LOWER BOUNDS FOR GROMOV WIDTH OF THE $U(n)$-COADJOINT ORBITS.

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Abstract. We use the Gelfand-Tsetlin pattern to construct an effective Hamiltonian, completely integrable action of a torus T on an open dense subset of a coadjoint orbit of the unitary group. We then identify a proper Hamiltonian T-manifold centered around a point in the dual of the Lie algebra of T. A theorem of Karshon and Tolman says that such a manifold is equivariantly symplectomorphic to a particular subset of $\mathbb{R}^{2D}$. This fact enables us to construct symplectic embeddings of balls into certain coadjoint orbits of the unitary group, and therefore obtain a lower bound for their Gromov width. Using the identification of the dual of the Lie algebra of the unitary group with the space of $n \times n$ Hermitian matrices, the main theorem states that for a coadjoint orbit through $\lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ in the dual of the Lie algebra of the unitary group, where at most one eigenvalue is repeated, the lower bound for Gromov width is equal to the minimum of the differences $\lambda_i - \lambda_j$, over all $\lambda_i > \lambda_j$. For a generic orbit (i.e. with distinct λ_i's), with additional integrality conditions, this minimum has been proved to be exactly the Gromov width of the orbit. For nongeneric orbits this lower bound is new.

1. Introduction

In 1985 Mikhail Gromov proved the nonsqueezing theorem which is one of the foundational results in the modern theory of symplectic invariants. The theorem says that a ball $B^{2N}(r)$ of radius r, in a symplectic vector space $\mathbb{R}^{2N}$ with the usual symplectic structure, cannot be symplectically embedded into $B^2(R) \times \mathbb{R}^{2N-2}$ unless $r \leq R$. This motivated the definition of the invariant called the Gromov width. Consider the ball of capacity a

$$B^{2N}_a = \Big\{ z \in \mathbb{C}^N \;\Big|\; \pi \sum_{i=1}^{N} |z_i|^2 < a \Big\},$$

with the standard symplectic form $\omega_{std} = \sum dx_j \wedge dy_j$. The **Gromov width** of a $2N$-dimensional symplectic manifold (M, ω) is the supremum of the set of a's such that B^{2N}_a can be symplectically embedded in (M, ω).

In this work we focus on the Gromov width of coadjoint orbits of Lie groups. A Lie group G acts on itself by conjugation

$$G \ni g : G \to G, \quad g(h) = ghg^{-1}.$$

Derivative at the identity element gives the action of G on its Lie algebra $\mathfrak{g}$, called adjoint action. This induces the action of G on $\mathfrak{g}^*$, the dual of its Lie algebra, called the coadjoint action. Each orbit $\mathcal{O}$ of the coadjoint action is naturally equipped with the Kostant-Kirillov symplectic form:

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle, \quad \xi \in \mathfrak{g}^*,\ X, Y \in \mathfrak{g}.$$

For example, when $G = U(n)$ the group of (complex) unitary matrices, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. With this identification, the coadjoint action of G on an orbit $\mathcal{O}$ is simply action by conjugation. It is Hamiltonian, and the momentum map is just inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

Choose a maximal torus $T \subset G$ and a positive Weyl chamber $\mathfrak{t}^*_+$. Every coadjoint orbit intersects the positive Weyl chamber in a single point. Therefore there is a bijection between the coadjoint orbits and points in the positive Weyl chamber. Points in the interior of the positive Weyl chamber are called **regular** points.

In this paper we consider coadjoint orbits of $U(n)$. Multiplying by a factor of i, we can identify the Lie algebra $\mathfrak{u}(n)$ with the space of Hermitian matrices. The pairing in $\mathfrak{u}(n)$

$$(A, B) = \text{trace}(AB)$$

gives us the identification of $\mathfrak{u}^*(n)$ with $\mathfrak{u}(n)$. From now on, we will identify $\mathfrak{u}^*(n)$ with the space of Hermitian matrices.

Given a Hamiltonian torus action one can construct embeddings of balls using the information from the moment polytope. Using this technique we prove the following theorem.

Theorem 1.1. *Consider the $U(n)$ coadjoint orbit $M := \mathcal{O}_\lambda$ in $\mathfrak{u}(n)^*$ through a point diag $(\lambda_1, \lambda_2, \dots, \lambda_n)$ where*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_l = \lambda_{l+1} = \ldots = \lambda_{l+s} > \lambda_{l+s+1} > \ldots > \lambda_n,\ s \geq 0.$$

The Gromov width of M is at least the minimum $\min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$.

Using Lie theory language we can reformulate Theorem 1.1 in the following way.

Theorem 1.2. (Reformulation of Theorem 1.1) *Consider the $U(n)$ coadjoint orbit $M := \mathcal{O}_\lambda$ in $\mathfrak{u}(n)^*$ through a point $\lambda =$ diag $(\lambda_1, \lambda_2, \dots, \lambda_n)$ where*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_l = \lambda_{l+1} = \ldots = \lambda_{l+s} > \lambda_{l+s+1} > \ldots > \lambda_n,\ s \geq 0.$$

The Gromov width of M is at least the minimum

$$\min\{\, \langle \alpha^\vee, \lambda \rangle \mid \alpha^\vee \text{ a coroot}, \ \langle \alpha^\vee, \lambda \rangle > 0\}.$$

Remark. In fact the hypothesis can be weakened. The only necessary condition is that the Gelfand-Tsetlin polytope associated to $\mathcal{O}_\lambda$ contains at least one good vertex. These notions will be explained in Section 3.3.

There are reasons to care about this particular lower bound. In the case of generic coadjoint orbits, i.e. when $\lambda_1 > \lambda_2 \ldots > \lambda_n$, Masrour Zoghi in [Z] had already obtained this lower bound. Moreover, with some additional integrability assumption on λ, he proved that this lower bound is precisely the Gromov width. He also proved a similar upper bound for Gromov width of generic coadjoint orbits (with some integrality conditions) of other simple compact Lie groups. This suggests that the lower bound for non-generic orbits that we provide here may in fact be the Gromov width.

To prove the Theorem 1.1 we will recall an action of the Gelfand-Tsetlin torus on an open dense subset of $\mathcal{O}_\lambda$. We will then use the theorem of Karshon and Tolman, [KT1], recalled here as Proposition 2.6, to obtain symplectic embeddings of balls. Masrour Zoghi also used the Karshon and Tolman's result, but applied to the standard coadjoint action of a maximal torus. He suggested that maybe the action of the Gelfand-Tsetlin torus could give stronger results for a wider class of orbits.

Organization. Section 2 provides background about centered actions and Gelfand-Tsetlin functions. In Section 3, we carefully analyze Gelfand-Tsetlin functions and the action they induce. Section 4 contains the proof of the main result. Section 5 has a "bookkeeping" character. There we summarize what is known about the Gromov width of $U(n)$ coadjoint orbits for small values of n.

Acknowledgments. The author is very grateful to Yael Karshon for suggesting this problem and helpful conversations during my work on this project. The author also would like to thank her advisor, Tara Holm, for useful discussions.

2. Preliminaries

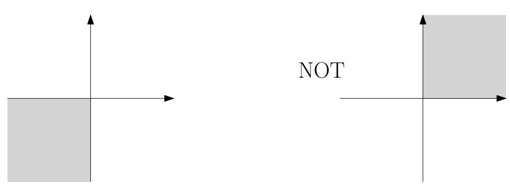
2.1. **Centered actions.** Centered actions were introduced in [KT2]. For completeness and to set notation we include the details here following [KT1]. Let (M,ω) be a connected symplectic manifold, equipped with an effective, symplectic action of a torus $T \cong (S^1)^{\dim T}$. The action of T is called **Hamiltonian** if there exists a T-invariant map $\Phi\colon M \to \mathfrak{t}^*$, called the **momentum map**, such that

$$\iota(\xi_M)\omega = d\,\langle \Phi, \xi\rangle \quad \forall\, \xi \in \mathfrak{t}, \tag{2.1}$$

where ξ_M is the vector field on M generated by $\xi \in \mathfrak{t}$. We will identify Lie(S^1) with $\mathbb{R}$ using the convention that the exponential map $exp : \mathbb{R} \cong$Lie$(S^1) \to S^1$ is given by $t \to e^{2\pi i t}$, that is $S^1 \cong \mathbb{R}/\mathbb{Z}$.
At a fixed point $p \in M^T$, we may consider the induced action of T on the tangent space T_pM. There exist $\eta_j \in \mathfrak{t}^*$, called the **isotropy weights** at p, such that this action is isomorphic to the action on $(\mathbb{C}^n, \omega_{std})$ generated by the moment map

$$\Phi_{\mathbb{C}^n}(z) = \Phi(p) + \pi \sum |z_j|^2(-\eta_j).$$

The isotropy weights are uniquely determined up to permutation. Note that with our sign convention in equation 2.1 the isotropy weights are pointing out of the momentum map image. For example, standard S^1 action on $\mathbb{C}^2$ by rotation with speed one gives the following momentum map image:



By the equivariant Darboux theorem, a neighborhood of p in M is equivariantly symplectomorphic to a neighborhood of 0 in $\mathbb{C}^n$. However, this theorem does not tell us how large we may take this neighborhood to be. Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex set which contains $\Phi(M)$. The quadruple $(M, \omega, \Phi, \mathcal{T})$ is a **proper Hamiltonian T-manifold** if Φ is proper as a map to $\mathcal{T}$, that is, the preimage of every compact subset of $\mathcal{T}$ is compact.

For any subgroup K of T, let $M^K = \{m \in M \mid a \cdot m = m \;\forall a \in K\}$ denote its fixed point set.

Definition 2.1. A proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ is **centered** about a point $\alpha \in \mathcal{T}$ if α is contained in the moment map image of every component of M^K, for every subgroup $K \subseteq T$.

We now quote several examples and non-examples, following [KT1].

Example 2.2. A compact symplectic manifold with a non-trivial T-action is never centered, because it has fixed points with different moment map images.

Example 2.3. Let a torus T act linearly on $\mathbb{C}^n$ with a proper moment map $\Phi_{\mathbb{C}^n}$ such that $\Phi_{\mathbb{C}^n}(0) = 0$. Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex subset containing the origin. Then $\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$ is centered about the origin.

