of S^3 under the identification $A \sim A \begin{bmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{bmatrix}$ with $A \in SU(2), \zeta = e^{2\pi i/p}$. Left multiplication gives a $SU(2)$ action on $L(p, 1)$. The stabilizer of $[I] \in L(p, 1)$ is the subgroup generated by $\begin{bmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{bmatrix}$ which maybe identified with $\mathbb{Z}_p \subset U(1)$ consisting of the pth roots of unity. Let $X_i = \sigma_i$ be the basis for the Lie algebra $\mathfrak{su}(2)$ given by the Pauli matrices. Let $\pi_*(\sigma_i) = \bar{X}_i$ denote the corresponding pushforwards under the natural projection $\pi : S^3 \to L(p, 1)$. We may compute

$$\begin{aligned}
Ad_{e^{it}}\sigma_1 &= \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} 0 & ie^{2it} \\ ie^{-2it} & 0 \end{bmatrix} && (6.108)\\
&= \cos(2t)\sigma_1 + \sin(2t)\sigma_2, && (6.109)\\
Ad_{e^{it}}\sigma_2 &= \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} 0 & -e^{2it} \\ e^{-2it} & 0 \end{bmatrix} && (6.110)\\
&= -\sin(2t)\sigma_1 + \cos(2t)\sigma_2 \quad \text{and} && (6.111)\\
Ad_{e^{it}}\sigma_3 &= \sigma_3. && (6.112)
\end{aligned}$$

Hence we see that the isotropy representation $\alpha : \mathbb{Z}_p \to SO(3)$ is the restriction to $\mathbb{Z}_p$ of the map $\alpha : U(1) \to SO(3)$

$$\alpha(e^{it}) = \begin{bmatrix} \cos(2t) & \sin(2t) & 0 \\ -\sin(2t) & \cos(2t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.113}$$

in the basis given by the $\bar{X}_i$'s.

First consider the case where p is odd. In this case $H^1(L(p,1), \mathbb{Z}_2) = 0$ and there is a unique spin structure. It corresponds to the lift of the isotropy representation given by the restriction to $\mathbb{Z}_p$ of the map

$$\alpha' : U(1) \to SU(2), \quad \alpha'(e^{it}) = \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix}. \tag{6.114}$$

We twist the corresponding spin bundle by the trivial Hermitian line bundle $\mathbb{C}$ equipped with the connection $d - ira$. Here a is the unique one form on $L(p,q)$ whose pullback $\pi^*(a) = X_3^*$ on S^3. We wish to compute the spectrum of the corresponding coupled Dirac operator D_{ra}.

The irreducible representations of $\mathbb{Z}_p$ are parametrized by elements of $\mathbb{Z}_p$. The representation W_l corresponding to $l \in \mathbb{Z}_p$ is the one dimensional representation given by $\pi_l : \mathbb{Z}_p \to U(1)$ with $\pi_l(\zeta) = \zeta^l$. The irreducible representations of $SU(2)$ are $V_k = Sym^k(\mathbb{C}^2)$ and are spanned by the $k+1$ monomials $v_a = x^a y^{k-a}$, $0 \leq a \leq k$. The spin representation is the standard representation $S = \mathbb{C}^2$ of $SU(2)$ with Clifford multiplication by $\bar{X}_i$ being given by the Pauli matrices σ_i in the standard basis $s_1, s_2 \in \mathbb{C}^2$. The space of L^2 sections of the spin bundle decomposes as

$$L^2(S) = \bigoplus_k V_k \otimes Hom_{\mathbb{Z}_p}(V_k, S) \tag{6.115}$$

and the restriction of the Dirac operator to the k-isotypical part is of the form $id \otimes D_k$ by proposition (6.2.5). It is easy to compute $\beta_1 = \beta_2 = \beta_3 = 0$ and $\alpha_{123} = \frac{3}{2}$. Hence D_k can be computed to be

$$D_k(A) = -\sum_i \sigma_i \cdot A(\pi_k)_*(X_i) - \frac{3}{2}A - ir\sigma_3 \tag{6.116}$$

for $A \in Hom_{\mathbb{Z}_p}(V_k, S)$. Following (6.114) we compute the action of $\mathbb{Z}_p$ on a monomial $\pi_k \circ \alpha'(\zeta)(v_a) = \zeta^{k-2a}v_a$. Hence the restriction to $\mathbb{Z}_p$ of V_k decomposes as

$$V_k = \bigoplus_{a=0}^{k} W_{k-2a}. \tag{6.117}$$

The restriction to $\mathbb{Z}_p$ of the spin representation similarly decomposes as $S = W_{-1} \oplus W_1$. Using these decompositions we may write

$$\begin{aligned} Hom_{\mathbb{Z}_p}(V_k, S) &= Hom_{\mathbb{Z}_p}(W_k, W_{-1}) \oplus Hom_{\mathbb{Z}_p}(W_{-k}, W_1) \oplus & (6.118) \\ & \bigoplus_{a=0}^{k-1} \left(Hom_{\mathbb{Z}_p}(W_{k-2a}, W_1) \oplus Hom_{\mathbb{Z}_p}(W_{k-2a-2}, W_{-1})\right). & (6.119) \end{aligned}$$

By Schur's lemma the first two summands are nonzero when $k \equiv -1 \pmod{p}$ while index a part of the third summand is nonzero when $k - 2a \equiv 1 \pmod{p}$. When nonzero, these summands are preserved by the operator D_k using (6.116) and the relations (6.18). The restriction of D_k to the first two is then given by the scalars $-k - \frac{3}{2} + r$ and $-k - \frac{3}{2} - r$ respectively. While its restriction to the index a part of the third is the matrix

$$D_k = \begin{bmatrix} k - 2a - \frac{3}{2} + r & 2(a+1) \\ 2(k-a) & 2a - k + \frac{1}{2} - r \end{bmatrix}. \tag{6.120}$$

The eigenvalues and multiplicities are now computed to be

$$\lambda = \begin{cases} -bp - \dfrac{1}{2} \pm r & \text{for } b \in \mathbb{N} \\ -\dfrac{1}{2} \pm \sqrt{(r+bp)^2 + (k+1)^2 - (bp)^2} & \text{for } b, k+1 \in \mathbb{N},\ k + bp \text{ odd}, \\ & -[\frac{k+1}{p}] \le b \le [\frac{k-1}{p}]. \end{cases}$$

Here the first eigenvalue has multiplicity bp while the second has multiplicity $k+1$.

In the case where $p = 2p_0$ is even $H^1(L(p,1),\mathbb{Z}_2) = \mathbb{Z}_2$ and there are two spin structures. The first corresponds to the lift of the isotropy representation given by (6.114) and gives the same spectrum as the odd p case. The second spin structure comes from the lift given by the restriction to $\mathbb{Z}_p$ of

$$\alpha' : U(1) \to SU(2), \quad \alpha'(e^{it}) = \begin{bmatrix} e^{-i(1+p_0)t} & 0 \\ 0 & e^{i(1+p_0)t} \end{bmatrix}. \tag{6.121}$$

The rest of the computation is now the same as the p odd case, the answer is as summarized below.

Theorem 6.3.1. *Let $L(p,1)$ be the Lens space. The spectrum of the coupled Dirac operator D_{ra} corresponding to the trivial spin structure is given by*

$$\lambda = \begin{cases} -bp - \dfrac{1}{2} \pm r & \text{for } b \in \mathbb{N} \\ -\dfrac{1}{2} \pm \sqrt{(r+bp)^2 + (k+1)^2 - (bp)^2} & \text{for } b, k+1 \in \mathbb{N},\ k+bp \text{ odd}, \\ & -[\frac{k+1}{p}] \le b \le [\frac{k-1}{p}]. \end{cases}$$

where the first eigenvalue has multiplicity bp while the second has multiplicity $k+1$.

For $p = 2p_0$ even the spectrum of the coupled Dirac operator D_{ra} corresponding to the non-trivial spin structure is given by

$$\lambda = \begin{cases} -(p_0 + bp) - \dfrac{1}{2} \pm r & \text{for } b \in \mathbb{N} \\ -\dfrac{1}{2} \pm \sqrt{(r+p_0+bp)^2 + (k+1)^2 - (p_0+bp)^2} & \text{for } b, k+1 \in \mathbb{N},\ k+p_0 \text{ odd}, \\ & -[\frac{k+p_0+1}{p}] \le b \le [\frac{k-p_0-1}{p}]. \end{cases}$$

where the first eigenvalue has multiplicity $p_0 + bp$ while the second has multiplicity $k+1$.

