SPECTRUM AND ABNORMALS IN SUB-RIEMANNIAN GEOMETRY: THE 4D QUASI-CONTACT CASE

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Abstract. We prove several relations between spectrum and dynamics including wave trace expansion, sharp/improved Weyl laws, propagation of singularities and quantum ergodicity for the sub-Riemannian (sR) Laplacian in the four dimensional quasi-contact case. A key role in all results is played by the presence of abnormal geodesics and represents the first such appearance of these in sub-Riemannian spectral geometry.

Contents

1. Introduction

Sub-Riemannian (sR) geometry is the study of metric subbundles $\left(E \subset TX, g^E\right)$ inside the tangent bundle of a manifold X that are bracket generating; we refer to [1, 4, 40] for some textbook references on the subject. The geometric/dynamical significance of the bracket-generating hypothesis is via the theorem of Chow-Rashevky on connectivity of points by horizontal curves. With the metric assigning lengths to horizontal curves, the manifold acquires a natural metric space structure. A geodesic is a horizontal length minimizing path. A peculiar feature of sub-Riemannian geometry, unlike Riemannian geometry, is that there are geodesics which do

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not satisfy any variational equation or equivalently are not projections of the corresponding Hamiltonian flow [6, Ch. 1], [39]; these geodesics are *abnormal.*

The choice of an auxiliary density μ_X allows for the definition of a sub-Riemannian Laplacian on the manifold which in general is not an elliptic operator. The analytic significance of the bracket-generating hypothesis is then via the classical theorem of Hörmander [24] saying that the sub-Riemannian Laplacian is hypoelliptic and as such has a discrete spectrum of real eigenvalues. Classical Riemannian results on spectral asymptotics where geodesic flow plays a role such as Weyl's law [3, 25, 32, 38], wave trace trace formulas [11, 13, 19], propagation of singularities [20] and quantum ergodicity [14, 48, 49] remain largely unexplored in sub-Riemannian geometry. It is in particular an interesting question whether abnormal geodesics would play a role in sR spectral geometry. The purpose of this article is to positively answer this question in one of the simplest cases where abnormals exist, namely the four dimensional quasi-contact case.

Let us now state our results more precisely. Let X^4 be a smooth, compact oriented four dimensional manifold. A nowhere vanishing one form $a \in \Omega^1(X)$ is called quasi-contact if the restriction $\text{rk}\, da|_E = 2$ is of maximal rank, where $E \coloneqq \ker a \subset TX$. The three dimensional distribution $E = \ker a \subset TX$ can be shown to be bracket generating and we equip it with a metric g^E. The characteristic line field is defined via $L^E = \ker(a \wedge da) \subset E$ and can be seen to only depend on $E = \ker a$. It carries a natural orientation, induced from that of X, and hence a positively oriented unit section $Z \in C^\infty(L^E)$. The set of integral curves of L^E, also called characteristics, contains the abnormal geodesics in this case.

Given an auxiliary volume form μ on X, the sR Laplacian acting on function is defined via

$$\Delta_{g^E,\mu} \coloneqq \left(\nabla^{g^E}\right)^*_\mu \nabla^{g^E} : C^\infty(X) \to C^\infty(X) \tag{1.1}$$

where $\nabla^{g^E} : C^\infty(X) \to C^\infty(X; E)$, $\left\langle \nabla^{g^E} f, e\right\rangle \coloneqq e(f)$, $\forall e \in E$, is the sR gradient and the adjoint (1.1) above is taken with respect to the natural L^2-inner products coming from μ. The Laplacian (1.1) is not elliptic with characteristic variety $\Sigma \subset T^*X, \Sigma \coloneqq \left\{\sigma\left(\Delta_{g^E,\mu}\right) = 0\right\} = \mathbb{R}[a]$ being given by the graph of the one form a. However being self-adjoint of Hörmander type, there is a complete orthonormal basis of $\{\varphi_j\}_{j=0}^\infty$ for $L^2(X,\mu)$ consisting of (real-valued) eigenvectors for (1.1) $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, $0 \le \lambda_0 \le \lambda_1 \le \ldots$.

Our first result on spectral asymptotics is then the following sharp Weyl law for the counting function $N(\lambda)$ of the number of eigenvalues of the sR Laplacian $\Delta_{g^E,\mu}$ of size at most λ. Below μ_{Popp}, $\nu_{\text{Popp}} = \frac{1}{\left(\int_X \mu_{\text{Popp}}\right)}\mu_{\text{Popp}}$ and a_{g^E} denote the unnormalized, normalized Popp volume and Popp one form respectively (see Section 2.1).

Theorem 1. *The Weyl counting function $N(\lambda)$ for the sR Laplacian $\Delta_{g^E,\mu}$ in the 4D quasi-contact case satisfies the sharp asymptotics*

$$N(\lambda) = \frac{1}{24\pi}\lambda^{5/2}\int_X \mu_{Popp} + O\left(\lambda^2\right) \tag{1.2}$$

Assuming the union of closed integrals curves of L^E to be of measure zero, one further has

$$N(\lambda) = \frac{1}{24\pi}\lambda^{5/2}\int_X \mu_{Popp} + o\left(\lambda^2\right). \tag{1.3}$$

By a usual Tauberian argument, the sharp Weyl law (1.2) above is proved using small time asymptotics of the wave trace. Below we denote by T^E_{abnormal} the length of the shortest closed integral curve of L^E. The (signed) lengths of normal closed geodesics are by definition the periods of closed integral curves for the Hamilton flow of $\sigma\left(\Delta_{g^E,\mu}\right)^{1/2}$ away from Σ. We denote the set of such by $\mathscr{L}_{\text{normal}}$.

Theorem 2. *The singular support of the wave trace satisfies*

$$\text{sing spt}\left(\text{tr } e^{it\sqrt{\Delta_{g^E,\mu}}}\right) \subset \{0\} \cup \left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right) \cup \mathscr{L}_{normal}. \tag{1.4}$$

Furthermore, the singularity at zero is described by the small time asymptotics

$$\begin{aligned} \text{tr } e^{it\sqrt{\Delta_{g^E,\mu}}} = &\sum_{j=0}^{N} c_{j,0}\left(t+i0\right)^{j-5} + \sum_{j=0}^{N} c_{j,1}\left(t+i0\right)^{j-3}\ln\left(t+i0\right) \\ &+ \sum_{j=0}^{N} c_{j,2}t^{j}\ln^2\left(t+i0\right) + O\left(t^{N-4}\right), \end{aligned} \tag{1.5}$$

$\forall N\in\mathbb{N}$, *as* $t\to 0$, *in the distributional sense with leading term* $c_{0,0} = \frac{1}{12}\int_X \mu_{Popp}$.

Note the presence of logarithmic terms in the wave trace expansion (1.5) is unlike on a Riemannian manifold. The singularities of the wave trace (1.4) at (isolated) lengths of non-degenerate normal geodesics in the interval $\left(-T^E_{\text{abnormal}}, T^E_{\text{abnormal}}\right)$ are described by the usual Duistermaat-Guillemin trace formula. Beyond this interval there is a possible density of lengths of $\mathscr{L}_{\text{normal}}$ inside $\left(-\infty, -T^E_{\text{abnormal}}\right] \cup \left[T^E_{\text{abnormal}}, \infty\right)$ for albeit non-degenerate characteristics, caused by closed Hamilton trajectories that approach the characteristic variety (see Prop. 13), making the description of these singularities less tractable.

The large time wave trace formula (1.4) is in turn related to the propagation of singularities for the corresponding wave equation. The classical theorem of [20] describes the propagation of singularities for the half wave equation outside the characteristic variety Σ. To describe the propagation of singularities on Σ we consider the blowup $[T^*X;\Sigma]$ of the cotangent bundle along the characteristic variety with corresponding blow-down map $\beta : [T^*X;\Sigma] \to T^*X$. This is a manifold with boundary $\partial\,[T^*X;\Sigma] = SN\Sigma$ being identified with the spherical normal bundle of Σ which in turn carries an $\mathbb{R}_+$ action extending the one on its interior. The boundary $SN\Sigma$ is equipped with a natural homogeneous and β fiber preserving circle action, by rotation of its symplectic directions, and corresponding generator $R_0 = \frac{d}{d\theta}\left(e^{i\theta}.p\right)\big|_{\theta=0}$. In Section 2.1 we shall define a homogeneous of degree zero section $\hat{Z} \in C^\infty\left(TSN\Sigma/\mathbb{R}\left[R_0\right]\right)$ and a refined circle invariant conic characteristic wave-front set $WF_\Sigma\left(u\right) \subset SN\Sigma$ associated to any distribution $u \in C^{-\infty}\left(X\right)$. These can be equivalently thought of as a homogeneous of degree zero vector field on and conic subset of the quotient $SN\Sigma/S^1$ by the circle action. They project

$$\begin{aligned} \left(\pi\circ\beta\right)_*\hat{Z} &\in L^E \\ \beta\left(WF_\Sigma\left(u\right)\right) &= WF\left(u\right)\cap\Sigma \end{aligned}$$

onto the characteristic line and intersection of the wavefront set of u with Σ respectively. The interval in (1.4) is furthermore related to the set of closed periods of the vector field $\hat{Z}$ (see Prop. 13).

We now have the following propagation of singularities.

Theorem 3. *For any* $u \in C^{-\infty}\left(X\right)$, *the characteristic wavefront set satisfies*

$$WF_\Sigma\left(e^{it\sqrt{\Delta_{g^E,\mu}}}u\right) = e^{t\hat{Z}}\left[WF_\Sigma\left(u\right)\right].$$

Our final result is quantum ergodicity for the sR Laplacian. The line field L^E is said to be ergodic if any union of closed integral curves of L^E is of zero or full measure. The ergodicity of the vector field $\hat{Z}$ is a stronger assumption implying the ergodicity of L^E. We now have the following.

Theorem 4. *Assume that $\hat{Z}$ is ergodic or L^E is ergodic and $L_Z\mu_{Popp} = 0$. Then one has quantum ergodicity for $\Delta_{g^E,\mu}$: there exists a density one subsequence $\{j_k\}_{k=0}^{\infty} \subset \mathbb{N}_0$ such that*

$$\langle B\varphi_{j_k}, \varphi_{j_k}\rangle \to \frac{1}{2}\int d\nu_{Popp}\left[b\left(x, a_{g^E}(x)\right) + b\left(x, -a_{g^E}(x)\right)\right].$$

as $j_k \to \infty$, for each $B \in \Psi^0_{cl}(X)$, with homogeneous principal symbol $b = \sigma(B) \in C^\infty(T^*X)$. In particular, the eigenfunctions get uniformly distributed $|\varphi_{j_k}|^2\mu \rightharpoonup \nu_{Popp}$ as $j_k \to \infty$.

We note the role played by characteristics or integral curves of L^E in all the results above. Under the natural projection, these correspond to isotropic directions along Σ and thereafter with abnormal geodesics via their microlocal characterization by Hsu [27]. Our results are restricted to dimension four as they rely on a normal form that is less workable in higher dimensions. Moreover, there is general lack of understanding of strictly abnormal geodesics in sub-Riemannian geometry; it is for instance outstanding whether they are necessarily smooth [40, Ch. 10], [2, 22, 31].

The leading term in the Weyl law Theorem 1 has been long known [35, 36, 37], the improvement here is in the two remainders. The only previous work treating a sharp Weyl law based on a wave trace expansion of a hypoelliptic operator is [34]. In the sub-Riemannian context [34] however only specializes to the three dimensional contact case; therein the characteristic variety Σ was assumed to be symplectic which is not the case here. There is one isotropic direction along Σ that projects onto L^E. A general result for propagation of singularities of hypoelliptic operators exists in the literature [30]. Our result Theorem 3 based on the characteristic wavefront set is a refinement of the aforementioned in the present context. Recently, quantum ergodicity for the sub-Riemannian Laplacian was established in the three dimensional contact case [17] and as such was the first result on quantum ergodicity for a hypoelliptic operator. Our technique here while partly borrowing from [17] also overcomes significant additional difficulties. In particular our proof of Theorem 4 requires the use of a more exotic second microlocal pseudo-differential calculus near the characteristic variety. Finally unlike here there are no abnormal geodesics in the three dimensional contact case.

The results here also tie in with the authors previous work [46, 47] wherein a trace formula was proved for the semiclassical (magnetic) Dirac operator on a metric contact manifold involving closed Reeb orbits; semiclassical analogs of quasi-contact characteristics of L^E. However there are also significant differences; there firstly seems to be at present no general analog of the Dirac operator, with good spectral properties, in sub-Riemannian geometry (see for example [23, 28]). This forces us to work with the non-(pseudo)differential square root $\sqrt{\Delta_{g^E,\mu}}$ and understand it in a more exotic pseudo-differential calculus. Secondly, the trace considered in [47, 46] was microlocalized on an $\sqrt{h}$ scale near the characteristic variety, using the intrinsic semiclassical parameter, cutting off the Hamilton trajectories away from it. This microlocalized trace formula subsequently does not see the dense accumulation of the Hamilton periods (1.4), involves contributions only from the Reeb orbits and works in higher dimension.

The paper is organized as follows. In Section 2 we begin with some preliminaries on sub-Riemannian geometry including certain specific features of the four dimensional quasi-contact case in Section 2.1. In Section 3 we develop the relevant second microlocal Hermite-Landau calculus on Euclidean space necessary for the proofs. In Section Section 4 we derive normal forms for the sR Laplacian. The normal form of 4.1 is then used in Section Section 5 to develop a global Hermite-Landau calculus on a quasi-contact manifold. The calculus is then used to prove the propagation theorem Theorem 3 in 5.3 and construct a parametrix for the wave operator in 5.4. The parametrix gives a proof of the Weyl laws in 1 via the wave trace expansion Theorem 2 in Section Section 6. In the final Section 7 the calculus is used to prove the quantum ergodicity Theorem 4.

2. sub-Riemannian geometry

Sub-Riemannian (sR) geometry is the study of (metric-)distributions in smooth manifolds. More precisely, a sub-Riemannian manifold is a triple $\left(X^n, E^k \subset TX, g^E\right)$ consisting of an n-dimensional manifold X with and a metric subbundle $\left(E, g^E\right)$ of rank k inside its tangent space. This sub-bundle is assumed to be *bracket generating:* sections of E generate all sections of TX under the Lie bracket. The metric g^E allows for the definition of a length function $l(\gamma) := \int_0^1 |\dot{\gamma}|\, dt$ on the space of horizontal paths of Sobolev regularity one

$$\Omega_E(x_0, x_1) := \left\{\gamma \in H^1([0,1]; X) \,|\, \gamma(0) = x_0,\ \gamma(1) = x_1,\ \dot{\gamma}(t) \in E_{\gamma(t)} \text{ a.e.}\right\}$$

connecting any two points $x_0, x_1 \in X$. This in turn defines the distance function between these points via

$$d^E(x_0, x_1) := \inf_{\gamma \in \Omega_E(x_0, x_1)} l(\gamma). \tag{2.1}$$

The theorem of Chow-Rashevsky [40, Thm 1.6.2] gives the existence of a horizontal path connecting x_0, x_1. This shows that the distance function above is finite and defines a metric space $\left(X, d^E\right)$.

Using the bracket generating condition for E, the canonical flag may be defined

$$\underbrace{E_0(x)}_{=\{0\}} \subset \underbrace{E_1(x)}_{=E} \subset \ldots \subsetneq E_{r(x)}(x) = TX \tag{2.2}$$

inductively via $E_j = E + [E, E_{j-1}]$, $j \geq 2$, as a flag of vector subspaces of TX at any point $x \in X$. Here $r(x)$ is the smallest number such that $E_{r(x)} = TX$ and called the degree of nonholonomy or step of the distribution E at x. The dual canonical flag is then

$$T^*X = \Sigma_0(x) \supset \underbrace{\Sigma_1(x)}_{=:\Sigma} \supset \ldots \supset \underbrace{\Sigma_{r(x)}(x)}_{=\{0\}}, \tag{2.3}$$

$\Sigma_j(x) = E_j^{\perp} := \ker \left[T^*X \to E_j^*\right]$, $1 \leq j \leq r(x)$. We further define the growth and weight vectors at the point $x \in X$ via

$$k^E(x) := \left(\underbrace{k_0^E}_{:=0}, \underbrace{k_1^E}_{=\dim E_1}, \underbrace{k_2^E}_{=\dim E_2}, \ldots, \underbrace{k_r^E}_{=n}\right) \tag{2.4}$$

$$w^E(x) := \left(\underbrace{1, \ldots, 1}_{k_1^E \text{ times}}, \underbrace{2, \ldots 2}_{k_2^E - k_1^E \text{ times}}, \ldots \underbrace{j, \ldots j}_{k_j^E - k_{j-1}^E \text{ times}}, \ldots, \underbrace{r, \ldots, r}_{k_r^E - k_{r-1}^E \text{ times}}\right) \tag{2.5}$$

respectively. The distribution E is called regular at the point $x \in X$ if each k_j^E is a locally constant function near x. The distribution E is said to be equiregular if it is regular at all points of X, in which case each element E_j of the canonical flag (2.2) is a vector bundle. Finally we set

$$Q(x) := \sum_{j=1}^{r(x)} j\left(k_j^E(x) - k_{j-1}^E(x)\right) = \sum_{j=1}^{n} w_j^E(x)$$

whose significance is given by the Mitchell measure theorem [40, Theorem 2.8.3]: $Q(x)$ is the Hausdorff dimension of $\left(X, d^E\right)$ at a regular point $x \in X$.

A canonical volume form on X (analogous to the Riemannian volume) can be defined in the equiregular case. To define this, first note that any surjection $\pi : V \to W$ between two vector spaces allows one to pushforward a metric g^V on V to another $\pi_* g^V$ on W. This is simply the metric on W induced via the identification $W \cong (\ker\pi)^\perp \subset V$, with the metric on $(\ker\pi)^\perp$ being the restriction of g^V . Now for each j we define the linear surjection

$$B_j : E^{\otimes j} \to E_j/E_{j-1}$$
$$B_j(e_1, \ldots e_j) := \mathrm{ad}_{\tilde{e}_1}\mathrm{ad}_{\tilde{e}_2}\ldots\mathrm{ad}_{\tilde{e}_{j-1}}\tilde{e}_j$$

with $\tilde{e}_j \in C^\infty(E)$ denoting local sections extending $e_j \in E$. The pushforward metrics are then well defined on E_j/E_{j-1} and hence define canonical volume elements

$$\det g_j^E \in \Lambda^*\left(E_j/E_{j-1}\right)^*. \tag{2.6}$$

The canonical isomorphism of determinant lines

$$\bigotimes_{j=1}^{r}\Lambda^*\left(E_j/E_{j-1}\right) = \Lambda^*\left(\bigoplus_{j=1}^{r} E_j/E_{j-1}\right) \cong \Lambda^* TX \tag{2.7}$$

along with its dual isomorphism to now gives a canonical smooth volume form

$$\mu_{\mathrm{Popp}} := \bigotimes_{j=1}^{r}\det g_j^E \in \Lambda^*\left(T^*X\right) \tag{2.8}$$

known as the *Popp volume* form. We remark that although the definition makes sense in general it only leads to a smooth form in the equiregular case.

In 7.1 we shall need the important notion of a privileged coordinate system. To define this let $U_1, U_2, \ldots U_k$ be a locally defined set of orthonormal, generating vector fields near $x \in X$. The E-order of a function at the point x is defined via

$$\mathrm{ord}_{E,x}(f) := \max\left\{ s \middle| \left(U_1^{s_1}\ldots U_k^{s_k} f\right)(x) = 0,\ \forall (s_1,\ldots,s_k) \in \mathbb{N}_0^k,\ \sum_{j=1}^{k} s_j = s \right\}.$$

Similarly the $E-$order of a differential operator P at the point x is defined via

$$\mathrm{ord}_{E,x}(P) := \max\left\{ s | \mathrm{ord}_{E,x}(f) \geq s' \Longrightarrow \mathrm{ord}_{E,x}(Pf) \geq s + s' \right\}.$$

It is clear from this definition that the defining vector fields U_j each have E-order at least -1. A coordinate system centered at x is said to be *privileged* if: the set $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{k_j^E}}$ forms a basis of $E_j(x)$ for each j and furthermore each x_j has E-order $w_j^E(x)$ at x. The order of the coordinate vector field $\frac{\partial}{\partial x_j}$ is then easily computed to be $-w_j^E(x)$. There exists a privileged coordinate system at centered at each point of X (see [4] pg. 30). Next define the privileged coordinate dilation $\delta_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, $\forall \varepsilon > 0$, using the weight vector (2.5) via $\delta_\varepsilon(x_1, \ldots, x_n) = (\varepsilon^{w_1} x_1, \ldots, \varepsilon^{w_n} x_n)$. A differential operator P is said to be homogeneous of E-order s if $(\delta_\varepsilon)_* P = \varepsilon^s P$. We may now Taylor expand each defining vector field in terms of homogeneous degrees

$$(\delta_\varepsilon)_* U_j = \varepsilon^{-1} \hat{U}_j^{(-1)} + \hat{U}_j^{(0)} + \varepsilon \hat{U}_j^{(1)} + \ldots, \tag{2.9}$$

, where each $\hat{U}_j^{(s)}$ is an ε-independent vector field with polynomial coefficients. The *nilpotentization* $\left(\hat{X}, \hat{E}, \hat{g}^E\right)$of the sR structure at $x \in X$ is now defined via $\hat{X} = \mathbb{R}^n$, $\hat{E} := \mathbb{R}\left[\hat{U}_1^{(-1)}, \ldots, \hat{U}_k^{(-1)}\right]$ and where the metric $\hat{g}^E$ makes $\left\{\hat{U}_j^{(-1)}\right\}_{j=1}^k$ orthonormal. For any smooth volume form μ on X, one may similarly define its nilpotentization $\hat{\mu} = \mu_0$ at x as the leading order part in its expansion under the privileged coordinate dilation

$$\delta_\varepsilon^* \mu = \varepsilon^{Q(x)} \left[\mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots\right]. \tag{2.10}$$

The nilpotentizations of the sR structure and the volume can be shown to be independent of the choice of privileged coordinates upto sR isometry ([4] Ch. 5).

At a regular point, an invariant definition of the nilpotentizations maybe given. First the sR structure defines a nilpotent Lie algebra at x via

$$\mathfrak{g}_x := (E_1)_x \oplus (E_2/E_1)_x \oplus \ldots \oplus (E_r/E_{r-1})_x \tag{2.11}$$

with the Lie bracket of vector fields inducing an anti-linear map $[.,.] : \mathfrak{g}_x \otimes \mathfrak{g}_x \to \mathfrak{g}_x$. The algebra is clearly graded with its jth graded component $(\mathfrak{g}_x)_j := (E_j/E_{j-1})_x$ and the bracket preserving the grading $\left[(\mathfrak{g}_x)_i, (\mathfrak{g}_x)_j\right] \subset (\mathfrak{g}_x)_{i+j}$. Associated to the nilpotent Lie algebra $\mathfrak{g}$ is a unique simply connected Lie group G with the exponential map giving a diffeomorphism $\exp : \mathfrak{g} \to G$. We define the *nilpotentization* of the sR structure $\left(\hat{X}, \hat{E}, \hat{g}^E\right)$ at x to be $\hat{X} := G$ with the metric distribution $\hat{E}, \hat{g}^E$ obtained via left translation. Given any volume form μ on X, the canonical identification $\Lambda^n \mathfrak{g}_x = \Lambda^n \left[(E_1)_x \oplus (E_2/E_1)_x \oplus \ldots \oplus (E_r/E_{r-1})_x\right] \cong \Lambda^n T_x X$ allows for a definition of the nilpotentization $\hat{\mu}$ of the volume form μ on $\hat{X}$.

Sub-Riemannian geometry may be viewed as a limit of Riemannian geometry. Namely, choose a metric complement $\left(F, g^F\right)$ for the sR distribution satisfying $E \oplus F = TX$. This gives a one parameter family of Riemannian metrics

$$g_\epsilon^{TX} = g^E \oplus \frac{1}{\epsilon} g^F \tag{2.12}$$

which converge $g_\epsilon^{TX} \to g^E$ as $\epsilon \to 0$. We call the above a family of Riemannian metrics *extending/taming* g^E. The corresponding Riemannian distance then converges $d^\epsilon(x_0, x_1) \to d^E(x_0, x_1)$ to the sR distance (2.1) for any $x_0, x_1 \in X$ as $\epsilon \to 0$ (see for eg. [33, Prop. 4]).

2.0.1. *sR Laplacian.* We now define the sub-Riemannian Laplacian and state some of its first properties. First given any function $f \in C^\infty(X)$, define its sR gradient $\nabla^{g^E} f \in C^\infty(E)$ by the equation

$$g^E \left\langle \nabla^{g^E} f, e \right\rangle := e(f), \quad \forall e \in C^\infty(E). \tag{2.13}$$

Fixing an arbitrary volume form μ defines the natural L^2- inner products on $C^\infty(X)$ and $C^\infty(X; E)$ giving the adjoint $(\nabla^g)_\mu^*$ to the gradient depending on μ. The sR Laplacian is now given by

$$\Delta_{g^E,\mu} := \left(\nabla^{g^E}\right)^*_\mu \circ \nabla^{g^E} : C^\infty(X) \to C^\infty(X). \tag{2.14}$$

In terms of a local frame $U_1, \ldots, U_k$ for E, the above maybe written

$$\Delta_{g^E,\mu} f = -U_i\left[g^{E,ij} U_j(f)\right] + g^{E,ij} U_j(f)\left(\left(\nabla^{g^E}\right)^*_\mu U_i\right) \tag{2.15}$$

where $g^E_{ij} = g^E(U_i, U_j)$ and $g^{E,ij}$ is the inverse metric. If the frame is orthonormal the formula simplifies to

$$\Delta_{g^E,\mu} f = \sum_{j=1}^{k}\left[-U_j^2(f) + U_j(f)\left(\left(\nabla^{g^E}\right)^*_\mu U_j\right)\right]. \tag{2.16}$$

To remark on how the choice of the auxiliary form μ affects the Laplacian, let $\mu' = h\mu$ denote another non-vanishing volume form where h is a positive smooth function on X. From the definition (2.14) it now follows easily that one has the relation

$$\Delta_{g^E,\mu'} = h^{-1}\Delta_{g^E,\mu} h + h^{-1}\left(\Delta_{g^E,\mu} h\right).$$

Thus the two corresponding Laplacians are conjugate modulo a zeroth-order term. The sR Laplacian $\Delta_{g^E,\mu}$ is self adjoint with respect to the obvious inner product $\langle f, g\rangle = \int_X fg\mu$. The principal symbol of $\Delta_{g^E,\mu}$ is easily computed to be the Hamiltonian

$$\sigma\left(\Delta_{g^E,\mu}\right)(x,\xi) = H^E(x,\xi) := |\xi|_E^2 \tag{2.17}$$

using the dual metric while its sub-principal symbol is zero. The characteristic variety

$$\Sigma = \left\{\sigma\left(\Delta_{g^E,\mu}\right) = 0\right\} = E^\perp := \{\xi \in T^*X | \xi(v) = 0, \forall v \in E\} \tag{2.18}$$

is the annihilator.

From the local expression (2.16) the sR Laplacian is seen to be a sum of squares operator of Hörmander type [24] and is thus hypoelliptic. Further it satisfies the following optimal sub-elliptic estimate [43]

$$\|f\|_{H^{s+2/r}} \le C\left[\left\|\Delta_{g^E,\mu} f\right\|_{H^s} + \|f\|_{H^s}\right], \quad \forall f \in C^\infty(X), \forall s \in \mathbb{R}, \tag{2.19}$$

where $r := \sup_{x\in X} r(x)$ is the maximal degree of non-holonomy. It now follows that $\Delta_{g^E,\mu}$ has a compact resolvent and thus there is a complete orthonormal basis of $\{\varphi_j\}_{j=0}^\infty$ for $L^2(X,\mu)$ consisting of (real-valued) eigenvectors $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, $0 \le \lambda_0 \le \lambda_1 \le \ldots$.

For each $p \in \Sigma$ on the characteristic variety, the fundamental matrix $F_p \in \mathrm{End}(T_pM)$, $M := T^*X$, is defined via

$$\omega(., F_p.) = \nabla^2\sigma(.,.),$$

where $\nabla^2\sigma$ denotes the Hessian of the symbol (2.17) and ω the symplectic form on T^*X. The fundamental matrix clearly satisfies $\omega(., F_p.) = -\omega(F_p., .)$ and we denote by $\mathrm{Spec}^+(iF_p)$ the set of real and positive eigenvalues of iF_p. Under the condition that

$$\mathrm{tr}^+ F_p := \sum_{\mu\in\mathrm{Spec}^+(iF_p)} \mu > 0 \tag{2.20}$$

the sR Laplacian $\Delta_{g^E,\mu}$ is known to satisfy the better sub-elliptic estimate with loss of one derivative [26]

$$\|f\|_{H^{s+1}} \le C\left[\left\|\Delta_{g^E,\mu} f\right\|_{H^s} + \|f\|_{H^s}\right], \quad \forall f \in C^\infty(X), \forall s \in \mathbb{R}. \tag{2.21}$$

In (5.33) we shall prove a further refined subelliptic estimate for $\Delta_{g^E,\mu}$ in the particular 4D quasi-contact case of our interest.

