WS 2017/2018 Dr. Milena Pabiniak

From Calculus to Cohomology

Homework 7

Exercise 1. Prove that a vector field X on M is smooth if and only if given any $f \in \mathcal{C}^\infty(M)$ the function Xf defined by $Xf(p) = X_{|p}(f)$ is smooth.

Exercise 2.
Let $\{(U_\alpha, \varphi_\alpha)\}$ be a differentiable structure on M and consider the maps

$$\Phi_\alpha \colon U_\alpha \times \mathbb{R}^n \longrightarrow TM$$
$$(x, v) \longmapsto (d\varphi_\alpha)_x(v) \in T_{\varphi_\alpha(x)}M.$$

(a) Show that the family $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ is a smooth atlas.

(b) Conclude that TM carries the structure of a differentiable manifold. What dimension has TM?

(c) If $f \colon M \to N$ is differentiable, then $df \colon TM \to TN$ is also differentiable.

Exercise 3.
Exercise 9.10 (orthoginal group is a Lie group) from the book.

Exercise 4.
Recall that given a Riemannian metric $\langle , \rangle$ on M we can associate to any $f \in \mathcal{C}^\infty(M)$ a smooth vector field $grad\, f$ defined by

$$\forall_{p\in M} \forall_{v \in T_pM} \langle grad\, f_{|p}, v\rangle = df_{|p}(v).$$

Prove that if a is a regular value of f then $grad\, f$ is normal to $f^{-1}(a)$, i.e. $\langle grad\, f_{|p}, v\rangle = 0$ for all $p \in f^{-1}(a)$, $v \in T_pf^{-1}(a)$.
For more about $grad\, f$ you may want to look at Ex. 9.14 from the book.

Suggested Exercise 1. Exercises 8.5 and 9.13 (Klein bottle).

These exercises are to be discussed on Thursday December 7th.