The proposition again allows us to compute the spectral flow function in each case. Considering the trivial spin structure it is clear that the only eigenvalues crossing the origin are of the type $-bp - \frac{1}{2} + r$. Hence the spectral flow function is

$$sf(D, D_{ra}) = \sum_{b=1}^{[\frac{2r-1}{2p}]} bp. \tag{6.122}$$

This is now seen to satisfy the asymptotics

$$sf(D, D_{ra}) = \frac{r^2}{2p} + O(r). \tag{6.123}$$

Appendix A

The semiclassical resolvent expansion

In this section we collect some facts from semiclassical analysis. The primary goal is proposition (A.5.5) where we prove the existence of a trace expansion for any function of an elliptic semiclassical operator. To do so we will first review some fact about the semiclassical pseudodifferential algebra. The main references here are [13] and [44]. We use this section to supplement these references and to modify some of their arguments to fit our purpose.

A.1 The Semiclassical Pseudodifferential Algebra

Here we shall recall the definition of a semiclassical pseudodifferential operator. We shall assume familiarity with the usual pseudodifferential algebra as in chapter 18 of [24] or chapter 2 of [33]. Although a semiclassical pseudodifferential operator is really a family of pseudodifferential operators it is still referred to as 'an' operator by abuse of language. The precise definition appears below.

Definition A.1.1. *A semiclassical pseudodifferential operator of order* $(m, 0)$ *on* $\mathbb{R}^n$ *is a* 1*-parameter family of psedodifferential operators* $A_h \in \Psi^m_{sl} \subset C^\infty((0, 1]_h; \Psi^m(\mathbb{R}^n; \mathbb{R}^l))$ *of the form*

$$A_h = a(x, hD, h) = (2\pi h)^{-n} \int e^{i(x-y).\xi/h} a(x, \xi, h) d\xi dy, \tag{A.1}$$

such that $a \in C^\infty([0,1]_h; S^m(\mathbb{R}^{2n}; \mathbb{R}^l))$.

We recall that the space of symbols $S^m(\mathbb{R}^{2n}; \mathbb{R}^l)$ is defined to be the space of smooth maps $a : \mathbb{R}^{2n} \to Mat_l(\mathbb{C})$ for which each seminorm

$$\sup_{x,\xi} \langle \xi \rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \tag{A.2}$$

is finite. This space is a Frechet space with these semi-norms and the smoothness in definition (A.1.1) means smoothness with respect to each of these seminorms. Following this definition on Euclidean space we define semiclassical operators on a compact manifold.

Definition A.1.2. *Let* E *be a vector bundle of rank* l *a compact manifold* X *of dimension* n. *A semiclassical pseudodifferential operator of order* $(m, 0)$ *is a* 1*-parameter family of pseudodifferential operators* $A_h \in \Psi^m_{sl}(X; E) \subset C^\infty((0,1]_h; \Psi^m(X))$ *such that*

i. *there exists an atlas* $\{(U_\alpha, \alpha)\}$ *of coordinate charts* $\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ *with respect to which* E *is locally trivial. Furthermore for each* $\phi, \psi \subset C_c^\infty(V_\alpha)$

$$\psi(\alpha^{-1})^* A_h \alpha^* \phi = a_\alpha(x, hD) \in \Psi^m_{sl}(\mathbb{R}^n; \mathbb{R}^k) \tag{A.3}$$

and

ii. *for each* $\phi_1, \phi_2 \in C^\infty(X)$ *satisfying* $supp(\phi_1) \cap supp(\phi_2) = \emptyset$ *we have that the kernel* K_h *of* $\phi_1 A_h \phi_2$ *is in* $C^\infty(X \times X)$ *and is* $O(h^\infty)$ *in each* C^k *norm on the product.*

A semiclassical pseudodifferential operator of order (m, k) is a 1-parameter family of pseudodifferential operators of the form $h^{-k}\Psi^m_{sl}$. The coordinate independence of pseudodifferential operators proves that if an operator has the form (A.3) with respect to one atlas it would have to the same with resect to any other (cf. theorem 9.10 in

[44]). These also form an algebra with respect to composition $h^{-k_1}\Psi_{sl}^{m_1} \circ h^{-k_2}\Psi_{sl}^{m_2} \subset h^{-(k_1+k_2)}\Psi_{sl}^{(m_1+m_2)}$.

A.2 Symbol maps, quantization and ellipticity

Next we define the symbol of a semiclassical operator. First define the semiclassical spaces of symbols on the cotangent bundle to be $S_{sl}^m(T^*X;E) = C^\infty([0,1];S^m(T^*X;E))$ where again smoothness is understood to be with respect to the Frechet space norms on $S^m(T^*X;E)$. In the semiclassical setting there are two versions of the symbol. The first is the usual symbol of each operator in the family

$$\sigma_m(A_h) \in C^\infty([0,1], S^m(T^*X;E)/S^{m-1}(T^*X;E)) = S_{sl}^m/S_{sl}^{m-1}. \tag{A.4}$$

The next is the semiclassical symbol $\sigma_{sl}(A_h) \in S_{sl}^m/hS_{sl}^m = S^m(T^*X;E)$. For a semiclassical operator $a(x,hD,h)$ on Euclidean space this is simply defined as $a(x,\xi,0) \in S^m$. This definition is now extended to manifolds using an appropriate partition of unity (cf. theorem 14.1 in [44]). The two symbols satisfy the compatibility relation

$$\sigma_m|_{h=0} = [\sigma_{sl}] \in S^m/S^{m-1}. \tag{A.5}$$

Both symbols are multiplicative in the sense $\sigma_{sl}(AB) = \sigma_{sl}(A)\sigma_{sl}(B)$ and $\sigma_m(AB) = \sigma_m(A)\sigma_m(B)$ (cf. theorem 14.1 in [44]). They also fit into the short exact sequences

$$0 \to h^{-k}\Psi_{sl}^{m-1} \to h^{-k}\Psi_{sl}^m \xrightarrow{\sigma_m} C^\infty([0,1];S^m/S^{m-1}) \to 0 \tag{A.6}$$

$$0 \to h^{-k+1}\Psi_{sl}^{m} \to h^{-k}\Psi_{sl}^m \xrightarrow{\sigma_{sl}} h^{-k}S^m \to 0. \tag{A.7}$$

The semiclassical notion of ellipticity is defined as

$$A_h \in \Psi^m_{sl} \quad \text{is semi-classically elliptic} \iff \tag{A.8}$$

$$\exists \text{ constants } C_1, C_2 > 0 \text{ such that } \sigma_{sl}(A)(x,\xi) > C_1|\xi|^m \text{ for } |\xi| > C_2. \tag{A.9}$$

We comment here that unlike the classical symbol the semiclassical symbol is not a homogeneous function on the cotangent bundle. There also exist a quatization map $Op : S^m_{sl}(T^*X) \to \Psi^m_{sl}(X)$ (cf. theorem 14.1 in [44]). This is a right inverse to the symbols in the sense that

$$\sigma_m(Op(a)) = [a] \in S^m_{sl}/S^{m-1}_{sl} \tag{A.10}$$

$$\sigma_{sl}(Op(a)) = a|_{h=0} \in S^m. \tag{A.11}$$

It follows from the short exact sequences (A.6) and (A.7) and multiplicativity of the symbols that if either A or B has a scalar symbol (i.e. the symbol has a scalar representative in the case of σ_m) then their commutator has lower order. More precisely let $A \in h^{-k_1}\Psi^{m_1}_{sl}$ and $B \in h^{-k_2}\Psi^{m_2}_{sl}$ then one has the following two implications

$$\sigma_{m_1}(A) \quad \text{or} \quad \sigma_{m_2}(A) \quad \text{is scalar} \implies [A,B] \in h^{-k_1-k_2}\Psi^{m_1+m_2-1}_{sl} \tag{A.12}$$

$$\sigma_{sl}(A) \quad \text{or} \quad \sigma_{sl}(A) \quad \text{is scalar} \implies [A,B] \in h^{-k_1-k_2+1}\Psi^{m_1+m_2}_{sl}. \tag{A.13}$$