As a first property for the sR Laplacian we prove the finite propagation speed for its half-wave equation.

Theorem 5. *(Finite propagation speed) Let $u(x;t)$ be the unique solution to the initial value problem*

$$\left(i\partial_t + \sqrt{\Delta_{g^E,\mu}}\right) u = 0$$

$$u(x,0) = u_0 \in C^{-\infty}(X). \tag{2.22}$$

Then the solution satisfies

$$spt\ u(x;t) \subset \left\{ y | \exists x \in sptu_0;\ d^E(x,y) \leq |t| \right\}.$$

Proof. The result maybe restated in terms of the Schwartz kernel $K_t = \left[e^{it\sqrt{\Delta_{g^E,\mu}}}\right]_\mu$ of the half-wave operator

$$\text{spt } K_t \subset \left\{ (x,y) | d^E(x,y) \leq |t| \right\}.$$

We choose a family of metrics g_ϵ^{TX} (2.12) extending g^E. The Riemannian Laplacian $\Delta_{g_\varepsilon^{TX},\mu}$ (still coupled to the form μ) is written

$$\Delta_{g_\epsilon^{TX},\mu} = \Delta_{g^E,\mu} + \epsilon \Delta_{g^F,\mu}$$

where $\Delta_{g^F,\mu}$ is the sR Laplacian on the complementary distribution F. The min-max principle for eigenvalues implies the L^2 convergence $\Pi_{[0,L]}^{\Delta_{g_\epsilon^{TX},\mu}} \to \Pi_{[0,L]}^{\Delta_{g^E,\mu}}$of the corresponding spectral projectors onto the interval $[0,L]$, $\forall L>0$. It now follows that $K_t^\varepsilon \rightharpoonup K_t$ weakly as $\varepsilon \to 0$ with $K_t^\varepsilon := \left[e^{it\sqrt{\Delta_{g_\epsilon^{TX},\mu}}}\right]_\mu$. Knowing that d^E is the limit of the Riemannian distance function for g_ϵ^{TX}, the theorem now follows from the finite propagation speed of $\Delta_{g_\varepsilon^{TX},\mu}$. □

2.1. **Quasi-contact case.** We now describe some sR geometric features in the particular four dimensional quasi-contact case of our interest. We now let X be a smooth, compact oriented four dimensional manifold. A nowhere vanishing one form $a \in \Omega^1(X)$ is called quasi-contact, sometimes referred to as even-contact, if the restriction rk $da|_E = 2$, $E := \ker a \subset TX$, is of maximal rank. The kernel $L^E := \ker(a \wedge da) \subset E$ is then seen to be one dimensional defining the characteristic line field which furthermore only depends on $E = \ker a$. Let $\left(L^E\right)^\perp \subset E$ denote the two dimensional orthogonal complement of the characteristic line on which the restriction $da|_{(L^E)^\perp}$ is non-degenerate by definition. In particular the bundle $\left(L^E\right)^\perp$ is orientable. A canonical Popp one form a_{g^E} (well-defined up to a sign) defining $E = \ker\left(a_{g^E}\right)$ may now be given by requiring that

$$da_{g^E}\big|_{(L^E)^\perp} = \text{vol}\left(g^E\big|_{(L^E)^\perp}\right) \tag{2.23}$$

agree with the metric volume form of $g^E\big|_{(L^E)^\perp}$ corresponding to some choice of orientation for $\left(L^E\right)^\perp$. It is now easy to check that the distributions

$$\left(L^E\right)_2^\perp := \left(L^E\right)^\perp + \left[\left(L^E\right)^\perp, \left(L^E\right)^\perp\right]$$

$$\left(L^E\right)^{\perp, da_{g^E}} := \left\{ v \in TX | da_{g^E}(v,e) = 0, \forall e \in \left(L^E\right)^\perp \right\}$$

are three and two dimensional respectively and both transverse to E. Thus their intersection is one-dimensional and transverse to E. We now define the quasi-contact Reeb vector field $R \in C^{\infty}(TX)$ to be the unique vector field satisfying $R \in \left(L^E\right)_2^{\perp} \cap \left(L^E\right)^{\perp, da_{g^E}}$, $i_R a_{g^E}=1$ (cf. [10], [7, Sec. 10.1]). Note again that the orientation of R depends on the choice of sign for a_{g^E}. However the orientation of $\left(L^E\right)^{\perp} \oplus \mathbb{R}[R]$ defined by $a_{g^E} \wedge da_{g^E}$ is clearly independent of the choice of sign. Furthermore, given that L^E is transverse to $\left(L^E\right)^{\perp} \oplus \mathbb{R}[R]$, the orientation defined by $a_{g^E} \wedge da_{g^E}$ combines with the μ-orientation of manifold to define an orientation of L^E. This defines the unique positively oriented vector field $Z \in C^{\infty}\left(L^E\right)$ such that $|Z| = 1$. We note that ergodicity of L^E is equivalent to the ergodicity of the vector field Z. Let $Z^* \in \Omega^1(X)$ denote the one form which satisfies $Z^*(Z) = 1$ and annihilates $\left(L^E\right)^{\perp} \oplus \mathbb{R}[R]$. The Popp volume form (2.8) in the quasi-contact case is now seen to be

$$\mu_{\text{Popp}} \coloneqq Z^* \wedge a_{g^E} \wedge da_{g^E} \tag{2.24}$$

and we may also define the normalized Popp volume $\nu_{\text{Popp}} \coloneqq \frac{1}{P(X)}\mu_{\text{Popp}}$, $P(X) \coloneqq \int \mu_{\text{Popp}}$. One now has the relations

$$\begin{aligned} L_Z a_{g^E} &= -da_{g^E}(R, Z)\, a_{g^E} \\ L_Z \mu_{\text{Popp}} &= -da_{g^E}(R, Z)\, \mu_{\text{Popp}}. \end{aligned} \tag{2.25}$$

In particular the Z-flow preserves $E = \ker\left(a_{g^E}\right)$.

The characteristic line L^E is said to be volume preserving if there exists a smooth volume on X that is invariant under some non-vanishing section of L^E; the existence does not depend on the choice of the section. In particular there exists a Z-invariant volume $L_Z\left(\underbrace{\hat{\rho}_Z \mu_{\text{Popp}}}_{=:\mu_Z}\right) = 0$ for some positive function $\hat{\rho}_Z$ which would in turn satisfy a similar equation $L_Z \hat{\rho}_Z = da_{g^E}(R, Z)\, \hat{\rho}_Z$; thus further giving $L_Z\left(\underbrace{\hat{\rho}_Z a_{g^E}}_{=\hat{a}_{g^E}}\right) = 0$. It now follows that the volume preserving condition is equivalent to the existence of a defining one form $a = \hat{a}_{g^E}$ for E with $a \wedge da$ closed. Furthermore it is also known to be equivalent to the existence of a defining one form a for E with rk $da = 2$ being constant [29, Lemma 2.3] or the existence of a vector field transverse to and preserving E [42, Prop. 2.1]. We note however that the volume preserving condition on L^E is quite restrictive and often violated (see Example 10 below).

Next, let $Y^3 \subset X$, $TY \pitchfork L^E$ be a locally defined transverse hypersurface near a point $x \in X$. The restriction of the one form a_{g^E} to Y is then a contact form and one has Darboux coordinates (x_1, x_2, x_3) on Y such that $a_{g^E}\big|_Y = \frac{1}{2}[dx_3 + x_1 dx_2 - x_2 dx_1]$. One now translates these coordinates by the flow of Z to obtain local coordinates (x_0, x_1, x_2, x_3) near the point x. Defining the positive function $\hat{\rho} \coloneqq \exp\left\{\int_0^{x_0} da_{g^E}(R, Z)\right\}$, satisfying $Z\hat{\rho} = \partial_{x_0}\hat{\rho} = da_{g^E}(R, Z)\,\hat{\rho}$, one now computes $\mathcal{L}_Z\left(\hat{\rho} a_{g^E}\right) = 0$ giving

$$\hat{\rho} a_{g^E} = \frac{1}{2}[dx_3 + x_1 dx_2 - x_2 dx_1] \tag{2.26}$$

$$Z = \partial_{x_0} \tag{2.27}$$

$\mu_{\text{Popp}} = \frac{1}{2}\hat{\rho}^{-2} dx$ and $Y = \{x_0 = 0\}$ locally.

The characteristic variety $\Sigma \subset T^*X$ of the Laplacian $\Sigma = E^{\perp} = \mathbb{R}[a]$ is clearly the graph of a defining one form by (2.18) in this case. A homogeneous function of degree one on the characteristic variety is then defined via

$$\rho : \Sigma \to \mathbb{R}$$
$$\rho\left(x, s a_{g^E}(x)\right) = s, \quad \forall s \in \mathbb{R}, \tag{2.28}$$

and equals the restriction of the symbol of the Reeb vector field $\rho = \sigma(R)|_{\Sigma}$. With $X, Y \in \left(L^E\right)^{\perp}$, $da_{g^E}(X, Y) = 1$, being a positively oriented orthonormal basis, the relations

$$\begin{aligned} \{\sigma(X), \sigma(Y)\}|_{\Sigma} &= \sigma([X,Y])|_{\Sigma} = \rho a_{g^E}([X,Y]) = \rho \\ \{\sigma(X), \sigma(Z)\}|_{\Sigma} &= \sigma([X,Z])|_{\Sigma} = \rho a_{g^E}([X,Z]) = 0 \\ \{\sigma(Y), \sigma(Z)\}|_{\Sigma} &= \sigma([Y,Z])|_{\Sigma} = \rho a_{g^E}([Y,Z]) = 0 \end{aligned} \tag{2.29}$$

as well as (2.16) show that $\rho = \mathrm{tr}^+ F_p$ is identifiable with (2.20) via the fundamental matrix in this case. This is seen to satisfy the equation

$$H_{\sigma(Z)}\rho\big|_{\Sigma} = \{\sigma(Z), \sigma(R)\}|_{\Sigma} = \sigma([Z,R])|_{\Sigma} = \rho a_{g^E}([Z,R]) = \rho d a_{g^E}(R, Z) \tag{2.30}$$

along the isotropic directions of Σ. From the above computations the following conditions are seen to be equivalent

$$da_{g^E}(R, Z) = 0,\ L_Z a_{g^E} = 0,\ L_Z \mu_{\mathrm{Popp}} = 0,\ H_{\sigma(Z)}\rho\big|_{\Sigma} = 0. \tag{2.31}$$

The Popp volume form pulls back under the natural projection to a four form on Σ which we denote by the same notation μ_{Popp}. It further defines a volume form on Σ via $\mu^{\Sigma}_{\mathrm{Popp}} := d\rho \wedge \mu_{\mathrm{Popp}}$. The Hessian of the symbol $\nabla^2\sigma$ gives a non-degenerate, positive-definite quadratic form on the normal bundle $N\Sigma := TM/T\Sigma$, $M := T^*X$, over the characteristic variety. Under the canonical isomorphism of determinant lines $\Lambda^* T^* M = (\Lambda^* T^*\Sigma) \otimes (\Lambda^* N^*\Sigma)$, the lift of the Popp volume is the unique volume satisfying $\frac{1}{4!}\omega^4 = \mu^{\Sigma}_{\mathrm{Popp}} \wedge \det\left(\nabla^2\sigma\right)$ (cf. [35, 36]).

Next we define the spherical normal bundle $SN\Sigma \xrightarrow{\pi_S} \Sigma$, $SN\Sigma := \{v \in N\Sigma | \nabla^2\sigma(v,v) = 1\}$. Let $T\Sigma^{\omega} \subset TM$ be the symplectic complement of $T\Sigma$. The image $N_1\Sigma$ of $T\Sigma^{\omega} \hookrightarrow TM \to TM/T\Sigma =: N\Sigma$ is two dimensional and equipped with an induced symplectic form ω_0. The bundle $N_1\Sigma$ has a one dimensional $\nabla^2\sigma$- orthocomplement $N_0\Sigma \subset N\Sigma$. This defines (the absolute value of) a homogeneous of degree zero function $\Xi_0 \in C^{\infty}(SN\Sigma)$ satisfying

$$|\Xi_0| := \|\pi_{N_0\Sigma}(v)\|, \quad \forall v \in SN\Sigma, \tag{2.32}$$

with respect to the orthogonal projection/decomposition $N\Sigma = N_0\Sigma \oplus N_1\Sigma$. A sign for this function will be defined shortly. An endomorphism $\mathfrak{J}$ of $N_1\Sigma$ is defined via $\nabla^2\sigma(.,\mathfrak{J}.) = \omega_0(.,.)$. This defines a circle action on $N_1\Sigma$ via $e^{i\theta}.v_0 = (\cos\theta)v_0 + (\sin\theta)\frac{\mathfrak{J}}{|\mathfrak{J}|}v_0$ and subsequently one on $SN\Sigma$ which fixes $N_0\Sigma$. We denote by $R_0 = \partial_\theta \in C^{\infty}(TSN\Sigma)$ the generating vector field satisfying $(\pi \circ \beta)_* R_0 = 0 \in TX$. The quotient $(SN\Sigma)/S^1$ is an interval $[-1,1]_{\Xi_0}$ bundle over Σ. The vertical fiber measure $\mu_V = \left(1 - \Xi_0^2\right) d\Xi_0$ again allows to lift the Popp volume via

$$\mu^{SN\Sigma}_{\mathrm{Popp}} := \mu_V \wedge \pi_S^* \mu^{\Sigma}_{\mathrm{Popp}}. \tag{2.33}$$

which may equivalently be thought of as a rotationally invariant volume on the spherical normal bundle $SN\Sigma$ satisfying

$$L_{R_0}\Xi_0 = 0,\ L_{R_0}\mu^{SN\Sigma}_{\mathrm{Popp}} = 0. \tag{2.34}$$

The blow-up of the cotangent space along the characteristic variety

$$[M;\Sigma] := (M \setminus \Sigma) \amalg SN\Sigma \tag{2.35}$$

and the corresponding blow-down map

$$\beta : [M;\Sigma] \to M$$
$$\beta(p) := \begin{cases} p; & p \in (M \setminus \Sigma) \\ \pi_S(p); & p \in SN\Sigma. \end{cases} \tag{2.36}$$

may now be defined. The blowup has the structure of a smooth manifold with boundary; its interior is $[M;\Sigma]^o = (M \setminus \Sigma)$ while the boundary $\partial [M;\Sigma] = SN\Sigma$ is identified with the spherical normal bundle. The boundary defining function is the square root of the symbol $\sigma^{1/2}$ (or its pullback to the blowup). There is a natural action of $\mathbb{R}_+$ on the blowup with the quotient $[M;\Sigma]/\mathbb{R}_+ = [S^*X; S^*\Sigma]$ canonically identified with the corresponding blowup of the cospheres $S^*X = T^*X/\mathbb{R}_+$, $S^*\Sigma := \Sigma/\mathbb{R}_+$. The cosphere of the characteristic variety

$$S^*\Sigma := \Sigma/\mathbb{R}_+ = \{\rho = \pm 1\} = \underbrace{X_+}_{=:\{(x, a_{g^E}(x))\}} \cup \underbrace{X_-}_{=:\{(x, -a_{g^E}(x))\}} \subset \Sigma$$

is identifiable with two copies of the manifold given a choice of sign for the Popp form a_{g^E} and thus carries the lift of the Popp volume μ_{Popp}. The spherical normal bundle carries a similar $\mathbb{R}_+$-action and we denote the quotient by $SNS^*\Sigma := SN\Sigma/\mathbb{R}_+$. The S^1 action on $SN\Sigma$ is homogeneous of degree zero and one may form the double quotient $SNS^*\Sigma/S^1$ as an $[-1,1]_{\Xi_0}$ bundle over X. In similar vein as (2.33) this now carries a lift of the Popp measure

$$\mu_{\text{Popp}}^{SNS^*\Sigma} := \mu_V \wedge \pi_S^* \mu_{\text{Popp}} \tag{2.37}$$

which is again equivalently thought of as a rotationally invariant volume on the spherical normal bundle $SNS^*\Sigma$. We also define the normalized versions ν_{Popp}, $\nu_{\text{Popp}}^{SNS^*\Sigma}$ of μ_{Popp}, $\mu_{\text{Popp}}^{SNS^*\Sigma}$ with total volume one.

In 4.1, Section 5 we shall show the existence of smooth function Ω, invariantly defined using the sR structure on a neighborhood of the characteristic variety Σ, whose Hamilton vector field restricts

$$H_\Omega|_{SN\Sigma} = R_0 \tag{2.38}$$

to the rotational derivative R_0. The Hamilton vector field $H_{\sigma^{1/2}}$ of the square root symbol $\sigma^{1/2} := \sigma(\Delta_{g^E,\mu})^{1/2}$ is well-defined on the complement $[M;\Sigma]^o = (M \setminus \Sigma)$ of the characteristic variety and hence on the interior of the blowup. Its singularity near the boundary is then captured by the rotational vector field R_0. In particular, the following will be proved in 4.1.

Proposition 6. *The Hamilton vector field $H_{\sigma^{1/2}}$ has a singular expansion*

$$H_{\sigma^{1/2}} = \frac{\sigma(R)}{\sigma^{1/2}} H_\Omega + \hat{Z} + o(1) \tag{2.39}$$

near the boundary of the blowup $[M;\Sigma]$. Here $\hat{Z} \in C^\infty(TS^*N\Sigma)$ projects

$$(\pi \circ \beta)_* \hat{Z} \in L^E \subset TX, \tag{2.40}$$

with $\left|(\pi \circ \beta)_* \hat{Z}\right| = |\Xi_0|$, onto the characteristic line with the lift of the Popp volume (2.37) *preserved under the flow of $\left[\hat{Z}\right] \in C^\infty(T(S^*N\Sigma/S^1)) = C^\infty(TS^*N\Sigma/\mathbb{R}[R_0])$*

$$L_{[\hat{Z}]} \mu_{Popp}^{SNS^*\Sigma} = 0. \tag{2.41}$$

We note that the above (2.40) also defines a signed version of (2.32) via $\Xi_0 = \left\langle Z, (\pi \circ \beta)_* \hat{Z} \right\rangle$.

3. Hermite Calculus

In this section we define the requisite Hermite-Landau calculus. We begin with the definition of the Hermite transform.

3.1. **Hermite transform.** Below we denote by (x_0, x_1, x_2, x_3) the coordinates on $\mathbb{R}^4$ and abbreviate $\underline{x} = (x_0, x_2, x_3)$. Let $(T^*\mathbb{R}^4)_+ = \{(x, \xi) \in T^*\mathbb{R}^4 | \xi_3 > 0\}$ and let $\left(\hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\right) = \left(\xi_3^{-1}\xi_0, \xi_3^{-1}\xi_1, \xi_3^{-1}\xi_2\right)$ to be the homogeneous variables on this cone. It shall also be useful to define the homogeneous variables

$$\hat{x}_3 := x_3 + \frac{1}{2}x_1\hat{\xi}_1$$
$$\Omega := \xi_3\left(x_1^2 + \hat{\xi}_1^2\right) \quad \text{satisfying}$$
$$\{\hat{x}_3, \Omega\} = 0. \tag{3.1}$$

Set $h_k(u) := \frac{\pi^{1/4}}{(2^k k!)}\left[-\partial_u + u\right]^k e^{-\frac{1}{2}u^2}$ to be the kth Hermite function and set $h_k(x_1, \xi_3) := |\xi_3|^{1/4} h_k\left(|\xi_3|^{1/2} x_1\right)$; $k \in \mathbb{N}_0$. The Hermite operators $H_k : \mathcal{S}'_c\left(\mathbb{R}^3_{\underline{x}}\right) \to \mathcal{S}'\left(\mathbb{R}^4_x\right)$, $H_k^* : \mathcal{S}'_c\left(\mathbb{R}^4_x\right) \to \mathcal{S}'\left(\mathbb{R}^3_{\underline{x}}\right)$ are then defined

$$(H_k u)(x) := (2\pi)^{-1}\int e^{ix_3\xi_3} h_k(x_1, \xi_3)\left(\mathcal{F}_{x_3}u\right)(\xi_3)\, d\xi_3 \tag{3.2}$$

$$(H_k^* u)(\underline{x}) := (2\pi)^{-1}\int e^{ix_3\xi_3} h_k(x_1, \xi_3)\left(\mathcal{F}_{x_3}u\right)(\xi_3)\, dx_1 d\xi_3 \tag{3.3}$$

where $\mathcal{F}_{x_3}(\xi_3) := \int e^{-ix_3\xi_3}u(x_3)\,dx_3$ denotes the partial Fourier transform in the x_3 variable. The above clearly maps $L^2\left(\mathbb{R}^3_{\underline{x}}\right)$, $L^2\left(\mathbb{R}^4_x\right)$ into each other and as such are adjoints satisfying

$$H_k^* H_l = \delta_{kl}. \tag{3.4}$$

It is then an easy exercise to show

$$WF(H_k u) = \left\{\left(0, \underline{x}; 0, \underline{\xi}\right) \mid \left(\underline{x}; \underline{\xi}\right) \in WF(u)\right\}$$
$$WF(H_k^* v) = \left\{\left(\underline{x}; \underline{\xi}\right) \mid \left(0, \underline{x}; 0, \underline{\xi}\right) \in WF(v)\right\}$$

$\forall u \in \mathcal{S}'_c\left(\mathbb{R}^3_{\underline{x}}\right)$, $v \in \mathcal{S}'_c\left(\mathbb{R}^4_x\right)$ and $k \in \mathbb{N}_0$. In particular distributions in $\mathcal{S}'_c\left(\mathbb{R}^3_{\underline{x}}\right)$ micro-supported in $\left\{\xi_3 > c\left|\underline{\xi}\right|\right\} \subset T^*\mathbb{R}^3$ are mapped into those micro-supported in $\{\xi_3 > c|\xi|\} \subset T^*\mathbb{R}^4$ under H_k for each $c > 0$ and vice versa under H_k^*. As acting on such one now has the identities

$$\left[-\xi_1 + x_1\xi_3\right] H_k = H_{k+1}\left[2(k+1)\xi_3\right]^{1/2}$$
$$\left[\xi_1 + x_1\xi_3\right] H_k = H_{k-1}\left[2k\xi_3\right]^{1/2},$$
$$\Omega H_k = H_k(2k+1)$$
$$\hat{x}_3 H_k = H_k x_3 \tag{3.5}$$

(cf. [9] Sec. 6). In particular

$$a\left(x_0, x_2, \hat{x}_3; \xi_0, \xi_2, \xi_3; x_1^2 + \hat{\xi}_1^2\right)^W H_k = H_k a\left(\underline{x}, \underline{\xi}; \xi_3^{-1}(2k+1)\right)^W \tag{3.6}$$

for any $a \in S^m\left(T^*\mathbb{R}^4_x\right)$ of the given form. The image of each H_k thus corresponds to an eigenspace of Ω by (3.5) and is referred to as a Landau level.

The *Hermite transform* is now defined

$$H^* : \mathcal{S}'_c\left(\mathbb{R}^4_x\right) \to \mathcal{S}'\left(\mathbb{R}^3_{\underline{x}}; \mathbb{C}^{\mathbb{N}_0}\right);$$
$$(H^*u)_k := H^*_k u \tag{3.7}$$

as the map from $\mathcal{S}'_c\left(\mathbb{R}^4_4\right)$ into $\mathcal{S}'\left(\mathbb{R}^3_{\underline{x}}\right)$-valued $\mathbb{N}_0$-sequences.

Next, set $h^s := \left\{u : \mathbb{N}_0 \to \mathbb{C} | \left\Vert u\right\Vert_s := \sum \langle k\rangle^{2s} \left|u\left(k\right)\right|^2 < \infty\right\} \subset \mathbb{C}^{\mathbb{N}_0}$ with the special notation $l^2 = h^0$. As just noted H maps $L^2_c\left(\mathbb{R}^4_x\right)$ into $L^2\left(\mathbb{R}^3_{\underline{x}}; l^2\right)$. More generally, we define $\forall s_1, s_2 \in \mathbb{R}$ the anisotropic Sobolev space

$$h^{s_1,s_2} = H^{s_2}\left(\mathbb{R}^3_{\underline{x}}; h^{s_1}\right) = \left\{u : \mathbb{N}_0 \to H^{s_2}\left(\mathbb{R}^3_{\underline{x}}\right) | \left\Vert u\right\Vert_{s_1,s_2} := \left(\sum_{k\in\mathbb{N}_0} \langle k\rangle^{2s_1} \left\Vert u\left(k\right)\right\Vert^2_{H^{s_2}}\right)^{1/2} < \infty\right\}.$$

For $s_1, s_2 \in \mathbb{N}_0$, it follows from (3.5) that the Hermite transform H is a isomorphism between h^{s_1,s_2} and the space

$$H^{s_1,s_2} := \left\{u \in \mathcal{S}'\left(\mathbb{R}^4_x\right) | \left(x_1\xi_3^{1/2}\right)^\alpha \left(\xi_3^{-1/2}\xi_1\right)^\beta \langle\underline{\xi}\rangle^{s_2} \hat{u} \in L^2\left(\mathbb{R}^4_\xi\right), \forall \left|\alpha\right| + \left|\beta\right| \leq 2s_1\right\} \subset \mathcal{S}'\left(\mathbb{R}^4_x\right). \tag{3.8}$$

3.2. **Symbol classes.** In this subsection we define classes of pseudo-differential operators on $\mathbb{R}^4$ using the Hermite transform (3.7).

First for each $\delta_1, \delta_2 > 0$ define the conic subsets

$$K_{\delta_1,\delta_2} := \left\{\xi_3 > 0; \left|\left(\hat{\xi}_2, x_0, x_2, \hat{x}_3\right)\right| < \delta_1, \left|\left(\hat{\xi}_0, \hat{\xi}_1, x_1\right)\right| < \delta_2\right\} \subset T^*\mathbb{R}^4$$
$$\Sigma_0 := \left\{(x,\xi) \in K_{\delta_1,\delta_2} | \xi_0 = x_1 = \xi_1 = 0\right\} \subset K_{\delta_1,\delta_2} \subset T^*\mathbb{R}^4 \tag{3.9}$$

containing the point $(0,0,0,0;0,0,0,1) \in T^*\mathbb{R}^4$. The corresponding spherical bundles for the cones above are $S^*K_{\delta_1,\delta_2} = \left\{(x,\xi) \in K_{\delta_1,\delta_2} | \left|\xi\right| = 1\right\}$, $S^*\Sigma_0 = \left\{(x,\xi) \in \Sigma_0 | \left|\xi\right| = 1\right\}$. Letting $\rho\left(x_0, x_2\hat{x}_3; \hat{\xi}_2, \xi_3\right) = \xi_3\hat{\rho}\left(x_0, x_2\hat{x}_3; \hat{\xi}_2\right) \in C^\infty\left(\Sigma_0\right)$, be a positive homogeneous function of degree one on the sub-cone Σ_0 set

$$d_\rho\left(x,\xi\right) = \xi_3\hat{d}_\rho := \sqrt{\xi_0^2 + \rho\Omega} \tag{3.10}$$

as a defining function for the respective sub-cone $\left\{d_\rho = 0\right\} = \Sigma_0 \subset K_{\delta_1,\delta_2}$ and subset $\left\{d_\rho = 0\right\} = S^*\Sigma_0 \subset S^*K_{\delta_1,\delta_2}$ above. It is further mapped to $d_k := \sqrt{\xi_0^2 + \rho\left(2k+1\right)}$ under the Hermite transform H^*_k for each $k \in \mathbb{N}_0$. The blowup along these sub-cones and corresponding blowdown map are defined via

$$\left[K_{\delta_1,\delta_2}; \Sigma_0\right] := \left\{(x,\xi) \in K_{\delta_1,\delta_2} | \hat{d}_\rho \geq 1\right\}$$
$$\beta : \left[K_{\delta_1,\delta_2}; \Sigma_0\right] \to K_{\delta_1,\delta_2};$$
$$\beta\left(x_1, \hat{\xi}_0, \hat{\xi}_1; x_0, x_2, x_3, \hat{\xi}_2, \xi_3\right) := \left(\frac{\hat{d}_\rho - 1}{\hat{d}_\rho}\left(x_1, \hat{\xi}_0, \hat{\xi}_1\right); x_0, x_2, x_3, \hat{\xi}_2, \xi_3\right). \tag{3.11}$$

The blowup $\left[K_{\delta_1,\delta_2}; \Sigma_0\right]$ is a manifold with boundary

$$\partial\left[K_{\delta_1,\delta_2}; \Sigma_0\right] = \left\{(x,\xi) \in K_{\delta_1,\delta_2},\ \hat{d}_\rho = 1\right\}$$

and interior $[K_{\delta_1,\delta_2};\Sigma_0]^o = \left\{(x,\xi)\in K_{\delta_1,\delta_2},\, \hat{d}_\rho > 1\right\}$. The boundary defining function is the pullback

$$\beta^*\hat{d}_\rho = \hat{d}_\rho - 1 \tag{3.12}$$

of (3.10) under the blowdown. A similar blowup $[S^*K_{\delta_1,\delta_2};S^*\Sigma_0]$ with interior $[S^*K_{\delta_1,\delta_2};S^*\Sigma_0]^o$ and corresponding blowdown map to $S^*K_{\delta_1,\delta_2}$ may also be defined. Let $C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]^o\right)$, $C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\right.$ $C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]\right)$, $C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right)$ denote smooth functions on the interior and those extending to the boundary respectively. Similarly,

$$\begin{aligned} C^\infty_{\text{inv}}\left([K_{\delta_1,\delta_2};\Sigma_0]^o\right) &\subset C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]^o\right)\\ C^\infty_{\text{inv}}\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]^o\right) &\subset C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]^o\right)\\ C^\infty_{\text{inv}}\left([K_{\delta_1,\delta_2};\Sigma_0]\right) &\subset C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]\right)\\ C^\infty_{\text{inv}}\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right) &\subset C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right) \end{aligned} \tag{3.13}$$

are subsets of those functions f which have the rotational symmetry

$$\{\Omega, f\} = \left(x_1\partial_{\hat{\xi}_1} - \hat{\xi}_1\partial_{x_1}\right) f = 0.$$

These are functions of the arguments

$$\left(x_1^2+\hat{\xi}_1^2, \hat{\xi}_0, \hat{\xi}_2, \xi_3; x_0, x_2, \hat{x}_3\right). \tag{3.14}$$

Further, let

$$\begin{aligned} \left(\beta^* d_\rho\right)^{-m_2} C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]\right) &\subset C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]^o\right)\\ \left(\beta^* d_\rho\right)^{-m_2} C^\infty_{\text{inv}}\left([K_{\delta_1,\delta_2};\Sigma_0]\right) &\subset C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]^o\right)\\ \left(\beta^*\hat{d}_\rho\right)^{-m_2} C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right) &\subset C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]^o\right)\\ \left(\beta^*\hat{d}_\rho\right)^{-m_2} C^\infty_{\text{inv}}\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right) &\subset C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]^o\right) \end{aligned} \tag{3.15}$$

denote the set of functions f in the interiors such that $\left(\beta^* d_\rho\right)^{m_2} f \in C^\infty\left([K_{\delta_1,\delta_2};\Sigma_0]\right)$, $\left(\beta^* d_\rho\right)^{m_2} f \in C^\infty_{\text{inv}}\left([K_{\delta_1,\delta_2};\Sigma_0]\right)$, $\left(\beta^*\hat{d}_\rho\right)^{m_2} f \in C^\infty\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right)$, $\left(\beta^*\hat{d}_\rho\right)^{m_2} f \in C^\infty_{\text{inv}}\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right)$ respectively. Finally denote by $\left(\beta^* d_\rho\right)^{-m_2} C^\infty_{c,\text{inv}}\left([K_{\delta_1,\delta_2};\Sigma_0]\right)$, $\left(\beta^*\hat{d}_\rho\right)^{-m_2} C^\infty_{c,\text{inv}}\left([S^*K_{\delta_1,\delta_2};S^*\Sigma_0]\right)$ the subset of those functions supported in $[K_{\delta_1-\varepsilon,\delta_2-\varepsilon};\Sigma_0]$, $[SK_{\delta_1-\varepsilon,\delta_2-\varepsilon};S^*\Sigma_0]$ respectively for some $\varepsilon>0$.