A Hamiltonian T action on M is called **toric** if $\dim T = \frac{1}{2}\dim M$.

Example 2.4. Let M be a compact symplectic toric manifold with moment map $\Phi\colon M \to \mathfrak{t}^*$. Then $\Delta := \text{Im } \Phi$ is a convex polytope. The orbit type strata in M are the moment map preimages of the relative interiors of the faces of Δ. Hence, for any $\alpha \in \Delta$,

$$\bigcup_{\substack{F \text{ face of } \Delta \\ \alpha \in F}} \Phi^{-1}(\text{rel-int } F)$$

is the largest subset of M that is centered about α.

When the dimension of the torus acting on a compact symplectic manifold is less then half of the dimension of the manifold, one can easily find a centered region from an x-ray of the Hamiltonian T-space M. The **x-ray** of (M, ω, ϕ) is the collection of convex polytopes $\phi(X)$ over all connected compontents X of M^K for some subtorus K of T (for more details see [To]). For the toric symplectic manifold, an x-ray is exactly the collection of faces of convex polytope that is the image of moment map. Figure 2.1 presents some examples of centered regions, that we can see directly from the x-rays of M.

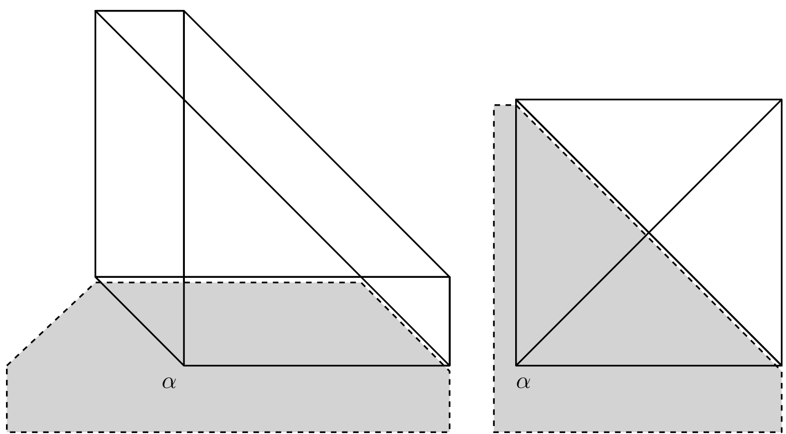


FIGURE 1. The regions centered around α.

Example 2.5. Let $(M, \omega, \Phi, \mathcal{T})$ be a proper Hamiltonian T-manifold. Then every point in $\mathfrak{t}^*$ has a neighborhood whose preimage is centered. This is a consequence of the local normal form theorem and the properness of the moment map.

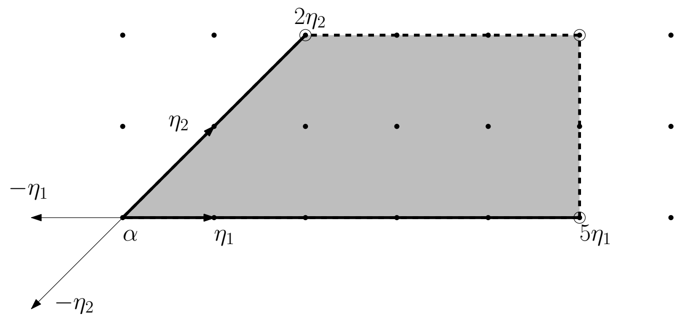
Proposition 2.6. (Karshon, Tolman, [KT1]) Let $(M, \omega, \Phi, \mathcal{T})$ be a proper Hamiltonian T-manifold. Assume that M is centered about $\alpha \in \mathcal{T}$ and that $\Phi^{-1}(\{\alpha\})$ consists of a single fixed point p. Let $-\eta_1, \ldots, -\eta_n$ be the isotropy weights of T action on T_pM. Then M is equivariantly symplectomorphic to

$$\left\{ z \in \mathbb{C}^n \mid \alpha + \pi \sum |z_j|^2 \eta_j \in \mathcal{T} \right\},$$

where T acts on $\mathbb{C}^n$ with weights $-\eta_1, \ldots, -\eta_n$.

Note that the above formlumation differs from the one in [KT1] by a minus sign. This is due to the fact that our definition of momentum map 2.1 also differs by a minus sign from the definition used in [KT1].

Example 2.7. Consider a compact symplectic toric manifold M whose momentum map image is the closure of the following region.



The weights of the torus action are $(-\eta_1)$ and $(-\eta_2)$, and the lattice lengths of edges starting from α are 5 and 2 (with respect to weight lattice). The largest subset of M that is centered about α, as described in Example 2.4, maps under the moment map to the shaded region. The above Proposition tells us that it is equivariantly symplectomorphic to

$$\{z \in \mathbb{C}^2 | \alpha + \pi(|z_1|^2 + |z_2|^2) \in \text{ shaded region }\}.$$

If $z \in B^4_2 = \{z \in \mathbb{C}^2 \,\Big|\, \pi(|z_1|^2 + |z_2|^2) < 2\}$ then $\alpha + \pi(|z_1|^2\eta_1 + |z_2|^2\eta_2)$ is in the shaded region. Therefore the 4-dimensional ball B^4_2 of capacity 2 embeds into M and the Gromov width of M is at least the minimum of lattice lengths of edges of the moment polytope, starting at α. Note also that the momentum map image of the embedded ball B^4_2 is the triangle with vertices α, $\alpha + 2\eta_1$ and $\alpha + 2\eta_2$.

2.2. **Standard torus action on a coadjoint orbit.** Under our identifications, the coadjoint action of $U(n)$ on $\mathfrak{u}(n)^*$ is by conjugation: $A \cdot \xi = A\xi A^{-1}$. Restricted to an orbit $\mathcal{O}_\lambda$, this action is Hamiltonian with momentum map the inclusion $\mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n)^*$. Let $T = T^n$ be the standard maximal torus in $U(n)$ (given by diagonal matrices). As explained in the introduction, we identify $\mathfrak{u}(n)^*$ with the space of $n \times n$ Hermitian matrices. We will use coordinates $\{e_{ij}\}$, with e_{ij} corresponding to (i,j)-th entry of a matrix. We choose the positive Weyl chamber, $(\mathfrak{t}^*)_+$, to be

$$(\mathfrak{t}^*)_+ := \{\text{diag}(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn});\ \lambda_{11} \geq \lambda_{22} \geq \ldots \geq \lambda_{nn}\}.$$

Then $\Delta = \{e_{ii} - e_{jj} \,|\, i \neq j\}$ is a root system and $\Sigma = \{e_{ii} - e_{i+1,i+1} \,|\, i = 1, 2, \ldots, n-1\}$ is the set of positive roots. The coadjoint orbits in $\mathfrak{u}(n)^*$ are in one-to-one correspondence with the points of $(\mathfrak{t}^*)_+$. Precisely, for any $(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn}) \in (\mathfrak{t}^*)_+$ the corresponding coadjoint orbit is the set of all Hermitian matrices with eigenvalues $(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn})$. Fix some $\lambda = (\lambda_{11} \geq \lambda_{22} \geq \ldots \geq \lambda_{nn}) \in (\mathfrak{t}^*)_+$ and denote by $\mathcal{O}_\lambda$ the coadjoint orbit through λ. The standard T^n action on $\mathcal{O}_\lambda$ is the action of the maximal torus $T^n \subset U(n)$. The fixed points of this action are the diagonal matrices. In particular, λ is a fixed point and the weights of T^n action on $T_\lambda\mathcal{O}_\lambda$ are given by the negative roots $-\Sigma$. The T^n action is Hamiltonian with moment map $\mu : \mathcal{O}_\lambda \to (\mathfrak{t}^n)^* \cong \mathbb{R}^n$ that maps a matrix $A = (a_{ij})$ to the diagonal $n \times n$ matrix $\text{diag}\,(a_{11}, \ldots, a_{nn})$.

However the dimension of torus acting effectively is less then half of the dimension of the coadjoint orbit, so this action is not toric. If $\mathcal{O}_\lambda$ is regular then this action is effective but $\dim T^n = n$ while $\dim \mathcal{O}_\lambda = \frac{1}{2}n(n-1)$. Let $\mathcal{Q} = \mu(\mathcal{O}_\lambda) \subset (\mathfrak{t}^n)^*$ denote the momentum map image for the standard T^n action. The vertices of $\mathcal{Q}$ correspond to the T^n-fixed points, that is, the diagonal matrices in $\mathcal{O}_\lambda$. If λ is generic, then the vertices correspond exactly to permutations on n elements. Thus there are exactly $n!$ of them. If λ is non-generic, say

$$\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{n-l_s+1} = \ldots = \lambda_n,$$

then the vertices correspond to cosets $S_n/(S_{l_1} \times \ldots \times S_{l_s})$, and there are exactly $\frac{n!}{l_1! \ldots l_s!}$ of them.

Recall that **GKM manifold** is a manifold M equipped with a faithful action of a torus K of dimension $l > 1$ such that the set of zero dimensional orbits in the orbit space M/K is zero dimensional and the set of one dimensional orbits in M/K is one dimensional (see [GKM], [GHZ], [TW]). The coadjoint orbit $\mathcal{O}_\lambda$ with the standard T^n action is an example of GKM manifold. In particular this means that the closure of every connected component of the set $\{x \in \mathcal{O}_\lambda; \dim(T^n \cdot x) = 1\}$ is a sphere. The closure of $\{x \in \mathcal{O}_\lambda; \dim(T^n \cdot x) = 1\}$ is called **1-skeleton** of $\mathcal{O}_\lambda$. Denote by $\mathcal{Q}_1$ the image of 1-skeleton under the momentum map. The GKM assumption forces $\mathcal{Q}_1$ to be a $(\frac{1}{2}\dim \mathcal{O}_\lambda)$-valent graph with vertices $Vert(\mathcal{Q}_1) = Vert(\mathcal{Q})$ corresponding to T^n-fixed points and edges corresponding to closures of connected components of the 1-skeleton. Note that not all edges in $\mathcal{Q}_1$ are edges of the polytope $\mathcal{Q}$. Images of two fixed points, F and F', are connected by an edge in $\mathcal{Q}_1$ if and only if they differ by one transposition of two different diagonal entries. Therefore there are exactly

$$D := [\, l_1(l_2 + \ldots l_s) + l_2(l_3 + \ldots + l_s) + \ldots + l_{s-1}l_s \,] = \sum_{i<j} l_i l_j$$

edges leaving any vertex of $\mathcal{Q}_1$ and thus dim $\mathcal{O}_\lambda = D \dim(S^2) = 2D$. In the case of generic λ, the moment polytope of $\mathcal{O}_\lambda$ is called a permutahedron.