A.3 Semiclassical Sobolev Spaces

The semiclassical Sobolev spaces $H^k_{sl}(X;E)$ are the defined as spaces whose elements are the same as the classical Sobolev spaces H^k. However their norms are rescaled as follows. Choose a set of vector fields $V_1, \cdots, V_J$ that span the tangent space T_xM at every point $x \in M$. Let ∇ be a fixed connection on E. Then $u \in H^k_{sl} \iff \nabla_{V_{i_1}} \cdots \nabla_{V_{i_l}} u \in L^2(X), \quad \forall (i_1, \cdots, i_k) \in \{1, \cdots, J\}^l$ with $0 \le l \le k$. Moreover the norm is defined as

$$\|u\|_{H^k_{sl}} = \sum_{l=0}^{k} \sum_{\substack{\alpha \in \mathbb{N}^l \\ 1 \le \alpha_i \le J}} h^{2l} \|\nabla_{V_{\alpha_1}} \cdots \nabla_{V_{\alpha_l}} u\|_{L^2}. \tag{A.14}$$

A semiclassical differential operator $A_h \in h^{-k}\Psi^m_{sl}$ is bounded on these Sobolev spaces in the sense

$$\|A_h\|_{H^{m+s}_{sl} \to H^m_{sl}} = O(h^{-k}), \text{ as } h \to 0. \tag{A.15}$$

Using the fact that semiclassical operators form an algebra this reduces to the L^2 boundedness of Ψ^0_{sl} which is theorem 14.2 of [44]. Finally we mention that semiclassical operators satisfy asymptotic summation. This means that for any set of semiclassical operators $A_j \in \Psi^{m-j}_{sl}, j \in \mathbb{N}_0$ there exists $A \in \Psi^m_{sl}$ such that

$$A \sim \sum_{j \ge 0} A_j \quad \text{or} \quad A - \sum_{j=0}^{N} A_j \in \Psi^{m-N-1}_{sl} \quad \forall N. \tag{A.16}$$

A.4 Semiclassical Elliptic regularity

Here we prove a semiclassical analogue of Gardings inequality or elliptic regularity. This will follow after the construction of a parametrix for an elliptic semiclassical operator.

Proposition A.4.1. *Let $A \in \Psi^m_{sl}(X)$ be an elliptic semiclassical pseudodifferential operator. Then there exists a semiclassical operator $B_h \in \Psi^{-m}_{sl}(X)$ such that*

$$AB - I \in \Psi^{-\infty}_{sl}(X) \quad \textit{and} \quad BA - I \in \Psi^{-\infty}_{sl}(X). \tag{A.17}$$

Proof. Since A is elliptic there exist constants C_1, C_2 such that $|\sigma_{sl}(A)(x,\xi)| \ge C_1|\xi|^m$ for $|\xi| \ge C_2$. Using the compatibility of the symbols (A.5) we may assume $|\sigma_m(A)(x,\xi,h) \ge C_1|\xi|^m$ for $|\xi| \ge C_2$ for uniform constants C_1, C_2 on some interval $h \in [0, h_0]$. Choose a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi = 0$ on $[-2C, 2C]$ and $\phi = 1$ outside $[-3C, 3C]$. Consider

$$B_{-m} = Op\left(\phi(|\xi|)(\sigma_m(A))^{-1}\right). \tag{A.18}$$

Using the multiplicativity of the symbol one has

$$\sigma_0(AB_{-m} - I) = 1 - \phi(|\xi|) = 0 \in S^0_{sl}/S^{-1}_{sl}. \tag{A.19}$$

Hence $AB_{-m} - I = R_{-1} \in \Psi^{-1}_{sl}$ from the symbol exact sequence for σ_0. Now choose

$$B_{-m-1} = -Op\left(\phi(|\xi|)(\sigma_m(A))^{-1}\sigma_{-1}(R_{-1})\right). \tag{A.20}$$

We then have

$$\sigma_{-1}(A(B_{-m}+B_{-m-1})-I) = \sigma_{-1}(AB_{-m}-I)-\sigma_{-1}(R_{-1})\phi(|\xi|) = 0 \in S^{-1}_{sl}/S^{-2}_{sl}. \tag{A.21}$$

Continuing iteratively we obtain $B_{-m-j} \in \Psi^{-m-j}_{sl}, j \geq 0$ such that $A(B_{-m} + \cdots + B_{-m-N}) - I \in \Psi{-N-1}_{sl}(X)$. Using the asymptotic summation property we now pick $B \sim \sum_{j\geq 0} B_j$ to be the required right parametrix for A. The construction of the left parametrix is similar. □

We now state the elliptic regularity lemma.

Proposition A.4.2. *Let $A \in \Psi^m_{sl}(X)$ be a semiclassical elliptic operator of order $m \geq 0$. Then one has the estimate*

$$\|u\|_{H^{s+m}} \leq C(\|Au\|_{H^s} + \|u\|_{H^s}) \tag{A.22}$$

for some constant C uniform in h.

Proof. This follows easily from the parametrix construction, namely let B be the left parametrix such that $BA - I = S \in \Psi^{-\infty}_{sl}$. Then

$$\begin{aligned} \|u\|_{H_{sl}^{s+m}} &= \|(BA+S)u\|_{H_{sl}^{s+m}} && \text{(A.23)} \\ &\leq \|BAu\|_{H_{sl}^{s+m}} + \|Su\|_{H_{sl}^{s+m}} && \text{(A.24)} \\ &\leq \|B\|\|Au\|_{H_{sl}^{s}} + \|S\|\|u\|_{H_{sl}^{s}} && \text{(A.25)} \\ &\leq C(\|Au\|_{H^s} + \|u\|_{H^s}) && \text{(A.26)} \end{aligned}$$

using the boundedness of semiclassical operators. □

A.5 Semiclassical Beals lemma and Resolvent estimates

In this section we state a characterization for semiclassical pseudodifferential operators knwon as Beals' lemma. This characterization will be useful in showing that the resolvent of an self-adjoint elliptic pseudodifferential operator is pseudodifferential. The proof we present below is a semiclassical modification of the one appearing in Beals' original paper [7].

Theorem A.5.1. *(Semiclassical Beals' Lemma) A family of operators $A_h : C_c^\infty(X) \to C^{-\infty}(X)$ is in $\Psi_{sl}^m(X; E)$ if and only if*

$$\|ad_{A_1} \dots ad_{A_N} ad_{B_1} \dots ad_{B_M} A\|_{H_{sl}^{m+s} \to H_{sl}^{s+N}} = O(h^{N+M}) \tag{A.27}$$

for all M, N, s and for all $A_i \in \Psi_{sl}^0, B_i \in \Psi_{sl}^1$ with scalar symbols. Moreover if

$$\|ad_{A_1} \dots ad_{A_N} ad_{B_1} \dots ad_{B_M} A\|_{H_{sl}^{m+s} \to H_{sl}^{s+N}} = O(\delta^{-N-M} h^{N+M}) \tag{A.28}$$

for some $\delta > 0$ then each amplitude a_α of A_h in (A.3) *can be taken to satisfy the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} \delta^{-|\alpha|-|\beta|} \langle \xi \rangle^{m-|\alpha|}. \tag{A.29}$$

Proof. The only if part follows since the equations (A.12) and (A.13) imply that

$$\|ad_{A_1} \dots ad_{A_N} ad_{B_1} \dots ad_{B_M} A\|_{H^{m+s}_{sl} \to H^{s+N}_{sl}} \in h^{M+N} \Psi^{M+N}_{sl} \tag{A.30}$$

and we then apply the boundedness of pseudodifferential operators on Sobolev spaces.

