WS 2017/2018 Dr. Milena Pabiniak

From Calculus to Cohomology

Homework 8

Exercise 1. As we already know, if M is a manifold then T^*M is also a manifold: if $\phi\colon U \to U' \subset \mathbb{R}^n$ a chart on M, then for any 1-form ξ on M its restriction to U can be expressed as $\xi = \sum \xi_j d\phi_j$ for some smooth functions $\xi_j\colon U \to \mathbb{R}$ and one can use

$$T^*U \ni (p, \xi) \mapsto (\phi_1(p), \dots, \phi_n(p), \xi_1(p), \dots, \xi_n(p)) \in U' \times \mathbb{R}^n$$

as a chart $T^*U \to U' \times \mathbb{R}^n$ for T^*M. By a slight abuse of notation, we will still use the symbol ϕ_j for a function $\phi_j \circ \pi$ on T^*U, with $\pi\colon T^*U \to U$ the projection onto U. Define a 1-form α on T^*M locally by the formula

$$\alpha = \sum \xi_j d\phi_j,$$

valid on T^*U.

Show that α is well defined, i.e. the definition does not depend on the choice of the chart.

Exercise 2. Here is a coordinate-free definition: for any $(p, \xi) \in T^*M$ and any $v \in T_{(p,\xi)}(T^*M)$ let

$$\alpha_{(p,\xi)}(v) = \xi_p(D_{(p,\xi)}\pi(v))$$

where $D_{(p,\xi)}\pi$ is the derivative of the projection π at the point (p, ξ).

Show that this really describes the 1 form α from the previous exercise.

Exercise 3. Show that given any 1 form μ on M (so that in particular μ is a smooth function $\mu\colon M \to T^*M$) if you use the function μ to pull back the 1 form α from previous exercises, you get μ back, i.e.

$$\mu^*(\alpha) = \mu.$$

Story behind these exercises: A 2 form $\omega \in \Omega^2(M)$ is called a *symplectic form* if it is closed and non-degenerate, i.e. for any $p \in M$ and any non-zero $v \in T_pM$ there exists $w \in T_pM$ such that $\omega(v, w) \neq 0$. The cotangent bundle of a manifold, with $\omega = -d\alpha$ for the above α, is a classical example of a manifold with a symplectic form. This example appeared naturally in physics and gave rise to Symplectic Geometry, i.e. a study of manifolds with symplectic forms.

Suggested Exercise 1. Exercises 9.15 and 9.16 from the book.

These exercises are to be discussed on Thursday December 14th.