Denote the diagonal entries of F by $F_{11}, \ldots, F_{nn}$. Let $p < q$ be indices from $\{1, \ldots n\}$ such that $F_{pp} \neq F_{qq}$ and F' is the matrix obtained from F by switching p-th and q-th entry. The edge joining $\mu(F)$ and $\mu(F')$ is an μ-image of a sphere in $\mathcal{O}_\lambda$. This sphere is the orbit of $SU(2)$ action on F and is obtained in the following way. Denote $F_{pp} = v_i,\ F_{qq} = v_k$. For any $z \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ let I_z be the matrix obtained from the identity matrix by changing four entries (j,k) with $j,k \in \{p,q\}$ in the way presented below and let $F_z = I_z F I_z^{-1}$ be the matrix obtained from F by conjugation with I_z. This means that F_z differs from F only at four entries (j,k) with $j,k \in \{p,q\}$. The matrices have the following shapes

$$I_z = \begin{bmatrix} I & \vdots & & \vdots & \\ \cdots & \frac{1}{Z} & \cdots & \frac{-\bar{z}}{Z} & \cdots \\ & \vdots & I & \vdots & \\ \cdots & \frac{z}{Z} & \cdots & \frac{1}{Z} & \cdots \\ & \vdots & & \vdots & I \end{bmatrix}, \quad F_z = \begin{bmatrix} \ddots & \vdots & 0 & \vdots & 0 \\ \cdots & \frac{(v_i+|z|^2 v_k)}{Z} & \cdots & \frac{\bar{z}(v_i - v_k)}{Z} & \cdots \\ 0 & \vdots & \ddots & \vdots & 0 \\ \cdots & \frac{z(v_i - v_k)}{Z} & \cdots & \frac{(v_k + |z|^2 v_i)}{Z} & \cdots \\ 0 & \vdots & 0 & \vdots & \ddots \end{bmatrix}$$

where $Z = \sqrt{1+|z|^2}$. Then

$$\mu(\{F_z;\ z \in \mathbb{CP}^1\}) = \overline{\mu(F)\,\mu(F')}.$$

Moment image of the standard torus action is also explained in [Ty],[MRS].

There are also other natural actions on $\mathcal{O}_\lambda$. For any $j = 1, \ldots, n$, we have a natural embedding $\iota_j : U(j) \to U(n)$

$$\iota_j(B) = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & Id \end{array} \right),$$

where $B \in U(j)$. Using this embedding we obtain a $U(j)$ action (and also an action of maximal torus T^j) on $\mathcal{O}_\lambda$: for $B \in U(j)$ and $\xi \in \mathcal{O}_\lambda$, we define

$$B \cdot \xi = \iota_j(B)\, \xi\, (\iota_j(B))^{-1}.$$

To simplify the notation, we will often write B instead of $\iota_j(B)$. Both of these actions are also Hamiltonian. The momentum map for the $U(j)$ action is the projection

$$\Phi^j : \mathcal{O}_\lambda \to \mathfrak{u}(j)^*$$

sending every matrix to its $j \times j$ top left minor.

2.3. **Gelfand-Tsetlin system.** In this subsection we recall the Gelfand-Tsetlin (sometimes spelled Gelfand-Cetlin, or Gelfand-Zetlin) system of action coordinates, which originally appeared in [GS1]. It is related to the classical Gelfand-Tsetlin polytope introduced in [GTs]. There are many references describing this system, for example [GS1], [K], [NNU], [H]. For the readers' convenience and to fix the notation, we follow Mikhai Kogan's construction for a coadjoint $U(n)$ orbit in $\mathfrak{u}(n)^*$, [K].

Consider the sequence of subgroups

$$U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1).$$

For each $U(j)$ in the sequence choose the maximal torus T_j to be the set of diagonal matrices in $U(j)$ and the positive Weyl chamber, $(\mathfrak{t}^j)^*_+$, to consist of diagonal Hermitian $j \times j$ matrices with non-increasing diagonal entries. Recall that the moment map for the $U(j)$ action on $\mathcal{O}_\lambda$ is denoted by Φ^j and maps $A \in \mathcal{O}_\lambda$ to $j \times j$ top left submatrix of A. Denote the eigenvalues of $\Phi^j(A)$, ordered in a non-increasing way, by

$$\lambda_1^{(j)}(A) \geq \lambda_2^{(j)}(A) \geq \ldots \geq \lambda_j^{(j)}(A).$$

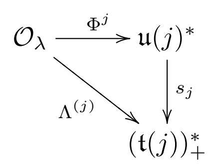
We will use the notation $\Lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_j^{(j)}) : \mathcal{O}_\lambda \to \mathbb{R}^j$, for a function sending A to $(\lambda_1^{(j)}(A), \ldots, \lambda_j^{(j)}(A)) \in \mathbb{R}^j$. For $j = n$, we just get $\Phi^n(A) = A$ and $\lambda_k^{(n)}(A) = \lambda_k$. The **Gelfand -Tsetlin system of action coordinates** is the collection of the functions $\lambda_k^{(j)}$ for $j = 1, \ldots, n-1$ and $k = 1, \ldots, j$. We will denote them by

$$\Lambda : \mathcal{O}_\lambda \to \mathbb{R}^N,$$

where

$$N := (n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}.$$

Notice that $\Lambda^{(j)}$ is the composition of Φ^j and a map $s_j : \mathfrak{u}(j)^* \to (\mathfrak{t}^j)^*_+ \subset \mathbb{R}^j$ sending a point in $\mathfrak{u}(j)^*$ to the unique point of intersection of its $U(j)$ orbit with the positive Weyl chamber.



Here we identify $(\mathfrak{t}^j)^*$ with $\mathbb{R}^j$ by $\mathrm{diag}(a_1, \ldots, a_j) \to (a_1, \ldots, a_j)$.

The components of s_j are $U(j)$ invariant, so they Poisson commute. After precomposing them with Φ^j, we get a family of Poisson commuting functions on $\mathcal{O}_\lambda$ (see Proposition 3.2 in [GS1]). These are exactly $\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)}$. For $l < j$ denote by $\kappa_{lj} : \mathfrak{u}(j)^* \to \mathfrak{u}(l)^*$ the transpose of the map $\mathfrak{u}(l) \to \mathfrak{u}(j)$ induced by the inclusion. The functions

$$\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)}, \lambda_1^{(l)} \circ \kappa_{lj}, \lambda_2^{(l)} \circ \kappa_{lj}, \ldots, \lambda_l^{(l)} \circ \kappa_{lj}$$

Poisson commute on $\mathfrak{u}(l)^*$ by Proposition 3.2 in [GS1] and the fact that first j of them are $U(j)$ invariant. Therefore the Gelfand-Tsetlin functions Poisson commute on $\mathcal{O}_\lambda$.

3. The action of the Gelfand-Tsetlin torus

3.1. **Smoothness of the Gelfand-Tsetlin functions.** The function $\lambda_k^{(j)}$ need not be smooth on the whole orbit $\mathcal{O}_\lambda$. The eigenvalues depend smoothly on the matrix entries, but this property is not preserved when reordering them in a non-increasing way. They are smooth, however, on a dense open subset of $\mathcal{O}_\lambda$. To identify this subset we will need the following result proved in [CDM]. This theorem is also true for orbifolds: see [LMTW, Theorem 3.1].

Theorem 3.1. *Let G be a compact connected Lie group with a maximal torus T. Suppose G acts on a compact connected symplectic manifold M in a Hamiltonian way, with moment map $\Phi : M \to \mathfrak{g}^*$. Then there exists a unique open wall σ^o of the Weyl chamber $\mathfrak{t}^*_+$ with the properties that $\Phi(M) \cap \mathfrak{t}^*_+ \subset \sigma = \overline{\sigma^o}$ and $\Phi(M) \cap \mathfrak{t}^*_+ \cap \sigma^o \neq \emptyset$.*

Let $\sigma^o = \sigma^o_j$ be the unique open wall from the above theorem applied to the standard $G = U(j)$ action on $M = \mathcal{O}_\lambda$. We call $\sigma = \overline{\sigma^o}$ the **principal face.** Any (closed) wall of positive Weyl chamber $(\mathfrak{t}^j)^*_+$ that contains σ is called a **special wall**, while all the others walls are called **regular walls**. Thus σ is the intersection of all special walls, and $\sigma^o = \sigma \setminus (\cup \text{ regular walls})$. Walls of $(\mathfrak{t}^j)^*_+$ are defined by a collection of equations of the form $\lambda_L^{(j)} = \lambda_{L+1}^{(j)}$. If a wall τ is special, i.e. $\sigma \subset \tau$, then its defining equations hold on the whole $\Lambda(\mathcal{O}_\lambda)$.

Proposition 3.2. The function $\Lambda^{(j)}$ is smooth on the set $U^{(j)} = (\Lambda^{(j)})^{-1}(\sigma^o)$.