Now we prove the if part. Since the definition (A.1.2) for a pseudodifferential operator is local, we may reduce to the case where $X = \mathbb{R}^n$ is Euclidean space and we have a smooth family of operators $A_h : \mathcal{S} \to \mathcal{S}'$. Choose $g \in \mathcal{S}(\mathbb{R})$ such that $g(0) = 1$, $\hat{g} \in C^\infty_c((-1, 1))$ and $g(x) = g(-x)$. Let $g_x(y) = g(y - x)$. We then have

$$\begin{aligned} u(x) = u(x)g_x(x) &= (2\pi h)^{-n} \int e^{i(x-y)\xi/h} g_x(y)u(y)dyd\xi && \text{(A.31)} \\ &= (2\pi h)^{-n} \int e^{-iy\xi/h} e_\xi(x)g_y(x)u(y)dyd\xi && \text{(A.32)} \end{aligned}$$

where $e_\xi(x) = e^{ix\xi/h}$. Now assume that we have a smooth family of operators

$$A : \mathcal{S}' \to \mathcal{S} \tag{A.33}$$

so that each A_h has kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. We may then compute

$$\begin{aligned} Au(x) &= (2\pi h)^{-n} \int e^{-iy\xi/h} A(e_\xi(x)g_y(x))u(y)dyd\xi && \text{(A.34)} \\ &= (2\pi h)^{-n} \int e^{i(x-y)\xi/h} a_0(x, y, \xi)u(y)dyd\xi && \text{(A.35)} \end{aligned}$$

where $a_0(x, y, \xi) = e_{-\xi}(x)A(e_\xi g_y)(x)$ and the integral converges for $u \in \mathcal{S}$ as kernel in A has kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. We now estimate

$$\|a_0(., y, \xi)\|_{L^2} = \|A(e_\xi g_y)\|_{L^2} \leq C\|e_\xi g_y\|_{H^m_{sl}} \tag{A.36}$$

for some constant C uniform in y, ξ and h. We also compute

$$D_{x_i} a_0 = e_{-\xi}[D_{x_i}, A](e_\xi g_y) + e_{-\xi} A(e_\xi D_{x_i} g_y) \tag{A.37}$$

where $D_{x_i} = \frac{1}{i}\partial_{x_i}$. Hence we have

$$\|D_{x_i} a_0\|_{L^2} \leq C(\|e_\xi g_y\|_{H^m_{sl}} + \|e_\xi D_{x_i} g_y\|_{H^m_{sl}}) \tag{A.38}$$

where C is again uniform in y, ξ and h. The identity $D_{\xi_i} a_0 = e_{-\xi}\frac{1}{h}[A, x_i](e_\xi g_y)$ gives the estimate

$$\|D_{\xi_i} a_0\|_{L^2} \leq C\|e_\xi g_y\|_{H^{m-1}_{sl}}. \tag{A.39}$$

Continuing in this way we get the estimate

$$\|D_x^\alpha D_y^\beta D_\xi^\gamma a_0\|_{L^2} \leq C_{\alpha\beta\gamma}\left(\sum_{\delta \leq \alpha} \|e_\xi D_x^\delta g_y\|_{H^{m-|\gamma|}_{sl}}\right) \tag{A.40}$$

for constants $C_{\alpha\beta\gamma}$ uniform in y, ξ and h. Next for any fixed g' with $\hat{g}'$ compactly supported in $(-1, 1)$ we may estimate

$$\begin{aligned}
\|e_\xi g'\|^2_{H^s_{sl}} &= (2\pi h)^{-n} \int \langle\eta\rangle^{2s} |\mathcal{F}_h(e_\xi g')(\eta)|^2 d\eta && \text{(A.41)}\\
&= (2\pi h)^{-n} \int \langle\eta\rangle^{2s} |\hat{g}'\left(\frac{\eta - \xi}{h}\right)|^2 d\eta && \text{(A.42)}\\
&= (2\pi)^{-n} \int \langle\xi + \alpha h\rangle^{2s} |\hat{g}'(\alpha)|^2 d\alpha && \text{(A.43)}\\
&\leq C_m \langle\xi\rangle^{2s}, && \text{(A.44)}
\end{aligned}$$

where $\mathcal{F}_h u(\xi) = \int e^{-ix.\xi/h} u(x)dx$ stands for the semiclassical Fourier transform. Hence (A.40) and (A.44) give

$$\|D_x^\alpha D_y^\beta D_\xi^\gamma a_0\|_{L^2} \leq C'_{\alpha\beta\gamma}\langle\xi\rangle^{m-|\gamma|} \tag{A.45}$$

for constants $C'_{\alpha\beta\gamma}$ uniform in y, ξ and h. Combining this with Sobolev's inequality gives

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a_0| \leq C''_{\alpha\beta\gamma} \langle \xi \rangle^{m-|\gamma|} \tag{A.46}$$

for constants $C''_{\alpha\beta\gamma}$ uniform in y, ξ and h. Hence (A.35) and (A.46) show that $A_h \in \Psi_{sl}^m$ as required. Finally to do away with assumption (A.33) we approximate a more general operator A_h by operators of this type. Namely we choose $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood around 0, and define

$$p^\epsilon(x, \xi) = \chi(\epsilon x) \quad , \qquad q^\epsilon(x, \xi) = \chi(\epsilon \xi) \tag{A.47}$$

$$P^\epsilon = p^\epsilon(x, hD) \quad , \qquad Q^\epsilon = q^\epsilon(x, hD) \tag{A.48}$$

$$A^\epsilon = P^\epsilon A Q^\epsilon. \tag{A.49}$$

Each A^ϵ maps $\mathcal{S}'$ to $\mathcal{S}$ and satisfies norm bounds (A.27) independent of ϵ. Thus we have $A^\epsilon = a^\epsilon(x, hD)$ with bounds on the amplitudes a^ϵ. The amplitudes converge in $C^\infty(\mathbb{R} \times \mathbb{R})$ to an amplitude $a \in S^m$ and $A_h = a(x, hD)$. The second part of the theorem, namely the bounds (A.29), follow after replacing the bounds (A.38)-(A.40), (A.45) and (A.46) by their analogues involving δ.

□

The above theorem and the pseudodifferential calculus can be used to obtain a resolvent expansion for an elliptic pseododifferential operator. This is the proposition below.

Proposition A.5.2. *Let $A_h \in \Psi_{sl}^m$ be a self-adjoint elliptic pseudodifferential operator. Then for each $z \in \mathbb{C}$ with $Imz \neq 0$ we have $(A - z)^{-1} \in \Psi_{sl}^{-m}$. This resolvent has an expansion in the sense that there exists a sequence of symbols $a_0^z, a_1^z, \ldots \in S^{-m}$ such that for each k*

$$h^{k+1} B_k^z = (A - z)^{-1} - Op(a_0^z + h a_1^z + \ldots + h^k a_k^z) \in h^{k+1} \Psi_{sl}^{-m}. \tag{A.50}$$

Moreover each a_i and the amplitudes of each B_i satisfy the estimates (A.29) *with $\delta = (Imz)^{k_i}$ for some $k_i > 0$.*

Proof. First we note that the elliptic regularity estimate and self-adjointness of A imply that

$$(Imz)\|u\|_{H^s_{sl}} \leq C\|(A-z)u\|_{H^{s-m}_{sl}}. \tag{A.51}$$

Hence we have

$$\|(A-z)^{-1}\|_{H^{-m+s}_{sl} \to H^s_{sl}} = O((Imz)^{-1}). \tag{A.52}$$

Next the computation $ad_{A_1}(A-z)^{-1} = -(A-z)^{-1} ad_{A_1} A (A-z)^{-1}$ gives

$$\|ad_{A_1}(A-z)^{-1}\|_{H^{-m+s}_{sl} \to H^{s+1}_{sl}} = O((Imz)^{-2}h). \tag{A.53}$$

Computing further in this fashion we obtain

$$\|ad_{A_1} \dots ad_{A_N} ad_{B_1} \dots ad_{B_M}(A-z)^{-1}\|_{H^{-m+s}_{sl} \to H^{N+s}_{sl}} = O((Imz)^{-N-M-1}h^{N+M}) \quad \forall M, N, s. \tag{A.54}$$

Hence we see that the resolvent satisfies the criterion of Beals' lemma with $\delta = Imz$ and we have $(A-z)^{-1} \in \Psi^{-1}_{sl}$ with the corresponding estimates (A.29) on its amplitudes. To derive the resolvent expansion first set

$$a^z_0 = (\sigma_{sl}(A) - z)^{-1}. \tag{A.55}$$

The self-adjointness of A, and hence its symbol, gaurantees that this inverse exists. We then compute

$$\sigma_{sl}((A-z)Op(a^z_0) - I) = 0 \quad \text{in} \quad S^0. \tag{A.56}$$

Hence from the symbol exact sequence for σ_{sl} we have $(A-z)Op(a^z_0) = I + hR^z_0$ for some $R^z_0 \in \Psi^0_{sl}$. We then set

$$a^z_1 = -(\sigma_{sl}(A) - z)^{-1}\sigma_{sl}(R_0). \tag{A.57}$$

Again we compute

$$\sigma_{sl}((A-z)Op(a_0^z + ha_1^z) - I) = 0 \quad \text{in} \quad hS^0 \tag{A.58}$$

and hence we must have $(A-z)Op(a_0^z + ha_1^z) = I + h^2R_1^z$ for some $R_1^z \in \Psi_{sl}^0$. This inductive procedure constructs the sequence of symbols a_i^z with the property that $(A-z)Op(a_0^z + ha_1^z + \cdots + h^k a_k^z) = I + h^{k+1}R_k^z$ for $R_k^z \in \Psi_{sl}^0$. Hence we see that this sequence of symbols a_i along with $B_i^z = (A-z)^{-1}h^{i+1}R_i^z \in h^{i+1}\Psi_{sl}^{-m}$ satisfies (A.50). The claimed estimates on the amplitudes follow from local computations. □

Next we show how this resolvent expansion implies an expansion for any function of the operator. Namely we show that given any Schwartz function $f \in \mathcal{S}(\mathbb{R})$ we have $f(A_h) \in \Psi_{sl}^{-\infty}$ and that there exists an expansion for its trace $trf(A_h) \sim a_0h^{-n} + a_1h^{-n+1} + \ldots$ in powers of h. This will be done by expressing such a function of the operator in terms of its resolvent. To do this we will first prove the existence of almost analytic extensions of a Schwartz function in the proposition below.