Next set $\partial_{\hat{d}_\rho} \coloneqq \frac{1}{\hat{d}_\rho}\left[\hat{\xi}_0\partial_{\hat{\xi}_0} + \hat{\rho}\left(x_1\partial_{x_1} + \hat{\xi}_1\partial_{\hat{\xi}_1}\right)\right]$ to be the homogeneous radial vector field on the blowups $[K_{\delta_1,\delta_2};\Sigma_0]$, $[S^*K_{\delta_1,\delta_2};S^*\Sigma_0]$. We now define the class of symbols $S^{m_1,m_2}\left(\mathbb{R}^4,\Sigma_0\right)$, $m_1,m_2\in\mathbb{R}$, as the set of functions $a\in\left(\beta^* d_\rho\right)^{-m_2} C^\infty_{c,\text{inv}}\left([K_{1,1};\Sigma_0]\right)$ satisfying

$$\|a\|_{\alpha,\beta,\gamma} \coloneqq \sup_{[K_{1,1};\Sigma_0]}\left|\left(\beta^*\hat{d}_\rho\right)^{m_2+\beta}\xi_3^{-m_1+\alpha}\partial^\alpha_{\xi_3}\partial^\beta_{\hat{d}_\rho}\left(T_1\ldots T_N\right)a\right| < \infty, \tag{3.16}$$

$\forall\,(\alpha,\beta)\in\mathbb{N}_0\times\mathbb{N}_0$ and any set of smooth, homogeneous of degree zero, vector fields $(T_1,\ldots,T_N)$ on the blowup that are tangent to the boundary. For any $a\in S^{m_1,m_2}\left(\mathbb{R}^4,\Sigma_0\right)$, we shall also define the associated sequence of functions $a_k\in C^\infty\left(\mathbb{R}^6_{\underline{x},\underline{\xi}}\right)$, $k\in\mathbb{N}_0$, via

$$a_k \coloneqq \left(\beta^{-1}\right)^* a\left((2k+1)\,\xi_3^{-1}; \hat{\xi}_0, \hat{\xi}_2, \xi_3; x_0, x_2, x_3\right), \tag{3.17}$$

where $(2k+1)\,\xi_3^{-1}$, x_3 replace the $x_1^2+\hat{\xi}_1^2$, $\hat{x}_3$ arguments (3.14) respectively.

3.3. **Quantization and calculus.** The quantization $a^H : \mathcal{S}(\mathbb{R}^4_x) \to \mathcal{S}(\mathbb{R}^4_x)$ of a symbol $a \in S^{m_1,m_2}(\mathbb{R}^4, \Sigma_0)$ is defined by the rule

$$H_k a^H H^*_{k'} = \delta_{kk'} a^W_k. \tag{3.18}$$

or alternately written

$$a^H = \sum_{k=0}^{\infty} H^*_k a^W_k H_k. \tag{3.19}$$

We denote by $\Psi^{m_1,m_2}(\mathbb{R}^4; \Sigma_0)$ the set of such quantizations. We remark that this class depends on the decorated cone (Σ_0, ρ), i.e. additionally on the homogeneous function $\rho \in C^\infty(\Sigma_0)$; we shall sometimes precise this with the notation $\Psi^{m_1,m_2}(\mathbb{R}^4; \Sigma_0, \rho)$ instead to avoid confusion. We note that the quantization above depends only on the value of the symbol at points of $[K_{1,1}; \Sigma_0]$ where $\xi_3 \beta^* \left(x_1^2 + \hat{\xi}_1^2\right) \in 2\mathbb{N}_0 + 1$. In particular the quantization a^H only depends on the restriction of a to the parabolic region

$$P = \left\{\beta^* \hat{d}_\rho \geq \xi_3^{-1}\right\}. \tag{3.20}$$

This gives the inclusions

$$\Psi^{m_1,m_2}\left(\mathbb{R}^4; \Sigma_0\right) \subset \Psi^{m_1+\frac{1}{2},m_2-1}\left(\mathbb{R}^4; \Sigma_0\right) \tag{3.21}$$

$$\Psi^{-\infty,m_2}\left(\mathbb{R}^4; \Sigma_0\right) \subset \Psi^{-\infty}\left(\mathbb{R}^4\right). \tag{3.22}$$

In the case where the symbol $a = \beta^* a_0$ happens to be the pullback of $a_0 \in C^\infty\left(\left(T^*\mathbb{R}^4_x\right)^+\right)$ under the blowdown one has

$$a^H = a_0^W \tag{3.23}$$

by (3.5), (3.6) and this partly motivates our definition (3.18), (3.19). This also gives the inclusion

$$\begin{aligned} &\Psi^m_{\text{inv}}\left(\mathbb{R}^4\right) \subset \Psi^{m,0}\left(\mathbb{R}^4; \Sigma_0\right) \quad \text{where} \\ &\Psi^m_{\text{inv}}\left(\mathbb{R}^4\right) \coloneqq \left\{A = a^W \in \Psi^m\left(\mathbb{R}^4\right) | \text{spt}(a) \subset K_{1,1},\ \{a, \Omega\} = 0\right\}. \end{aligned} \tag{3.24}$$

Next define a subclass $S^{m_1,m_2}_{\text{cl}}(\mathbb{R}^4, \Sigma_0) \subset S^{m_1,m_2}(\mathbb{R}^4, \Sigma_0)$ of classical symbols. This is the subset of those symbols a for which there exist $a_j \in \left(\beta^* \hat{d}_\rho\right)^{-m_2} C^\infty_{c,\text{inv}}([S^*K_{1,1}; S^*\Sigma_0])$, $j = 0, 1, \ldots$, $\chi \in C^\infty_c(\mathbb{R})$, such that

$$a = [1 - \chi(|\xi|)]\, \xi_3^{m_1} \left[a_0 + (\beta^* d_\rho)^{-1} a_1 + \ldots + (\beta^* d_\rho)^{-N} a_N\right] + S^{m_1 - \frac{N+1}{2}, m_2}\left(\mathbb{R}^4, \Sigma_0\right), \tag{3.25}$$

$\forall N \in \mathbb{N}_0$. Here the remainder estimate is understood on the parabolic region P (3.20). Further time dependent symbol classes $S^{m_1,m_2}_t(\mathbb{R}^4, \Sigma_0)$, $S^{m_1,m_2}_{\text{cl},t}(\mathbb{R}^4, \Sigma_0)$ are defined as follows: $S^{m_1,m_2}_t(\mathbb{R}^4, \Sigma_0)$ is the set of time-dependent functions $a \in C^\infty(\mathbb{R}_t \times [K_{1,1}; \Sigma_0]^o)$ such that each $a(t; .) \in (\beta^* d_\rho)^{-m_2} C^\infty_{c,\text{inv}}([K_{1,1}; \Sigma_0])$, $t \in \mathbb{R}$, with each estimate (3.16) being uniform on compact intervals of time. Finally $S^{m_1,m_2}_{\text{cl},t}(\mathbb{R}^4, \Sigma_0)$ is the subset of those symbols $a \in S^{m_1,m_2}_t(\mathbb{R}^4, \Sigma_0)$ for which there exist time independent $a_j \in S^{m_1-j,m_2+j}_{\text{cl}}(\mathbb{R}^4, \Sigma_0)$, $j = 0, 1, \ldots$, such that

$$a = \sum_{j=0}^{N} t^j a_j + t^{N+1} S^{m_1 - \frac{N+1}{2}, m_2}\left(\mathbb{R}^4, \Sigma_0\right), \tag{3.26}$$

$\forall N \in \mathbb{N}_0$. We denote by $\Psi^{m_1,m_2}_{\text{cl}}(\mathbb{R}^4; \Sigma_0)$, $\Psi^{m_1,m_2}_{\text{cl},t}(\mathbb{R}^4; \Sigma_0)$ the set of quantizations of the classical symbols (3.25), (3.26).

Standard application of Borel's lemma gives asymptotic summation: for any set of operators $A_j \in \Psi^{m_1-j,m_2+j}$, $B_j \in \Psi^{m_1,m_2-j}$, $j \in \mathbb{N}_0$, there exists $A, B \in \Psi^{m_1,m_2}$ such that

$$A - \sum_{j=0}^{N} A_j \in \Psi^{m_1-\frac{N}{2},m_2},$$

$$B - \sum_{j=0}^{N} B_j \in \Psi^{m_1,m_2-N}, \forall N \in \mathbb{N}_0 \tag{3.27}$$

and respectively for the classes $\Psi_{\mathrm{cl}}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$, $\Psi_{\mathrm{cl},t}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$.

Below we show that these classes are well behaved under composition and adjoint.

Proposition 7. *For $a^H \in \Psi^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$, $b^H \in \Psi^{m_1',m_2'}(\mathbb{R}^4;\Sigma_0)$ we have*

$$a^H b^H \in \Psi^{m_1+m_1',m_2+m_2'}\left(\mathbb{R}^4;\Sigma_0\right)$$

$$a^H b^H = (ab)^H + \Psi^{m_1+m_1'-1,m_2+m_2'+1}\left(\mathbb{R}^4;\Sigma_0\right)$$

$$\left(a^H\right)^* = \bar{a}^H \tag{3.28}$$

and respectively for the classes $\Psi_{cl}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$, $\Psi_{cl,t}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$.

Proof. We first prove that the corresponding symbols a_k & b_k (3.17) compose for each $k \in \mathbb{N}_0$. From (3.19) and composition of Weyl symbols, the composed symbol $a_k \circ b_k$ has an asymptotic expansion

$$a_k^W \circ b_k^W \sim \left[\sum_{|\alpha|=0}^{\infty} \frac{i^\alpha}{\alpha!}\left(\underbrace{D_x D_\eta - D_y D_\xi}_{=A(D)}\right)^\alpha \left[a_k\left(\underline{x},\underline{\xi}\right) b_k\left(\underline{y},\underline{\eta}\right)\right]_{\underline{x}=\underline{y};\underline{\xi}=\underline{\eta}}\right]^W. \tag{3.29}$$

Each successive term above then corresponds to a symbol in $S^{m_1+m_1'-|\alpha|,m_2+m_2'+|\alpha|}(\mathbb{R}^4;\Sigma_0) \subset S^{m_1+m_1'-\frac{|\alpha|}{2},m_2+m_2'}(\mathbb{R}^4;\Sigma_0)$ (3.21) and can be asymptotically summed (3.27). The residual term above is then in $\Psi^{-\infty,m_2}(\mathbb{R}^4) \subset \Psi^{-\infty}(\mathbb{R}^4)$ (3.22). The support condition for the composed symbol follows from a standard integral representation formula for the symbol of the composition $a_k \circ b_k$ ([26] Sec. 18.1). The adjoint property is an immediate consequence of the usual adjoint property $\left(a_k^W\right)^* = \bar{a}_k^W$ of Weyl quantization for each k. □

The principal symbol of $A = a^H \in \Psi_{\mathrm{cl}}^{m_1,m_2}$ is now defined via

$$\sigma_{m_1,m_2}^H(A) = a_0 \in \left(\beta^* \hat{d}_\rho\right)^{-m_2} C_{c,\mathrm{inv}}^\infty\left([S^*K_{1,1};S^*\Sigma_0]\right) \tag{3.30}$$

to be the leading term in the expansion (3.25) above. One has the symbol short exact sequence

$$0 \to \Psi_{\mathrm{cl}}^{m_1-1,m_2+1}\left(\mathbb{R}^4;\Sigma_0\right) \to \Psi_{\mathrm{cl}}^{m_1,m_2}\left(\mathbb{R}^4;\Sigma_0\right) \xrightarrow{\sigma_{m_1,m_2}^H} \left(\beta^* \hat{d}_\rho\right)^{-m_2} C_{c,\mathrm{inv}}^\infty\left([S^*K_{1,1};S^*\Sigma_0]\right) \to 0. \tag{3.31}$$

From (3.28), it follows that the symbol (3.30) is multiplicative and closed under adjoints

$$\sigma_{m_1+m_1',m_2+m_2'}^H(AB) = \sigma_{m_1,m_2}^H(A)\,\sigma_{m_1',m_2'}^H(B), \tag{3.32}$$

$$\sigma_{m_1,m_2}^H(A^*) = \overline{\sigma_{m_1,m_2}^H(A)}, \tag{3.33}$$

$\forall A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$, $B\in\Psi_{\mathrm{cl}}^{m_1',m_2'}(\mathbb{R}^4;\Sigma_0)$. The symbol exact sequence (3.31) gives

$$[A,B]\in\Psi_{\mathrm{cl}}^{m_1+m_1'-1,m_2+m_2'+1}\left(\mathbb{R}^4;\Sigma_0\right)\quad\text{with}$$
$$\sigma^H_{m_1+m_1'-1,m_2+m_2'+1}\left([A,B]\right)=i\left\{\sigma^H_{m_1,m_2}\left(A\right),\sigma^H_{m_1',m_2'}\left(B\right)\right\}$$

following from (3.29).

Next we define the generalized Sobolev spaces as the subspace of tempered distributions $u\in H^{s_1,s_2}(\mathbb{R}^4_x;\Sigma_0)\subset\mathcal{S}_c'(\mathbb{R}^4_x)$, $s_1,s_2\in\mathbb{R}$, micro-supported in $K_{1,1}$ satisfying

$$\|u\|_{H^{s_1,s_2}}\coloneqq\int\sum_{k\in\mathbb{N}_0}\left|(2k+1)^{-\frac{1}{2}s_2}\left\langle\xi_3\right\rangle^{s_1+\frac{1}{2}s_2}H_k^*u\right|^2dx<\infty.\tag{3.34}$$

Following (3.5), (3.6), (3.16) and the Calderon-Vaillancourt inequality, these can be equivalently characterized as $u\subset\mathcal{S}_c'(\mathbb{R}^4_x)$ micro-supported in $K_{1,1}$ satisfying

$$A\in\Psi^{s_1,s_2}\left(\mathbb{R}^4;\Sigma_0\right)\implies Au\in L^2.\tag{3.35}$$

In light of the inclusions (3.22), (3.24) this gives

$$\begin{aligned}H^{s_1+\frac{1}{2},s_2-1}\left(\mathbb{R}^4_x;\Sigma_0\right)&\subset H^{s_1,s_2}\left(\mathbb{R}^4_x;\Sigma_0\right)\\ H^{s_1,0}\left(\mathbb{R}^4_x;\Sigma_0\right)&=\left\{u\in H^{s_1}\left(\mathbb{R}^4_x\right)|WF\left(u\right)\subset K_{1,1}\right\}.\end{aligned}\tag{3.36}$$

One further has Sobolev boundedness

$$u\in H^{s_1,s_2}\left(\mathbb{R}^4_x;\Sigma_0\right),\,A\in\Psi^{m_1,m_2}\left(\mathbb{R}^4;\Sigma_0\right)\implies Au\in H^{s_1-m_1,s_2-m_2}\left(\mathbb{R}^4_x;\Sigma_0\right).\tag{3.37}$$

Next, we define the characteristic wavefront $WF_{\Sigma_0}(A)\subset\partial\left[S^*K_{1,1};S^*\Sigma_0\right]$ of an operator $A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$ in the exotic class as a subset of the boundary of the blowup. This is the intersection $\left[\bigcap_{j=0}^{\infty}\operatorname{spt}(a_j)\right]\cap\partial\left[S^*K_{1,1};S^*\Sigma_0\right]$, of the supports of the symbols in its symbolic expansion (3.25). The characteristic wavefront $WF_{\Sigma_0}(u)\subset\partial\left[S^*K_{1,1};S^*\Sigma_0\right]$ of a distribution $u\subset\mathcal{S}'(\mathbb{R}^4_x)$ micro-supported in $K_{1,1}$ is also defined via

$$(x,\xi)\notin WF_{\Sigma_0}(u)\iff\exists A\in\Psi_{\mathrm{cl}}^{0,0}\left(\mathbb{R}^4;\Sigma_0\right),\text{s.t. }(x,\xi)\in WF_{\Sigma_0}(A),\,Au\in C^\infty\tag{3.38}$$

or equivalently via

$$(x,\xi)\notin WF_{\Sigma_0}(u)\iff\exists A\in\Psi_{\mathrm{cl}}^{0,0}\left(\mathbb{R}^4;\Sigma_0\right),\text{s.t. }\sigma^H_{0,0}(A)(x,\xi)\neq0,\,Au\in C^\infty.\tag{3.39}$$

The wavefronts can also be considered as conic subsets of $\partial\left[K_{1,1};\Sigma_0\right]$ and are again rotationally invariant under the action of $x_1\partial_{\hat{\xi}_1}-\hat{\xi}_1\partial_{x_1}$ by definition. The following are easily established

$$\begin{aligned}WF_{\Sigma_0}(A+B)&\subset WF_{\Sigma_0}(A)\cup WF_{\Sigma_0}(B)\\ WF_{\Sigma_0}(AB)&\subset WF_{\Sigma_0}(A)\cap WF_{\Sigma_0}(B)\\ WF_{\Sigma_0}(Au)&\subset WF_{\Sigma_0}(A)\cap WF_{\Sigma_0}(u)\end{aligned}\tag{3.40}$$

$\forall A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(\mathbb{R}^4;\Sigma_0)$, $B\in\Psi_{\mathrm{cl}}^{m_1',m_2'}(\mathbb{R}^4;\Sigma_0)$ and $u\subset\mathcal{S}'(\mathbb{R}^4_x)$ micro-supported in $K_{1,1}$. Finally using (3.23), (3.24) and a partition of unity argument one shows

$$\beta\left(WF_{\Sigma_0}(u)\right)=WF(u)\cap\Sigma_0\tag{3.41}$$

under the blowdown map (3.11), for each $u\subset\mathcal{S}'(\mathbb{R}^4_x)$ micro-supported in $K_{1,1}$.

4. Birkhoff normal forms

In this section we obtain two Birkhoff normal forms for $\Delta_{g^E,\mu}$. The first near points on the characteristic cone Σ and the second near any closed characteristic.

4.1. **Normal form near Σ.** Choose the canonical quasi-contact form a_{g^E} (2.23) defining E and let $x \in X$. As before one then has a system of local Darboux coordinates (x_0, x_1, x_2, x_3) centered at x such that

$$\hat{\rho} a_{g^E} = \frac{1}{2}\left[dx_3 + x_1 dx_2 - x_2 dx_1\right] \tag{4.1}$$

$$U_0 = Z = \partial_{x_0}. \tag{4.2}$$

The distribution E is locally generated by the vector fields U_0, $U_1 = \partial_{x_1} + x_2\partial_{x_3}$ and $U_2 = \partial_{x_2} - x_1\partial_{x_3}$ and we let g_{ij} denote the components of the metric in this basis. Further let $X, Y \in \left(L^E\right)^{\perp}$ be orthonormal. The relations $da_{g^E}(X,Y) = \hat{\rho}\, da_{g^E}(U_1, U_2) = 1$ imply the existence of locally defined functions δ_1, δ_2 and $\Lambda = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$ with the latter taking values in $\mathfrak{sp}(2)$ such that

$$Z = U_0$$
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \delta_1 U_0 \\ \delta_2 U_0 \end{bmatrix} + \hat{\rho}^{1/2} e^{\Lambda} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

The symbol of the Laplacian is then calculated

$$\begin{aligned} \sigma\left(\Delta_{g^E,\mu}\right) &= \sigma(X)^2 + \sigma(Y)^2 + \sigma(Z)^2 \\ &= \left(1 + \delta_1^2 + \delta_2^2\right)\xi_0^2 + 2\xi_0 \hat{\rho}^{1/2}\begin{bmatrix}\delta_1 & \delta_2\end{bmatrix} e^{\Lambda}\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} + \hat{\rho}\begin{bmatrix}\eta_1 & \eta_2\end{bmatrix} e^{\Lambda^t} e^{\Lambda}\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} \end{aligned}$$

with η_0, η_1, η_2 denoting the symbols

$$\begin{aligned} \eta_0 &= \sigma(U_0) = \xi_0 \\ \eta_1 &= \sigma(U_1) = \xi_1 + x_2\xi_3 \\ \eta_2 &= \sigma(U_2) = \xi_2 - x_1\xi_3, \end{aligned}$$

in terms of the induced coordinates (x, ξ) on the cotangent bundle. The characteristic variety or vanishing locus of the symbol is given by

$$\Sigma = \left\{\left(x, s a_{g^E}(x)\right) | s \in \mathbb{R}\right\}.$$

Now if $(x,\xi) \in (\Sigma \setminus 0) \cap \pi^{-1}(x)$, we clearly have from (4.1) that $(x_0, x_1, x_2, x_3; \xi_0, \xi_1, \xi_2) = 0$ while $\xi_3 \neq 0$. We may assume

$$(x,\xi) = c\underbrace{(0,0,0,0;0,0,0,1)}_{=p_0}, \tag{4.3}$$

$c > 0$, is a positive homogeneous multiple of the given point. The homogeneous coordinates x_j, $\hat{\xi}_j = \frac{\xi_j}{\xi_3}$, $0 \leq j \leq 2$, are then well defined on $C \setminus 0$ for a conic neighborhood C of (x,ξ). We set

$$f_0 = \xi_3\left(x_1 x_2 + \hat{\xi}_1\hat{\xi}_2\right) \tag{4.4}$$

and compute

$$e^{\frac{\pi}{4}H_{f_0}}\left(x_1,\hat{\xi}_1;x_2,\hat{\xi}_2\right) = \left(\frac{x_1+\hat{\xi}_2}{\sqrt{2}},\frac{-x_2+\hat{\xi}_1}{\sqrt{2}};\frac{x_2+\hat{\xi}_1}{\sqrt{2}},\frac{-x_1+\hat{\xi}_2}{\sqrt{2}}\right),$$

$$e^{\frac{\pi}{4}H_{f_0}}\left(x_0,x_3;\xi_0,\xi_3\right) = \left(x_0,x_3+\frac{1}{2}\left(x_1x_2-\hat{\xi}_1\hat{\xi}_2\right);\xi_0,\xi_3\right). \tag{4.5}$$

We further compute

$$\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\sigma\left(\Delta_{g^E,\mu}\right)=\xi_3^2\left[a_0\hat{\xi}_0^2+\hat{\xi}_0B_0\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}+2\hat{\rho}\begin{bmatrix}\hat{\xi}_1 & x_1\end{bmatrix}e^{\Lambda_0^t}e^{\Lambda_0}\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}\right]$$

where $a_0=\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*(1+\delta_1^2+\delta_2^2)$, $B_0=2\sqrt{2}\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\hat{\rho}^{1/2}\begin{bmatrix}\delta_1 & \delta_2\end{bmatrix}e^{\Lambda}$ and $\Lambda_0=\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\Lambda$.

Next denote by $O_\Sigma(k)$ homogeneous (of degree 2) functions on T^*X which vanish to order k along $\Sigma=\left\{x_1=\hat{\xi}_1=\hat{\xi}_0=0\right\}$. We also denote by $O_\Sigma(k)$ the Weyl quantizations on $\mathbb{R}^4$ of such symbols. A Taylor expansion gives

$$\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\sigma\left(\Delta_{g^E,\mu}\right)=\xi_3^2\left[\bar{a}\hat{\xi}_0^2+\hat{\xi}_0\bar{B}\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}+2\hat{\rho}\begin{bmatrix}\hat{\xi}_1 & x_1\end{bmatrix}e^{\bar{\Lambda}^t}e^{\bar{\Lambda}}\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}\right] + O_\Sigma(3)$$

where $\bar{a}>0,\bar{B}$ and $\bar{\Lambda}\in\mathfrak{sp}(2)$ may now be considered as functions of $(x_0,x_2,x_3;\xi_2)$. Next we consider another function f_1 of the form

$$f_1=\frac{\xi_3^2}{2}\begin{bmatrix}\hat{\xi}_1 & x_1\end{bmatrix}\begin{bmatrix} & -1\\ 1 & \end{bmatrix}\bar{\Lambda}\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}.$$

and compute

$$\left(e^{H_{f_1}}\right)^*\xi_3\begin{bmatrix}\hat{\xi}_0\\\hat{\xi}_1\\x_1\end{bmatrix} = \xi_3\begin{bmatrix}\hat{\xi}_0\\ e^{-\bar{\Lambda}}\begin{bmatrix}\hat{\xi}_1\\x_1\end{bmatrix}\end{bmatrix}+O_\Sigma(2).$$

Following this we may further compute

$$\left(e^{H_{f_1}}\right)^*\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\sigma\left(\Delta_{g^E,\mu}\right)=\xi_3^2\left[a_1\hat{\xi}_0^2+b_1\hat{\xi}_0x_1+c_1\hat{\xi}_0\hat{\xi}_1+2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right)\right]+O_\Sigma(3)$$

for some functions $a_1>0,b_1$ and c_1 of $(x_0,x_2,x_3;\xi_2)$. By a symplectic change of coordinates in the $\left(\underline{x},\underline{\xi}\right)$ variables we may set $a_1=1$ following which

$$\kappa_0^*\sigma\left(\Delta_{g^E,\mu}\right)=\xi_0^2+2\xi_3^2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right)+O_\Sigma(3) \tag{4.6}$$

for $\kappa_0=\left(e^{H_{f_2}}\right)\left(e^{H_{f_1}}\right)\left(e^{\frac{\pi}{4}H_{f_0}}\right)$ with $f_2=\frac{1}{2}\left[c_1\xi_0x_1-b_1\xi_0\xi_1\right]$. Here $\rho=\xi_3\hat{\rho}$, $\hat{\rho}=\hat{\rho}(x_0,x_2,x_3;\xi_2)$ is homogeneous of degree one, and is identifiable with the only positive eigenvalue of the fundamental matrix (2.20).