Proof. To simplify the notation, we will denote $U(j)$ by G, and the maximal torus in $U(j)$ simply by T. Recall that the function $\Lambda^{(j)}$ is a composition of a smooth function Φ^j and projection $\pi : \mathfrak{g}^* = \mathfrak{u}(j)^* \to \mathfrak{t}^*_+$. Therefore we only need to prove smoothness of the projection π on $\Phi^j(U^{(j)}) = \pi^{-1}(\sigma^o)$. Note that all points in σ^o have the same G-stabilizer (under the coadjoint action of G). Denote it by H. Let S be the subset of $\mathfrak{g}^*$ equal to $\pi^{-1}(\sigma^o)$. This means that $S = (\mathfrak{g}^*)_{(H)}$ is an orbit-type stratum and therefore it is a submanifold of $\mathfrak{g}^*$. Consider the smooth, G-equivariant, surjective map:

$$\begin{array}{ccc} G \times \sigma^o & \to & S \\ (g, x) & \to & g \cdot x \end{array}$$

This map induces G-equivariant bijective map

$$\begin{array}{ccc} \Theta : G/H \times \sigma^o & \to S, & \\ ([g], x) & \to & g \cdot x \end{array}$$

which is also a diffeomorphism. Notice that the composition, $\pi \circ \Theta$

$$\begin{array}{ccc} G/H \times \sigma^o & \to & \mathfrak{t}^*_+ \\ ([g], x) & \to & x \end{array}$$

is just the projection onto second factor, therefore it is smooth. This means that on S, π is smooth, as a composition of Θ^{-1} and a smooth projection. It follows that the function $\Lambda^{(j)}$ is smooth on the set $(\Phi^j)^{-1}(S) = (\Lambda^{(j)})^{-1}(\sigma^o) = U^{(j)}$. □

Remark. The set of smooth points for $\Lambda^{(j)}$ may be strictly bigger than $U^{(j)}$. For example, consider the orbit $\mathcal{O}_\lambda$ with $\lambda = \text{diag}(3, 3, 2, 1)$ and matrix $A = \lambda$. Then A is not in $U^{(j)}$ because $\lambda_1^{(3)} = \lambda_2^{(3)}$, so $\Phi^3(A)$ is in $\sigma_3 \setminus \sigma_3^o$. Note however that the function $\lambda_1^{(3)}$ is smooth as it is constant (equal to 3) on $\mathcal{O}_\lambda$. Therefore the function $\lambda_2^{(3)} = \text{trace} \circ \Phi^3 - \lambda_1^{(3)} - \lambda_3^{(3)}$ is also smooth. To be more general, suppose that a function $\lambda_k^{(j)}$ is constant on the whole orbit $\mathcal{O}_\lambda$, and let A be a point in $\mathcal{O}_\lambda$ such that $\lambda_k^{(j)}(A) = \lambda_{k+1}^{(j)}(A)$. Suppose further that if for any $l \neq k$ we also have

$\lambda_l^{(j)}(A) = \lambda_{l+1}^{(j)}(A)$ then $\lambda_l^{(j)}$ and $\lambda_{l+1}^{(j)}$ are equal on the whole $\mathcal{O}_\lambda$. In this case, the function

$$\lambda_{k+1}^{(j)} = \text{trace} \circ \Phi^j - \sum_{l \neq k+1} \lambda_l^{(j)}$$

is smooth at the point A, as a difference of smooth functions, although A is not in the set $U^{(j)}$ as defined above. We will later define a *good vertex* as a point $q \in \Lambda(\mathcal{O}_\lambda)$, with $\Lambda^{-1}(q)$ – single point, such that Λ is smooth in its neighborhood. The hypothesis of the main theorem can be weakened to the existence of a good vertex. Proving the smoothness of the Gelfand-Tsetlin functions on a set bigger then $U^{(j)}$ would allow us to apply the proof of the main theorem to a wider class of non-generic coadjoint orbits. The theorem holds if only there is a T^n-fixed point with a neighborhood equipped with a smooth action of Gelfand-Tsetlin torus T^D. Therefore our techniques may be extended to coadjoint orbits with an additional eigenvalue repeating twice. The technical details became more cumbersome, though, so we do not include them here.

3.2. **The torus action induced by the Gelfand-Tsetlin system.** At the points where $\Lambda^{(j)}$ is smooth, it induces a smooth action of $T^j \hookrightarrow T_{U(j)}$, a subtorus of $T_{U(j)}$. The process of obtaining this new action, which we denote by $*$, is often referred to as the **Thimm trick**. If λ is regular then $T^j = T_{U(j)}$. An element $t \in T^j$ acts on a point $A \in \mathcal{O}_\lambda$ by the standard, coadjoint $U(j)$ action of $B^{-1}\, t\, B$, where $B \in U(j)$ is such that $Ad^*(B)\, \Phi^j(A) \in (\mathfrak{t}_{U(j)})^*_+$ is the unique point of intersection of $(\mathfrak{t}_{U(j)})^*_+$ and the $U(j)$-coadjoint orbit through $\Phi^j(A)$. That is

$$t * A = Ad^* \left(\left[\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right] \right) (A).$$

In this thesis we consider only matrix groups, and for them the coadjoint action is the action by conjugation. Therefore we will simplify the notation and write conjugation in place of the coadjoint action:

$$t * A = \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right) A \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right)^{-1}. \tag{3.1}$$

Recall that for regular λ, a matrix $A \in U^{(j)}$ if $B\Phi^j(A)\, B^{-1} \in \; int\, (\mathfrak{t}_{U(j)})^*_+$, so the stabilizer of $B\Phi^j(A)\, B^{-1}$ in $U(j)$ is precisely $T_{U(j)} = T^j$. The fact that T^j commutes with the stabilizer of $B\Phi^j(A)\, B^{-1}$ implies that the action is well defined, as explained below.

If λ is not regular then some of the functions $\lambda_*^{(j)}$ may be constant on the whole orbit. Let $T^j \hookrightarrow T_{U(j)}$ be the subtorus defined by

$$\{(t_1, \ldots, t_j) \in T_{U(j)};\ t_i = 1 \text{ if } \lambda_i^{(j)} \text{ constant on the whole orbit } \}.$$

(This definition gives $T^j = T_{U(j)}$ if none of the functions $\lambda_*^{(j)}$ is constant on the whole orbit). Let σ_j be the unique wall of the positive Weyl chamber $(\mathfrak{t}_{U(j)})^*_+$ from Theorem 3.1. All points in σ_j^o have the same stabilizer. Note that the torus T^j commutes with the stabilizer in $U(j)$ of points in σ_j^o. The stabilizer in $U(j)$ of points in σ_j^o is a product of circles and of groups $U(m)$ (various $m \leq j$ whose sum is at most j), one for each longest sequence $\lambda_i^{(j)} = \lambda_{i+1}^{(j)} = \ldots = \lambda_{i+m-1}^{(j)} \equiv \lambda_i$ of the functions $\lambda_*^{(j)}$ that are constant on the whole orbit. Elements of the torus T^j are diagonal matrices with diagonal entries equal to 1 in blocks corresponding to the $U(m)$ factors of the stabilizer. For example, if $\lambda_1^{(j)} = \lambda_2^{(j)} = \ldots = \lambda_m^{(j)} \equiv \lambda_1$, then the stabilizer in $U(j)$ of points in σ_j^o is $U(m) \times S^1 \times \ldots \times S^1$, while elements of T^j are of the form $(1, \ldots, 1, t_{m+1}, \ldots, t_n)$ and thus commute with the stabilizer. The action of $t \in T^j$ on $A \in \mathcal{O}_\lambda$ is given by equation (3.1), where

$B \in U(j)$ is such that $B\Phi^j(A)\,B^{-1} \in \sigma_j \subset (\mathfrak{t}_{U(j)})^*_+$. If C is another element of $U(j)$ such that $C\,\Phi^j(A)C^{-1} \in (\mathfrak{t}_{U(j)})^*_+$, then

$$B\,\Phi^j(A)B^{-1} = C\,\Phi^j(A)C^{-1} = CB^{-1}\,B\,\Phi^j(A)B^{-1}BC^{-1},$$

so $CB^{-1} \in Stab_{U(j)}(B\Phi^j(A)B^{-1})$. Therefore for $t \in T^j$ have

$$C^{-1}tC = C^{-1}tC\,B^{-1}\,t^{-1}\,t\,B = C^{-1}\,t\,t^{-1}\,C\,B^{-1}\,t\,B = B^{-1}\,t\,B,$$

what implies that the action is well defined.

Proposition 3.3. The new T^j action defined above is Hamiltonian on the subset $U^{(j)} = (\Lambda^{(j)})^{-1}(\sigma_j^o)$, with momentum map $\Lambda^{(j)}$. (For non-regular orbits the momentum map consists only of non-constant coordinates of $\Lambda^{(j)}$).

Proof. To simplify the notation, we will denote $U^{(j)}$ simply by U, and let $\mathfrak{t}^j$ be the Lie algebra of T^j. Take any $X \in \mathfrak{t}^j$ and denote by X_{new} the vector field on U generated by X with $*$ action, and by X_{std} the vector field on U generated by X using the standard action by conjugation. As usual, for any function $\varphi : \mathcal{O}_\lambda \to \mathfrak{u}(j)^*$, and any $X \in \mathfrak{u}(j)$, we denote by φ^X a function from $\mathcal{O}_\lambda$ to $\mathbb{R}$ defined by $\varphi^X(p) = \langle \varphi(p), X\rangle$, where $\langle , \rangle$ is the standard $U(j)$ invariant pairing between $\mathfrak{u}(j)^*$ and $\mathfrak{u}(j)$. Take any $A \in U$. We want to prove that for any vector $Y \in T_A\mathcal{O}_\lambda = T_AU$

$$\omega(X_{new}, Y)|_A = d\,(\Lambda^{(j)})^X\,(Y)|_A. \tag{3.2}$$

Denote by N the connected symplectic submanifold $N := (\Phi^j)^{-1}(\sigma^o) \subset \mathcal{O}_\lambda$, where σ is the principal face. We refer to N as the **principal cross-section**. Note that $U = (\Lambda^{(j)})^{-1}(\sigma^o) = U(j)\cdot N$, and so every $A \in U$ can be $U(j)$ conjugated to an element of N. We first prove equation (3.2) for $A \in N$.

The proof of theorem 3.8 in [LMTW] implies that

$$T_A\mathcal{O}_\lambda = T_AN\, +\, T_A(U(j)\cdot A).$$

This is not a direct sum. Thus to prove the equation (3.2) for $A \in N$, it is enough to consider two cases: when vector Y is tangent to the principal cross-section, and when it is tangent to $U(j)$ orbit (for the standard action).

Before we start considering the cases, we fix some notation. For any vector field V on $\mathcal{O}_\lambda$, denote by Ψ^V its flow. Recall that $\Psi^V_{-t} = (\Psi^V_t)^{-1}$. Therefore, for example $\Psi^{X_{std}}_t(Q) = X_tQX_t^{-1}$ and $\Psi^{X_{std}}_{-t}(Q) = X_t^{-1}QX_t$.