Proposition A.5.3. *If $f \in \mathcal{S}(\mathbb{R})$ then there exists a function on the complex plane $\tilde{f} \in \mathcal{S}(\mathbb{C})$ such that*

i. $\tilde{f}|_{\mathbb{R}} = f$

ii. $supp(\tilde{f}) \subset \{z||Imz| \leq 1\}$

iii. *For each $M, N > 0$ we have*

$$|\bar{\partial}\tilde{f}(z)| \leq C_{M,N}(Rez)^{-M}(Imz)^N \tag{A.59}$$

for some constant $C_{M,N}$.

Proof. Pick a cutoff $\chi \in C_c^\infty(-1,1)$ such that $\chi = 1$ on $(-\frac{1}{2}, \frac{1}{2})$ and set

$$\tilde{f}(z) = \frac{1}{2\pi}\chi(y)\int_{\mathbb{R}} \chi(y\xi)\hat{f}(\xi)e^{i\xi(x+iy)}d\xi. \tag{A.60}$$

The Fourier inversion formula checks property iwhile iifollows because of the $\chi(y)$ term. We compute

$$\begin{aligned} x^M y^{-N}\bar\partial(f)| &= x^M y^{-N}\frac{i\chi'(y)}{2\pi}\int_{\mathbb{R}}\chi(y\xi)\hat f(\xi)e^{i\xi(x+iy)}d\xi \\ &\quad + x^M y^{-N}\frac{\chi(y)}{2\pi}\int_{\mathbb{R}} i\xi\chi'(y\xi)\hat f(\xi)e^{i\xi(x+iy)}d\xi. \end{aligned} \tag{A.61}$$

Next we write $x^M e^{i\xi x} = (-i\partial_\xi)^M e^{i\xi x}$ and integrate by parts in ξ. The first summand on the right hand side of (A.61) now gives a sum of terms of the type

$$\frac{iy^{-k}\chi'(y)}{2\pi}\int_{\mathbb{R}}\chi(y\xi)\hat f(\xi)e^{-\xi y}e^{i\xi x}d\xi \tag{A.62}$$

each of which can be bound in absolute value by a constant multiple of $\|y^{-k}\chi'(y)\|_{C^0}\|\chi(y)e^{-y}\|_{C^0}\|\hat f\|_{L^1}$. The first summand gives a sum of terms of the type

$$\frac{\chi(y)}{2\pi}\int_{\mathbb{R}} y^{-k} i\xi\chi'(y\xi)\hat f(\xi)e^{-\xi y}e^{i\xi x}d\xi \tag{A.63}$$

each of which can be bound in absolute value by $\|\xi^{k+1}\hat f\|_{L^1}\|y^{-k}\chi'(y)e^{-y}\|_{C^0}$.

□

Now we write the function of an operator in terms of its resolvent. The corresponding formula (cf. theorem 14.8 in [44]) appears in the proposition below.

Proposition A.5.4. *Given any function $f \in \mathcal{S}(\mathbb{R})$ we have*

$$f(A_h) = \frac{1}{\pi}\int_{\mathbb{C}}\bar\partial\tilde f(z)(A_h - z)^{-1}dxdy, \tag{A.64}$$

where $dxdy$ stands for the Lebesgue measure on $\mathbb{C}$.

In the proposition above both sides are defined using functional calculus. The right hand side makes sense because $1/z$ is locally integrable on $\mathbb{C}$. Equation (A.64) reduces to the fact that $\frac{1}{\pi z}$ is the fundamental solution of $\bar\partial$. We are now ready to prove the existence of a functional trace expansion for an elliptic semiclassical operator.

Proposition A.5.5. *Let A_h be an elliptic self-adjoint semiclassical operator on a compact manifold X. For any function $f \in \mathcal{S}(\mathbb{R})$ one has that $f(A_h) \in \Psi_{sl}^{-\infty}(X)$.*

Moreover the trace of $f(A_h)$ has a trace expansion

$$trf(A_h) \sim c_0 h^{-n} + c_1 h^{-n+1} + \dots \tag{A.65}$$

for some constants c_i.

Proof. By proposition (A.5.2) the resolvent $(A - z)^{-1} \in \Psi_{sl}^{-m}$ for $Imz \neq 0$. Using this and the formula (A.64) we see that $f(A_h)$ has the form (A.3) with amplitudes given by

$$f_\alpha = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) a_\alpha^z dxdy. \tag{A.66}$$

Here a_α^z are the corresponding amplitudes of the resolvent. From proposition (A.5.2) we know that the amplitudes a_α^z satisfy the bounds (A.29) with $\delta = Imz$. Combining this with (A.59) we have that each amplitude f_α satisfies uniform bounds $|\partial_x^\alpha \partial_\xi^\beta f_\alpha| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}$ and hence $f(A_h) \in \Psi_{sl}^{-m}$. To see that $f(A_h) \in \Psi_{sl}^{-\infty}$ note that $f(A_h) = (1+A_h^2)^{-k} g(A_h)$ where $g(x) = (1+x^2)^k f(x)$ and hence $f(A_h) \in \Psi_{sl}^{-\infty}$ from the algebra property of pseudodifferential operators. To derive the trace expansion set

$$F_i = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) Op(a_i^z) dxdy \tag{A.67}$$

where a_i^z are the coefficients in the resolvent expansion (A.5.2). Again we have that $F_i \in \Psi_{sl}^{-\infty}$ and the trace expansion (A.65) now follows fron the resolvent expansion (A.5.2) with $c_i = trF_i$. □

The coefficients in the trace expansion (A.65) $c_i(f)$ all depend on the function f and so do the remainders $R_i(f)$ defined via

$$h^{i+1} R_{i+1}(f) = trf(A_h) - (c_0(f) h^{-n} + \dots + c_i(f) h^{-n+i}). \tag{A.68}$$

We shall need the fact that each coefficient $c_i(f)$ defines a tempered distribution and a similar statement about the remainders. This is done in the proposition below.

Proposition A.5.6. *For a fixed operator A_h the trace coefficients $c_i(f)$ in the expansion* (A.65) *define tempered distributions. Further each remainder $R_i(f)$ defined via* (A.68) *satisfies the estimate*

$$|R_i(f)| \leq C \sum_{\alpha,\beta \leq N} \|x^\alpha \partial_x^\beta f\|_{C^0} \tag{A.69}$$

for some N and C independent of h.

Proof. Following proposition (A.5.5) we have that each $c_i(f) = trG_i$ with

$$G_i = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{g}(z)(1 + A^2)^{-k} Op(a_i^z) dxdy, \tag{A.70}$$

$g = (1 + x^2)^k f$ and a_i^z denote the coefficients of the resolvent expansion. Combining this with proposition (A.5.3) we have that the amplitudes g_α of G_i are bounded by

$$|g_\alpha| \leq C \langle \xi \rangle^{-m-k} \left(\sum_{\alpha \leq N} \|\widehat{\partial_x^\alpha g}\|_{L^1} \right). \tag{A.71}$$

Here the constant C is independent of g while N may depend on i and k. Letting k be large we get a bound

$$|c_i(f)| = |trG_i| \leq C \left(\sum_{\alpha \leq N} \|\widehat{\partial_x^\alpha g}\|_{L^1} \right) \tag{A.72}$$

for some N. The right hand side of (A.72) can now be bound by some multiple of a Schwartz norm of f. The proof of the bounds (A.69) is similar. □

Appendix B

The semiclassical wave kernel

In this section we review some facts about the semiclassical wave equation. The main goal is proposition (B.4.2) which shows that a certain wave evolution operator admits a representation as an oscillatory integral. The main references here are [13] and [21]. We shall first define the required notion of an oscillatory density in the next section.