Next we claim that for some Hamiltonian diffeomorphism $\kappa_1=e^{H_{\xi_3f_3}}$, $f_3\left(x;\hat{\xi}\right)\in O_\Sigma(1)$, and $\hat{\xi}_0$-independent function $R\left(x_1^2+\hat{\xi}_1^2;x_0,x_2,\hat{x}_3;\hat{\xi}_2\right)\in O_\Sigma(4)$, $[R,\Omega]=0$, we have

$$\kappa_1^*\kappa_0^*\sigma\left(\Delta_{g^E,\mu}\right)=\xi_3^2\left[\hat{\xi}_0^2+2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right)\right]+R. \tag{4.7}$$

To this end, we first define the complex variables $z_1=\frac{1}{\sqrt{2}}\left(x_1+i\hat{\xi}_1\right)$, $\bar{z}_1=\frac{1}{\sqrt{2}}\left(x_1-i\hat{\xi}_1\right)$ and a grading on monomials in the variables $\left(\hat{\xi}_0,z_1,\bar{z}_1\right)$ via $\text{gr}\left(\hat{\xi}_0^az_1^b\bar{z}_1^c\right)=2a+b+c$. Further define by $\mathcal{O}_\Sigma(k)$ the set of homogeneous (of degree 2) functions defined near Σ whose Taylor series involves monomials of grading at least k. We first prove that for each $N \geq 3$ there exists $g_N\left(x;\hat{\xi}\right) \in \mathcal{O}_\Sigma(1)$, $R_N\left(x_1^2+\hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \hat{\xi}_2\right) \in \mathcal{O}_\Sigma(4)$ such that

$$\begin{aligned}
\left(e^{H_{\xi_3^{-1}g_N}}\right)^* \kappa_0^* \sigma\left(\Delta_{g^E,\mu}\right) &= \xi_3^2\left[\hat{\xi}_0^2 + 2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right)\right] + R_N + \mathcal{O}_\Sigma(N)\\
g_N - g_{N-1} &= \mathcal{O}_\Sigma(N-3)\,.\\
R_N - R_{N-1} &= \mathcal{O}_\Sigma(N)\,.
\end{aligned} \tag{4.8}$$

The case $N=3$ is (4.6). To complete the induction step write

$$\left(e^{H_{\xi_3^{-1}g_N}}\right)^* \kappa_0^* \sigma\left(\Delta_{g^E,\mu}\right) = \xi_3^2\left[\hat{\xi}_0^2 + 2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right)\right] + R_N + \xi_3^2 \sum_{2a+b+c=N} r_{abc}\hat{\xi}_0^a z_1^b \bar{z}_1^c + \mathcal{O}_\Sigma(N+1) \tag{4.9}$$

for complex functions $r_{abc}\left(x_0, x_2, \hat{x}_3; \hat{\xi}_2\right)$ satisfying $\bar{r}_{abc} = r_{acb}$. Define

$$\begin{aligned}
g_{N+1} &= g_N + \xi_3^2\left[\sum_{\substack{2a+b+c=N\\ b\neq c}} s_{abc}\hat{\xi}_0^a z_1^b \bar{z}_1^c + \sum_{2a+2b=N-2} s_{abb}\hat{\xi}_0^a \left(z_1\bar{z}_1\right)^b\right],\\
s_{abc} &= \frac{1}{4i\left(b-c\right)\hat{\rho}} r_{abc}; \quad b \neq c,\\
s_{abb} &= -\frac{1}{2}\int_0^{x_0} r_{(a-1)bb}; \quad a \geq 1.
\end{aligned}$$

A simple computation from (4.9) then gives

$$\begin{aligned}
\left(e^{H_{\xi_3^{-1}g_{N+1}}}\right)^* \kappa_0^* \sigma\left(\Delta_{g^E,\mu}\right) &= \xi_3^2\left[\hat{\xi}_0^2 + 2\hat{\rho}\underbrace{\left(x_1^2+\hat{\xi}_1^2\right)}_{=\xi_3^{-1}\Omega} + R_{N+1}\right] + O_\Sigma(N+1)\\
R_{N+1} &= R_N + \xi_3^2 r_{0\frac{N}{2}\frac{N}{2}}\left(z_1\bar{z}_1\right)^{\frac{N}{2}}
\end{aligned}$$

where the term involving $\frac{N}{2}$ above is understood to be zero for N odd. This completes the induction step. An application of Borel's lemma then gives $g\left(x;\hat{\xi}\right) \in \mathcal{O}_\Sigma(1)$, $R\left(x_1^2+\hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \hat{\xi}_2\right) \in \mathcal{O}_\Sigma(4)$, $R_\infty \in \mathcal{O}_\Sigma(\infty)$ such that

$$\left(e^{H_{\xi_3^{-1}g}}\right)^* \kappa_0^* \sigma\left(\Delta_{g^E,\mu}\right) = \underbrace{\xi_3^2\left[\hat{\xi}_0^2 + 2\hat{\rho}\left(x_1^2+\hat{\xi}_1^2\right) + \xi_3^{-2}R\right]}_{=\sigma_0} + \xi_3^2 R_\infty. \tag{4.10}$$

We shall now eliminate the last infinite order error term $\xi_3^2 R_\infty$ above by the following lemma.

Lemma 8. *There exists a smooth, homogeneous of degree one function f_∞ defined in a conic neighborhood of p_0 (4.3) satisfying*

$$\left(e^{H_{f_\infty}}\right)^*\left(\xi_3^2\sigma_0 + \xi_3^2 R_\infty\right) = \xi_3^2\sigma_0. \tag{4.11}$$

Proof. Without loss of generality assume that the conic neighborhood in which (4.7) holds to be of the form $C_\varepsilon = \left\{\left|\left(x_0, x_1, x_2, x_3, \hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\right)\right| \leq \varepsilon\right\}$ for some $\varepsilon > 0$. Next with $\chi \in C_c^\infty(-1,1)$

with $\chi = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, define the microlocal cutoff $\chi_{\varepsilon} := \chi\left(\frac{\left|\left(x_0,x_1,x_2,x_3,\hat{\xi}_0,\hat{\xi}_1,\hat{\xi}_2\right)\right|}{\varepsilon}\right)$. Further define the function

$$\tilde{\sigma}_0 := \sigma_0 + \chi_{\varepsilon} R_{\infty} = \begin{cases} \sigma_0 + R_{\infty}; & \text{on } C_{\varepsilon/2} \\ \sigma_0; & \text{on } C_{\varepsilon}^c \end{cases}$$

$$\text{satisfying} \quad \xi_3^2\left(\tilde{\sigma}_0 - \sigma_0\right) = \mathcal{O}_{\Sigma}\left(\infty\right). \tag{4.12}$$

We may then compute the Hamilton vector field

$$H_{\xi_3\tilde{\sigma}_0^{1/2}} = \tilde{\sigma}_0^{-1/2}\left[2\hat{\xi}_0\partial_{x_0} + 2\xi_3^{-1}\Omega H_{\rho} + 2\hat{\rho}H_{\Omega} + \xi_3^{-1}H_{\xi_3^2(R+R_{\infty})}\right]$$

which is well-defined on $\{\tilde{\sigma}_0 \neq 0\} = T^*\mathbb{R}^4 \setminus \Sigma$. From $R + R_{\infty} = O\left(\tilde{\sigma}_0^2\right)$ one may calculate

$$e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}\left(x_0,x_1,x_2,x_3,\hat{\xi}_0,\hat{\xi}_1,\hat{\xi}_2\right) := \left(x_0\left(t\right),x_1\left(t\right),x_2\left(t\right),x_3\left(t\right),\hat{\xi}_0\left(t\right),\hat{\xi}_1\left(t\right),\hat{\xi}_2\left(t\right)\right), \quad \text{with}$$

$$x_0\left(t\right) = x_0 + 2t\tilde{\sigma}_0^{-1/2}\hat{\xi}_0 + O\left(\tilde{\sigma}_0^{3/2}\right)$$
$$x_3\left(t\right) = x_3 + 2t\tilde{\sigma}_0^{-1/2}\xi_3^{-1}\Omega + O\left(\tilde{\sigma}_0^{3/2}\right).$$

The above shows that there exists a uniform c_1 such that any point $p \in C_{\varepsilon}$ flows out of the cone

$$e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}p \notin C_{\varepsilon} \text{ for time } t > c_1\tilde{\sigma}_0\left(p\right)^{-1/2}. \tag{4.13}$$

Outside of the cone C_{ε} the flows of $e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}$ and $e^{tH_{\xi_3\sigma_0^{1/2}}}$ agree by (4.12).

Now we define the symplectomorphism

$$\kappa_{\infty} : T^*\mathbb{R}^4 \setminus \Sigma \to T^*\mathbb{R}^4 \setminus \Sigma$$
$$\kappa_{\infty} := \lim_{t\to\infty} e^{-tH_{\xi_3\sigma_0^{1/2}}} \circ e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}.$$

The limit exists, and is in fact attained in finite time, since

$$e^{-t'H_{\xi_3\sigma_0^{1/2}}} \circ e^{t'H_{\xi_3\tilde{\sigma}_0^{1/2}}}p = e^{-t'H_{\xi_3\sigma_0^{1/2}}} \circ e^{(t'-t)H_{\xi_3\tilde{\sigma}_0^{1/2}}} \circ e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}p = e^{-tH_{\xi_3\sigma_0^{1/2}}} \circ e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}p$$

for $t' > t > c_1\tilde{\sigma}_0\left(p\right)^{-1/2}$ using (4.13) and the fact that $e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}} = e^{tH_{\xi_3\sigma_0^{1/2}}}$ outside C_{ε}. It is thus a Hamiltonian symplectomorphism $\kappa_{\infty} = e^{H_{f_{\infty}}}$ and clearly satisfies

$$\left(e^{H_{f_{\infty}}}\right)_* H_{\xi_3\tilde{\sigma}_0^{1/2}} = H_{\xi_3\sigma_0^{1/2}} \tag{4.14}$$

by definition. Finally to prove that it extends to the characteristic variety, first define $\tilde{x}_0\left(t\right) := \left(e^{-tH_{\xi_3\sigma_0^{1/2}}} \circ e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}\right)^* x_0$ which equals $\kappa_{\infty}^* x_0$ on $T^*\mathbb{R}^4 \setminus \Sigma$ for $t > c_1\tilde{\sigma}_0\left(p\right)^{-1/2}$. We then compute the time derivative

$$\frac{d}{dt}\tilde{x}_0\left(t\right) = \frac{d}{dt}\left(e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}\right)^*\left(e^{-tH_{\xi_3\sigma_0^{1/2}}}\right)^* x_0$$
$$= \left\{\xi_3\tilde{\sigma}_0^{1/2} - \left(e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}\right)^*\xi_3\sigma_0^{1/2}, \tilde{x}_0\left(t\right)\right\}$$
$$= \left\{\left(e^{tH_{\xi_3\tilde{\sigma}_0^{1/2}}}\right)^*\left(\xi_3\tilde{\sigma}_0^{1/2} - \xi_3\sigma_0^{1/2}\right), \tilde{x}_0\left(t\right)\right\} = \xi_3^{-2}\mathcal{O}_{\Sigma}\left(\infty\right)$$

uniformly on compact intervals of time following (4.12). It now follows that the function $\kappa_{\infty}^{*}x_0 = x_0 + \int_0^{c_1\tilde{\sigma}_0(p)^{-1/2}} \frac{d}{dt}\tilde{x}_0(t)\,dt$ extends smoothly by the identity to the characteristic variety. A similar argument for the other coordinate functions along with (4.14) completes the proof. □

The proof of the lemma above follows the 'scattering trick' of Nelson [41, 16]; as already pointed out in [17, Sec. 5] its requisite analog is missing from [34].

We now prove a Birkhoff normal form for the total symbol of $\Delta_{g^E,\mu}$. Below let $C_\kappa \subset T^*X \times T^*\mathbb{R}^4$ denote the canonical relation associated to the symplectomorphism $\kappa := \kappa_0 \circ \kappa_1$ in (4.7) and the pullback $(\kappa^*\rho)(x_0, x_2, \hat{x}_3; \xi_2, \xi_3)$ by the same notation $\rho(x_0, x_2, \hat{x}_3; \xi_2, \xi_3)$.

Theorem 9. *There exists a Fourier integral operator* $U \in I^0_{cl}(X, \mathbb{R}^4; C_\kappa)$ *and* ξ_0*-independent symbols* $R\left(x_1^2 + \hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \xi_2, \xi_3\right) \in O_\Sigma(4)$, $r\left(x_1^2 + \hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \xi_2, \xi_3\right) \in S^0_{cl}$ *satisfying*

$$U\Delta_{g^E,\mu}U^* = \underbrace{\xi_0^2 + 2\rho\underbrace{\left(\xi_3 x_1^2 + \xi_3^{-1}\xi_1^2\right)}_{=\Omega} + R + r}_{=:\Delta_{\rho,R,r}} + \Psi^{-\infty}\left(\mathbb{R}^4\right), \tag{4.15}$$

and $UU^* = 1$ *microlocally on some open conic neighborhood* $C \supset (\Sigma \setminus 0) \cap \pi^{-1}(x)$.

Proof. If $U_1 : L^2(X) \to L^2(\mathbb{R}^4)$ denotes a unitary Fourier integral operator quantizing the symplectomorphism $\kappa_0 \circ e^{H_{\xi_3 g}} \circ e^{H_{f_\infty}}$ in (4.7), (4.11) one has

$$\sigma\left(U_1\Delta_{g^E,\mu}U_1^*\right) = \xi_0^2 + 2\xi_3\Omega + R$$

by Egorov's theorem. By an argument of Weinstein (see Prop. 6 of [17]) the quantization U_1 may be further chosen so that the sub-principal symbol of the composition is zero and we may rewrite

$$U_1\Delta_{g^E,\mu}U_1^* = \xi_0^2 + \xi_3\Omega + R + \Psi^0\left(\mathbb{R}^4\right) \tag{4.16}$$

at the operator level; we drop the Weyl quantization symbol above for simplicity.

Next we prove by induction that $\forall N \geq 0$, there exists $r_N\left(x_1^2 + \hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \xi_2, \xi_3\right) \in S^0_{\mathrm{cl}}$ and Fourier integral operator e^{if_N}, $f_N \in S^{-1}(\mathbb{R}^4)$ such that

$$\begin{aligned} e^{if_N}U_1\Delta_{g^E,\mu}U_1^*e^{-if_N} &= \xi_0^2 + \xi_3\Omega + R + r_N + \Psi^{-N}\left(\mathbb{R}^4\right)\\ f_{N+1} - f_N &\in S^{-N-1}\left(\mathbb{R}^4\right). \end{aligned} \tag{4.17}$$

The base case of the induction is (4.16) with $f_0 = r_0 = 0$. For the inductive step, we first write

$$e^{if_N}U_1\Delta_{g^E,\mu}U_1^*e^{-if_N} = \xi_0^2 + 2\xi_3\Omega + R + r_N + \xi_3^{-N}s_{N+1}\left(x_0, x_1, x_2, \hat{x}_3; \hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\right) + \Psi^{-N-1}\left(\mathbb{R}^4\right).$$

Then with $f_{N+1} = f_N + \xi_3^{-N-1}g_N\left(x_0, x_1, x_2, \hat{x}_3; \hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\right)$ we compute

$$e^{if_{N+1}}U_1\Delta_{g^E,\mu}U_1^*e^{-if_{N+1}} = \xi_0^2 + 2\xi_3\Omega + R + r_N + \xi_3^{-N}\left\{s_{N+1} + 2\hat{\xi}_0\partial_{x_0}g_N + \left(4 + 2\partial_\varrho R\right)\partial_\theta g_N\right\} + \Psi^{-N-1}\left(\mathbb{R}^4\right)$$

in polar coordinates $\left(x_1,\hat{\xi}_1\right)=\left(\varrho^{1/2}\cos\theta,\varrho^{1/2}\sin\theta\right)$. We may then choose

$$\begin{aligned}g_N:=&\frac{1}{2}\int_0^{x_0}\bar{s}_{N+1,1}\left(x_0',\hat{\xi}_0,\varrho,\theta'\right)dx_0'\\&+\frac{1}{(4+2\partial_\varrho R)}\int_0^{\theta}d\theta'\left[s_{N+1}\left(x_0',\hat{\xi}_0,\varrho,\theta'\right)-\bar{s}_{N+1}\left(x_0',\hat{\xi}_0,\varrho,\theta'\right)\right]\\ \text{with}\quad\bar{s}_{N+1}\left(x_0',\hat{\xi}_0,\varrho\right):=&\frac{1}{2\pi}\int_0^{2\pi}d\theta'\,s_{N+1}\left(x_0',\hat{\xi}_0,\varrho,\theta'\right)\\ \bar{s}_{N+1}\left(x_0,\hat{\xi}_0,\varrho\right):=&\bar{s}_{N+1}\left(x_0,0,\varrho\right)+\hat{\xi}_0\bar{s}_{N+1,1}\left(x_0,\hat{\xi}_0,\varrho\right)\end{aligned}$$

to compute

$$\begin{aligned}e^{if_{N+1}}U_1\Delta_{g^E,\mu}U_1^*e^{-if_{N+1}}=\xi_0^2+2\xi_3\Omega+R+r_{N+1}+\Psi^{-N-1}\left(\mathbb{R}^4\right)\quad\text{with}\\ r_{N+1}=r_N+\xi_3^{-N}\bar{s}_{N+1}\left(x_0,0,\varrho\right).\end{aligned}$$

Finally an application of Borel's lemma following (4.17) completes the proof. □

In the normal form above since $R\in O_\Sigma(4)$ in (4.15) we may write $R=\left(x_1^2+\hat{\xi}_1^2\right)R_0$, with $R_0=R_0\left(x_1^2+\hat{\xi}_1^2;x_0,x_2,\hat{x}_3;\xi_2,\xi_3\right)\in O_\Sigma(2)$. Given $\varepsilon>0$, one may thus arrange

$$\left\|H_lRH_l^*\right\|_{L^2(\mathbb{R}^3)\to H^{-1}(\mathbb{R}^3)}\leq\varepsilon\,(2l+1)\,,\quad\forall l\in\mathbb{N}_0,\tag{4.18}$$

$$\left\|B\left[\left(\xi_0^2+r\right)^W,U\right]\right\|_{L^2(X)\to H^{-1}(\mathbb{R}^4)}\leq\varepsilon\quad;\tag{4.19}$$

$\forall B\in\Psi^0\left(\mathbb{R}^4\right)$, $WF(B)\subset C'$, $WF(1-B)\subset C$, on choosing $C\subset C'$ to be sufficiently small neighborhoods of $(\Sigma\setminus 0)\cap\pi^{-1}(x)$.

As an immediate corollary of the normal form 9 above we now prove the existence part of 6, the invariance of Ω (3.1) will be proved later in 5.1.

Proof of 6. At the symbolic level (4.15) reads $\sigma=\xi_0^2+2\rho\left(\xi_3x_1^2+\xi_3^{-1}\xi_1^2\right)+O_\Sigma(4)$. This gives

$$\begin{aligned}H_{\sigma^{1/2}}&=\sigma^{-1/2}\left[\xi_0\partial_{x_0}-\left(\rho_{x_0}\Omega\right)\partial_{\xi_0}+\rho H_\Omega+O_\Sigma(2)\right]\\&=\Xi_0\partial_{x_0}-\frac{1}{2}\rho^{-1}\rho_{x_0}\left(1-\Xi_0^2\right)\partial_{\Xi_0}+\sigma^{-1/2}\rho H_\Omega+\sigma^{-1/2}O_\Sigma(2)\end{aligned}\tag{4.20}$$

in terms of the new coordinates $\left(x_0,\Xi_0:=\frac{\xi_0}{\sqrt{\xi_0^2+2\rho\Omega}},x_1,\dots\right)$ and where the $O_\Sigma(2)$ term above denotes a vector field vanishing to second order along Σ. The blowup and its boundary are locally modeled by $[M;\Sigma]=\left\{\xi_0^2+2\xi_3\rho\left(\xi_3x_1^2+\xi_3^{-1}\xi_1^2\right)\geq 1\right\}$, $SN\Sigma=\left\{\xi_0^2+2\xi_3\rho\left(\xi_3x_1^2+\xi_3^{-1}\xi_1^2\right)=1\right\}$ while the function $\Xi_0:=\frac{\xi_0}{\sqrt{\xi_0^2+2\rho\Omega}}$ is identified with (2.32). The Hamiltonian vector field on the interior of the blowup is identified with (4.20) where $O_\Sigma(2)$ now denotes a vector field vanishing to second order near the boundary of the blowup. One may then rewrite

$$H_{\sigma^{1/2}}=\sigma^{-1/2}\sigma(R)\,H_\Omega+\hat{Z}+O_\Sigma(1)$$

with

$$\hat{Z}:=\Xi_0\partial_{x_0}-\frac{1}{2}\rho^{-1}\rho_{x_0}\left(1-\Xi_0^2\right)\partial_{\Xi_0}+\sigma^{-1/2}\left[\rho-\sigma(R)\right]H_\Omega\tag{4.21}$$

The invariance property (2.41) follows from the definition, (2.24), (2.26) and (2.37) via the further identifications

$$H_\Omega|_{SN\Sigma} = R_0 \text{ and}$$
$$\mu_{\text{Popp}}^{SNS^*\Sigma} = \hat{\rho}^{-2}(1-\Xi_0^2)d\Xi_0 dx_0 dx_2 d\hat{x}_3 d\hat{\xi}_2. \tag{4.22}$$

□

4.2. **Normal form near a closed characteristic.** We next obtain a normal form for $\Delta_{g^E,\mu}$ near a primitive closed characteristic γ assuming that the characteristic line L^E is volume preserving. Before proceeding we however note that there exists a large class of quasi-contact structures where L^E is not volume preserving as below.

Example 10. Let $(Y, F\subset TY)$ be a contact manifold with contact vector field $H\in C^\infty(TY)$, $\left(e^{tH}\right)_* F = F$. The mapping torus $X := Y\times[0,1]_{x_0}/\left\{(0,y)\sim\left(1,e^H(y)\right)\right\}$ carries the quasi-contact structure $E = F\oplus\mathbb{R}[\partial_{x_0}+H]$ whose characteristic line field is $L^E = \mathbb{R}[Z] = \mathbb{R}[\partial_{x_0}+H]$ (cf. [12, Lemma 2.5]). The Poincare section $Y\times\{0\}$ however cannot carry an H invariant volume such as $\hat{a}_{g^E}\wedge d\hat{a}_{g^E}$ in the case when the time one flow of H is a strictly expanding/contracting map on some region; say near one of its zeros. An explicit example of such an H is quite easily constructed; choose Darboux coordinates on an ε-ball $B_p(\varepsilon)$ centered at a point $p\in Y$ in which $F = \ker\left(\underbrace{x_1dx_2 - x_2dx_1 + dx_3}_{=a_0}\right)$. Letting $\chi\in C_c^\infty([-1,1];[0,1])$, $\chi = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, define the contact Hamiltonian vector field

$$H = H_\varphi = \begin{cases}(\varphi_{x_1}+x_2\varphi_{x_3})\,\partial_{x_2} - (\varphi_{x_2}-x_1\varphi_{x_3})\,\partial_{x_1} + (2\varphi - x_1\varphi_{x_1} - x_2\varphi_{x_2})\,\partial_{x_3}; & x\in B_p(\varepsilon)\\ 0; & x\notin B_p(\varepsilon)\end{cases},$$

$\varphi := x_3\chi\left(\frac{|(x_1,x_2,x_3)|}{\varepsilon}\right)$, satisfying $L_{H_\varphi}a_0 = 2\varphi_{x_3}a_0$ on $B_p(\varepsilon)$, and which has a strictly expanding time one flow near the origin where $H_\varphi|_{B_p(\varepsilon/2)} = x_1\partial_{x_1} + x_2\partial_{x_2} + 2x_3\partial_{x_3}$.

We now show that in the volume preserving case, the normal structure of E is described as such a mapping torus of an x_3-independent contact Hamiltonian φ near a non-degenerate closed characteristic. To this end, as mentioned before, in the volume preserving case one has a Z-invariant defining one form $L_Z\left(\underbrace{\hat{\rho}_Z a_{g^E}}_{=:\hat{a}_{g^E}}\right) = 0$. The linearized Poincare return map P_γ of Z is seen to be symplectic on $\left(E/L^E, d\hat{a}_{g^E}\right)$. We call the characteristic elliptic if the eigenvalues of P_γ are of the form $e^{\pm i\alpha}(2\pi>\alpha\geq 0)$ and (positive) hyperbolic if of the form $e^{\pm\beta}(\beta\geq 0)$. The characteristic is said to be non-degenerate iff $\frac{\alpha}{2\pi}\notin\mathbb{Q}$ or $\beta\neq 0$. For each γ we then define the model quadratic on $\mathbb{R}^2$ via

$$Q = \begin{cases}\frac{\alpha}{2}\left(x_1^2+x_2^2\right); & \gamma \text{ is elliptic,}\\ \beta x_1x_2; & \gamma \text{ is hyperbolic.}\end{cases} \tag{4.23}$$

We first begin describing the normal structure of the Popp form $\hat{a}_{g^E}$ near a non-degenerate γ. In the theorem below we let $\gamma^0 := S^1\times\{0\}\subset S^1_{x_0}\times\mathbb{R}^3$ and T_γ the length of γ.

Proposition 11. *There exists a diffeomorphism $\kappa : \Omega_\gamma^0 \to \Omega_\gamma$ between some neighborhood of $\gamma^0 \subset \Omega_\gamma^0$ and some neighborhood of the closed characteristic $\gamma \subset \Omega_\gamma$ such that*

$$\kappa^* \hat{a}_{g^E} = \underbrace{\varphi dx_0 + \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1 + dx_3\right]}_{=:a_\varphi} \tag{4.24}$$

$$\begin{aligned} \left|\tilde{Z}\right| &\coloneqq \left|-\partial_{x_0} + \varphi_{x_1}\partial_{x_2} - \varphi_{x_2}\partial_{x_1} + \left[2\varphi - \left(x_1\varphi_{x_1} + x_2\varphi_{x_2}\right)\right]\partial_{x_3}\right| \\ &= T_\gamma + O\left(\left|(x_1,x_2)\right|^2\right) + O\left(x_3\right) \end{aligned} \tag{4.25}$$

modulo $O\left(Q^\infty\right)$. Here

$$\varphi = \varphi\left(Q\right) = Q + O\left(Q^2\right) \tag{4.26}$$

above (4.24) *is a function on $\mathbb{R}^2$ of the quadratic* (4.23) *with linear term Q.*

Proof. Choose a Poincare section Y transversal to Z through a point $p \in \gamma$ with Poincare return map and return time functions $P_Y : Y \to Y$, $T_Y : Y \to \mathbb{R}$. Having $P_Y = e^{\tilde{Z}}$; $\tilde{Z} = T_Y Z$, we may compute

$$\begin{aligned} P_Y^* \hat{a}_{g^E} - \hat{a}_{g^E} &= \int_0^1 \left(\mathcal{L}_{\tilde{Z}} \hat{a}_{g^E}\right) dt \\ &= \int_0^1 \left(d i_{\tilde{Z}} \hat{a}_{g^E} + i_{\tilde{Z}} d\hat{a}_{g^E}\right) dt \\ &= 0. \end{aligned} \tag{4.27}$$

The one form a is contact on Y with contact hyperplane $F = TY \cap E$ and we choose a set of Darboux coordinates (x_1, x_2, x_3) with $\hat{a}_{g^E}\big|_Y = \frac{1}{2}\left[dx_3 + x_1 dx_2 - x_2 dx_1\right]$ as (2.26). By (4.27), the return map P_Y is a contactomorphism with its linearization at 0 being identified with P_γ. We now claim that such a contactomorphism is given by

$$\begin{aligned} &P_Y = e^{H_\varphi}, \quad \text{with} \\ &H_\varphi \coloneqq \varphi_{x_1}\partial_{x_2} - \varphi_{x_2}\partial_{x_1} + \left[-2\varphi + \left(x_1\varphi_{x_1} + x_2\varphi_{x_2}\right)\right]\partial_{x_3} \end{aligned} \tag{4.28}$$

under the non-degeneracy assumption. To see the above let $P_Y = (P_1, P_2, P_3)$. Since the Reeb vector field is mapped to itself, $\partial_{x_3} = \frac{\partial P_1}{\partial x_3}\partial_{P_1} + \frac{\partial P_2}{\partial x_3}\partial_{P_2} + \frac{\partial P_3}{\partial x_3}\partial_{P_3} = \partial_{P_3}$ giving $\frac{\partial P_1}{\partial x_3} = \frac{\partial P_2}{\partial x_3} = 0$ and thus P_1, P_2 are independent of x_3. The map $P_Y^0 \coloneqq (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is then symplectic with respect to $dx_1 dx_2$; the eigenvalues of its linearization are those of P_γ^+ not equal to 1. Thus $P_Y^0 = e^{H_\varphi}$ for some $\varphi\left(Q\right)$ of the form (4.26) under the degeneracy assumption. Next define $\left(P_1\left(t\right), P_2\left(t\right)\right) \coloneqq e^{tH_\varphi}\left(x_1, x_2\right)$ and calculate

$$\begin{aligned} \frac{d}{dt}\left(e^{tH_\varphi}\right)^* \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] &= \mathcal{L}_{H_\varphi}\frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] \\ &= \left(i_{H_\varphi} d + d i_{H_\varphi}\right)\frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] \\ &= d\left[-\varphi + \frac{1}{2}\left(x_1\varphi_{x_1} + x_2\varphi_{x_2}\right)\right] \end{aligned}$$

to obtain

$$\left(e^{H_\varphi}\right)^* \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] - \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] = d\left[-\varphi + \frac{1}{2}\left(x_1\varphi_{x_1} + x_2\varphi_{x_2}\right)\right].$$

This gives

$$0 = P_Y^* \hat{a}_{g^E} - \hat{a}_{g^E} = \frac{1}{2} d\left(P_3 - x_3\right) + \left(e^{H_\varphi}\right)^* \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right] - \frac{1}{2}\left[x_1 dx_2 - x_2 dx_1\right]$$
$$= \frac{1}{2} d\left(P_3 - x_3\right) + d\left[-\varphi + \frac{1}{2}\left(x_1 \varphi_{x_1} + x_2 \varphi_{x_2}\right)\right]$$

and thus $P_3 = x_3 - 2\varphi + \left(x_1 \varphi_{x_1} + x_2 \varphi_{x_2}\right)$ on knowing $P_Y(0) = 0$, proving the claim (4.28). Now, noting that Poincare map is also given via $P_\Sigma = e^{\tilde{Z}}$; with

$$\tilde{Z} = -\partial_{x_0} + H_\varphi \tag{4.29}$$

satisfying $i_{\tilde{Z}} a_\varphi = i_{\tilde{Z}} da_\varphi = 0$ for the model form a_φ (4.24) proves (4.24).