Case 1: Take $Y \in T_AN \subset T_A\mathcal{O}_\lambda$. We want to compute $\omega(X_{new}, Y)|_A = \langle A, [X_{new}, Y]\rangle$. Notice that on the principal cross section functions Φ^j and $\Lambda^{(j)}$ are equal, and the standard and the new actions of T^j coincide. Therefore the vector fields X_{std} and X_{new} have equal values and flows on N. Using the formula

$$[X_{new}, Y] = \lim_{t\to 0} \frac{(\Psi^{X_{new}}_{-t})_*(Y) - Y}{t} = [X_{std}, Y].$$

we have that, if $Y \in T_AN$, then $\langle A, [X_{new}, Y]\rangle = \langle A, [X_{std}, Y]\rangle$. The fact that functions Φ^j and $\Lambda^{(j)}$ agree on all of the N, means also that for $Y \in T_AN$ we have

$$d(\Phi^j)^X(Y) = d(\Lambda^{(j)})^X(Y).$$

Therefore

$$\begin{aligned}\omega(X_{new}, Y)|_A &= \langle A, [X_{new}, Y]\rangle = \langle A, [X_{std}, Y]\rangle \\ &= \omega(X_{std}, Y)|_A = d(\Phi^j)^X(Y)|_A \\ &= d(\Lambda^{(j)})^X(Y)|_A.\end{aligned}$$

Case 2: Take $Y \in T_A(U(j)\cdot A)$. That is $Y = Y_{std}$ for some $Y = \frac{d}{dt} Y_t|_{t=0} \in \mathfrak{u}(j)$ and the integral curve of Y through A is $\Psi_t^Y(A) = Y_t\, A\, Y_t^{-1}$. As before, we start by analyzing $[X_{new}, Y]$ at A. We have:

$$[X_{new}, Y]|_A = \lim_{t\to 0} \frac{(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} - Y|_A}{t}.$$

The point A is in N, so $\Psi_t^{X_{new}}(A) = X_t \cdot A = X_t\, A\, X_t^{-1}$. Now we need to understand the expression:

$$(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} = \frac{d}{dv}\, \Psi_{-t}^{X_{new}}(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1})\, |_{v=0}.$$

To compute the value of $\Psi_{-t}^{X_{new}}$ on $Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}$, we need to find an element C of $U(j)$ that would conjguate $\Phi^j(\Psi_{-t}^{X_{new}})$ to some element in $(\mathfrak{t}^j)^*_+$. We have

$$\begin{aligned}\Phi^j(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}) &= \Phi^j(Y_v\, X_t A\, X_t^{-1}\, Y_v^{-1}) \\ &= Y_v\, X_t\, \Phi^j(A) X_t^{-1} Y_v^{-1}.\end{aligned}$$

Therefore, for

$$C = X_t^{-1}\, Y_v^{-1}$$

we have that

$$C \Phi^j(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1})\, C^{-1} = \Phi^j(A) \in (\mathfrak{t}^j)^*_+.$$

This means that the new action of X_t at a point $Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}$ is the same as standard action of

$$C^{-1}\, X_t\, C = Y_v X_t\, X_t\, X_t^{-1}\, Y_v^{-1} = Y_v X_t\, Y_v^{-1},$$

so

$$\begin{aligned}&\Psi_{-t}^{X_{new}}(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}) \\ &= (Y_v X_t^{-1}\, Y_v^{-1})(\, Y_v\, X_t\, A\, X_t^{-1}\, Y_v^{-1})(Y_v X_t Y_v^{-1}) \\ &= Y_v\, A\, Y_v^{-1}.\end{aligned}$$

Therefore

$$[X_{new}, Y]|_A = \lim_{t\to 0} \frac{(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} - Y|_A}{t} = \lim_{t\to 0} \frac{Y|_A - Y|_A}{t} = 0,$$

and

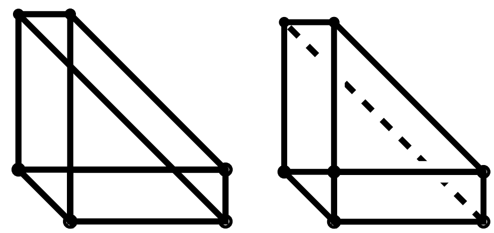
$$\omega(X_{new}, Y)|_A = \langle A, [X_{new}, Y]\rangle = 0.$$

Notice that the function $\Lambda^{(j)}$ is constant on $U(j)$ orbits, because Φ^j is $U(j)$-equivariant and the whole $U(j)$ orbit intersects $(\mathfrak{t}^j)^*_+$ in a unique point. Thus, for $Y \in T_A(U(j)\ \cdot\ A)$,

$$d\,(\Lambda^{(j)})^X\,(Y) = 0.$$

and equation (3.2) for A in N follows.

Now we want to prove equation (3.2) for all $C \in U$. Let B be an element of $U(j)$ such that $BCB^{-1} = A \in \mathfrak{t}^*_+$. Take any $X \in \mathfrak{t}$ and $Y \in T_C U$. Using the $U(j)$ invariance of ω and of $\Lambda^{(j)}$, and equation (3.2) at the principal cross section, we have



$\mathcal{Q} = \mu(\mathcal{O}_\lambda) \subset \mathbb{R}^2$ $\mathcal{P} = \Lambda(\mathcal{O}_\lambda) \subset \mathbb{R}^3$

FIGURE 2. The momentum map images for the standard and Gelfand-Tsetlin actions on a regular $SU(3)$ coadjoint orbit.

$$\begin{aligned}
\omega(X_{new}, Y)_{|B^{-1}AB} &= \omega(\ \frac{d}{dt}(B^{-1}X_tB \cdot C)|_{t=0}, \ \frac{d}{dt}(\Psi^Y_t(C))|_{t=0} \ ) \\
&= \omega(\ \frac{d}{dt}B(B^{-1}X_tB \cdot C)B^{-1}|_{t=0}, \ \frac{d}{dt}B(\Psi^Y_t(C))B^{-1}|_{t=0}) \\
&= \omega(\ \frac{d}{dt}(X_tBB^{-1}ABB^{-1}X_t^{-1})|_{t=0}, \ \frac{d}{dt}(\Psi^{BYB^{-1}}_t(A))|_{t=0}) \\
&= \omega(X_{new}, BYB^{-1})|_A = d\,(\Lambda^{(j)})^X(BYB^{-1})|_A \\
&= \frac{d}{dt}\left[\,(\Lambda^{(j)})^X(B\Psi^Y_t(C)B^{-1}))\,\right]|_{t=0} = \frac{d}{dt}\left[\,(\Lambda^{(j)})^X(\Psi^Y_t(C))\,\right]|_{t=0} \\
&= d\,(\Lambda^{(j)})^X(Y)|_C,
\end{aligned}$$

which is exactly what we needed to show. □

Putting the actions together we obtain the Hamiltonian action of the **Gelfand-Tsetlin torus** in $U(n)$ case, $T = T_{GT} = T'_{U(n-1)} \oplus \ldots \oplus T'_{U(1)} \cong (S^1)^D$, $D = \sum_{i=1}^{n-1} \dim(T'_{U(i)})$, on the dense open subset

$$U := \cap_j U^{(j)}$$

of the coadjoint orbit $\mathcal{O}_\lambda$ where all functions $\Lambda^{(j)}$ are smooth. This action is called the **Gelfand-Tsetlin action** and its momentum map is Λ. If the orbit is regular then $D = N = \frac{1}{2}n(n-1)$.

Notice that the standard action of T^n, described in the Section 2.2, is a part of the T^N action on U. One can easily compute the T^n-momentum map μ, which mapps a matrix to its diagonal entries, from Λ. Of course $\lambda^{(1)}_1(A) = a_{11}$. Using the fact that the trace of $\Phi^2(A)$ is $a_{11} + a_{22} = \lambda^{(2)}_1(A) + \lambda^{(2)}_2(A)$ we compute the value a_{22}. Continuing this process we obtain all the diagonal entries of A, that is we obtain $\mu(A)$. This defines the projection $pr : (\mathfrak{t}^N)^* \to (\mathfrak{t}^n)^*$, which on the image of Λ is given by the following formula

$$pr(\{\lambda^{(j)}_l\}) = \Big(\lambda^{(1)}_1, \, \lambda^{(2)}_1 + \lambda^{(2)}_2 - \lambda^{(1)}_1, \ldots, \sum_i \lambda^{(n-1)}_i - \sum_i \lambda^{(n-2)}_i, \sum_i \lambda^{(n)}_i - \sum_i \lambda^{(n-1)}_i\Big).$$

This means $\mu = pr \circ \Lambda$. Under this projection, the Gelfand-Tsetlin polytope $\mathcal{P} := \Lambda(\mathcal{O}_\lambda)$, described below, maps to the momentum map image, $\mathcal{Q}$, of the standard maximal torus action. Here is an example for a regular $SU(3)$ coadjoint orbit, $\mathcal{O}_\lambda$.

Proposition 3.4. The Gelfand-Tsetlin action on a $U(n)$-coadjoint orbit $\mathcal{O}_\lambda$ is effective for all λ.