B.1 Oscillatory densities

Before defining oscillatory densities we shall need some relevent notions from symplectic geometry. We first review some functorial properties of Lagrangian submanifolds. Given a symplectic manifold (M, ω) denote by M^- the symplectic manifold $(M, -\omega)$. A Lagrangian $\Gamma_{12} \subset M_1^- \times M_2$ is called a canonical relation between M_1 and M_2. Given two canonical relations $\Gamma_{12} \subset M_1^- \times M_2$ and $\Gamma_{23} \subset M_2^- \times M_3$ the subset

$$\Gamma_{12} \circ \Gamma_{23} = \pi_{13}(\Gamma_{12} \star \Gamma_{23}) \subset M_1^- \times M_3 \tag{B.1}$$

$$\text{with} \quad \Gamma_{12} \star \Gamma_{23} = (\Gamma_{12} \times \Gamma_{23}) \cap (M_1 \times \Delta_{M_2} \times M_3) \tag{B.2}$$

is an immersed Lagrangian assuming the intersection in (B.2) to be transversal. Under the additional assumption that $\pi_{13} : \Gamma_{12} \star \Gamma_{23} \to \Gamma_{12} \circ \Gamma_{23}$ is proper with connected and simply connected fibers, (B.1) gives a submanifold (cf. chapter 4 in [21]). In this case we say that Γ_{12} and Γ_{23} are composible and call $\Gamma_{12} \circ \Gamma_{23}$ their composition. An exact Lagrangian $(\Lambda, \psi) \subset T^*X$ of the cotangent bundle is one equipped with a phase function ψ satisfying $\alpha|_\Lambda = d\psi$, with α being the tautological one form on T^*X. An exact canonical relation (Γ, ψ) is an exact Lagrangian of $(T^*X)^- \times (T^*Y)$. The composition of exact canonical relations $(\Gamma_{12} \circ \Gamma_{23}, \psi_{12} \circ \psi_{23})$ is defined as in (B.1) with the phase function being defined via

$$\pi_{13}^*(\psi_{12} \circ \psi_{23}) = \pi_{12}^*\psi_{12} + \pi_{23}^*\psi_{23}. \tag{B.3}$$

Given a smooth map $f : X \to Y$ define the canonical relation

$$\Gamma_f = (\varsigma \times id)(N^*(\text{graph} f)) \subset (T^*X)^- \times (T^*Y), \tag{B.4}$$

where $N^*(\text{graph} f)$ is the conormal bundle to the graph of f and

$$\varsigma : T^*X \to T^*X, \quad \varsigma(x, \xi) = (x, -\xi). \tag{B.5}$$

Using these constructions we may now define the pushforward of a Lagrangian $\Lambda \subset T^*X$ under a smooth map $f : X \to Y$ via

$$f_*\Lambda = \Lambda \circ \Gamma_f. \tag{B.6}$$

Here we think of $\Gamma \subset \text{pt} \times T^*X$ as a canonical relation and again assume that the composition in (B.6) is well defined. We shall be particularly interested in the case when $f = \pi : Z \to X$ is a fibration and $\Lambda = d\varphi \subset T^*Z$ is a horizontal Lagrangian. In this case the trasversality hypothesis is the same as requiring that $d\varphi$ and $H^*Z = d\pi^*(T^*X)$ intersect transversally inside T^*Z. Now we have an exact sequence

$$0 \to H^*Z \to T^*Z \to V^*Z \to 0, \qquad \text{where} \tag{B.7}$$

$$V^*Z = T_z^*(\pi^{-1}(x)), \qquad x = \pi(z) \tag{B.8}$$

denotes the cotangent space to the fiber through z. The section $d\varphi$ gives a section $d_v\varphi$ of V^*Z via (B.7) and $d\varphi \pitchfork H^*Z$ if an only if $d_v\varphi$ intersects the zero section of V^*Z transversally. In this case

$$C_\varphi = \{z \in Z | d_v\varphi = 0\} \tag{B.9}$$

is a submanifold of Z. Moreover for each $z \in C_\varphi$ we have $d\varphi(z) = d\pi^*\eta$ for a unique $\eta \in T^*_{\pi(z)}X$ and hence we have an embedding

$$\lambda_\varphi : C_\varphi \hookrightarrow T^*X, \quad \lambda_\varphi(z) = (\pi(z), \eta). \tag{B.10}$$

We shall denote the image of this embedding λ_φ by $\Lambda_\varphi \subset T^*X$. Each point $z \in C_\varphi$ is a critical point of the restriction of φ to $\pi^{-1}(z)$. Let $\mathrm{sgn}^\sharp(z) : C_\varphi \to \mathbb{Z}$ be the function where $\mathrm{sgn}^\sharp(z)$ denotes the signature of the Hessian at z of $\varphi|_{\pi^{-1}(z)}$. We may carry over this function to Λ_φ via

$$\mathrm{sgn}_\varphi : \Lambda_\varphi \to \mathbb{Z}, \quad \mathrm{sgn}_\varphi = \mathrm{sgn}^\sharp \circ \lambda_\varphi^{-1}. \tag{B.11}$$

To define oscillatory density we shall need the notion of a generating function for an exact Lagrangian via the definition below.

Definition B.1.1. *Let (Λ, ψ) be an exact Lagrangian submanifold of T^*X. Let $p = (x, \xi) \subset T^*X$ be a point on the Lagrangian and U_x an open neighbourhood of x. Let $Z \xrightarrow{\pi} U_x$ be a fibration whose fibers are identified with some open subset of $\mathbb{R}^d$. We say that the function $\varphi : Z \to \mathbb{R}$ is a generating function for Λ with respect to the fibration π if*

i. *$d\varphi \pitchfork H^*Z$ and Λ_φ gives an open neighbourhood of p in Λ*
ii. *$\varphi = \lambda_\varphi^*\psi$ on C_φ.*

Part (i) of the definition already implies that $d\varphi = d(\lambda_\varphi^*\psi)$ and hence it is enough to check (ii) at some point on C_φ. Proposition 35 in [21] shows that one can find a generating function near any point of a given exact Lagrangian. Now given a generating function $\varphi : Z \to \mathbb{R}$ as in the above definition we define the class $I^k(U_x, \Lambda_\varphi, \varphi; \mathbb{C}^l)$ of oscillatory densities to be the space of all $\mathbb{C}^l$ valued compactly supported $\frac{1}{2}$-densities $\mu \in C_c^\infty(U_x; \mathbb{C}^l \otimes |TU_x|^{\frac{1}{2}})$ which are of the form

$$\mu = h^{k-d/2} \pi_*(a e^{i\frac{\varphi}{h}} \tau). \tag{B.12}$$

Here $a = a(z,h) \in C_c^\infty(Z \times \mathbb{R}; \mathbb{C}^l)$, τ is a nowhere vanishing $\frac{1}{2}$-density on Z and π_* denotes the pushforward of a $\frac{1}{2}$-density as defined in section 6.6 of [21]. The space of oscillatory $\frac{1}{2}$-densities associated to a Lagrangian is now defined below.

Definition B.1.2. *Let $E \to X$ be a complex vector bundle and (Λ, ψ) be an exact Lagrangian in T^*X. Let $\varphi_i : Z_i \to \mathbb{R}$ be a collection of generating functions for Λ, with respect to fibrations $\pi_i : Z_i \to U_i$, such that the Λ_{φ_i}'s all cover Λ and each $E|_{U_i}$ is trivial. The space $I^k(X, \Lambda, \psi; E)$ consists of all smooth sections $\mu \in C^\infty(X; E \otimes |TX|^{\frac{1}{2}})$ such that for each $\rho \in C_c^\infty(X)$ we can write $\rho\mu$ as a finite sum*

$$\rho\mu = \sum_{i=1}^{N} \mu_i \quad \textit{with} \quad \mu_i \in I^k(U_i, \Lambda_{\varphi_i}, \varphi_i; \mathbb{C}^l), \tag{B.13}$$

modulo $O(h^\infty)$.