To prove (4.25), first note

$$\left|\tilde{Z}\right| = T_Y \tag{4.30}$$

and compute

$$P_Y^* dT_Y - dT_Y = \int_0^1 \left(\mathcal{L}_{\tilde{Z}} dT_Y\right) dt$$
$$= \int_0^1 d\tilde{Z}\left(T_Y\right) dt = 0$$

by definition; T_Y is defined on a neighborhood of γ using the flow of $\tilde{Z}$. This gives

$$P_Y^* T_Y = \left(e^{H_\varphi}\right)^* T_Y = T_Y$$

on knowing $P_Y(0) = 0$. Comparing the coefficients in the last equation using (4.26), (4.28) shows that the linear (x_1, x_2) terms in T_Y must vanish under the non-degeneracy assumption. □

The distribution E is now locally generated by the vector fields $U_0 = -\partial_{x_0} + 2\varphi\partial_{x_3}$, $U_1 = \partial_{x_1} + x_2\partial_{x_3}$ and $U_2 = \partial_{x_2} - x_1\partial_{x_3}$. The generator of the characteristic line (4.29) maybe written

$$\tilde{Z} = U_0 - \varphi_{x_2} U_1 + \varphi_{x_1} U_2.$$

We may again choose $X, Y \in \left(L^E\right)^\perp$ satisfying $|X| = 1$, $da_{g^E}(Y, X) = \hat{\rho}_Z da_{g^E}(U_1, U_2) = 1$ and

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \delta_1 \tilde{Z} \\ \delta_2 \tilde{Z} \end{bmatrix} + \hat{\rho}_Z^{1/2} e^\Lambda \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

for some set of functions δ_1, δ_2 and $\Lambda = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix} \in \mathfrak{sp}(2)$.

The symbol of the Laplacian is then calculated

$$\sigma\left(\Delta_{g^E,\mu}\right) = \frac{1}{\left|\tilde{Z}\right|^2} \sigma\left(\tilde{Z}\right)^2 + \sigma(X)^2 + \sigma(Y)^2$$
$$= a_0 \tilde{\eta}_0^2 + 2\hat{\rho}_Z^{1/2} \tilde{\eta}_0 \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix} e^\Lambda \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \hat{\rho}_Z \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} e^{\Lambda^t} e^\Lambda \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \xi_3^2 O_\Sigma(2)\, O_\gamma(1).$$

Here $\tilde{\eta}_0, \eta_1, \eta_2$ denote the symbols

$$\begin{aligned}\tilde{\eta}_0 &:= \sigma\left(\tilde{Z}\right) = -\xi_0 + 2\varphi\xi_3\\ \eta_1 &:= \sigma\left(U_1\right) = \xi_1 + x_2\xi_3\\ \eta_2 &:= \sigma\left(U_2\right) = \xi_2 - x_1\xi_3,\end{aligned}$$

of the given vector fields while $O_\Sigma(k)$, $O_\gamma(k)$, denote homogeneous degree zero symbols which vanish to order k in the variables $\left(\xi_3^{-1}\tilde{\eta}_0, \xi_3^{-1}\eta_1, \xi_3^{-1}\eta_2\right)$ and (x_1, x_2, x_3) respectively.

Setting $f_0 = \xi_3\left(x_1x_2 + \hat{\xi}_1\hat{\xi}_2\right)$ as before, we again calculate

$$\begin{aligned}\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\sigma\left(\Delta_{g^E,\mu}\right) = \xi_3^2&\left[a_0\left(-\hat{\xi}_0 + 2\bar{\varphi}\right)^2 + 2\hat{\rho}_Z^{1/2}\left(-\hat{\xi}_0 + 2\bar{\varphi}\right)\begin{bmatrix}\delta_1 & \delta_2\end{bmatrix}e^{\Lambda}\begin{bmatrix}\hat{\xi}_1\\ x_1\end{bmatrix} + 2\hat{\rho}_Z\begin{bmatrix}\hat{\xi}_1 & x_1\end{bmatrix}e^{\Lambda^t}e^{\Lambda}\begin{bmatrix}\hat{\xi}_1\\ x_1\end{bmatrix}\right]\\ &+ \xi_3^2O_\Sigma(2)\,O_\gamma(1) + \xi_3^2O_\Sigma(3)\end{aligned}$$

where $\bar{\varphi} =a_0, \delta_1, \delta_2, \Lambda$ are functions of $\left(x_0, x_2, x_3; \hat{\xi}_2\right)$ while $O_\Sigma(k)$, $O_\gamma(k)$, denote homogeneous degree zero symbols which vanish to order k in the variables $\left(\hat{\xi}_0 + 2\varphi, x_1, \hat{\xi}_1\right)$ and $\left(x_2, x_3; \hat{\xi}_2\right)$ respectively.

Further, with f_1 of the form

$$f_1 = \frac{\xi_3^2}{2}\begin{bmatrix}\hat{\xi}_1 & x_1\end{bmatrix}\begin{bmatrix} & -1\\ 1 & \end{bmatrix}\bar{\Lambda}\begin{bmatrix}\hat{\xi}_1\\ x_1\end{bmatrix}$$

we compute

$$\left(e^{H_{f_1}}\right)^*\xi_3\begin{bmatrix}\hat{\xi}_1\\ x_1\end{bmatrix} = \xi_3e^{-\Lambda_0}\begin{bmatrix}\hat{\xi}_1\\ x_1\end{bmatrix} + \xi_3O_\Sigma(2),$$

giving

$$\begin{aligned}\left(e^{H_{f_1}}\right)^*\left(e^{\frac{\pi}{4}H_{f_0}}\right)^*\sigma\left(\Delta_{g^E,\mu}\right) = \xi_3^2&\left[a_0\left(-\hat{\xi}_0 + 2\bar{\varphi}\right)^2 + \left(-\hat{\xi}_0 + 2\bar{\varphi}\right)\left(b_0x_1 + c_0\hat{\xi}_1\right) + 2\hat{\rho}_Z\left(x_1^2 + \hat{\xi}_1^2\right)\right]\\ &+ \xi_3^2O_\Sigma(2)\,O_\gamma(2) + \xi_3^2O_\Sigma(3)\end{aligned}$$

for $\begin{bmatrix}\delta_1^1\\ \delta_2^1\end{bmatrix} \in O_\gamma(1)$. Finally $f_2 = \frac{1}{2}\left[c_0\xi_0x_1 - b_0\xi_0\xi_1\right]$ we also have

$$\kappa_0^*\sigma\left(\Delta_{g^E,\mu}\right) = a_0\xi_3^2\left(-\hat{\xi}_0 + 2\bar{\varphi}\right)^2 + 2\hat{\rho}_Z\xi_3^2\left(x_1^2 + \hat{\xi}_1^2\right) + \xi_3^2O_\Sigma(2)\,O_\gamma(2) + \xi_3^2O_\Sigma(3)$$

for $\kappa_0 = \left(e^{H_{f_2}}\right)\left(e^{H_{f_1}}\right)\left(e^{\frac{\pi}{4}H_{f_0}}\right)$ and for some $a_0\left(x_0, x_2, \hat{x}_3; \hat{\xi}_2\right) > 0$. Finally, by another Hamiltonian diffeomorphism we may set $\hat{\rho}_Z = 1$ and $a_0 = a_0\left(x_2, \hat{x}_3; \hat{\xi}_2\right)$ independent of x_0 and satisfying

$$a_0(0,0;0) = \frac{1}{T_\gamma^2}.$$

Following the preliminary normal form above the rest of the normal form procedure proceeds as in the previous section. We then first have a Hamiltonian diffeomorphism $\kappa_1 = e^{H_{\xi_3f_3}}$, $f_3\left(x;\hat{\xi}\right) \in O_\Sigma(1)$, and function $R\left(x_1^2 + \hat{\xi}_1^2; x_0, x_2, \hat{x}_3; \hat{\xi}_0 + \bar{\varphi}, \hat{\xi}_2\right) \in \xi_3^2O_\Sigma(2)\,O_\gamma(2) + \xi_3^2O_\Sigma(3)$ such that

$$\kappa_1^*\kappa_0^*\sigma\left(\Delta_{g^E,\mu}\right) = \xi_3^2\left[a_0\left(-\hat{\xi}_0 + \bar{\varphi}\right)^2 + 2\left(x_1^2 + \hat{\xi}_1^2\right) + R\right] + O_\Sigma(\infty). \tag{4.31}$$

The normal form for the symbol is now given next.

Theorem 12. *There exists a Hamiltonian symplectomorphism* $\kappa : T^*\left(S^1_{x_0} \times \mathbb{R}^3\right) \to T^*X$ *and symbol* $R\left(x_1^2+\hat{\xi}_1^2; x_0, x_2, \hat{x}_3; -\hat{\xi}_0+\bar{\varphi}, \hat{\xi}_2\right) \in \xi_3^2 O_\Sigma(2) O_\gamma(2) + \xi_3^2 O_\Sigma(3)$

$$\kappa^*\sigma\left(\Delta_{g^E,\mu}\right) = \xi_3^2\left[a_0\left(-\hat{\xi}_0+\bar{\varphi}\right)^2 + 2\left(x_1^2+\hat{\xi}_1^2\right) + R\right] + O_\Sigma(\infty) \tag{4.32}$$

on some open conic neighborhood $C \supset (\Sigma \setminus 0) \cap \pi^{-1}(\gamma)$.

We refer to a (nondegenerate) closed characteristic γ as flat if there exists a normal form as above with $R=0$, $a_0 = T_\gamma$ (constant).

We next compute $\mathscr{L}_{\hat{Z}}$ the set of closed periods of the vector field $\hat{Z}$, in both the volume preserving and non-preserving cases. First note that in the volume preserving case since $L_Z\hat{\rho}_Z = da_{g^E}(R,Z)\,\hat{\rho}_Z$ for some positive function $\hat{\rho}_Z$ one has

$$L_Z(\ln\hat{\rho}_Z) = da_{g^E}(R,Z) \qquad \text{and hence} \tag{4.33}$$

$$\int_0^{T_\gamma} dt\left(e^{tZ}\right)^* da_{g^E}(R,Z)\big|_\gamma = 0 \tag{4.34}$$

along any closed characteristic γ with period T_γ. Motivated by this we say that L^E *is volume preserving along* γ iff the equation (4.34) above holds. In this case we may define a unique positive function $\hat{\rho}_Z^\gamma \in C^\infty(\gamma;(0,1])$ satisfying (4.33) along γ and $\sup_\gamma \hat{\rho}_Z^\gamma = 1$. In the case when L^E is globally volume preserving this would equal $\hat{\rho}_Z^\gamma := \frac{\hat{\rho}_Z|_\gamma}{\sup_\gamma \hat{\rho}_Z}$ for any globally defined function satisfying (4.33). Viewing $\hat{\rho}_Z^\gamma$ as a periodic function on $\mathbb{R}$ with period T_γ, we define $\hat{T}_\gamma > T_\gamma$ as the smallest positive number for which$\int_0^{\hat{T}_\gamma} \frac{1-\hat{\rho}_Z^\gamma}{1+\hat{\rho}_Z^\gamma} = T_\gamma$. Here we use the convention that $\hat{T}_\gamma = \infty$ if $\hat{\rho}_Z^\gamma \equiv 1$, in which case $L_Z\mu_{\text{Popp}}|_\gamma = 0$. Finally, we extend this definition to the case when L^E is not volume-preserving along γ by simply setting $\hat{T}_\gamma = T_\gamma$. Below we denote by $\mathbb{N}[I]$ the set of all positive integer multiples of elements in any given interval $I \subset \mathbb{R}$. We now have the following.

Proposition 13. *(Density of periods) The set of periods*

$$\mathscr{L}_{\hat{Z}} = \bigcup_{\gamma \text{ closed characteristic}} \mathbb{N}\left[-\hat{T}_\gamma, -T_\gamma\right] \cup \mathbb{N}\left[T_\gamma, \hat{T}_\gamma\right] \tag{4.35}$$

In particular if $L_Z\mu_{Popp} = 0$ *along the shortest closed characteristic, the set of periods*

$$\mathscr{L}_{\hat{Z}} = \left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right). \tag{4.36}$$

Finally, if the shortest (nondegenerate) closed characteristic γ *is flat one has the density of normal periods*

$$\mathscr{L}_{normal} \supset \left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right). \tag{4.37}$$

Proof. Clearly by (2.40), a closed integral curve of $\hat{Z}$ lies over a closed characteristic; say $\gamma(t) := e^{tZ}$, $\gamma(T_\gamma) = \gamma(0)$. The restriction of $(SN\Sigma/\mathbb{R}_+)/S^1$ to γ is a $[-1,1]_{\Xi_0}$ bundle on which $\hat{Z} = \Xi_0\partial_t - \underbrace{\frac{1}{2}da_{g^E}(R,Z)\Big|_\gamma}_{=:A(t)}\left(1-\Xi_0^2\right)\partial_{\Xi_0}$, following the computation (4.21), which we may further view as a vector field on $\mathbb{R}_t \times [-1,1]_{\Xi_0}$ that is periodic in t. The flow of the above can be explicitly computed

$$e^{t\hat{Z}}\left(0,\Xi_0\left(0\right)\right) = \left(\int_0^t \Xi_0\left(s\right)ds, \underbrace{\frac{1+\Xi_0\left(0\right)-\left(1-\Xi_0\left(0\right)\right)e^{-2\int_0^t A(s)ds}}{1+\Xi_0\left(0\right)+\left(1-\Xi_0\left(0\right)\right)e^{-2\int_0^t A(s)ds}}}_{=:\Xi_0(t)}\right). \tag{4.38}$$

It is clear that the second coordinate above represents a periodic function only if $\int_0^{T_\gamma} A\left(s\right)ds = 0$ (i.e. L^E is volume-preserving along γ) or $\Xi_0\left(0\right) = \pm 1$. Thus in the non-volume preserving case we must have $\Xi_0\left(0\right) = \pm 1$, which gives the periods of the $\hat{Z}$ to be the same as those of Z. On the other hand if $\int_0^{T_\gamma} A\left(s\right)ds = 0$, (4.38) is periodic with its periods at the two initial extreme conditions $\Xi_0\left(0\right) = 0, 1$ computed to be $\hat{T}_\gamma, T_\gamma$ respectively. The second equality (4.36) is an immediate specialization of the first while the last (4.37) is an easy computation from of the normal form(4.32). □

5. Global calculus

We now define a global calculus of Hermite operators using the local calculus of Section 3 and the normal form 9. To give a definition independent of choices one needs an invariance lemma in the upcoming section.

5.1. **Invariance.** Below $p = (0,0,0,0;0,0,0,1) \in T^*\mathbb{R}^4$ is as before (4.3) while $\kappa : T^*\mathbb{R}^4 \to T^*\mathbb{R}^4$ denotes a local conic symplectomorphism fixing p and Σ_0. Let $C_\kappa \subset (T^*\mathbb{R}^4) \times (T^*\mathbb{R}^4)^-$ be the associated canonical relation. We denote by the same notation κ the induced local diffeomorphisms of $S^*\mathbb{R}^4_x$ as well as the blowup $[S^*\mathbb{R}^4_x; S^*\Sigma_0]$. Furthermore $\Delta_{R,r}$ is as in (4.15) and $C \subset C'$ are conic neighborhoods of $(\Sigma \setminus 0) \cap \pi^{-1}\left(x\right)$ satisfying (4.18), (4.19).

Lemma 14. *Let $U \in I^0_{cl}(\mathbb{R}^4, \mathbb{R}^4; C_\kappa)$ be a local Fourier integral and $\rho, \rho' = \kappa^*\rho \in C^\infty\left(\Sigma_0\right)$, $R, R' \in O_\Sigma\left(4\right)$, $r, r' \in S^0_{cl}$ as in Theorem 9 satisfying*

$$U\Delta_{\rho,R,r}U^* = \Delta_{\rho',R',r'} \tag{5.1}$$

$$UU^* = U^*U = 1 \tag{5.2}$$

microlocally on a conic neighborhood C of p.

Then one has

$$U\Omega U^* = \Omega \text{ microlocally on } C, \text{ and} \tag{5.3}$$

$$UAU^* \in \Psi^{m_1,m_2}_{cl}\left(\mathbb{R}^4; \Sigma_0\right), \tag{5.4}$$

$$\sigma^H_{m_1,m_2}\left(UAU^*\right) = \kappa^*\sigma^H_{m_1,m_2}\left(A\right), \tag{5.5}$$

$\forall A \in \Psi^{m_1,m_2}_{cl}(\mathbb{R}^4; \Sigma_0)$ with microsupport in $C \times C$.

Proof. First from Thm. 9 one has $\Delta_{\rho,R,r}H_k = H_k\left[\xi_0^2 + (2k+1)\rho + R + r\right]^W$. For $B \in \Psi^0\left(\mathbb{R}^4\right)$, $WF\left(B\right) \subset C'$, $B = 1$ on C one computes

$$\begin{aligned}
&c_0\left|(l-k)\right|\left(1+O\left(\varepsilon\right)\right)\left\|H_l B U H_k^*\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \\
=&\left\|\left(H_l B\left[(2l+1)\rho + R^W\right]UH_k^*\right) - \left(H_l BU\left[(2k+1)\rho' + R'^W\right]H_k^*\right)\right\|_{L^2(\mathbb{R}^3)\to H^{-1}(\mathbb{R}^3)} \\
=&\left\|H_l B\left(\left[U, \left(\xi_0^2\right)^W\right] + Ur^W - r'^W U\right)H_k^*\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \\
\leq&O\left(\varepsilon\right)\left\|H_l B U H_k^*\right\|_{L^2(\mathbb{R}^3)\to H^{-1}(\mathbb{R}^3)}
\end{aligned}$$

using the ellipticity of ρ, ρ' near p and (4.18), (4.19), (5.1). This gives

$$H_l B U H_k^* = 0, \quad \forall l \neq k; \tag{5.6}$$

i.e. U microlocally preserves the Landau levels.

Next, for $A = a^W \in \Psi^m(\mathbb{R}^4)$ with $WF(A) \subset C$ the above and $\Omega = \xi_3 x_1^2 + \xi_3^{-1}\xi_1^2 = \sum_{k=0}^{\infty}(2k+1) H_k^* H_k$ gives

$$[U, \Omega] = 0 \quad \text{microlocally on } C \tag{5.7}$$

$$\left[a^W, \Omega\right] = 0 \implies \left[U^* a^W U, \Omega\right] = 0 \tag{5.8}$$

proving (5.3). In other words, for a symbol $a \in C^\infty_{\text{inv}}(T^*\mathbb{R}^4_x)$, the conjugate $a^W_U := U^* a^W U$ is again of the same form $a_U \in C^\infty_{\text{inv}}(T^*\mathbb{R}^4_x)$. Furthermore by an Egorov argument as in [21, Ch. 10], the conjugate has the form $a_U \sim \kappa^* \left(\underbrace{P_j^U a}_{\in S^{m-j}} \right)$; where each P_j is a differential operator of homogeneous degree $-j$ mapping S^m to S^{m-j} with $P_0 = 1$. The last implies that each of $\left\{ \underbrace{x_1^2 + \hat{\xi}_1^2}_{=:\varrho^2}, \hat{\xi}_0, \hat{\xi}_2, \xi_3; x_0, x_2, \hat{x}_3 \right\}$ maps under κ to a function of the same set of variables. Thus each

$$P_j^U = \sum_{\alpha \in \mathbb{N}_0^7} c_{\alpha,j}\left(\varrho, \hat{\xi}_0, \hat{\xi}_2, \xi_3; x_0, x_2, \hat{x}_3\right)(\varrho\partial_\varrho)^{\alpha_1} \partial^{\alpha_0}_{\xi_0}\partial^{\alpha_2}_{\xi_2}\partial^{\alpha_3}_{\xi_3}\partial^{\beta_2}_{x_0}\partial^{\beta_2}_{x_2}\partial^{\beta_3}_{\hat{x}_3} \tag{5.9}$$

is also a differential operator in the given set of variables.

Finally for $A = a^H \in \Psi^{m_1,m_2}_{\text{cl}}(\mathbb{R}^4; \Sigma_0)$; with $a \in (\beta^* d)^{-m_2} C^\infty_{c,\text{inv}}([K_{1,1}; \Sigma_0])$ supported in the lift of C, it now follows using (3.19), (5.6) that

$$\begin{aligned} U^* a^H U &= \sum_{k=0}^{\infty} U^* H_k^* a_k^W H_k U \\ &= \sum_{k=0}^{\infty} H_k^* U^* a_k^W U H_k \\ &= \sum_{k=0}^{\infty} H_k^* a_{U,k}^W H_k \\ &= a_U^H \end{aligned} \tag{5.10}$$

Here $a_U \in (\beta^* d)^{-m_2} C^\infty_{c,\text{inv}}([K_{1,1}; \Sigma_0])$ satisfies

$$a_U \sim \kappa^* \left(\underbrace{\tilde{P}_j^U a}_{\in S^{m_1 - \frac{j}{2}, m_2}} \right). \tag{5.11}$$

where $\tilde{P}_j^U$ denotes the lift to the blowup $[K_{1,1}; \Sigma_0]$ of the differential operator obtained by deleting the terms in (5.9) involving a $\varrho\partial_\varrho$ derivative (with $\alpha_1 \geq 1$) . The necessary symbolic estimates and and expansion for the conjugate symbol a_U in S^{m_1,m_2}_{cl} now follow from (5.11) and the corresponding estimates for $a \in S^{m_1,m_2}_{\mathrm{cl}}$. In order to obtain the symbolic expansion we note $U\rho U^* = \underbrace{\rho'}_{=\kappa^*\rho} + S^0$, (5.1) and (5.3) give $U\xi_0^2U^* = \xi_0^2 + O\left(\xi_3^2\left(\hat{d}_{\rho'}\right)^4\right)$. Then

$$Ud_\rho U^* = d_{\rho'} + O\left(\xi_3^2\left(\hat{d}_{\rho'}\right)^4\right)$$

(5.10) and symbolic calculus in the ρ' calculus give the necessary symbolic expansion for a_U. □

We note that (5.3) establishes the invariance of Ω, completing the proof of 6.

5.2. **Calculus.** Following the invariance Lemma 14, one may now construct a global calculus of Hermite operators. To this end, we choose a collection of points $\{p_j \in \Sigma\}_{j=1}^M$ along with diagonalizing Fourier integral operators $\{U_j : L^2(X) \to L^2(\mathbb{R}^4)\}_{j=1}^M$ associated to symplectomorphisms $\kappa_j : T^*X \to T^*\mathbb{R}^4$ which put $\Delta_{g^E,\mu}$ in normal form (4.15) in conic neighborhoods $\{p_j \in C_j\}_{j=1}^M$ covering Σ.

Definition 15. An operator $T : C^\infty(X) \to C^{-\infty}(X)$ is said to lie in the class $\Psi^{m_1,m_2}_{\mathrm{cl}}(X,\Sigma)$ iff it can be written $T = T_0 + \sum_{j=1}^M T_j$ where

(1) $WF(T_0) \subset (T^*X \times T^*X) \setminus (\Sigma \times \Sigma)$ with $T_0 \in \Psi^{m_1}_{\mathrm{cl}}(X)$
(2) $WF(T_j) \subset C_j \times C_j$ with $U_jT_jU_j^* \in \Psi^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^4;\Sigma_0)$, $j = 1,\dots,M$.

It is an easy exercise from Lemma 14 that the definition above is independent of the choice of diagonalizing Fourier integral operators $\{U_j : L^2(X) \to L^2(\mathbb{R}^4)\}_{j=1}^M$.

The symbol of $T \in \Psi^{m_1,m_2}_{\mathrm{cl}}(X,\Sigma)$, $m_2 \le 0$, is then defined via

$$\sigma^H_{m_1,m_2}(T) := \sigma(T_0) + \kappa_j^*\sigma^H_{m_1,m_2}\left(U_jT_jU_j^*\right) \in C^\infty([T^*X;\Sigma]) \tag{5.12}$$

and is again invariantly defined by virtue of (5.5). Much like (3.30), the symbol above has an invariance property. First note that by (5.7), the pseudo-differential operator $\Omega \in \Psi^1_{\mathrm{cl}}(X)$ and its homogeneous symbol are microlocally and invariantly defined on a conic neighborhood $C_\Omega \subset \Sigma$ of the the characteristic variety. We also denote by Ω its pullback to the blowup defined on the neighborhood $\beta^{-1}(C_\Omega)$ of the boundary. Furthermore its Hamilton vector field H_Ω has a lift to the blowup, that is tangent to the boundary and homogeneous of degree zero, which we denote by the same notation $H_\Omega \in C^\infty(T[T^*X;\Sigma])$. Its restriction to the boundary is the rotational vector field

$$H_\Omega|_{SN\Sigma} = R_0 \tag{5.13}$$

is the rotational vector field following the identification (4.22).

We then define the space of invariant symbols

$$\begin{aligned} C^\infty_{\mathrm{inv}}([(S^*X);S^*\Sigma]) := \{ f \in C^\infty([(S^*X);S^*\Sigma]) \,|\, f = f_0 + f_1,\ f_0 \in C^\infty_c([(S^*X);S^*\Sigma]^o), \\ f_1 \in C_c\left(\beta^{-1}(C_\Omega)\right),\ H_\Omega f_1 = 0 \}. \end{aligned} \tag{5.14}$$

The above may also be considered as homogeneous functions of degree zero on $[T^*X;\Sigma]$. We may then similarly define $C^\infty_{\mathrm{inv},m}$, $m \in \mathbb{Z}$, by requiring homogeneity of degree m; this space is however non-canonically identified with (5.14) on choosing positive function in (5.14). It follows from definition that the sR Laplacian $\Delta_{g^E,\mu} \in \Psi^{2,-2}(X,\Sigma)$. Further, it easy to see from the normal form (4.15) that with

$$d := \left[\sigma^H_{2,-2}\left(\Delta_{g^E,\mu}\right)\right]^{1/2}\Big|_{S^*X} \tag{5.15}$$

being homogeneous of degree one, $(\beta^* d)$ defines an element of the symbol space (5.14). The symbol of a general $T \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$ is defined by the same formula 5.12 and is now an element of

$$\sigma^H_{m_1,m_2}(T) \in (\beta^* d)^{-m_2} C^\infty_{\mathrm{inv}}([(S^*X);S^*\Sigma]) . \tag{5.16}$$

We shall say that an element $T \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$ is elliptic in the exotic calculus if and only if

$$c(\beta^* d)^{-m_2} \le \sigma^H_{m_1,m_2}(T) \le C(\beta^* d)^{-m_2} \tag{5.17}$$

for some constants $c, C > 0$. Similar to (3.22), (3.24) one then has the inclusions

$$\begin{aligned} \Psi_{\mathrm{cl}}^{m_1,m_2}(X;\Sigma) &\subset \Psi_{\mathrm{cl}}^{m_1+\frac{1}{2},m_2-1}(X;\Sigma) \\ \Psi_{\mathrm{inv,cl}}^{m}(X) &\subset \Psi_{\mathrm{cl}}^{m,0}(X;\Sigma) \end{aligned} \tag{5.18}$$

where

$$\Psi^m_{\mathrm{inv,cl}}(X) \coloneqq \{A = A_0 + A_1 \in \Psi^m_{\mathrm{cl}}(X) | WF(A_0) \subset C_\Omega,\, WF([A_0,\Omega]) \subset T^*X \setminus \Sigma,\, WF(A_1) \subset T^*X \setminus \Sigma\} .$$

One similarly defines the generalized Sobolev spaces $H^{s_1,s_2}(X,\Sigma)$ via $u \in H^{s_1,s_2}(X,\Sigma)$ if and only if $u = u_0 + \sum_{j=1}^M u_j$ where 1.$WF(u_0) \subset T^*X \setminus \Sigma$ with $u_0 \in H^{s_1}$ and 2. $WF(u_j) \subset C_j$ with $u_j \in H^{s_1,s_2}(\mathbb{R}^4_x;\Sigma_0)$. A pseudo-differential characterization of $H^{s_1,s_2}(X,\Sigma)$ is given using (3.35) by

$$u \in H^{s_1,s_2}(X,\Sigma) \iff Au \in L^2, \forall A \in \Psi^{s_1,s_2}(X,\Sigma) . \tag{5.19}$$

Following (5.18) this now gives

$$\begin{aligned} H^{s,0}(X,\Sigma) &= H^s(X) \\ H^{s_1+\frac{1}{2},s_2-1}(X,\Sigma) &\subset H^{s_1,s_2}(X,\Sigma) . \end{aligned} \tag{5.20}$$

The characteristic wavefront set $WF_\Sigma(T) \subset \partial[(S^*X);S^*\Sigma]$ of an operator $T \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$ is defined via $(x,\xi) \in WF_\Sigma(T) \iff \kappa(x,\xi) \in WF(UTU^*)$. Here $U : L^2(X) \to L^2(\mathbb{R}^4)$ is a diagonalizing FIO near $\beta(x,\xi)$ associated to a homogeneous symplectomorphisms $\kappa : T^*X \to T^*\mathbb{R}^4$ mapping Σ to Σ_0 and with lift $\kappa : [(S^*X);S^*\Sigma] \to [(S^*\mathbb{R}^4);S^*\Sigma_0]$ being denoted by the same notation. The characteristic wavefront set $WF_\Sigma(u) \subset \partial[(S^*X);S^*\Sigma]$ of any distribution $u \in C^{-\infty}(X)$ is then defined via

$$(x,\xi) \notin WF_\Sigma(u) \iff \exists A \in \Psi^{0,0}_{\mathrm{cl}}(X,\Sigma), \text{s.t. } (x,\xi) \in WF_\Sigma(A),\ Au \in C^\infty . \tag{5.21}$$

or equivalently

$$(x,\xi) \notin WF_{\Sigma_0}(u) \iff \exists A \in \Psi^{0,0}_{\mathrm{cl}}\left(\mathbb{R}^4;\Sigma_0\right), \text{s.t. } \sigma^H_{0,0}(A)(x,\xi) \neq 0,\ Au \in C^\infty . \tag{5.22}$$

The wavefront projects to restriction of the wavefront $\beta(WF_\Sigma(u)) = WF(u) \cap \Sigma$ under the blowdown map (3.41).