Proof. Suppose that

$$R = (R_{n-1}, \ldots, R_1) \in T'_{U(n-1)} \oplus \ldots \oplus T'_{U(1)} = T_{GT},$$

$R_j = diag(r_{j,1}, \ldots, r_{j,j}, 1, \ldots, 1)$, is a global stabilizer. Let

$$\widetilde{R} := \left(\begin{array}{c|c} R_{n-1} & \\ \hline & 1 \end{array}\right) \cdots \left(\begin{array}{c|c} R_1 & \\ \hline & I_{n-1} \end{array}\right) = \begin{pmatrix} \prod_j r_{1,j} & & & & \\ & \prod_j r_{2,j} & & & \\ & & \ddots & & \\ & & & r_{n-1,n-1} & \\ & & & & 1 \end{pmatrix}.$$

Note that for any $k = 1, \ldots, n-1$

$$(\Phi^{n-1})^{-1}(\sigma^o_{n-1}) \subset (\Phi^k)^{-1}(\sigma^o_k).$$

Therefore for any $A = \left(\begin{array}{c|c} \Phi^{n-1}(A) & X \\ \hline X^* & c \end{array}\right) \in (\Phi^{n-1})^{-1}(\sigma^o_{n-1})$ have

$$R * A = \widetilde{R}\, A\, \widetilde{R}^{-1} = \widetilde{R} \left(\begin{array}{c|c} \Phi^{n-1}(A) & X \\ \hline X^* & c \end{array}\right) \widetilde{R}^{-1} = \left(\begin{array}{c|c} \Phi^{n-1}(A) & \widetilde{R}X \\ \hline X^* \widetilde{R}^{-1} & c \end{array}\right).$$

Let λ be of the form

$$(\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{l_1+\ldots+l_{s-1}+1} = \ldots = \lambda_{l_1+\ldots+l_s}).$$

Denote $\lambda_{l_1+\ldots+l_j}$ by w_j. In this notation

$$\lambda = \begin{pmatrix} w_1\, I_{l_1} & & \\ & \ddots & \\ & & w_s\, I_{l_s} \end{pmatrix}.$$

Fix any $j = 1, \ldots, s$ and take $\varepsilon > 0$ such that $w_j - \varepsilon > w_{j+1}$. For any

$$X = (0, \ldots, 0, x_{l_1+\ldots+l_{j-1}+1}, \ldots, x_{l_1+\ldots+l_j}, 0, \ldots, 0)^T$$

such that $|x_{l_1+\ldots+l_{j-1}+1}|^2 + \ldots + |x_{l_1+\ldots+l_j}|^2 = l_j(\, w_j^2 - (w_j - \varepsilon)^2\,)$, the matrix

$$\left(\begin{array}{ccccc|c} w_1\, I_{l_1} & & & & & \\ & \ddots & & & & \\ & & (w_j - \varepsilon)\, I_{l_j} & & & X \\ & & & \ddots & & \\ & & & & w_s\, I_{l_s} & \\ \hline & & X^* & & & \sum \lambda_i - l_j \varepsilon \end{array}\right)$$

is in $(\Phi^{n-1})^{-1}(\sigma^o_{n-1})$ (see Lemma 3.6). Therefore R stabilizes this matrix if and only if

$$\widetilde{R}X = X.$$

As R is a global stabilizer, considering similar matrices for other j we see that $\widetilde{R} = I$. In particular this means that in R_{n-1} the coordinate $r_{n-1,n-1}$ must be equal to 1.

Now consider matrices of the form

$$\left(\begin{array}{ccccc|c|c} w_1\, I_{l_1} & & & & & & \\ & \ddots & & & & & \\ & & (w_j-\varepsilon)\, I_{l_j} & & & X & 0 \\ & & & \ddots & & & \\ & & & & w_s\, I_{l_s} & & \\ \hline & & X^* & & & c_1 & 0 \\ \hline & & 0 & & & 0 & c_2 \end{array}\right).$$

Torus $T'_{U(n-1)}$ acts trivially on such matrices. Therefore $R=(R_{n-1},\dots,R_1)$ acts in the same way as $(I, R_{n-2},\dots,R_1)$. Using similar argument as above we show that

$$\left(\begin{array}{c|c} R_{n-2} & \\ \hline & I_2 \end{array}\right)\cdots\left(\begin{array}{c|c} R_1 & \\ \hline & I_{n-1} \end{array}\right)=I_n.$$

In particular in R_{n-2} the coordinate $r_{n-2,n-2}$ must be equal to 1. Together with the condition $\widetilde{R}=I_n$ this means that $r_{n-1,n-2}=1$. Repeating these steps consecutively one shows that $R_i=I$ for all i. Therefore $R=I\in T_{GT}$ is the unique global stabilizer and the action is effective. ☐

3.3. **The Gelfand-Tsetlin polytope.** In this subsection we analyze the image $\Lambda(\mathcal{O}_\lambda)$ in $\mathbb{R}^N$, where $N:=n(n-1)/2$. The classical mini max principle (see for example Chapter I.4 in [CH]) implies that for any $A\in\mathcal{O}_\lambda$

$$\lambda_j^{(l+1)}(A)\geq\lambda_j^{(l)}(A)\geq\lambda_{j+1}^{(l+1)}(A).$$

We use the following notation for these inequalities:

$$\begin{array}{ll} A_{l,j}: & \lambda_j^{(l+1)}(A)\geq\lambda_j^{(l)}(A), \\ B_{l,j}: & \lambda_j^{(l)}(A)\geq\lambda_{j+1}^{(l+1)}(A). \end{array}\tag{3.3}$$

The inequalities (3.3) cut out a polytope in $\mathbb{R}^N$, which we denoted by $\mathcal{P}$, and $\Lambda(\mathcal{O}_\lambda)$ is contained in this polytope.

Proposition 3.5. The image $\Lambda(\mathcal{O}_\lambda)$ is exactly $\mathcal{P}$.

Proof. The Proposition follows from successive applications of the following lemma (Lemma 3.5 in [NNU], see also [GS2]), as explained below.

Lemma 3.6. For any real numbers $a_1\geq b_1\geq a_2\geq\ldots\geq a_k\geq b_k\geq a_{k+1}$ there exist $x_1,\dots,x_k$ in $\mathbb{C}$ and x_{k+1} in $\mathbb{R}$ such that the Hermitian matrix

$$A:=\begin{pmatrix} b_1 & & 0 & \bar{x}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \bar{x}_k \\ x_1 & \dots & x_k & x_{k+1} \end{pmatrix},$$

has eigenvalues $a_1,\dots,a_{k+1}$.

Now let $c_1,\dots,c_{k-1}$ be numbers such that $b_1\geq c_1\geq b_2\ldots\geq b_{k-1}\geq c_{k-1}\geq b_k$. Applying Lemma 3.6 again, we get that there exist $y_1,\dots,y_{k-1}$ in $\mathbb{C}$ and y_k in $\mathbb{R}$ such that the Hermitian matrix

$$B:=\begin{pmatrix} c_1 & & 0 & \bar{y}_1 \\ & \ddots & & \vdots \\ 0 & & c_{k-1} & \bar{y}_{k-1} \\ y_1 & \dots & y_{k-1} & y_k \end{pmatrix},$$

has eigenvalues $b_1, \dots, b_k$. Therefore there is an invertible matrix $C \in U(k)$ such that $CBC^{-1} = \operatorname{diag}(b_1, \dots, b_k)$. Denote by X the column vector $(x_1, \dots, x_k)^T$. Notice that

$$\left(\begin{array}{c|c} C & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 \ \dots \ 0 & 1\end{array}\right) \left(\begin{array}{c|c} B & C^{-1}\overline{X} \\ \hline X^T C & x_{k+1}\end{array}\right) \left(\begin{array}{c|c} C^{-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 \ \dots \ 0 & 1\end{array}\right) = \left(\begin{array}{c|c} CBC^{-1} & C\,C^{-1}\overline{X} \\ \hline X^T C\,C^{-1} & x_{k+1}\end{array}\right) = A$$

Therefore the Hermitian matrix

$$\left(\begin{array}{c|c} B & C^{-1}\overline{X} \\ \hline X^T C & x_{k+1}\end{array}\right)$$

has desired values of the Gelfand-Tsetlin functions $\lambda_*^{(k+1)}, \lambda_*^{(k)}, \lambda_*^{(k-1)}$. Continuing this process, we construct a matrix A in $\mathcal{O}_\lambda$ such that $\Lambda(A) = L$, for any chosen point L in the polytope $\mathcal{P}$. $\square$

The polytope $\mathcal{P} \subset \mathbb{R}^N$ is called the **Gelfand-Tsetlin polytope**. We think of $\mathbb{R}^N$ as having coordinates $\{x_k^{(j)}\}$, indexed by pairs (j,k), for $j = 1, \dots, n-1$, and $k = 1, \dots, j$, so that $x_k^{(j)}$-th coordinate of $\Lambda(A)$ is $\lambda_k^{(j)}(A)$.

Lemma 3.7. Let $\Lambda = \Lambda(A)$, $A \in \mathcal{O}_\lambda$, be a point in the polytope $\mathcal{P}$, with coordinates $\{\lambda_k^{(j)}(A)\}$. Suppose that for any (j,k), $j = 1, \dots, n-1$, $k = 1, \dots, j$, we have that

$$\lambda_k^{(j)}(A) = \lambda_k^{(j+1)}(A) \text{ or } \lambda_k^{(j)}(A) = \lambda_{k+1}^{(j+1)}(A).$$

Then Λ is a vertex of the polytope $\mathcal{P}$.

Proof. For any pair (j,k) pick one equality, $A_{j,k}$ or $B_{j,k}$, that is satisfied by Λ (if both are satisfied pick either one of them). Arrange these inequalities to be of the form:

$$(\text{linear combination of variables } x_k^{(j)}) \leq \text{real constant}.$$

Sum all of these N inequalities together, forming the inequality

$$CX \leq Z,$$

where $X = (x_1^{(n-1)}, \dots, x_1^{(1)}) \in \mathbb{R}^N$ is the variable, and $Z, C \in \mathbb{R}^N$ are constants. Every $X \in \mathcal{P}$ has to satisfy $CX \leq Z$, as this is just a sum of N of the $2N$ inequalities defining $\mathcal{P}$. Therefore $\mathcal{P} \cap \{X;\, CX = Z\}$ is a face of $\mathcal{P}$, (see Definition 2.1 in [Zi]). Note that $X \in \mathcal{P}$ satisfies $CX = Z$ if and only if all of the N inequalities defining $\mathcal{P}$ we have summed, are equalities for X. This determines the values of all $x_k^{(j)}$ in terms of $\lambda_1, \dots, \lambda_n$. Therefore

$$\mathcal{P} \cap \{X;\, CX = Z\} = \{\Lambda(A)\}$$

is a 0-dimensional face, in other words a vertex of $\mathcal{P}$. $\square$

To emphasize the main idea of this proof, we give the following example.