In section 8.1 of [21] it is shown that the class of functions $I^k(X, \Lambda, \psi; E)$ defined above is independent of the choice of the generating functions φ_i. Oscillatory densities form an algebra over the ring of semiclassical pseudodifferential operators $h^{k_1}\Psi^m_{sl}(X; E) \circ I^{k_2}(X, \Lambda, \psi; E) \subset I^{k_1+k_2}(X, \Lambda, \psi; E)$ (cf. chapter 8 in [21]). We shall often drop parts of the notation $I^k(X, \Lambda, \psi; E)$ when they are understood.

B.2 Maslov line bundle and the symbol map

Here we will define the symbol of an oscillatory density. First we shall need the definition of the Maslov line bundle. Given an exact Lagrangian (Λ, ψ) we first cover it with open sets of the form Λ_φ corresponding to all generating functions φ for Λ. The Maslov line bundle $\mathbb{L}_{Maslov} \to \Lambda$ over the Lagrangian is now defined via the transition functions

$$e^{\frac{i\pi}{4}(\mathrm{sgn}_\varphi - \mathrm{sgn}_{\varphi'})} : \Lambda_\varphi \cap \Lambda_{\varphi'} \to \mathbb{C}. \tag{B.14}$$

The intrinsic line bundle over Λ is defined as $\mathbb{L} = \mathbb{L}_{Maslov} \otimes |T\Lambda|^{\frac{1}{2}}$. From section 8.3 of [21] we have a symbol map $\sigma_k : I^k(X, \Lambda, \psi) \to C^\infty(X; \mathbb{L})$. We may now extend this symbol to a symbol map

$$\sigma_k : I^k(X, \Lambda, \psi; E) \to C^\infty(\Lambda; \mathbb{L} \otimes \pi^* E), \tag{B.15}$$

where $\pi : T^*X \to X$ is the projection onto the cotangent fibers, via the isomorphism $I^k(X, \Lambda, \psi; E) = I^k(X, \Lambda, \psi) \otimes C^\infty(X; E)$. This definition is now extended to all oscillatory densities as in chapter 8 of [21]. This symbol is multiplicative in the sense that

$$\sigma_{k_1+k_2}(A\mu) = \sigma_{sl}(A)|_\Lambda . \sigma_{k_2}(\mu) \quad \text{for} \quad A \in h^{k_1} \Psi^{m_1}_{sl}(X; E \otimes |TX|^{\frac{1}{2}}), \mu \in I^{k_2}(X, \Lambda; E). \tag{B.16}$$

Here we have taken the operator A to act on E valued $\frac{1}{2}$-densities and we shall use this convention for the rest of this appendix. The symbol fits into the short exact sequence

$$0 \to I^{k+1}(X, \Lambda) \to I^k(X, \Lambda) \xrightarrow{\sigma_k} C^\infty_{cf}(\Lambda, \mathbb{L}) \to 0, \tag{B.17}$$

where we have now dropped E from the notation assuming it is understood. The symbol also posesses a right inverse quantization $Op : C^\infty_{cf}(\Lambda, \mathbb{L}) \to I^k(X, \Lambda, \psi)$ satisfying

$$\sigma_k(Op(s)) = s \in C^\infty_{cf}(\Lambda, \mathbb{L}) \tag{B.18}$$

(cf. chapter 8 in [21]). Here the space C^∞_{cf} denotes the space of all smooth sections compactly supported in the fibre directions

$$C^\infty_{cf}(\Lambda,\mathbb{L}) = \{s \in C^\infty(\Lambda,\mathbb{L}) | supp(s) \cap T^*_x X \text{ is compact for each } \ x \in X\}. \quad \text{(B.19)}$$

B.3 Product with vanishing symbol

Here we describe another important part of the calculus that we shall need to construct the wave expansion. Consider a operator $A \in \Psi^m_{sl}$ with scalar semiclassical symbol such that $\sigma_{sl}(A)|_\Lambda = 0$. The multiplicativity of the symbol (B.16) gives that for $\mu \in I^k(X,\Lambda)$ we have

$$\sigma_k(A\mu) = \sigma_{sl}(A)|_\Lambda.\sigma_k(\mu) = 0 \quad \text{(B.20)}$$

and hence we have $A\mu \in I^{k+1}(X,\Lambda)$. Now if μ' is another element of $I^k(X,\Lambda)$ with $\sigma_k(\mu) = \sigma_k(\mu')$, so that $\mu - \mu' \in I^{k+1}(X,\Lambda)$, then multiplicativity of the symbol again implies $A(\mu - \mu') \in I^{k+2}(X,\Lambda)$. Hence $\sigma_{k+1}(A\mu) = \sigma_{k+1}(A\mu')$ depends only on $\sigma_k(\mu)$. We have thus defined an operator

$$L_A : C^\infty(\Lambda,\mathbb{L}) \to C^\infty(\Lambda,\mathbb{L}) \quad \text{(B.21)}$$

satisfying

$$\mu \in I^k(X,\Lambda), A \in \Psi^m_{sl} \quad \text{with} \quad \sigma_{sl}(A)|_\Lambda = 0 \implies \sigma_{k+1}(A\mu) = L_A\sigma_k(\mu). \quad \text{(B.22)}$$

We call L_A the semiclassical transport operator and shall now describe it more closely. Let $f \in C^\infty(\Lambda)$ and $\sigma_k(\mu) = s$. Pick $B \in \Psi^0_{sl}$ with scalar symbol such that $\sigma_{sl}(B)|_\Lambda = f$. We then have

$$
\begin{aligned}
L_A(fs) &= L_A(\sigma_k(B\mu)) && \text{(B.23)}\\
&= \sigma_{k+1}(AB\mu) && \text{(B.24)}\\
&= \sigma_{k+1}(BA\mu) + \sigma_{k+1}([A,B]\mu) && \text{(B.25)}\\
&= fL_As + \sigma_{sl}([A,B])|_\Lambda.\sigma_k(\mu) && \text{(B.26)}\\
&= fL_As + \frac{1}{i}\{\sigma_{sl}(A), \sigma_{sl}(B)\}|_\Lambda s. && \text{(B.27)}
\end{aligned}
$$

However since $\sigma(A)|_\Lambda = 0$ we have that the Hamilton vector field H_a of $\sigma_{sl}(A)$ is tangent to Λ. Hence

$$
L_A(fs) = fL_As + \frac{1}{i}(H_af)s. \tag{B.28}
$$

Now if we fix a connection ∇ on $\mathbb{L} \otimes \pi^*E$, (B.28) along with the Leibniz rule for ∇ implies

$$
\left(L_A - \frac{1}{i}\nabla_{H_a}\right)(fs) = f\left(L_A - \frac{1}{i}\nabla_{H_a}\right)s. \tag{B.29}
$$

Hence $\left(L_A - \frac{1}{i}\nabla_{H_a}\right)$ represents multiplication by a function

$$
\left(L_A - \frac{1}{i}\nabla_{H_a}\right)s = \sigma_{sub}(A, \nabla)s \tag{B.30}
$$

which we call the sub-principal symbol of A. Finally, we have that the transport operator can be written as

$$
L_A = \frac{1}{i}\nabla_{H_a} + \sigma_{sub}(A, \nabla). \tag{B.31}
$$

B.4 The wave kernel

We are now ready to describe the kernel of the wave operator and show that it is an oscillatory density. We first construct the corresponding exact Lagrangian below.