Following their pseudo-differential characterizations (5.19), (5.21) it is clear that $H^{s_1,s_2}(X,\Sigma)$ and $WF_\Sigma(u)$ are also defined independently of the choice of diagonalizing Fourier integral operators. The properties of the Hermite calculus from Section 3 then easily carry over globally. We state them below.

(1) (Adjoint & Composition) The class 15 is closed under composition and adjoint

$$\begin{aligned} A \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma),\, B \in \Psi_{\mathrm{cl}}^{m_1',m_2'}(X,\Sigma) &\implies AB \in \Psi_{\mathrm{cl}}^{m_1+m_1',m_2+m_2'}(X,\Sigma) \\ A \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma) &\implies A^* \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma) . \end{aligned} \tag{5.23}$$

(2) (Characterization of residual terms) One has the inclusions and characterization of residual terms and in particular the characterization of residual terms

$$\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)\subset\Psi_{\mathrm{cl}}^{m_1+\frac{1}{2},m_2-1}(X,\Sigma) \tag{5.24}$$

$$\Psi^{-\infty,m_2}(X,\Sigma)=\Psi_{\mathrm{inv,cl}}^{-\infty,m_2}(X,\Sigma)\subset\Psi^{-\infty}(X). \tag{5.25}$$

(3) (Principal symbol) There exists a multiplicative principal symbol map

$$\sigma^H_{m_1,m_2}:\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)\to(\beta^*d)^{-m_2}C^\infty_{\mathrm{inv}}([(S^*X);S^*\Sigma])$$

satisfying

$$\begin{aligned}\sigma^H_{m_1+m_1',m_2+m_2'}(AB)&=\sigma^H_{m_1,m_2}(A)\,\sigma^H_{m_1',m_2'}(B)\\ \sigma^H_{m_1,m_2}(A^*)&=\overline{\sigma^H_{m_1,m_2}(A)}\end{aligned} \tag{5.26}$$

for every $A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$, $B\in\Psi_{\mathrm{cl}}^{m_1',m_2'}(X,\Sigma)$.

(4) (Symbol exact sequence) The principal symbol fits into the exact sequence below

$$0\to\Psi_{\mathrm{cl}}^{m_1-1,m_2+1}(X,\Sigma)\hookrightarrow\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)\xrightarrow{\sigma^H_{m_1,m_2}}(\beta^*d)^{-m_2}C^\infty_{\mathrm{inv}}([(S^*X);S^*\Sigma])\to 0. \tag{5.27}$$

(5) (Quantization) There exists a surjective quantization map

$$\mathrm{Op}^H:(\beta^*d)^{-m_2}C^\infty_{\mathrm{inv}}[(S^*X);S^*\Sigma]\to\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$$

which is a left-inverse to the principal symbol

$$\begin{aligned}\sigma^H_{m_1,m_2}\left(\mathrm{Op}^H a\right)&=a\\ \mathrm{Op}^H\left[\sigma^H_{m_1,m_2}(A)\right]&=A\quad\left(\mathrm{mod}\ \Psi_{\mathrm{cl}}^{m_1-1,m_2+1}(X,\Sigma)\right).\end{aligned} \tag{5.28}$$

(6) (Symbol of commutator) For $A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$, $B\in\Psi_{\mathrm{cl}}^{m_1',m_2'}(X,\Sigma)$ the commutator $[A,B]\in\Psi_{\mathrm{cl}}^{m_1+m_1'-1,m_2+m_2'+1}(X,\Sigma)$ with symbol

$$\sigma^H_{m_1+m_1'-1,m_2+m_2'+1}([A,B])=i\left\{\sigma^H_{m_1,m_2}(A),\sigma^H_{m_1',m_2'}(B)\right\}. \tag{5.29}$$

(7) (Asymptotic summation) For any set of operators $A_j\in\Psi^{m_1-j,m_2+j}(X,\Sigma)$, $B_j\in\Psi^{m_1,m_2-j}(X,\Sigma)$, (resp. $\Psi_{\mathrm{cl}}(X,\Sigma)$), $j\in\mathbb{N}_0$, there exists $A,B\in\Psi^{m_1,m_2}$ such that

$$\begin{aligned}A-\sum_{j=0}^{N}A_j&\in\Psi^{m_1-\frac{N}{2},m_2}(X,\Sigma),\\ B-\sum_{j=0}^{N}B_j&\in\Psi^{m_1,m_2-N}(X,\Sigma),\forall N\in\mathbb{N}_0\end{aligned} \tag{5.30}$$

(8) (Sobolev boundedness) For any $A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$ and $u\in H^{s_1,s_2}(X,\Sigma)$ one has $Au\in H^{s_1-m_1,s_2-m_2}(X,\Sigma)$.

(9) (Microlocality) For any $A\in\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$, $B\in\Psi_{\mathrm{cl}}^{m_1',m_2'}(X,\Sigma)$ and $u\in C^{-\infty}(X)$ one has

$$\begin{aligned}WF_\Sigma(A+B)&\subset WF_\Sigma(A)\cup WF_\Sigma(B)\\ WF_\Sigma(AB)&\subset WF_\Sigma(A)\cap WF_\Sigma(B)\\ WF_\Sigma(Au)&\subset WF_\Sigma(A)\cap WF_\Sigma(u).\end{aligned} \tag{5.31}$$

As a first application of the calculus we construct parametrices for elliptic elements of $\Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$.

Proposition 16. *Let* $P \in \Psi^{m_1,m_2}_{cl}(X,\Sigma)$ *be elliptic. Then there exists* $Q \in \Psi^{-m_1,-m_2}_{cl}(X,\Sigma)$ *satisfying* $PQ - I \in \Psi^{-\infty}_{cl}(X)$, $QP - I \in \Psi^{-\infty}_{cl}(X)$.

Proof. This is a usual application of the pseudo-differential calculus albeit in the exotic class 15. Since $\sigma^H_{m_1,m_2}(P) \in (\beta^* d)^{-m_2} C^\infty_{\text{inv}}([(S^*X); S^*\Sigma])$ satisfies (5.17), its inverse $\left[\sigma^H_{m_1,m_2}(P)\right]^{-1} \in (\beta^* d)^{m_2} C^\infty_{\text{inv}}([(S^*X); S^*\Sigma])$ can be seen to lie in the given space and maybe quantized $Q_0 := \text{Op}^H\left(\left[\sigma^H_{m_1,m_2}(P)\right]^{-1}\right) \in \Psi^{-m_1,-m_2}_{\text{cl}}(X,\Sigma)$. We now compute $\sigma_{0,0}(PQ_0 - I) = 0$ using (5.26), (5.28) and thus $PQ_0 - I \in \Psi^{-1,1}_{\text{cl}}(X,\Sigma)$. We then set

$$Q_1 := -\text{Op}^H\left(\left[\sigma^H_{m_1,m_2}(P)\right]^{-1} \sigma_{-1,1}(PQ_0 - I)\right) \in \Psi^{-m_1-1,-m_2+1}_{\text{cl}}(X,\Sigma)$$

and again compute $P(Q_0 + Q_1) - I \in\in \Psi^{-2,2}_{\text{cl}}(X,\Sigma)$. Continuing in this fashion gives a sequence $Q_j \in \Psi^{-m_1-j,-m_2+j}_{\text{cl}}(X,\Sigma)$, $j = 0, 1, \ldots$ such that

$$P\left(\sum_{j=0}^{N} Q_j\right) - I \in \Psi^{-N-1,N+1}_{\text{cl}}(X,\Sigma) \subset \Psi^{-\frac{1}{2}(N+1),0}_{\text{cl}}(X,\Sigma).$$

The asymptotic summation $A \sim \sum_{j=0}^{\infty} Q_j$ (5.30) then satisfies $PQ - I \in \Psi^{-\infty,0}_{\text{cl}}(X,\Sigma) \subset \Psi^{-\infty}_{\text{cl}}(X)$ as required. The construction of the left parametrix Q' satisfying $Q'P - I \in \Psi^{-\infty}_{\text{cl}}(X)$ is similar. Seeing these to agree $Q - Q' = Q'(I - PQ) + (Q'P - I)Q \in \Psi^{-\infty}_{\text{cl}}(X)$ modulo residual terms gives the result. ☐

As an application we improve a subelliptic estimate.

Proposition 17. *Let* $P \in \Psi^{m_1,m_2}_{cl}(X,\Sigma)$ *be elliptic. Then there exists* $C > 0$ *such that*

$$\|f\|_{H^{s_1+m_1,s_2+m_2}} \le C\left[\|Pf\|_{H^{s_1,s_2}} + \|f\|_{H^{s_1,s_2}}\right] \tag{5.32}$$

$\forall f \in C^\infty(X)$, $s_1, s_2 \in \mathbb{R}$.

Proof. With Q being the parametrix 16 for P, write $f = QPf + (I - QP)f$ and use the Sobelev boundedness $\|Q\|_{H^{s_1,s_2}(X,\Sigma)\to H^{s_1+m_1,s_2+m_2}(X,\Sigma)} < \infty$, $\|I - QP\|_{H^{s_1,s_2}(X,\Sigma)\to H^{s_1+m_1,s_2+m_2}(X,\Sigma)} < \infty$. ☐

Since $\Delta_{g^E,\mu}$ is clearly elliptic in $\Psi^{2,-2}_{\text{cl}}(X,\Sigma)$ by definition, the above proposition gives

$$\|f\|_{H^{s_1+2,s_2-2}} \le C\left[\left\|\Delta_{g^E,\mu} f\right\|_{H^{s_1,s_2}} + \|f\|_{H^{s_1,s_2}}\right] \tag{5.33}$$

$\forall f \in C^\infty(X)$, $s_1, s_2 \in \mathbb{R}$. In light of the inclusions (5.20) the above refines the subelliptic estimate for the sR Laplacian (2.21) in our particular 4D quasi-contact case.

Remark 18. Although the notation suppresses it, the class of pseudo-differential operators $\Psi^{m_1,m_2}_{\text{cl}}(X,\Sigma)$ is depends on the Laplacian $\Delta_{g^E,\mu}$ and not just the characteristic variety. This class differs from the more well-known class of operators defined in [8, 9] wherein the corresponding classes depend only on the characteristic variety and their symbols do not necessarily satisfy any invariance condition.

5.3. **Egorov and propagation.** In this section we explore some immediate consequences of the global calculus of the previous subsection. We first begin by showing that the square root of the Laplacian lies in the given class.

Proposition 19. *The square root lies in the given class* $\sqrt{\Delta_{g^E,\mu}} \in \Psi^{1,-1}_{cl}(X,\Sigma)$ *with symbol* $\sigma^H_{1,-1}\left(\sqrt{\Delta_{g^E,\mu}}\right) = d$ (5.15).

Proof. This is another application of the pseudo-differential calculus 15. As noted before $\beta^* d$ lies in the symbol space and can be quantized $A_0 := \mathrm{Op}^H(\beta^* d) \in \Psi_{\mathrm{cl}}^{1,-1}(X,\Sigma)$. It squares principally $\sigma^H_{2,-2}\left(\Delta_{g^E,\mu} - \mathrm{Op}^H(\beta^* d)^2\right) = 0$ by (5.15),(5.26), (5.28) and thus $\Delta_{g^E,\mu} - A_0^2 \in \Psi_{\mathrm{cl}}^{1,-1}(X,\Sigma)$ by (5.27). Now define

$$A_1 := \frac{1}{2}\mathrm{Op}^H\left[(\beta^* d)^{-1}\sigma_{1,-1}\left(\Delta_{g^E,\mu} - \mathrm{Op}^H(\beta^* d)^2\right)\right] \in \Psi_{\mathrm{cl}}^{0,0}(X,\Sigma)$$

and again calculate $\Delta_{g^E,\mu} - (A_0 + A_1)^2 \in \Psi_{\mathrm{cl}}^{0,0}(X,\Sigma)$. Continuing in this fashion we inductively construct a sequence $A_j \in \Psi_{\mathrm{cl}}^{1-j,-1+j}(X,\Sigma)$, $j = 0,1,2,\ldots$ such that

$$\Delta_{g^E,\mu} - \left(\sum_{j=0}^{N} A_j\right)^2 \in \Psi_{\mathrm{cl}}^{1-N,-1+N}(X,\Sigma).$$

The asymptotic summation $A \sim \sum_{j=0}^{\infty} A_j \in \Psi_{\mathrm{cl}}^{1,-1}(X,\Sigma)$ (5.30) then satisfies $\Delta_{g^E,\mu} - A^2 \in \Psi_{\mathrm{cl}}^{-\infty,0}(X,\Sigma) \subset \Psi_{\mathrm{cl}}^{-\infty}(X)$. The symbol $\sigma^H_{1,-1}(A) = \beta^* d$ shows that A is elliptic, satisfying the subelliptic estimate(5.32), and hence has a compact resolvent by (5.20). It thus has only finitely many non-positive eigenvalues and can be altered, by projecting off the negative eigenvalues, to a positive operator. We now write the difference

$$\sqrt{\Delta_{g^E,\mu}} - A = \frac{1}{2\pi i}\int_\Gamma dz\, z^{-1/2}\left(\Delta_{g^E,\mu} - z\right)^{-1}\underbrace{\left(\Delta_{g^E,\mu} - A^2\right)}_{\in \Psi_{\mathrm{cl}}^{-\infty,0}(X)}\left(A^2 - z\right)^{-1}$$

with Γ representing a contour around the positive real axis, to see that the difference above is also in $\Psi_{\mathrm{cl}}^{-\infty}(X)$ and complete the proof. □

We next prove an Egorov theorem for conjugation by the half wave operator $e^{it\sqrt{\Delta_{g^E,\mu}}}$. Below and it what follows we note that the evolution $\left(e^{tH_d}\right)^* \sigma^H_{m_1,m_2}(P) \in\in (\beta^* d)^{-m_2} C^\infty_{\mathrm{inv}}([(S^*X);S^*\Sigma])$ (5.16) lies in the same class on account of (2.39) and circular invariance of symbol (5.14).

Theorem 20. *For any $P \in \Psi_{cl}^{m_1,m_2}(X,\Sigma)$ the conjugate $P(t) := e^{-it\sqrt{\Delta_{g^E,\mu}}} P e^{it\sqrt{\Delta_{g^E,\mu}}} \in \Psi_{cl}^{m_1,m_2}(X,\Sigma)$ lies in the same pseudo-differential class with $\sigma^H_{m_1,m_2}(P(t)) = \left(e^{-tH_d}\right)^* \sigma^H_{m_1,m_2}(P)$.*

Proof. We again use symbolic calculus in the class 15. Since the conjugate satisfies the differential equation $\partial_t P + \left[\sqrt{\Delta_{g^E,\mu}}, P\right] = 0$, we first solve this equation symbolically modulo residual terms. First define $A_0(t) = \mathrm{Op}^H\left[\left(e^{-tH_d}\right)^* \sigma^H_{m_1,m_2}(P)\right] \in \Psi_{\mathrm{cl}}^{m_1,m_2}(X,\Sigma)$; it is easy to check that $\left(e^{-tH_d}\right)^* \sigma^H_{m_1,m_2}(P) \in (\beta^* d)^{-m_2} C^\infty_{\mathrm{inv}}[(S^*X);S^*\Sigma]$. We then compute $\sigma^H_{m_1,m_2}\left(\partial_t A_0 + \left[\sqrt{\Delta_{g^E,\mu}}, A_0\right]\right) = -H_d P + H_d P = 0$ using (5.15), (5.29) and thus $\partial_t A_0 + \left[\sqrt{\Delta_{g^E,\mu}}, A_0\right] \in \Psi_{\mathrm{cl}}^{m_1-1,m_2+1}(X,\Sigma)$ by (5.27). Now define

$$\begin{aligned} A_1 :=& \mathrm{Op}^H\Big[\sigma^H_{m_1-1,m_2+1}\left(P - \sigma^H_{m_1,m_2}(P)\right) \\ &+ \int_0^t ds\, \left(e^{-(t-s)H_d}\right)^* \sigma^H_{m_1-1,m_2+1}\left(\partial_t A_0 + \left[\sqrt{\Delta_{g^E,\mu}}, A_0\right]\right)(s)\Big] \in \Psi_{\mathrm{cl}}^{m_1-1,m_2+1}(X,\Sigma) \end{aligned}$$

and again compute $\sigma^H_{m_1-1,m_2+1}\left(\partial_t(A_0 + A_1) + \left[\sqrt{\Delta_{g^E,\mu}}, (A_0 + A_1)\right]\right) = 0$ using the Duhamel's principle (5.15), (5.29) and thus $\partial_t(A_0 + A_1) + \left[\sqrt{\Delta_{g^E,\mu}}, (A_0 + A_1)\right] \in \Psi_{\mathrm{cl}}^{m_1-2,m_2+2}(X,\Sigma)$. Continuing in this fashion we inductively construct a sequence $A_j \in \Psi_{\mathrm{cl}}^{m_1-j,m_2+j}(X,\Sigma)$, $j =$

$0,1,2,\ldots$ such that

$$\partial_t\left(\sum_{j=0}^{N}A_j\right)+\left[\sqrt{\Delta_{g^E,\mu}},\left(\sum_{j=0}^{N}A_j\right)\right]\in\Psi_{\text{cl}}^{m_1-N-1,m_2+N+1}\left(X,\Sigma\right)$$
$$P-\left.\left(\sum_{j=0}^{N}A_j\right)\right|_{t=0}\in\Psi_{\text{cl}}^{m_1-N-1,m_2+N+1}\left(X,\Sigma\right).$$

Thus again the asymptotic summation $A\sim\sum_{j=0}^{\infty}A_j\in\Psi_{\text{cl}}^{m_1,m_2}\left(X,\Sigma\right)$ (5.30) then satisfies $R\left(t\right)\coloneqq\partial_tA+\left[\sqrt{\Delta_{g^E,\mu}},A\right]\in\Psi_{\text{cl}}^{-\infty,m_2}\left(X\right)$, $R_0\coloneqq P-A|_{t=0}\in\Psi_{\text{cl}}^{-\infty,m_2}\left(X\right)$. Finally Duhamel's principle gives

$$P\left(t\right)-A=\underbrace{R_0}_{\Psi_{\text{cl}}^{-\infty,m_2}(X)}+\int_0^t ds\,e^{-is\sqrt{\Delta_{g^E,\mu}}}\underbrace{R\left(t-s\right)}_{\in\Psi_{\text{cl}}^{-\infty,m_2}(X)}e^{is\sqrt{\Delta_{g^E,\mu}}},$$

showing that the difference is in $\Psi_{\text{cl}}^{-\infty}\left(X\right)$ and completing the proof. □

As an immediate application of Egorov theorem we have propagation of singularities (5.21).

Proof of Theorem 3. By (5.22), for $(x,\xi)\notin WF_\Sigma\left(u\right)$ there exists $A\in\Psi_{\text{cl}}^{0,0}\left(X,\Sigma\right)$ with $\sigma_{0,0}^{H}\left(A\right)\left(x,\xi\right)\neq 0$ and $Au\in C^\infty$. Thus with $A\left(t\right)\coloneqq e^{it\sqrt{\Delta_{g^E,\mu}}}Ae^{-it\sqrt{\Delta_{g^E,\mu}}}\in\Psi_{\text{cl}}^{0,0}\left(X,\Sigma\right)$ we have $\sigma_{0,0}^{H}\left(A\left(t\right)\right)\left(e^{t\hat{Z}}\left(x,\xi\right)\right)=\sigma_{0,0}^{H}\left(A\right)\left(x,\xi\right)\neq0$ by (2.38), (2.39), Theorem 20 and rotational invariance of the symbol. Then $A\left(t\right)e^{it\sqrt{\Delta_{g^E,\mu}}}u=e^{it\sqrt{\Delta_{g^E,\mu}}}Au\in C^\infty$ shows $e^{t\hat{Z}}\left(x,\xi\right)\notin WF_\Sigma\left(e^{it\sqrt{\Delta_{g^E,\mu}}}u\right)$ as required. □

5.4. **Parametrix.** In this section we construct a small time parametrix for the half wave operator; we work more generally to construct a parametrix for $Ae^{it\sqrt{\Delta_{g^E,\mu}}}$, $A\in\Psi_{\text{cl}}^{0,0}\left(X\right)$. The operators $\sqrt{\Delta_{g^E,\mu}}$ and A being pseudo-differential, and $\sqrt{\Delta_{g^E,\mu}}$ elliptic outside the characteristic variety, the parametrix construction is achieved by standard Hamilton-Jacobi theory in the complement of Σ. It shall then suffice to construct for each $p\in\Sigma$ a microlocal solution to

$$\begin{aligned}i\partial_tP+\sqrt{\Delta_{g^E,\mu}}P&\in\Psi_{\text{cl}}^{-\infty,0}\\ P|_{t=0}=A&\in\Psi_{\text{cl}}^{0,0},\end{aligned}\tag{5.34}$$

where we may further suppose $A\in\Psi_{\text{cl}}^{0,0}\left(X\right)$ to be micro-supported in a microlocal chart near p where (4.32) holds. We shall look for a solution of the form

$$P\coloneqq\left[\sum_{k\in\mathbb{N}_0}H_k^*P_kH_k\right]A.\tag{5.35}$$

Here each

$$P_k=\left[I_\varphi\left(a\right)\right]_k\coloneqq\int e^{i\left(\varphi_k-\underline{y}.\underline{\xi}\right)}a_k\left(t;\underline{x},\underline{\xi}\right)d\underline{\xi},\quad k\in\mathbb{N}_0,\tag{5.36}$$

with $a\in S_{\text{cl},t}^{0,0}$, $\varphi\in\underline{x}.\underline{\xi}+tS_{\text{cl},t}^{1,-1}$, and each φ_k solving the Hamilton-Jacobi equation

$$\begin{aligned}\partial_t\varphi_k&=d_k\left(x,\partial_x\varphi_k\right)\\ \varphi_k|_{t=0}&=\underline{x}.\underline{\xi}.\end{aligned}\tag{5.37}$$

We first show that the above has a solution.

Proposition 21. *There exists a sufficiently small conic neighborhood of $p \in C \subset [(T^*X);\Sigma]$, $T>0$ and*

$$\varphi \in \underline{x}.\underline{\xi} + tdC^{\infty}_{inv}\left(C \times (-T,T)\right), \tag{5.38}$$

of homogeneous of degree one such that each corresponding φ_k , $k \in \mathbb{N}_0$, solves the Hamilton Jacobi equation (5.37).

Proof. This is a modification of usual Hamilton-Jacobi theory. From the computation (4.20), we may then choose to work in a microlocal chart C' at p such that $e^{tH_d}(C)$, $t \in (-T,T)$, stays in the chart for some sufficiently small conic neighborhood $C \subset C'$ and $T>0$. With the notation of (4.15), $d\left(\underline{x},\underline{\xi},\Omega\right)$ being a function of the given variables with $\{\Omega, d\}=0$, the function Ω is preserved under the flow of H_d. One thus has

$$\begin{aligned} e^{tH_d}\left(\underline{x},\underline{\xi};x_1,\xi_1\right) &= \left(e^{tH_{d,\Omega}}\left(\underline{x},\underline{\xi}\right);*,*\right) \quad \text{for} \\ H_{d,\Omega} &:= \left(\partial_{\underline{\xi}}d\right)\left(\underline{x},\underline{\xi},\Omega\right)\partial_{\underline{x}} - \left(\partial_{\underline{x}}d\right)\left(\underline{x},\underline{\xi},\Omega\right)\partial_{\underline{\xi}}. \end{aligned} \tag{5.39}$$

The vector field $H_{d,\Omega}$ above extends smoothly to the boundary of the blowup $[(T^*X);\Sigma]$.

Given $\alpha \geq 0$,$\underline{\xi} \in \mathbb{R}^3$, we now define the flow-out

$$\Lambda_{\alpha,\underline{\xi}} := \left\{\left(e^{tH_{d,\alpha}}\left(\underline{x},\underline{\xi}\right),t,d\left(\underline{x},\underline{\xi},\alpha\right)\right) \mid \left(\underline{x},\xi_3^{-1/2}\alpha^{1/2},\underline{\xi},0\right) \in C,\, t \in (-T,T)\right\} \subset T^*\mathbb{R}^3_{\underline{x}} \times T^*_{(t,\tau)}\mathbb{R}.$$

By (4.22) and an application of Gronwall's lemma, for C and T sufficiently small, the flow-out $\Lambda_{\alpha,\underline{\xi}}$ is horizontal above $\mathbb{R}^3_{\underline{x}} \times \mathbb{R}_t$ for $t \in (-T,T)$. Hence one may find a solution $\varphi_{\alpha,\underline{\xi}}\left(\underline{x},t\right)$ to

$$\text{graph}\left(d\varphi_{\alpha,\underline{\xi}}\right) := \left\{x,t,d_{(\underline{x},t)}\varphi_{\alpha,\underline{\xi}}\right\} = \Lambda_{\alpha,\underline{\xi}} \tag{5.40}$$

The function $\varphi(x,\xi,t) := \varphi_{\Omega,\underline{\xi}}\left(\underline{x},t\right)$ is then the required solution to (5.37); its smoothness follows from the smooth extension of (5.39) to the boundary. To see that the solution lies in the given space (5.38) one needs to check $\varphi - \underline{x}.\underline{\xi}$, or its pullback to the graph (5.40), vanishes on the boundary $\partial\left[(T^*\mathbb{R}^4);\Sigma_0\right]$ at all time $t \in (-T,T)$. This follows from computing

$$\pi^* d\,\varphi|_{\partial[(T^*\mathbb{R}^4);\Sigma_0]} = \underbrace{d|_{\partial[(T^*\mathbb{R}^4);\Sigma_0]}}_{=0}(dt) + \underbrace{\left(\beta^{-1}\right)^*\alpha\left(H_{d,\Omega}\right)\big|_{\partial[(T^*\mathbb{R}^4);\Sigma_0]}}_{=0}$$

from (4.22), with α denoting the tautological one form on $T^*\mathbb{R}^4$. □

Next we solve for the amplitude in (5.36). Differentiation of (5.36) using the symbolic expansion of $\sqrt{\Delta_{g^E,\mu}} = d + \Psi^{0,-1/2}_{\text{cl}}$ gives

$$\begin{aligned} i\partial_t P + \sqrt{\Delta_{g^E,\mu}}P &= I_{\varphi}(b) \quad \text{with} \\ b &= \underbrace{\left(\partial_t + H_d\right)a + Ra}_{=:La} \end{aligned}$$

and where R maps S^{m_1,m_2}_{cl} to $S^{m_1-1,m_2+1}_{\text{cl}}$, $\forall m_1,m_2$. One may then write down a solution to the transport equation $La=0$ (mod $S^{-\infty}_{\text{cl}}$) as

$$a(t) \sim \sum_{j=0}^{\infty} a_j(t) \in S^{0,0}_{\text{cl},t}, \tag{5.41}$$

$a_j(t) \in S_{\mathrm{cl}}^{-j,j}$, starting from the symbolic expansion $a \sim \sum_{j=0}^{\infty} a_j$, $a_j \in S_{\mathrm{cl}}^{-j,j}$ for A (5.34) by inductively solving

$$a_0(t) = \left(e^{-tH_d}\right)^* a_0;$$

$$(\partial_t + H_d) a_j(t) = -R[a_0 + \ldots + a_{j-1}], \; a_j(0) = a_j, \quad j \geq 1. \tag{5.42}$$

6. Poisson relations

In this section we prove the Poisson relation Theorem 2. We more generally analyze the behavior of the microlocal wave trace tr $Ae^{it\sqrt{\Delta_{g^E,\mu}}}$, $A \in \Psi_{\mathrm{cl}}^{0,0}(X)$, for small time using the parametrix (5.36). It again suffices to consider the wave trace near characteristic variety and we may assume $A \in \Psi_{\mathrm{cl}}^{0,0}(X)$ to be micro-supported in a microlocal chart near p where (4.32) holds. By (5.37), (5.38) we have $\varphi - \underline{x}.\underline{\xi} = t\varphi^0$ with $\varphi^0 = \xi_3 \hat{d}_\rho \left[1 + t\underbrace{R\left(t; \underline{x}, \underline{\xi}, \hat{d}_\rho\right)}_{\in S_{\mathrm{cl}}^{0,0}}\right]$. On changing the ξ_3, ξ_0 variables to the new variables $r = \xi_3 d_k$, $\Xi_0 = \frac{\xi_0}{d_k}$ the wave trace

$$\mathrm{tr}\; Ae^{it\sqrt{\Delta_{g^E,\mu}}} = \sum_{k \in \mathbb{N}_0} \int e^{i\left(tr + t^2 r R_k\right)} a_k(t) \frac{\left(1 - \Xi_0^2\right) r^4}{\hat{\rho}^2 (2k+1)^2} dr\, d\Xi_0 d\hat{\xi}_2 d\underline{x} \quad (\mathrm{mod}\; t^\infty),$$

in the distributional sense. Since the amplitude was shown to be in the class $a \in S_{\mathrm{cl},t}^{0,0}$, the wave trace mod t^{N-5} is then a finite sum of terms of the form

$$\int \sum_{k \in \mathbb{N}_0} e^{irt} t^{2\alpha} r^{\alpha - \beta} a_{\alpha,\beta}\left(\underline{x}, \hat{\xi}_2, \Xi_0; d_k\right) \frac{\left(1 - \Xi_0^2\right) r^4}{\hat{\rho}^2 (2k+1)^2} dr\, d\Xi_0 d\hat{\xi}_2 d\underline{x};$$

$$\alpha, \beta \in \mathbb{N}_0, \; 0 \leq \alpha + \beta \leq N, \tag{6.1}$$

with

$$a_{\alpha,\beta}\left(\underline{x}, \hat{\xi}_2, \Xi_0; d_k\right) = b_{\alpha,\beta}\left(\underline{x}, \hat{\xi}_2, \Xi_0 (d_k + 1), d_k + 1\right) \tag{6.2}$$

for $b_{\alpha,\beta}\left(\underline{x}, \hat{\xi}_2, p_0, d\right) \in C_c^\infty\left(\mathbb{R}^5_{\underline{x}, \hat{\xi}_2, p_0} \times [1, \infty)_d\right)$. Furthermore the leading part

$$a_{0,0} = \sigma^H(A)\left(\underline{x}, \hat{\xi}_2, \Xi_0; d_k\right). \tag{6.3}$$

We now show how to sum the above in k with the help of the proposition below.