Example 3.8. Let $n = 3$, $\lambda = (5,5,4)$ and $\Lambda(A) = (\lambda_1^{(2)}(A), \lambda_2^{(2)}(A), \lambda_1^{(1)}(A)) = (5,4,5)$. We need to choose inequalities $A_{j,k}, B_{j,k}$, one for each pair (j,k), that are equalities for $\Lambda(A)$. For

$\lambda_1^{(2)}(A)$ we have a choice as both of them are equations. Say we pick $B_{2,1}$, $B_{2,2}$ and $A_{1,1}$. The set of rearranged inequalities is

$$\begin{aligned} -x_1^{(2)} &\leq -\lambda_2 = -5 \\ -x_2^{(2)} &\leq -\lambda_3 = -4 \\ x_1^{(1)} - x_1^{(2)} &\leq 0 \end{aligned}$$

Summing these inequalities together we obtain

$$-2x_1^{(2)} - x_2^{(2)} + x_1^{(1)} \leq -9.$$

This inequality is satisfied on all $\mathcal{P}$. An element $X \in \mathcal{P}$ satisfies $-2x_1^{(2)} - x_2^{(2)} + x_1^{(1)} = -9$ if and only if

$$\begin{aligned} -x_1^{(2)} &= -5 \\ -x_2^{(2)} &= -4 \\ x_1^{(1)} &= x_1^{(2)}. \end{aligned}$$

Thus, we see that $(5, 4, 5)$ is the unique solution to these inequalities in $\mathcal{P}$.

Lemma 3.9. The map Λ sends every T^n-fixed point to a vertex of $\mathcal{P}$.

Proof. For a diagonal matrix $F = \operatorname{diag}(F_{1,1}, \dots, F_{n,n})$, the set of eigenvalues of $F_{j+1} := \Phi^{j+1}(F)$ is obtained from the set of eigenvalues of $F_j := \Phi^j(F)$ by adding $F_{j+1,j+1}$. Let s be such that

$$\lambda_s^{(j)}(F) \geq F_{j+1,j+1} > \lambda_{s+1}^{(j)}(F).$$

Then

$$\begin{aligned} \forall_{l \leq s} \quad \lambda_l^{(j)}(F) &= \lambda_l^{(j+1)}(F) \\ \forall_{l > s} \quad \lambda_l^{(j)}(F) &= \lambda_{l+1}^{(j+1)}(F). \end{aligned}$$

Therefore $\Lambda(F)$ is a vertex of $\mathcal{P}$, by Lemma 3.7. □

Lemma 3.10. Let $\Lambda = \Lambda(A)$, for $A \in \mathcal{O}_\lambda$, be a point in the polytope $\mathcal{P}$, with coordinates $\{\lambda_k^{(j)}(A)\}$. Suppose that there exists exactly one pair of indices (j_0, k_0) such that both inequalities A_{j_0,k_0} and B_{j_0,k_0} at the point A are strict. That is, for all $(j,k) \neq (j_0.k_0)$, $j = 1, \dots, n-1$, $k = 1, \dots, j$, we have one of the equalities

$$\lambda_k^{(j)}(A) = \lambda_k^{(j+1)}(A) \text{ or } \lambda_k^{(j)}(A) = \lambda_{k+1}^{(j+1)}(A).$$

Then $\Lambda(A)$ is contained in the interior of an edge of $\mathcal{P}$.

Proof. Proceed similarly as in the proof of Lemma 3.7. For any $(j,k) \neq (j_0.k_0)$ choose one of the inequalities $A_{j,k}, B_{j,k}$ that is equality for $\Lambda(A)$. Arrange these inequalities to be of the form:

$$(\text{linear combination of variables } x_k^{(j)}) \leq \text{real constant}.$$

Sum all of these $N-1$ inequalities together forming the inequality

$$CX \leq Z.$$

As before, this gives an inequality valid for $\mathcal{P}$, and $\mathcal{P} \cap \{X; CX = Z\}$ is a face of $\mathcal{P}$. The equation $CX = Z$ determines the values of all $x_k^{(j)}$, with $(j,k) \neq (j_0, k_0)$, in terms of $\lambda_1, \dots, \lambda_n$ and $x_{k_0}^{(j_0)}$. These uniquely determined values are $x_k^{(j)} = \lambda_k^{(j)}(A)$. For any assignment of the value for $x_{k_0}^{(j_0)}$, the equation $CX = Z$ will still hold. In order to have $X \in \mathcal{P}$ we need to pick the value for $x^{(j_0)}_{k_0}$ in the open interval $(x^{(j_0+1)}_{k_0}, x^{(j_0+1)}_{k_0+1}) = (\lambda^{(j_0+1)}_{k_0}(A), \lambda^{(j_0+1)}_{k_0+1}(A))$. Note that $\lambda^{(j_0+1)}_{k_0}(A) \neq \lambda^{(j_0+1)}_{k_0+1}(A)$ because if they were equal, then they would also be equal to $\lambda^{(j_0)}_{k_0}(A)$ what contradicts our assumptions. Thus we really are choosing the value for $x^{(j_0)}_{k_0}$ from the open, non-degenerate interval $(\lambda^{(j_0+1)}_{k_0}(A), \lambda^{(j_0+1)}_{k_0+1}(A))$. Therefore

$$\mathcal{P} \cap \{X;\, CX = Z\} \cong (\lambda^{(j_0+1)}_{k_0}(A), \lambda^{(j_0+1)}_{k_0+1}(A))$$

is a 1-dimensional face of $\mathcal{P}$. □

Proposition 3.11. For any λ, the dimension of the polytope $\mathcal{P}$ is half of the dimension of $\mathcal{O}_\lambda$.

Proof. Fix $\lambda \in (\mathfrak{t}^n)^*_+$, not necessarily generic. Let $l_1, \dots, l_s$ be the integers such that $l_1 + \ldots + l_s = n$ and

$$\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{n-l_s+1} = \ldots = \lambda_n.$$

Consider the coadjoint orbit $M := \mathcal{O}_\lambda$ in $U(n)$. The dimension of $\mathcal{O}_\lambda$ was already computed in Section **??** and is equal to

$$2D := 2\,[\,l_1(l_2 + \ldots l_s) + l_2(l_3 + \ldots + l_s) + \ldots + l_{s-1}l_s\,] = 2\sum_{i<j} l_i l_j.$$

If some $l_j > 1$, then the $(l_j - 1)$ functions $\lambda^{(1)}_{l_1+\ldots+l_{j-1}+1} = \ldots = \lambda^{(1)}_{l_1+\ldots+l_j-1}$ have to be equal to $\lambda_{l_1+\ldots+l_{j-1}+1}$ due to inequalities (3.3). Lemma 3.6 implies that the image $\Lambda^{(1)}(\mathcal{O}_\lambda)$ in $(\mathfrak{t}^{n-1})^* \cong \mathbb{R}^{n-1}$ has dimension equal to the number of non-constant functions from $\lambda^{(1)}_*$ that is

$$n - 1 - \sum_{j=1}^{s} (l_j - 1).$$

Inequalities (3.3) force also $(l_j - 2)$ of functions $\lambda^{(2)}_*$ to be equal to $\lambda_{l_1+\ldots+l_{j-1}+1}$, as well as $l_j - 3$ of functions $\lambda^{(3)}_*$, etc. The number of our functions λ^*_* that are constant is

$$\frac{l_1(l_1-1)}{2} + \ldots + \frac{l_s(l_s-1)}{2}.$$

The remaining functions form the system of action coordinates, consisting of

$$\frac{n(n-1)}{2} - \left(\frac{l_1(l_1-1)}{2} + \ldots + \frac{l_s(l_s-1)}{2}\right) = \sum_{i<j} l_i l_j = D$$

independent functions (see Proposition 3.5 and its proof). Therefore the dimension of the image $\Lambda(\mathcal{O}_\lambda)$ is D. Recall from Section 3.2 that the Gelfand-Tsetlin torus $T_{GT} \cong (S^1)^D$ is a subtorus of $T_{U(n-1)} \oplus \ldots \oplus T_{U(1)} \cong (S^1)^N$ corresponding to D non-constant functions $\lambda^{(*)}_*$. Therefore $\mathcal{P} \subset (\mathfrak{t}_{GT})^* \subset \mathbb{R}^N$. □

If $\mathcal{F}$ is a face of $\mathcal{P}$ containing some $x \in \Lambda(U)$, then, by the definition of U, x is not on any regular wall. Therefore any point of the interior $\mathcal{F}$ also cannot be on any regular wall, so it is in U.

Lemma 3.12. If λ is generic, then the images of fixed points of standard T^n action are in U. If λ is non generic but there is only one eigenvalue that is repeated - then there is a T^n-fixed point that is in U.

Proof. If λ is generic, then for any T^n-fixed point F and any k, the matrix $\Phi^j(F)$ is a diagonal matrix with all diagonal entries distinct. Therefore $\Lambda(F)$ is not on any regular wall, so it is in U.
Now assume that λ is of the form

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{l_1} = \lambda_{l_1+1} = \ldots = \lambda_{l_1+s} > \lambda_{l_1+s+1} > \ldots > \lambda_n.$$

Let $\{v_1 > v_2 > \ldots > v_{n-s}\} = \{\lambda_1 > \lambda_2 > \ldots > \lambda_l > \lambda_{l_1+s+1} > \ldots > \lambda_n\}$ be the set of distinct eigenvalues. Consider the T^n-fixed point

$$F = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \lambda_{l_1}\mathrm{Id}_s \end{array}\right)$$

where A is any diagonal $(n-s)\times(n-s)$ matrix with spectrum $\{v_1, v_2, \ldots, v_{n-s}\}$. The figure below presents the values of Gelfand-Tsetlin functions $\lambda_k^{(j)}$ at F, for $j \geq n-s$ For $j \leq n-s$ the values $\lambda_1^{(j)}(F), \ldots, \lambda_j^{(j)}(F)$ are all distinct.

$$\begin{array}{ccccccccccccccccccc}
v_1 & & \ldots & & v_{l_1-1} & & v_l & & & \ldots & & v_{l_1} & & v_{l_1+1} & & \ldots & & v_{n-s} \\
 & v_1 & & \ldots & & v_{l_1-1} & & v_l & & \ldots & v_{l_1} & & v_{l_1+1} & & \ldots & & v_{n-s} & \\
 & & \ddots & & \ddots & & & & & \vdots & & & & \iddots & & & & \\
 & & & v_{l_1} & & \ldots & & v_{l_1-1} & & v_{l_1} & v_{l_1+1} & & \ldots & & v_{n-s} & & &
\end{array}$$

Therefore $\lambda_j^{(k)} = \lambda_{j+1}^{(k)}$ at F if and only if this equation is valid for the whole orbit. This shows that the fixed point F of the form described above is in the set U. □

We call Λ images of such T^n-fixed points, $\mathcal{O}_\lambda^{T^n} \cap U$, **good vertices** of $\mathcal{P}$. For example, in the case of regular $SU(3)$ orbit the Gelfand-Tsetlin polytope (see Figure 2) has 6 good vertices. The unique vertex with 4 adjacent edges is not a good vertex. In fact, preimage of this vertex is $\mathcal{O}_\lambda \setminus U$.