Proposition B.4.1. *For each $p \in C^\infty(T^*X)$, the embedding $i_\Lambda : T^*X \times \mathbb{R} \hookrightarrow T^*X \times T^*X \times T^*\mathbb{R}$*

$$i_\Lambda(x,\xi,t) = \big((x,-\xi), e^{tH_p}(x,\xi), t, -p(x,\xi)\big) \tag{B.32}$$

gives an exact Lagrangian with phase function $\psi \in C^\infty(T^*X \times \mathbb{R})$ given by

$$\psi = \int_0^t \left(e^{sH_p}\right)^* (i_{H_p}\alpha) ds - tp. \tag{B.33}$$

Proof. The tautological one form on $T^*X \times T^*X \times T^*\mathbb{R}$ is $\tilde{\alpha} = \pi_1^*\alpha + \pi_2^*\alpha + \tau dt$. We can compute

$$\begin{aligned} i_\Lambda^*\pi_1^*\alpha &= -\alpha, & \text{(B.34)}\\ i_\Lambda^*\pi_2^*\alpha &= \left(e^{tH_p}\right)^*\alpha + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt, & \text{(B.35)}\\ i_\Lambda^*(\tau dt) &= -pdt & \text{(B.36)}\end{aligned}$$

and hence

$$i_\Lambda^*\tilde{\alpha} = -\alpha + \left(e^{tH_p}\right)^*\alpha + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt - pdt. \tag{B.37}$$

Next we compute the differential of the phase function to be

$$\begin{aligned} d\psi &= \int_0^t \left(e^{sH_p}\right)^*(di_{H_p}\alpha)ds + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt - tdp - pdt & \text{(B.38)}\\ &= -\int_0^t \left(e^{sH_p}\right)^*(i_{H_p}d\alpha)ds + \int_0^t \left(e^{sH_p}\right)^*(L_{H_p}\alpha)ds + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt \\ &\qquad -tdp - pdt & \text{(B.39)}\\ &= tdp + \int_0^t \left(e^{sH_p}\right)^*(L_{H_p}\alpha)ds + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt - tdp - pdt & \text{(B.40)}\\ &= \int_0^t \left(e^{sH_p}\right)^*(L_{H_p}\alpha)ds + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt - pdt & \text{(B.41)}\\ &= -\alpha + \left(e^{tH_p}\right)^*\alpha + \left(e^{tH_p}\right)^*(i_{H_p}\alpha)dt - pdt. & \text{(B.42)}\end{aligned}$$

Hence (B.37) and (B.42) imply that $i_\Lambda^* \tilde{\alpha} = d\psi$ and thus Λ is an exact Lagrangian with phase function ψ. □

The next proposition now describes the wave kernel.

Proposition B.4.2. *Let $P_h \in \Psi_{sl}^m(X; E \otimes |TX|^{\frac{1}{2}})$ be elliptic and self-adjoint with scalar semiclassical symbol $p(x,\xi)$. Let $f \in C_c^\infty(\mathbb{R})$ be any compactly supported function. The kernel of the operator $f(P)e^{-\frac{itP}{h}}$ lies in $I^{-\frac{n}{2}}(X \times X \times \mathbb{R}, \Lambda, \psi; \pi_1^* E \otimes \pi_2^* E)$ where (Λ, ψ) is the exact Lagrangian given by proposition* (B.4.1).

Proof. Begin with the expansion given by proposition (A.5.5)

$$f(P_h) \sim h^{-n} P_0 + h^{-n+1} P_1 + \dots \tag{B.43}$$

where each $P_i = Op(p_i)$. Let $supp(f) \subset [-C, C]$ and $K \subset T^*X$ be a compact subset of cotangent space such that the elliptic symbol

$$\sigma_{sl}(P_h)(x,\xi) > C \quad \text{for} \quad (x,\xi) \in T^*X \backslash K. \tag{B.44}$$

Following the proof of proposition (A.5.5) we may assume $supp(p_i) \in K$ for each i. Now we pick $s_0 \in C^\infty(\Lambda, \mathbb{L})$ such that

$$-\frac{1}{i}\nabla_{\partial_t} s_0 + \frac{1}{i}\nabla_{H_p} s_0 + \sigma_{sub}(\tilde{P}) s_0 = 0, \quad s_0|_{t=0} = p_0. \tag{B.45}$$

Here $\tilde{P} = -ih\partial_t + P$ and ∇ is a fixed connection on $\mathbb{L} \otimes \pi^* E$ with respect to which the sub-principal symbol in (B.45) is computed. Since p_0 is compactly supported, $s_0 \in C_{cf}^\infty$ and can be quantized to $\mu_0 = Op(s_0) \in I^{-\frac{n}{2}}$. By construction the symbol $\sigma_{sl}(\tilde{P}) = \sigma_{sl}(-ih\partial_t + P) = \tau + p$ vanishes on Λ. Hence by (B.22), (B.31) and (B.45) we have

$$\sigma_{-\frac{n}{2}+1}(\tilde{P}\mu_0) = 0 \tag{B.46}$$

and $\tilde{P}\mu_0 \in I^{-\frac{n}{2}+2}$. Similarly we choose $s_1 \in C_{cf}^\infty(\Lambda, \mathbb{L})$ such that

$$-\frac{1}{i}\nabla_{\partial_t}s_1 + \frac{1}{i}\nabla_{H_p}s_1 + \sigma_{sub}(\tilde{P})s_1 = -\sigma_{-\frac{n}{2}+2}(\tilde{P}\mu_0), \quad s_1|_{t=0} = p_1 \tag{B.47}$$

and set $\mu_1 = Op(s_1) \in I^{-\frac{n}{2}+1}$. Again we may compute

$$\sigma_{-\frac{n}{2}+2}(\tilde{P}(\mu_0 + \mu_1)) = 0 \tag{B.48}$$

and hence $\tilde{P}(\mu_0 + \mu_1) \in I^{-\frac{n}{2}+3}$. By induction we construct $s_i \in C^\infty_{cf}(\Lambda, \mathbb{L})$ such that $s_i|_{t=0} = p_i$ and $\mu_i = Op(s_i) \in I^{-\frac{n}{2}+2}$ satisfy $\tilde{P}(\mu_0 + \ldots + \mu_i) \in I^{-\frac{n}{2}+i+2}, \forall i$. Next we choose, as in chapter 2 of [13], $\mu \in I^{-\frac{n}{2}}$ such that

$$\mu \sim \sum_{j\geq 0}\mu_j \quad \text{or} \quad \mu - \sum_{j=0}^{N}\mu_j \in I^{N+1-\frac{n}{2}} \quad \forall N. \tag{B.49}$$

If we let $k(x, y, t)$ denote the kernel of $f(P)e^{\frac{itP}{h}}$, we then have

$$(-ih\partial_t + P)(\mu - k) = \tilde{P}\mu = r \in I^\infty. \tag{B.50}$$

The initial conditions $s_i|_{t=0} = p_i$ and (B.43) imply that

$$(\mu - k)|_{t=0} = O(h^\infty). \tag{B.51}$$

Finally (B.50) and (B.51) imply via Duhamel's principle that $\mu - k = O(h^\infty)$ and hence $k \in I^{-\frac{n}{2}}$.

□

We shall use the result above to derive trace expansions. For this purpose we shall require explicit generating functions for the Lagrangian in the above proposition near time $t = 0$. The result below will be useful in this regard and appears as proposition (IV-14) in [36].

Proposition B.4.3. *Given $p(x, \xi) \in S^m(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, there exists $T > 0$ sufficiently small such that the Hamilton-Jacobi equation*

$$\begin{aligned} \partial_t S + p(x, \partial_x S) &= 0 \quad \text{(B.52)} \\ S|_{t=0} = x.\xi & \quad \text{(B.53)} \end{aligned}$$

admits a unique solution $S \in C^\infty([-T,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$*. Moreover* S *satisfies*

$$(x, \partial_x S) = e^{tH_p}(\partial_\xi S, \xi). \quad \text{(B.54)}$$

Proof. Define the Hamiltonian trajectory

$$(x(t), \xi(t)) = e^{tH_p}(x_0, \xi). \quad \text{(B.55)}$$

Clearly $x(0)$ is the identity and hence $x(t)$ is a diffeomorphism for $t < T$ sufficiently small. Define S via

$$S(x(t), t) = x(0).\xi + \int_0^t \{\dot{x}(\tau)\xi(\tau) - p(x(\tau), \xi(\tau))\}d\tau. \quad \text{(B.56)}$$

Now let $(x_s(t), \xi_s(t))$ be another Hamiltonian trajectory with initial condition $(x_s(0), \xi)$ chosen such that $x_s(t) = x(t) + s\alpha$. We may then compute the variation $\frac{\partial}{\partial s} S(x_s(t))|_{s=0}$ in two ways to get

$$\alpha.S_x(x(t), t) = \frac{\partial x_s(0)}{\partial s}.\xi - \frac{\partial x_s(0)}{\partial s}.\xi + \alpha.\xi(t). \quad \text{(B.57)}$$

Hence we get

$$S_x(x(t), t) = \xi(t) \quad \text{(B.58)}$$

which proves (B.54). Next differentiate (B.56) with respect to t to get

$$\partial_t S + \dot{x} S_x = \dot{x}\xi(t) - p(x(t), \xi(t)), \quad \text{(B.59)}$$

which combined with (B.58) gives (B.52). □

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