Proposition 22. *Given $b \in C_c^\infty\left(\mathbb{R}^5_{\underline{x}, \hat{\xi}_2, p_0} \times [1, \infty)_d\right)$ and a defined as in* (6.2)*, the expression*

$$I(r) := \int d\Xi_0 d\hat{\xi}_2 d\underline{x} \left[\sum_{k \in \mathbb{N}_0} \frac{\left(1 - \Xi_0^2\right)}{\hat{\rho}^2 (2k+1)^2} a\left(\underline{x}, \hat{\xi}_2, \Xi_0; d_k\right)\right]$$

$$\sim c r^{-1} \ln r + c_0 + c_1 r^{-1} + c_2 r^{-2} + \ldots; \tag{6.4}$$

$$c_0 = \frac{\pi^2}{8} \int d\Xi_0 d\hat{\xi}_2 d\underline{x} \frac{\left(1 - \Xi_0^2\right)}{\hat{\rho}^2} b\left(\underline{x}, \hat{\xi}_2, \Xi_0, 0\right)$$

$$c = \frac{1}{2} \int d\Xi_0 d\hat{\xi}_2 d\underline{x} \frac{\left(1 - \Xi_0^2\right)}{\hat{\rho}} \left(\Xi_0 \partial_{p_0} + \partial_d\right) b\left(\underline{x}, \hat{\xi}_2, \Xi_0, 0\right) \tag{6.5}$$

is the sum of a classical symbol in r of order 0 *and a log term $r^{-1} \ln r$.*

Proof. The symbolic estimates are easily seen on differentiation and noting $d_k = \frac{\hat{\rho}(2k+1)}{r\left(1-\Xi_0^2\right)}$ to be a symbol of order -1 in r in the region $d_k \lesssim 1$, $\Xi_0 \lesssim 1$. To show a classical expansion, we perform the change of variables $\alpha = \left(1-\Xi_0^2\right)^{-1}$ in the Ξ_0 integration to obtain

$$I(r) = \sum_{k\in\mathbb{N}_0} \int \frac{d\hat{\xi}_2 d\underline{x}}{\hat{\rho}^2 (2k+1)^2} \left[\underbrace{\int_1^\infty \frac{d\alpha}{\alpha^3\sqrt{1-\alpha^{-1}}} a\left(\underline{x}, \hat{\xi}_2, \sqrt{1-\alpha^{-1}}; \frac{\hat{\rho}(2k+1)}{r}\alpha\right)}_{=:I_0\left(\underline{x},\hat{\xi}_2;\frac{\hat{\rho}(2k+1)}{r}\right)} \right]$$

with the integral in parentheses above seen to be $I_0\left(\underline{x}, \hat{\xi}_2; \frac{\hat{\rho}(2k+1)}{r}\right) \in C_c^\infty\left(\mathbb{R}^5_{\underline{x},\hat{\xi}_2,\frac{\hat{\rho}(2k+1)}{r}}\right)$.

From here the proposition follows from [34, Prop. 7.20] but we give a shorter argument. Differentiating $I_1(\varepsilon) := \sum_{k\in\mathbb{N}_0} \frac{1}{\hat{\rho}^2(2k+1)^2} I_0\left(\underline{x}, \hat{\xi}_2; \varepsilon\hat{\rho}(2k+1)\right)$, $\varepsilon := \frac{1}{r}$, gives

$$\begin{aligned} \partial_\varepsilon^2 I_1 &= \sum_{k\in\mathbb{N}_0} \left(\partial_d^2 I_0\right)\left(\underline{x}, \hat{\xi}_2; \varepsilon\hat{\rho}(2k+1)\right) \\ &= \int_0^\infty dk\, \left(\partial_d^2 I_0\right)\left(\underline{x}, \hat{\xi}_2; \varepsilon\hat{\rho}(2k+1)\right) \\ &\quad + \frac{1}{2}\sum_{l\in\mathbb{N}\setminus\{0\}} \int dk\, e^{i2\pi kl} \left(\partial_d^2 I_0\right)\left(\underline{x}, \hat{\xi}_2; \varepsilon\hat{\rho}(2k+1)\right) \end{aligned}$$

by the Poisson summation formula. By repeated integration by parts the second term in the last line above is seen to be $O(\varepsilon^\infty)$, while the first term is evaluated to be

$$\int_0^\infty dk\, \left(\partial_d^2 I_0\right)\left(\underline{x}, \hat{\xi}_2; \varepsilon\hat{\rho}(2k+1)\right) \sim c\varepsilon^{-1} + c_0 + c_1\varepsilon + \ldots$$

$$c = \frac{1}{2}\hat{\rho}^{-1}\left[\partial_d I_0\left(\underline{x}, \hat{\xi}_2; 0\right)\right] \tag{6.6}$$

to complete the proof. □

Following this proposition, we may further simplify (6.1) as being mod t^{N-5} a sum of terms of the form

$$\begin{aligned} &t^{2\alpha}\int_0^\infty dr\, e^{irt} r^{4-j+\alpha-\beta}; \\ \text{or}\quad &t^{2\alpha}\int_0^\infty dr\, e^{irt} r^{3-j+\alpha-\beta} \ln r\,; \end{aligned}$$

$\alpha, \beta, j \in \mathbb{N}_0$, $\alpha+\beta \leq N$. Using the identifications (4.22), the knowledge of these elementary Fourier transforms and identifying the constants we now have the following.

Theorem 23. *For any $A \in \Psi_{cl}^{0,0}$, the microlocal wave trace in the 4D quasi-contact case has the asymptotics*

$$tr\, Ae^{it\sqrt{\Delta_{g^E,\mu}}} = \sum_{j=0}^N c_{j,0}^A (t+i0)^{j-5} + \sum_{j=0}^N c_{j,1}^A (t+i0)^{j-4}\ln(t+i0) + \sum_{j=0}^N c_{j,2}^A t^j \ln^2(t+i0) + O\left(t^{N-4}\right) \tag{6.7}$$

$\forall N \in \mathbb{N}$, as $t \to 0$, in the distributional sense with leading term $c_{0,0}^A = \frac{1}{32\pi}\int \sigma(A)|_{SNS^*\Sigma}\, \mu_{Popp}^{SNS^*\Sigma}$.

In the case when $A = 1$ one has $b_{0,0} = 1$ in (6.2) which following (6.5) gives that the first logarithmic term above vanishes $c^1_{0,1} = 0$ proving Theorem 2. Pairing (6.7) with $\theta \in C^\infty_c(-C_0, C_0)$, for C_0 sufficiently small, gives

$$\text{tr } A\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right) = \sum_{j=0}^{N} \tilde{c}^A_{j,0}\lambda^{4-j} + \sum_{j=0}^{N} \tilde{c}^A_{j,1}\lambda^{3-j}\ln\lambda + O\left(\lambda^{4-N}\right) \tag{6.8}$$

$$\text{tr } \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right) = \sum_{j=0}^{N} \tilde{c}_{j,0}\lambda^{4-j} + \sum_{j=0}^{N} \tilde{c}_{j,1}\lambda^{2-j}\ln\lambda + O\left(\lambda^{4-N}\right) \tag{6.9}$$

$\forall N \in \mathbb{N}$ as $\lambda \to \infty$ with leading terms

$$\begin{aligned}
\tilde{c}^A_{j,0} &= \frac{\theta(0)}{32\pi}\int \sigma(A)|_{SNS^*\Sigma}\,\mu^{SNS^*\Sigma}_{\text{Popp}},\\
\tilde{c}^1_{j,0} &= \frac{\theta(0)}{32\pi}\int\left[\int_{-1}^{1} d\Xi_0\left(1-\Xi_0^2\right)\right]\mu_{\text{Popp}} = \frac{\theta(0)}{24\pi}\mu_{\text{Popp}}.
\end{aligned}$$

We now prove the Weyl laws Theorem 1.

Proof of Theorem 1. Following a standard Tauberian theorem for Fourier transforms (cf. [19, Sec. 2]) (6.9) gives (1.2). To prove (1.3) one needs to prove (6.9) at leading order for $\theta \in C^\infty_c(\mathbb{R})$ of arbitrary support under the dynamical assumption.

We first consider the trace norm of $A\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)$, $A = a^H \in \Psi^{0,0}_{\text{cl}}$, $a \in C^\infty_{c,\text{inv}}\left([(S^*X);S^*\Sigma];[0,1]\right)$, for $\theta \in C^\infty_c(-C_0, C_0)$ assuming $\check{\theta} \geq 0$. To this end, let $\tilde{a} \in C^\infty_{c,\text{inv}}\left([(S^*X);S^*\Sigma];[0,1]\right)$ such that $\tilde{a} = 1$ on $\cup_{t\in(-C_0,C_0)} e^{tH_d}(\text{spt}a)$. Then an Egorov type argument Theorem 20 gives

$$\begin{aligned}
&a^H e^{it\left(\sqrt{\Delta_{g^E,\mu}}-\lambda\right)}\left(1-\tilde{a}^H\right) \in \Psi^{-\infty}_{\text{cl}}, \quad \forall t \in (-C_0, C_0),\\
&\left\|a^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\left(1-\tilde{a}^H\right)\right\|_{\text{tr}} = O\left(\lambda^{-\infty}\right)
\end{aligned} \tag{6.10}$$

and thus

$$\begin{aligned}
\left\|a^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\text{tr}} &= \left\|a^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}^H\right\|_{\text{tr}} + O\left(\lambda^{-\infty}\right)\\
&= \left\|a^H\tilde{a}^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}^H\right\|_{\text{tr}} + O\left(\lambda^{-\infty}\right)
\end{aligned}$$

Now since $|a| < 1+\varepsilon$, $\forall\varepsilon > 0$, we may use symbolic calculus to write $a^H = 1+\varepsilon-\left(b^H\right)^2 + \Psi^{-\infty}_{\text{cl}}$. This gives

$$\begin{aligned}
\left\|a^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\text{tr}} &= \left\|\left(1+\varepsilon-\left(b^H\right)^2\right)\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}^H\right\|_{\text{tr}} + O\left(\lambda^{-\infty}\right)\\
&\leq (1+\varepsilon)\left\|\tilde{a}^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}^H\right\|_{\text{tr}} + O\left(\lambda^{-\infty}\right)
\end{aligned}$$

Next for $\check{\theta} \geq 0$, the operator $\tilde{a}^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}^H$ being positive and self-adjoint, its trace norm coincides with its trace which is in turn analyzed in a similar fashion to (6.8). Hence

$$\left\|a^H\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\text{tr}} \leq (1+\varepsilon)\lambda^4\left[C_\theta\int |\tilde{a}|^2|_{SNS^*\Sigma}\,\nu^{SNS^*\Sigma}_{\text{Popp}}\right] + O_{a,\tilde{a},\theta,\varepsilon}\left(\lambda^3\right). \tag{6.11}$$

To remove the condition $\check{\theta} \geq 0$ on the Fourier transform of the cutoff, one may choose $\phi \in C^\infty_c(-C_0, C_0)$ satisfying $\phi > 0$ on $\text{spt}(\theta)$ and $\check{\phi} \geq 0$. Then writing $\theta = g\phi$, $g \in C^\infty_c(-C_0, C_0)$

gives $\check{\theta} = \check{g} * \check{\phi}$ and

$$\left\| a^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\mathrm{tr}} = \int_{|\lambda'|<\lambda^{1/2}} d\lambda' \left|\check{g}\left(\lambda'\right)\right| \left\| a^H \check{\phi}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda - \lambda'\right)\right\|_{\mathrm{tr}} + \int_{|\lambda'|>\lambda^{1/2}} d\lambda' \left|\check{g}\left(\lambda'\right)\right| \left\| a^H \check{\phi}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda - \lambda'\right)\right\|_{\mathrm{tr}}$$

The second integral above is $O\left(\lambda^{-\infty}\right)$. The first integral is then estimated following the corresponding estimate (6.11) for $\check{\phi}$. To remove the condition on sptθ, we may write a function of arbitrary support as a sum of translates $\theta_c(s) = \theta(s-c) \in C_c(\mathbb{R})$, $c \in \mathbb{R}$, of functions supported near zero. Then

$$\left\| a^H \check{\theta}_c\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\mathrm{tr}} = \left\| a^H e^{-ic\left(\sqrt{\Delta_{g^E,\mu}}-\lambda\right)} \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\mathrm{tr}} = \left\| a^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right\|_{\mathrm{tr}}$$

gives (6.11) for any arbitrary $\theta \in C_c^\infty(\mathbb{R})$.

We now come to estimating the trace (6.9) for arbitrary $\theta \in C_c^\infty(\mathbb{R})$. Splitting $\theta = \underbrace{\vartheta}_{\in C_c(-C_0,C_0)} + \underbrace{(\theta-\vartheta)}_{\in C_c\left(\mathbb{R}\setminus\left(-\frac{C_0}{2},\frac{C_0}{2}\right)\right)}$, with the trace tr $\check{\vartheta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)$ expanded as (6.9), we next estimate tr $\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)$. Under the assumption on L^E, we may find $\forall \varepsilon > 0$ a microlocal partition of unity $\left\{A_j = a_j^H\right\}_{j=-1}^N \in \Psi_{\mathrm{cl}}^{0,0}(X)$, $\left\{a_j\right\}_{j=-1}^N \in C_{\mathrm{inv}}^\infty\left(\left[\left(S^*X\right); S^*\Sigma\right]; [0,1]\right)$, $\sum_{j=-1}^N a_j = 1$, satisfying

$$\begin{aligned} \mathrm{spt} a_{-1} \cap SNS^*\Sigma &= \emptyset, \\ \mu_{\mathrm{Popp}}^{SNS^*\Sigma}\left(\mathrm{spt} a_0 \cap SNS^*\Sigma\right) &\leq \varepsilon, \\ \left[\cup_{t\in \mathrm{spt}(\theta-\vartheta)} e^{tH_d}\left(\mathrm{spt} a_j\right)\right] \cap \left(\mathrm{spt} a_j\right) &= \emptyset, \quad 1 \leq j \leq N. \end{aligned} \tag{6.12}$$

The estimate (6.11) gives

$$\left|\mathrm{tr}\ \left(a_{-1}^H + a_0^H\right)\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right| \leq \varepsilon\left(1+\varepsilon\right)\lambda^4 + O_{\theta,\varepsilon}\left(\lambda^3\right), \ \forall \varepsilon > 0. \tag{6.13}$$

Furthermore, choosing $\tilde{a}_j \in C_{\mathrm{inv}}^\infty\left(\left[\left(S^*X\right); S^*\Sigma\right]; [0,1]\right)$, $1 \leq j \leq N$, with

$$\begin{aligned} \mathrm{spt}\tilde{a}_j \cap \mathrm{spt} a_j &= \emptyset \\ \tilde{a}_j = 1 \text{ on } &\left[\cup_{t\in \mathrm{spt}(\theta-\vartheta)} e^{tH_d}\left(\mathrm{spt} a_j\right)\right] \end{aligned} \tag{6.14}$$

gives

$$\begin{aligned} \mathrm{tr}\ \left[a_j^H\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right] &= \mathrm{tr}\ \left[a_j^H\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}_j^H\right] \\ &\quad + \mathrm{tr}\ \left[a_j^H\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\left(1 - \tilde{a}_j^H\right)\right] \\ &= \mathrm{tr}\ \left[\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}_j^H a_j^H\right] \\ &\quad + \mathrm{tr}\ \left[a_j^H\left(\check{\theta} - \check{\vartheta}\right)\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\left(1 - \tilde{a}_j^H\right)\right] \\ &= O\left(\lambda^{-\infty}\right) \end{aligned} \tag{6.15}$$

following a similar Egorov argument as in (6.10) and (6.14). Thus finally combining (6.8), (6.13) and (6.15) we have

$$\mathrm{tr}\ \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right) = \lambda^4 \frac{\theta(0)}{24\pi}\mu_{\mathrm{Popp}} + o\left(\lambda^4\right)$$

for any $\theta \in C_c^\infty(\mathbb{R})$, under the assumption on the closed integral curves of L^E. Following the above the usual Tauberian argument of continues to prove (1.3); cf. [19, Sec. 2] or [18, Ch. 11]. □

Next we prove the large time Poisson relation (1.4).

Proof of (1.4). We shall infact prove the stronger statement

$$\begin{aligned}\text{sing spt}\left(\text{tr } e^{it\sqrt{\Delta_{g^E,\mu}}}\right) &\subset \{0\} \cup \mathscr{L}_{\hat{Z}} \cup \mathscr{L}_{\text{normal}} \\ &\subset \{0\} \cup \left(-\infty, -T^E_{\text{abnormal}}\right] \cup \left[T^E_{\text{abnormal}}, \infty\right) \cup \mathscr{L}_{\text{normal}} \end{aligned} \tag{6.16}$$

with $\mathscr{L}_{\hat{Z}}$ as in the computation (4.35). Equivalently stated, the above (6.16) amounts to

$$\text{tr } \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right) = O\left(\lambda^{-\infty}\right)$$

for $\text{spt}(\theta) \Subset \mathbb{R} \setminus (\{0\} \cup \mathscr{L}_{\hat{Z}} \cup \mathscr{L}_{\text{normal}})$. We may then again choose a microlocal partition of unity $\left\{A_j = a_j^H\right\}_{j=1}^N \in \Psi^{0,0}_{\text{cl}}(X)$, $\{a_j\}_{j=1}^N \in C^\infty_{\text{inv}}([(S^*X); S^*\Sigma]; [0,1])$, $\sum_{j=1}^N a_j = 1$, satisfying

$$\left[\cup_{t \in \text{spt}(\theta)} e^{tH_d}(\text{spt} a_j)\right] \cap (\text{spt} a_j) = \emptyset, \quad 1 \le j \le N. \tag{6.17}$$

Furthermore, again choosing $\tilde{a}_j \in C^\infty_{\text{inv}}([(S^*X); S^*\Sigma]; [0,1])$, $1 \le j \le N$, with

$$\begin{aligned} \text{spt}\tilde{a}_j \cap \text{spt} a_j &= \emptyset \\ \tilde{a}_j = 1 \text{ on } &\left[\cup_{t \in \text{spt}(\theta)} e^{tH_d}(\text{spt} a_j)\right] \end{aligned} \tag{6.18}$$

gives

$$\begin{aligned}\text{tr }\left[a_j^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\right] &= \text{tr }\left[a_j^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}_j^H\right] + \text{tr }\left[a_j^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\left(1 - \tilde{a}_j^H\right)\right] \\ &= \text{tr }\left[\check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\tilde{a}_j^H a_j^H\right] + \text{tr }\left[a_j^H \check{\theta}\left(\sqrt{\Delta_{g^E,\mu}} - \lambda\right)\left(1 - \tilde{a}_j^H\right)\right] \\ &= O\left(\lambda^{-\infty}\right) \end{aligned} \tag{6.19}$$

following a similar Egorov argument as in (6.10) and (6.18). □

7. Quantum ergodicity

In this section we prove the quantum ergodicity theorem for the sR Laplacian Theorem 4. As usual (see for instance [50]), it is enough to establish a microlocal Weyl law

$$\begin{aligned} E(B) &:= \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \langle B\varphi_j, \varphi_j \rangle \\ &= \frac{1}{2} \int d\nu_{\text{Popp}} \left[b(x, a_g(x)) + b(x, -a_g(x))\right], \end{aligned} \tag{7.1}$$

and variance estimate

$$V(B) := \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \left|\langle [B - E(B)] \varphi_j, \varphi_j \rangle\right|^2 = 0, \tag{7.2}$$

given $B \in \Psi^0_{\text{cl}}(X)$, with $b = \sigma(B)$.

7.1. **Microlocal Weyl laws.** We begin with the microlocal Weyl law (7.1). The upcoming Lemma 25 in fact works more generally on any equiregular sR manifold; a more detailed discussion of it including some singular (non-equiregular) analysis will appear in [15].

We first prove a localization result for the heat kernel of the sR Laplacian on a general sR manifold X of dimension n. To state this, given point a $x \in X$ we choose a privileged coordinate chart contained inside the open ball $B_{\varrho}^{g^{TX}}(x) := \left\{x'|d^{g^{TX}}(x,x') < \varrho\right\}$; where g^{TX} denotes a fixed Riemannian metric on X and ϱ depends on x. Let $\chi \in C_c^{\infty}([-1,1];[0,1])$ with $\chi = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Choose a local orthonormal frame $U_1,\ldots,U_k$ for E and define

$$\begin{aligned}
\tilde{U}_j &= \hat{U}_j^{(-1)} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)\left(U_j - \hat{U}_j^{(-1)}\right), \quad \forall 1 \le j \le k,\\
\tilde{\mu} &= \hat{\mu} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)(\mu - \hat{\mu}),
\end{aligned}$$

to be the modified vector fields and volume on $\mathbb{R}^n$. Here $\hat{U}_j^{(-1)}$, $\hat{\mu}$ are the first terms in the homogeneous privileged coordinate expansions of U_j (2.9) and the volume μ (2.10) respectively. For ϱ sufficiently small, the $\tilde{U}_j$'s are linearly independent and bracket generating with degree of nonholonomy being $r(x)$. A formula similar to (2.16) now gives an sR Laplacian on $\mathbb{R}^n$ via $\tilde{\Delta}_{g,\mu} f = \sum_{j=1}^k \left[-\left(\tilde{U}_j\right)^2(f) + \tilde{U}_j(f)\left(\mathrm{div}_{\tilde{\mu}}\tilde{U}_j\right)\right]$. The operator $\tilde{\Delta}_{g,\mu}$ is again essentially self-adjoint with a resolvent that maps $\left(\tilde{\Delta}_{g,\mu} - z\right)^{-1} : H^s_{\mathrm{loc}} \to H^{s+\frac{1}{r}}_{\mathrm{loc}}$ (2.19) and has a well defined functional calculus. We now have the following localization lemma.

Lemma 24. *The heat kernel satisfies*

$$\left[e^{-t\Delta_{g^E,\mu}}(x,x')\right]_{\mu} = ct^{-2nr-1}e^{-\frac{d^E(x,x')^2}{4t}} \tag{7.3}$$

uniformly for $t \le 1$.

Moreover, there exists $\varrho_1(x) > 0$ such that

$$\left\|\left[e^{-t\Delta_{g^E,\mu}}\right]_{\mu}(.,x) - \left[e^{-t\tilde{\Delta}_{g,\mu}}\right]_{\tilde{\mu}}(.,0)\right\|_{C^k(X)} = C_{x,k}e^{-\frac{\varrho_1^2}{16t}} \tag{7.4}$$

have the same asymptotics for $d^E(x,x') \le \varrho_1$ as $t \to 0$.

Proof. Both claims follow from the finite propagation result Theorem 5 and the Fourier transformation formula

$$\begin{aligned}
\left[\Delta^q_{g^E,\mu}e^{-t\Delta_{g^E,\mu}}\right]_{\mu}(x,x') &= \frac{1}{2\pi}\int d\xi\,\left[e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\right]_{\mu} D_{\xi}^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}}\\
&= \frac{1}{2\pi}\int d\xi\,\left[e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\right]_{\mu}\chi\left(\frac{\xi}{\varrho_1}\right) D_{\xi}^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}}\\
&\quad + \frac{1}{2\pi}\int d\xi\,\left[e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\right]_{\mu}\left[1 - \chi\left(\frac{\xi}{\varrho_1}\right)\right] D_{\xi}^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}},
\end{aligned} \tag{7.5}$$

$\forall q \in \mathbb{N}_0$, $\varrho_1 > 0$. By finite propagation, the integral maybe restricted to $|\xi| \geq d^E(x,x')$. Now the integral estimate

$$\left|\int_{|\xi|\geq d^E(x,x')} e^{i\xi s} D_\xi^{2q} e^{-\frac{\xi^2}{4t}} d\xi\right| \leq c t^{-2q-\frac{1}{2}} e^{-\frac{d^E(x,x')^2}{4t}}$$

gives the bound on $\left\|\Delta^q_{g,\mu} e^{-t\Delta_{g^E,F,\mu}}\right\|_{L^2\to L^2} \leq c t^{-2q-\frac{1}{2}} e^{-\frac{d^E(x,x')^2}{8t}}$. This combined with the subelliptic estimate (2.19) gives (7.3). For (7.4), note that the second summand of (7.5) is exponentially decaying $O\left(\exp(-\frac{\varrho_1^2}{16t})\right)$. Next for ϱ_1 sufficiently small, $B^{g^E}_{\varrho_1}(x) \subset B^{g^{TX}}_{\varrho}(x)$. Thus finite propagation and $\Delta_{g^E,\mu} = \tilde{\Delta}_{g^E,\mu}$ on $B^{g^E}_{\varrho_1}(x)$ give that the corresponding first summands for $\Delta_{g^E,\mu}$, $\tilde{\Delta}_{g^E,\mu}$ agree for $d^E(x,x') \leq \varrho_1$. ☐

We now prove the microlocal Weyl law in the equiregular case; below let $\left[e^{-\hat{\Delta}_{g,\mu}}\right]_{\hat{\mu},x}$ we denote the heat kernel of the Laplacian on the nilpotentization (2.11) at a point x. Denote by

$$\int_{(E_r/E_{r-1})} dy\, e^{iy.\xi} \left[e^{-\hat{\Delta}_{g,\mu}}\right]_{\hat{\mu}} (0;y) \tag{7.6}$$

its partial Fourier transform in (E_r/E_{r-1}) variables and evaluation at 0 in the remaining $(E_1)\oplus(E_2/E_1)\oplus\ldots\oplus(E_{r-1}/E_{r-2})$ variables. We now have the following.

Theorem 25. *Let $\left(X,E,g^E\right)$ be an equiregular sR manifold. For $B\in\Psi^0_{cl}(X,\Sigma)$ with $\sigma(B) = b_0$ we have*

$$E(B) = \frac{1}{(2\pi)^{k_r^E}\,\mathcal{P}} \int_{E^\perp_{r-1}} d\hat{\mu} d\xi\;\, b(x,\xi)|_{E^\perp_{r-1}} \left\{\int_{(E_r/E_{r-1})} dy\, e^{iy.\xi}\left[e^{-\hat{\Delta}_{g,\mu}}\right]_{\hat{\mu},x}(0;y)\right\}, \quad \textit{with} \tag{7.7}$$

$$\mathcal{P} := \int_X \left[e^{-\hat{\Delta}_{g,\mu}}\right]_{\hat{\mu}}(0,0)\, d\hat{\mu}. \tag{7.8}$$

Here the fiber $d\xi$-integral on the annihilator $E^\perp_{r-1} = (E_r/E_{r-1})^*$ is with respect to the canonical volume elements (2.6)*.*

Proof. By a standard Tauberian argument, it suffices to prove that one has an on diagonal asymptotic expansion for the heat kernel

$$\left[Be^{-t\Delta_{g^E,\mu}}\right]_\mu (x,x) = t^{-Q/2}\left[\sum_{j=0}^N b_j(x)\, t^j + O\left(t^{N+1}\right)\right] \tag{7.9}$$

that is uniform in $x\in X$ with leading term

$$b_0 = \frac{1}{(2\pi)^{k_r^E}} \int_{E^\perp_{r-1}} d\mu_{\text{Popp}} d\xi\, a(x,\xi)\left\{\int_{(E_r/E_{r-1})} dy\, e^{iy.\xi}\left[e^{-\hat{\Delta}_{g,\mu}}\right]_{\hat{\mu},x}(0;y)\right\}. \tag{7.10}$$

First consider the case $B=1$. By Lemma 24, it suffices to demonstrate the expansion for the localized kernel $\left[e^{-t\tilde{\Delta}_{g,\mu}}\right]_{\tilde{\mu}}(0,0)$ on $\mathbb{R}^n$. To this end, consider the rescaled sR-Laplacian and measure

$$\tilde{\Delta}^\varepsilon_{g^E,\mu} := \varepsilon^2 (\delta_\varepsilon)_* \tilde{\Delta}_{g^E,\mu}$$
$$\mu_\varepsilon := \varepsilon^{Q(x)} (\delta_\varepsilon)_* \tilde{\mu}$$

using the privileged coordinate dilation from Section 2 . It is now clear that the Schwartz kernels satisfy the relation

$$\left[e^{-t\tilde{\Delta}^{\varepsilon}_{g,\mu}}\right]_{\mu_{\varepsilon}}(x',x) \;=\; \varepsilon^{Q(x)}\left[e^{-t\varepsilon^{2}\tilde{\Delta}_{g,\mu}}\right]_{\tilde{\mu}}(\delta_{\varepsilon}x',\delta_{\varepsilon}x)\,. \tag{7.11}$$

Rearranging and setting $x=x'=0$, $t=1$; gives

$$\varepsilon^{-Q(x)}\left[e^{-\tilde{\Delta}^{\varepsilon}_{g,\mu}}\right]_{\mu_{\varepsilon}}(0,0)=\left[e^{-\varepsilon^{2}\tilde{\Delta}_{g,\mu}}\right]_{\tilde{\mu}}(0,0)$$

and hence it suffices to compute the expansion of the left hand side above as the dilation $\varepsilon\to0$. To this end, first note that the rescaled Laplacian has an expansion

$$\tilde{\Delta}^{\varepsilon}_{g,\mu}=\left(\sum_{j=0}^{N}\varepsilon^{j}\hat{\Delta}^{j}_{g,\mu}\right)+\varepsilon^{N+1}R^{\varepsilon}_{N},\quad\forall N\in\mathbb{N} \tag{7.12}$$

Here each $\hat{\Delta}^{j}_{g^{E},\mu}$ is an ε-independent second order differential operator of homogeneous $E-$order $j-2$. While each $R^{(N)}_{\varepsilon}$ is an ε-dependent second order differential operators on $\mathbb{R}^{n}$ of E-order at least $N-1$. The coefficient functions of $\hat{\Delta}^{(j)}_{g^{E},\mu}$ are polynomials (of degree at most $j+2r$) while those of $R^{(N)}_{\varepsilon}$ are uniformly (in ε) C^{∞}-bounded. The first term is a scalar operator given in terms of the nilpotent approximation

$$\hat{\Delta}^{0}_{g^{E},\mu}=\Delta_{\hat{g}^{E},\hat{\mu};x}=\sum_{j=1}^{m}\left(\hat{U}^{(-1)}_{j}\right)^{2}. \tag{7.13}$$

at the point x. This expansion (7.12) along with the subelliptic estimates now gives

$$\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}-z\right)^{-1}-\left(\hat{\Delta}^{0}_{g^{E},\mu}-z\right)^{-1}=O_{H^{s}_{\mathrm{loc}}\to H^{s+1/r-2}_{\mathrm{loc}}}\left(\varepsilon\left|\mathrm{Im}z\right|^{-2}\right),$$

$\forall s\in\mathbb{R}$. More generally, we let $I_{j}\coloneqq\{p=(p_{0},p_{1},\dots)\,|p_{\alpha}\in\mathbb{N},\sum p_{\alpha}=j\}$ denote the set of partitions of the integer j and define

$$\mathtt{C}^{z}_{j}\coloneqq\sum_{p\in I_{j}}\left(\hat{\Delta}^{0}_{g^{E},\mu}-z\right)^{-1}\left[\prod_{\alpha}\hat{\Delta}^{p_{\alpha}}_{g^{E},F,\mu}\left(\hat{\Delta}^{0}_{g^{E},\mu}-z\right)^{-1}\right]. \tag{7.14}$$

Then by repeated applications of the subelliptic estimate we have

$$\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}-z\right)^{-1}-\sum_{j=0}^{N}\varepsilon^{j}\mathtt{C}^{z}_{j}=O_{H^{s}_{\mathrm{loc}}\to H^{s+N(1/r-2)}_{\mathrm{loc}}}\left(\varepsilon^{N+1}\left|\mathrm{Im}z\right|^{-2Nw^{E}_{n}-2}\right),$$

$\forall s\in\mathbb{R}$. A similar expansion as (7.12) for the operator $\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}+1\right)^{M}\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}-z\right)$, $M\in\mathbb{N}$, also gives

(7.15)

$$\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}+1\right)^{-M}\left(\tilde{\Delta}^{\varepsilon}_{g^{E},\mu}-z\right)^{-1}-\sum_{j=0}^{N}\varepsilon^{j}\mathtt{C}^{z}_{j,M}=O_{H^{s}_{\mathrm{loc}}\to H^{s+N(1/r-2)+\frac{M}{r}}_{\mathrm{loc}}}\left(\varepsilon^{N+1}\left|\mathrm{Im}z\right|^{-2Nw^{E}_{n}-2}\right)$$

for operators $\mathtt{C}^{z}_{j,M}=O_{H^{s}_{\mathrm{loc}}\to H^{s+N(1/r-2)+\frac{M}{r}}_{\mathrm{loc}}}\left(\varepsilon^{N+1}\left|\mathrm{Im}z\right|^{-2Nw^{E}_{n}-2}\right)$, $j=0,\dots,N$, with

$$\mathtt{C}^{z}_{0,M}=\left(\hat{\Delta}^{0}_{g^{E},\mu}+1\right)^{-M}\left(\hat{\Delta}^{0}_{g^{E},\mu}-z\right)^{-1}.$$

For $M\gg0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels of (7.15) in $C^{0}\left(\mathbb{R}^{n}\times\mathbb{R}^{n}\right)$. By plugging the resolvent expansion into the

Helffer-Sjöstrand formula and noting $\mu_\varepsilon \sim \hat{\mu} + \sum_{j=1}^{\infty} \varepsilon^j \mu_j$ gives the diagonal heat kernel expansion

$$\left[e^{-t\Delta_{g^E,\mu}}\right]_\mu (x,x) = \sum_{j=0}^{\infty} e_j(x)\, t^{j/2} \quad \text{with}$$
$$e_0(x) = \left[e^{-\Delta_{\hat{g}^E,\hat{\mu}}}\right]_{\hat{\mu}}.$$

Finally, to see that the expansion only involves only even powers of $t^{1/2}$, note that the operators $\hat{\Delta}^j_{g^E,\mu}$ in the expansion (7.12) change sign by $(-1)^j$ under the rescaling δ_{-1}. The integral expression (7.14) corresponding to $\mathtt{C}^z_j(0,0)$ then changes sign by $(-1)^j$ under this change of variables giving $\mathtt{C}^z_j(0,0) = 0$ for j odd.

We now come to the expansion for general $B \in \Psi^0_{\mathrm{cl}}$. By a partition of unity and Lemma 24, we may assume B to be supported in the privileged coordinate chart. That is it has an integral representation $[B]_\mu(0,x) = \left[b^W\right]_\mu(0,x) = \frac{1}{(2\pi)^n}\int d\xi e^{-ix.\xi} b\left(\frac{1}{2}x,\xi\right)$ in the privileged coordinate chart with symbol b being compactly supported in x . Next letting

$$\delta_{t^{1/2}} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n,$$
$$\delta_{t^{1/2}}(x,\xi) := \left(\delta_{t^{1/2}}x, \delta_{t^{-1/2}}\xi\right) \tag{7.16}$$

denote the induced symplectic dilation of phase space, we note

$$\left(\delta_{t^{-1/2}}\right)_* b^W := \delta^*_{t^{1/2}} b^W \delta^*_{t^{-1/2}} = \left(\delta^*_{t^{1/2}} b\right)^W. \tag{7.17}$$

Furthermore; the classical symbolic expansion for $b \in S^0_{\mathrm{cl}}$ gives

$$\begin{aligned}\left(\delta^*_{t^{1/2}} b\right)(x;\xi) &= b\left(\delta_{t^{1/2}}x; \delta_{t^{-1/2}}\xi\right) = b\left(\delta_{t^{1/2}}x; t^{-w_1/2}\xi_1, \dots, t^{-w_n/2}\xi_n\right)\\ &= \underbrace{b_0\Bigg(0;0,0,\dots,0,\underbrace{\xi_{n-k_r+1},\dots,\xi_n}_{\xi'=}\Bigg)}_{\mathtt{b}_0:=} + O_{S^0_{\mathrm{cl}}}(t)\end{aligned} \tag{7.18}$$

We now finally compute

$$\begin{aligned}&\left[Be^{-t\tilde{\Delta}_{g,\mu}}\right]_\mu (0,0)\\ =&t^{-Q/2}\left[\left(\delta_{t^{-1/2}}\right)_*\left(Be^{-t\tilde{\Delta}_{g,\mu}}\right)\right]_{\mu_{t^{1/2}}}(0,0)\\ =&t^{-Q/2}\left[\left(\delta_{t^{-1/2}}\right)_* B \left(\delta_{t^{-1/2}}\right)_* e^{-t\tilde{\Delta}_{g,\mu}}\right]_{\mu_{t^{1/2}}}(0,0)\\ =&t^{-Q/2}\left[1+o(1)\right]\left[\mathtt{b}_0^W e^{-\hat{\Delta}^0_{g^E,\mu}}\right]_{\hat{\mu}}(0,0)\\ =&t^{-Q/2}\left[1+o(1)\right]\int e^{-iy'\xi'} b_0\left(0;0,\xi'\right) e^{-\Delta_{\hat{g}^E,\hat{\mu}}}\left(0,y';0\right)\end{aligned} \tag{7.19}$$

following (7.11), (7.17) and (7.18). The theorem now follows on noting leading term above to agree with (7.10) in privileged coordinates. □

The rescaling arguments in the proof above are also analogous to those in local index theory cf. [46, Sec. 7] or [45, 44] and references therein. A local Weyl law for the semiclassical (magnetic) analogue of sR Laplacian was also recently explored in [33, Sec. 3].

One still needs to identify the right hand side of (7.7) with (7.1) in the 4D quasi-contact case. First note that (7.7), (7.8) are unchanged on replacing $\hat{\mu}$ by μ_{Popp}. A model for the nilpotentization is given in terms of the Darboux coordinates of 4.1 this case is $\hat{X} = \mathbb{R}^4$ with $\hat{E} = \mathbb{R}\left[\partial_{x_0}, \partial_{x_1} + x_2\partial_{x_3}, \partial_{x_2} - x_1\partial_{x_3}\right]$ being the span of the given (orthonormal) vector fields. The partial Fourier transform in x_3 of the nilpotent Laplacian is computed

$$\mathcal{F}_{x_3}\hat{\Delta}^0_{g,\mu}\mathcal{F}^{-1}_{x_3} = -\partial^2_{x_0} \underbrace{- \left(\partial_{x_1} + ix_2\xi_3\right)^2 - \left(\partial_{x_2} - ix_1\xi_3\right)^2}_{=\hat{\Delta}_{\xi_3}} \tag{7.20}$$

while the Popp volume $\mu_{\text{Popp}} = \frac{1}{2}dx$ is Euclidean. Mehler's formula ([5] Sec. 4.2) now gives the partial Fourier transform (7.6) of the heat kernel to be

$$\begin{aligned}\int dx_3\, e^{-ix_3.\xi_3}\left[e^{-\hat{\Delta}^0_{g,\mu_{\text{Popp}}}}\right]_{\mu_{\text{Popp}}}(0,0) &= \left[\exp - \left\{\partial^2_{x_0} + \left(\partial_{x_1} + ix_2\xi_3\right)^2 + \left(\partial_{x_2} - ix_1\xi_3\right)^2\right\}\right]_{\frac{1}{2}dx}(0,0)\\ &= \frac{1}{4\pi^{3/2}}\frac{|2\xi_3|}{\sinh|2\xi_3|}, \qquad \text{while}\end{aligned}$$

$$f\left(\hat{\Delta}_{\xi_3}\right) = \left\langle f(s), \frac{|\xi_3|}{\pi}\sum_{k=0}^{\infty}\delta\left(s - 2|\xi_3|(2k+1)\right)\right\rangle \tag{7.21}$$

cf. [46, Sec. 7.].

We then calculate

$$\mathcal{P} = \int d\mu_{\text{Popp}}\left[e^{-\hat{\Delta}^0_{g,\mu_{\text{Popp}}}}\right]_{\mu_{\text{Popp}}}(0,0) = \frac{1}{2\pi}\int d\mu_{\text{Popp}}\left\{\int_{-\infty}^{\infty} d\xi_3\,\frac{1}{4\pi^{3/2}}\frac{|2\xi_3|}{\sinh|2\xi_3|}\right\} = \frac{1}{32\sqrt{\pi}}P(X)$$

while the leading term of (7.7) is

$$\begin{aligned}E(B) =& \frac{1}{2\pi\mathcal{P}}\int d\mu_{\text{Popp}}\left[b\left(x, a_g(x)\right) + b\left(x, -a_g(x)\right)\right]\left\{\int_0^{\infty} d\xi_3\,\frac{1}{8\pi^{3/2}}\frac{|2\xi_3|}{\sinh|2\xi_3|}\right\}\\ =& \frac{1}{64\sqrt{\pi}\mathcal{P}}\int d\mu_{\text{Popp}}\left[b\left(x, a_g(x)\right) + b\left(x, -a_g(x)\right)\right]\\ =& \frac{1}{2}\int d\nu_{\text{Popp}}\left[b\left(x, a_g(x)\right) + b\left(x, -a_g(x)\right)\right]\end{aligned}$$

proving (7.1).

The expression above may be rewritten

$$\begin{aligned}E(B) &= \int b|_{S^*\Sigma}\,\nu_{\text{Popp}}\\ &= \int \pi_S^*\, b|_{S^*\Sigma}\,\nu^{SNS^*\Sigma}_{\text{Popp}}\end{aligned}$$

in terms of the lifts of the normalized Popp volume to the unit sphere of the characteristic variety and its blowup (2.37). We now generalize the above expression to prove a microlocal Weyl law in $\Psi^{0,0}_{\text{cl}}(X,\Sigma)$, via the heat kernel method here, agreeing with Theorem 23.

Theorem 26. *For X quasi-contact and $B \in \Psi^{0,0}_{cl}(X,\Sigma)$, $\sigma^H(B) = b_0$ we have*

$$E(B) = \int b|_{SNS^*\Sigma}\,\nu^{SNS^*\Sigma}_{Popp}. \tag{7.22}$$

Proof. Since B is microlocally a classical pseudo-differential operator 15 away from the characteristic variety, where the microlocal Weyl measure (7.1) vanishes, it suffices to prove (7.22) for B micro-supported near Σ. In particular we may work in the microlocal chart C where (4.15) holds. Note that in the quasi-contact case, the Darboux coordinates (4.1), (4.2) used in the normal form for $\Delta_{g^E,\mu}$ and thereafter used in the definition 15 of $\Psi_{\mathrm{cl}}^{0,0}(X,\Sigma)$ are in particular privileged. Furthermore, the privileged dilation of phase space (7.16) extends to the blow up

$$\begin{aligned}\delta_{t^{1/2}} : \left[T^*\mathbb{R}^4, \Sigma_0\right] &\rightarrow \left[T^*\mathbb{R}^4, \Sigma_0\right] \\ \delta_{t^{1/2}} :=& \beta^{-1}\delta_{t^{1/2}}\beta\end{aligned}$$

and one has the relation

$$(7.23)\qquad \left(\delta_{t^{-1/2}}\right)_* b^H = \delta^*_{t^{1/2}} b^H \delta^*_{t^{-1/2}} = \left(\delta^*_{t^{1/2}} b\right)^H,$$

$\forall b \in S_{\mathrm{cl}}^{0,0}$ similar to (7.17). Furthermore; the classical symbolic expansion for $b \in S_{\mathrm{cl}}^{0,0}$ gives

$$(7.24)\qquad \delta^*_{t^{1/2}} b = \underbrace{b_0\left(0, d^{-2}\left(x_1^2+\hat{\xi}_1^2\right), 0, 0; d^{-1}\hat{\xi}_0, 0\right)}_{\mathrm{b}_0:=} + O_{S_{\mathrm{cl}}^{0,0}}(t).$$

The equation (4.4) and Duhamel's principle give

$$(7.25)\qquad U_t := \left(\delta_{t^{-1/2}}\right)_* U = \delta^*_{t^{1/2}} U \delta^*_{t^{-1/2}} = e^{i\frac{\pi}{4}f_0^W} + O_{L^2_{\mathrm{loc}}\rightarrow H^{-1}_{\mathrm{loc}}}(t)$$

for the diagonalizing FIO in 9, while (4.5), (7.20) gives

$$(7.26)\qquad e^{i\frac{\pi}{4}f_0^W} e^{-\hat{\Delta}^0_{g^E,\mu}} e^{-i\frac{\pi}{4}f_0^W} = \left[e^{-\xi_0^2+2\rho\xi_3\left(x_1^2+\hat{\xi}_1^2\right)}\right]^H.$$

We may then compute

$$
\begin{aligned}
&\left[U^{*}BUe^{-t\Delta_{g^{E},\mu}}\right]_{\mu}(0,0)\\
&=t^{-5/2}\left[\left(\delta_{t^{-1/2}}\right)_{*}U^{*}BUe^{-t\Delta_{g^{E},\mu}}\right]_{\mu_{t^{1/2}}}(0,0)\\
&=t^{-5/2}\left[U_{t}^{*}B_{t}U_{t}\left(\delta_{t^{-1/2}}\right)_{*}e^{-t\Delta_{g^{E},\mu}}\right]_{\mu_{t^{1/2}}}(0,0)\\
&=t^{-5/2}\left[1+o\left(1\right)\right]\left[e^{-i\frac{\pi}{4}f_{0}^{W}}\mathsf{b}_{0}^{H}e^{i\frac{\pi}{4}f_{0}^{W}}e^{-\hat{\Delta}_{g^{E},\mu}^{0}}\right]_{dx}(0,0)\\
&=\frac{t^{-5/2}}{2\pi}\left[1+o\left(1\right)\right]\left[e^{-i\frac{\pi}{4}f_{0}^{W}}\sum_{k=0}^{\infty}H_{k}^{*}.\right.\\
&\qquad\left.\left.\cdot\left\{\int d\xi_{0}e^{-\xi_{0}^{2}+2\rho(2k+1)}b_{0}\left(d_{k}^{-1}\hat{\xi}_{0}\right)\right\}H_{k}e^{i\frac{\pi}{4}f_{0}^{W}}\right]_{dx}\right|_{x_{1}=x_{2}=x_{3}=0}\\
&=\frac{t^{-5/2}}{2\pi}\left[1+o\left(1\right)\right]\left[e^{-i\frac{\pi}{4}f_{0}^{W}}\sum_{k=0}^{\infty}H_{k}^{*}\right.\\
&\quad\left.\left.\cdot\left\{\int_{-1}^{1}d\Xi_{0}\frac{\rho^{1/2}\left(2k+1\right)^{1/2}}{\left(1-\Xi_{0}^{2}\right)^{3/2}}e^{-\rho(2k+1)/\left(1-\Xi_{0}^{2}\right)}b_{0}\left(\Xi_{0}\right)\right\}H_{k}e^{i\frac{\pi}{4}f_{0}^{W}}\right]_{dx}\right|_{x_{1}=x_{2}=x_{3}=0}\\
&=\frac{t^{-5/2}}{8\pi^{2}}\left[1+o\left(1\right)\right]\int d\Xi_{0}\int_{0}^{\infty}d\xi_{3}b_{0}\left(\Xi_{0}\right)\left(1-\Xi_{0}^{2}\right)^{-1}\left[\hat{\Delta}_{\hat{\rho}\xi_{3}/\left(1-\Xi_{0}^{2}\right)}^{1/2}e^{-\hat{\Delta}_{\hat{\rho}\xi_{3}/\left(1-\Xi_{0}^{2}\right)}}\right]_{dx}(0,0)\\
&=\frac{t^{-5/2}}{8\pi^{2}}\left[1+o\left(1\right)\right]\left[\int_{-1}^{1}d\Xi_{0}b_{0}\left(\Xi_{0}\right)\int_{0}^{\infty}d\xi_{3}\sum_{k=0}^{\infty}\hat{\rho}\xi_{3}\frac{\rho^{1/2}\left(2k+1\right)^{1/2}}{\left(1-\Xi_{0}^{2}\right)^{3/2}}e^{-\rho(2k+1)/\left(1-\Xi_{0}^{2}\right)}\right]\\
&=\frac{t^{-5/2}}{8\pi^{2}}\left[1+o\left(1\right)\right]\left[\left(\int_{0}^{\infty}r^{3/2}e^{-r}dr\right)\left(\sum_{k=0}^{\infty}\frac{1}{\left(2k+1\right)^{2}}\right)\int_{-1}^{1}d\Xi_{0}\left(1-\Xi_{0}^{2}\right)b_{0}\left(\Xi_{0}\right)\right]\\
&=\frac{3t^{-5/2}}{256\sqrt{\pi}}\left[1+o\left(1\right)\right]\int_{\pi^{-1}(x)}b|_{SNS^{*}\Sigma}\,\mu_{\text{Popp}}^{SNS^{*}\Sigma}
\end{aligned}
$$

following (7.21), (7.23), (7.24) and (7.25) and on identifying the right hand side of (7.22) here in terms of privileged coordinates. □

7.2. **Variance estimate.** We now prove the variance estimate (7.2), specializing again to the 4D quasi-contact case. By replacing $B\in\Psi_{\text{cl}}^{0}\left(X\right)$ by the operator $B-E\left(B\right)\in\Psi_{\text{cl}}^{0}\left(X\right)$ with the same variance, we may assume that $E\left(B\right)=0$. Furthermore we have

$$V\left(B_{1}\right)\leq E\left(B_{1}^{*}B_{1}\right)\tag{7.27}$$

$$V\left(B_{1}+B_{2}\right)\leq2\left[V\left(B_{1}\right)+V\left(B_{2}\right)\right]\tag{7.28}$$

$$V\left(B_{1}\right)=0\implies V\left(B_{1}+B_{2}\right)=V\left(B_{2}\right),\tag{7.29}$$

$\forall B_{1},B_{2}\in\Psi_{\text{cl}}^{0}\left(X\right)$, (see [17, Lemma 4.1]) and

$$\left.\sigma\left(B_{1}\right)\right|_{\Sigma}=0\implies E\left(B_{1}^{*}B_{1}\right)=0\tag{7.30}$$

by Lemma 25 giving

$$\left.\sigma\left(B_{1}\right)\right|_{\Sigma}=\left.\sigma\left(B_{2}\right)\right|_{\Sigma}\implies V\left(B_{1}\right)=V\left(B_{2}\right).\tag{7.31}$$

From (7.28), (7.31) and a partition of unity it now suffices to prove (7.2) for $B\in\Psi_{\text{cl}}^{0}\left(X\right)$ micro-supported in a conic neighborhood of $p\in\Sigma$ with $E\left(B\right)=0$. By a Taylor expansion in the coordinates $(x_0, x_1, x_2, \hat{x}_3; \xi_0, \xi_1, \xi_2, \xi_3)$ of the normal form (4.15), we may write $B = B_0 + B_1$ where $B_0, B_1 \in \Psi^0_{\rm cl}(X)$ with $\sigma(B_1)|_\Sigma = 0$ and $B_0 = [b_0(x_0, x_2, \hat{x}_3, \xi_2, \xi_3)]^W \in \Psi^0_{\rm cl}(X)$. Furthermore, the homogeneous symbol $\sigma(B_0) = \pi^* b_0$, $b_0 \in C^\infty(X)$, is the pull back of a function from the base. From (7.27) and (7.30) we have $V(B_1) = 0$ and it suffices to show $V(B_0) = 0$ by (7.28). Clearly $E(B_0) = E(B) = 0$ by (7.31) and $[B_0, \Omega] = 0$ (3.24) showing

$$B_0 \in \Psi^0_{\rm inv,cl}(X) \subset \Psi^{0,0}_{\rm cl}(X;\Sigma).$$

Next $V(B_0) = V\left(e^{-it\sqrt{\Delta_{g^E,\mu}}} B_0 e^{it\sqrt{\Delta_{g^E,\mu}}}\right)$ by definition, while Egorov's theorem Theorem 20 gives

$$D := e^{-it\sqrt{\Delta_{g^E,\mu}}} B_0 e^{it\sqrt{\Delta_{g^E,\mu}}} - \mathrm{Op}^H\left[\underbrace{\left(e^{-tH_d}\right)^* \sigma_{0,0}(B_0)}_{=:b_t}\right] \in \Psi^{-1,1}_{\rm cl}(X,\Sigma).$$

In particular the difference above $D : L^2(X) \to H^{1,-1}(X,\Sigma) \hookrightarrow H^{\frac{1}{2},0}(X,\Sigma) \hookrightarrow L^2(X)$ (5.20) being a compact operator, its variance vanishes $V(D) = 0$ [17, Lemma 4.2]. Hence $V(B_0) = V\left(\mathrm{Op}^H[b_t]\right)$, $\forall t$, and also

$$\begin{aligned} V(B_0) &= V\left(\underbrace{\mathrm{Op}^H\left[\underbrace{\frac{1}{T}\int_0^T dt\, b_t}_{=:b_T}\right]}_{=:\bar{B}_T}\right) \\ &\leq E\left(\bar{B}_T^* \bar{B}_T\right) = \int |b_T|_{SNS^*\Sigma}|^2 \nu^{SNS^*\Sigma}_{\rm Popp}, \end{aligned} \tag{7.32}$$

$\forall T > 0$, by (7.22) and (7.27). Finally $b_T|_{SNS^*\Sigma} \to 0$ in $L^2\left(SNS^*\Sigma; \nu^{SNS^*\Sigma}_{\rm Popp}\right)$ as $T \to \infty$ under the ergodicity assumption on $\hat{Z}$ by the von Neumann mean ergodic theorem to prove the first part of Theorem 4.

Next to prove the second part of Theorem 4, suppose that L^E is ergodic and $L_Z \mu_{\rm Popp} = 0$. From the equivalent conditions (2.31) and the computation (4.21) it follows that the function Ξ_0 is now preserved under the $\hat{Z}$-flow. Furthermore, the level sets $(SNS^*\Sigma/S^1)_c := \{\Xi_0 = c\}$, $c \in (-1, 1)$, are copies of X with the $\hat{Z}$-flow $\left(\hat{Z} b_t\right)\Big|_{(SNS^*\Sigma/S^1)_c} = (\pi_S \circ \pi)^* cZb_{t,c}$ being simply lifted from the base

$$b_t|_{(SNS^*\Sigma)_c} = (\pi_S \circ \pi)^* \left(e^{-tcZ}\right)^* b_0.$$

Setting, $(b_0)_T := \frac{1}{T}\int_0^T dt\, \left(e^{-tZ}\right)^* b_0$, we may then compute

$$\begin{aligned} \int |b_T|_{SNS^*\Sigma}|^2 \nu^{SNS^*\Sigma}_{\rm Popp} &= \int_{-1}^1 \left(1 - c^2\right) dc \int \nu_{\rm Popp} |(b_0)_{cT}|^2 \\ &= \frac{1}{T}\int_{-T}^T \left(1 - \frac{c'^2}{T^2}\right) dc' \int \nu_{\rm Popp} |(b_0)_{c'}|^2. \end{aligned} \tag{7.33}$$

As noted before the ergodicity assumption on L^E is equivalent to the ergodicity of the vector field $Z \in C^\infty\left(L^E\right)$. Since $E(B_0) = \int b_0 \nu_{\rm Popp} = 0$, the von Neumann mean ergodic theorem applied to e^{tZ} gives $\int \nu_{\rm Popp} |(b_0)_T|^2$ and hence (7.33) converges to zero as $T \to \infty$.

We finally remark that the ergodicity of L^E alone , which is a topological condition, does not suffice to prove the variance estimate and hence quantum ergodicity in the general volume preserving case. In this case, following the computation (4.21) and (4.33), functions of the form $f\left(\left(1-\Xi_0^2\right)/\hat{\rho}_Z\right)$ are seen to be invariant under the $\hat{Z}$ flow. The last line of (7.32) now converging to the projection of b_0 onto the $\hat{Z}$ invariant functions, such a projection $\int \nu_{\text{Popp}}^{SNS^*\Sigma} f\left(\left(1-\Xi_0^2\right)/\hat{\rho}_Z\right)\left(\beta^*\pi^* b_0\right)$ of the symbol b_0 , $\int b_0 \mu_{\text{Popp}} = 0$, might be non-zero unless $\hat{\rho}_Z = 1$.

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