Now consider a non-regular example: $\lambda = (5,4,4,4,3,1)$. Here is the T^n-fixed point and its Gelfand-Tsetlin functions (the bold ones are constant on the whole orbit)

$$F = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 5 & & & & \\ & & 3 & & & \\ \hline & & & 4 & & \\ & & & & 4 & \\ & & & & & 4 \end{array}\right), \qquad \begin{array}{ccccccccc} 5 & & \mathbf{4} & & \mathbf{4} & & 3 & & 1 \\ & 5 & & \mathbf{4} & & 3 & & 1 & \\ & & 5 & & 3 & & 1 & & \\ & & & 5 & & 1 & & & \\ & & & & 1 & & & & \end{array}$$

Note that the vertex $\Lambda(\mathrm{diag}(1,4,4,4,3,5))$ is not a good vertex.

Proposition 3.13. For any good vertex $V_F = \Lambda(F)$ there are exactly D edges in $\mathcal{P}$ emanating from $\Lambda(F)$.

Proof. All the Λ preimages of interiors of faces containing $\Lambda(F)$, are also in U. Thus around F we have a smooth, effective, Hamiltonian action of T^D on U. The local normal form theorem, (see for example [KT2]), gives that, in a suitably chosen basis, the image of moment map is a D dimensional orthant. In particular this proves that there are exactly D edges starting from this point. □

Note that there may be more then D edges starting from vertices of $\mathcal{P}$ that are not good vertices.

4. Proof of the main theorem

Let $\mathcal{O}_\lambda$ be a coadjoint orbit such that the Gelfand-Tsetlin polytope $\mathcal{P}$ contains at least one good vertex. In particular λ can be of the form

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{l_1} = \lambda_{l_1+1} = \ldots = \lambda_{l_1+s} > \lambda_{l_1+s+1} > \ldots > \lambda_n,\ s \geq 0.$$

Recall that D denotes half of the dimension of $\mathcal{O}_\lambda$, which is equal to the dimension of the Gelfand-Tsetlin torus T_{GT}, and that $\mathcal{P} \subset (\mathfrak{t}_{GT})^* \subset (\mathfrak{t}^N)^* \cong \mathbb{R}^N$. We are to show that the Gromov width of $\mathcal{O}_\lambda$ is at least $\min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$.

Proof. (of Theorem 1.1) Let $\Lambda(F) = V_F$ be a good vertex of $\mathcal{P}$ and $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$ such that

$$\Lambda(\mathcal{O}_\lambda) \cap \mathcal{T} = \bigcup_{\substack{\mathcal{F} \text{ face of } \mathcal{P} \\ V_F \in \mathcal{F}}} (\text{rel-int } \mathcal{F})$$

and let $\mathcal{W} = \Phi^{-1}(\mathcal{T})$. Then $\mathcal{W}$ is the largest subset of M centered around V_F (compare with Example 2.4). According to the Proposition 2.6 there is an equivariant symplectomorphism

$$\Psi : \left\{ z \in \mathbb{C}^D \mid V_F + \pi \sum |z_j|^2 \eta_j \in \mathcal{T} \right\} \xrightarrow{\cong} \mathcal{W},$$

where $-\eta_1, \ldots, -\eta_D$ are the isotropy weights of T^D action on $T_F\mathcal{O}_\lambda$. The vectors $\eta_1, \ldots, \eta_D$ span D edges of $\mathcal{P}$ starting from V_F. We call them the **edge generators**. For the edge in the direction of η_l, there is a number $c_l \in \mathbb{R}$ such that the edge is precisely $c_l\, \eta_l$. This is equivalent to saying that the edge is of lattice length c_l with respect to the weight lattice, because for the coadjoint $U(n)$ action all isotropy weights are primitive with respect to the lattice they span. Let

$$\{v_1 > v_2 > \ldots > v_{n-s}\}$$

be the set of distinct eigenvalues. Proposition 4.1, proved below, shows that for any edge, c_l is at least the minimum $\min\{v_i - v_{i+1}\} = \min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$. Therefore a ball of capacity $\min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$ embeds symplectically into $\mathcal{W} \subset \mathcal{O}_\lambda$, as explained in the Example 2.7. □

Moreover, we will show that for any good vertex there is an edge leaving from this vertex, with the length equal to the minimum of $v_i - v_{i+1}$ times the length of the edge generator. This means that the lower bound we prove is the best possible we can get from this particular action. Let us emphasize that there might exist symplectic embeddings of bigger balls, however this method fails to find them.

Proposition 4.1. The length of any edge in $\mathcal{P}$ starting from a good vertex V_F is at least $\min\{v_i - v_{i+1}\}$ times the length of the edge generator. Moreover, there is an edge with length exactly the $\min\{v_i - v_{i+1}\}$ times the length of its generator.

Proof. Recall from Section 3.2 that the momentum maps for the standard and the Gelfand-Tsetlin torus actions are related through projection pr, $\mu = pr \circ \Lambda$. We continue to denote the polytope $\mu(\mathcal{O}_\lambda)$ by $\mathcal{Q}$ and its one-skeleton (image of points whose orbits have dimension at most 1) by $\mathcal{Q}_1$ We will show that for any edge $e \in \mathcal{P}$ starting from V_F there is an edge e' in $\mathcal{Q}_1$ (possibly not and edge but just a line segment in $\mathcal{Q}$) such that $pr(e) \subset e'$. This will help us to analyze edges of $\mathcal{P}$.

Denote the diagonal entries of F by $F_{11}, \ldots, F_{nn}$. Let $p < q$ be indices from $\{1, \ldots, n\}$ such that $v_i = F_{pp} \neq F_{qq} = v_k$ and F' is the matrix obtained from F by switching p-th and q-th entry. There is an edge in $\mathcal{Q}_1$ joining $\mu(F)$ and $\mu(F')$, and it is an μ-image of a sphere $S := \{F_z;\, z \in \mathbb{C} \cup \{\infty\}\}$ in $\mathcal{O}_\lambda$ defined in the Section **??**. We will analyze $\Lambda(S)$.

Assume that $v_k < v_i$. The other case is proved in a similar way. First observe that for $j < p$ the matrices $(F_z)_j := \Phi^j(F_z)$ and $(F)_j := \Phi^j(F)$ are both equal to diag $(F_{1,1}, \ldots, F_{j,j})$. Also for $j \geq q$ the matrices $(F_z)_j$ and F_j have the same eigenvalues. This is because the eigenvalues of this 2×2 matrix

$$\begin{bmatrix} \frac{(v_i+|z|^2 v_k)}{Z} & \frac{\bar{z}(v_i - v_k)}{Z} \\ \frac{z(v_i-v_k)}{Z} & \frac{(v_k+|z|^2 v_i)}{Z} \end{bmatrix},$$

where $Z = \sqrt{1+|z|^2}$, are v_i and v_k. Therefore, for $j < p$ or $j \geq q$, we have

$$\forall_{F_z \in S} \;\; \lambda_m^{(j)}(F_z) = \lambda_m^{(j)}(F), \tag{4.1}$$

for any $m = 1, \ldots, n-j$. Denote by $\rho(|z|) = \frac{(v_i+|z|^2 v_k)}{Z}$. While a goes to ∞, ρ decreases its value from v_i to v_k. Let

$$i' = \min\{l\,;\; v_l \in \{F_{11}, \ldots, F_{qq}\}, \;\; v_i > v_l\}.$$

This implies that $i+1 \leq i' \leq k$. Note that i' is not necessarily $i+1$, as it might happen that v_{i+1} is a diagonal entry of F that does not sit in a submatrix $(F)_q$. Lemmas 4.2 and 4.3 below show that the set $\Lambda(\{F_z \mid \rho(|z|) \in [v_{i'}, v_i]\})$ is an edge of $\mathcal{P}$ starting from V_F. Now we need to compute it's length relative to the length of the edge generator (= $-$isotropy weight). Notice that the projection pr (induced by inclusion $T^n \hookrightarrow T_{GT}$) maps the isotropy weights of T_{GT} action to the isotropy weights of T^n action. If $e = c_l \eta_l$ is the edge of $\mathcal{P}$, then $pr(e) = c_l pr(\eta_l)$ is the part of the corresponding edge e' of $\mathcal{Q}_1$ starting from the vertex $\mu(F)$. The edge generator in the direction $pr(\eta_l)$ is $-e_{pp} + e_{qq}$ (because the isotropy weight of the standard action of maximal torus is $e_{pp} - e_{qq}$). We will denote $\widetilde{Z} := \{F_z \,|\, \rho(|z|) = v_{i'}\}$ and $\widetilde{V} := \Lambda(\widetilde{Z})$, regardless of the fact if it is a vertex or an interior point of and edge in $\mathcal{P}$. Notice that $\widetilde{V}$, has values of Λ that are different from those of F in exactly $(q-p)$ places. Precisely, for every $p \leq j < q$, there is exactly one s such that $\lambda_s^{(j)}(F) = v_i$ while $\lambda_s^{(j)}(\widetilde{Z}) = v_{i'}$. Recall from section 3.2 that the $k-th$ coordinate of $pr(\{\lambda_*^{(*)}\})$ is given by

$$(\, pr(\{\lambda_*^{(*)}\})\,)_k = \sum_{s=1}^{k} \lambda_s^{(k)} - \sum_{s=1}^{k-1} \lambda_s^{(k-1)}$$

for $k > 1$ and is equal to $\lambda_1^{(1)}$ for $k = 1$. Therefore $\mu(F) = pr(\Lambda(F))$ and $\mu(\widetilde{Z}) = pr(\Lambda(\widetilde{Z}))$ differ only at p-th and q-th coordinates:

$$\begin{aligned}(\, pr(\Lambda(F))\,)_p &= \sum_{s=1}^{p} \lambda_s^{(p)}(F) - \sum_{s=1}^{p-1} \lambda_s^{(p-1)}(F) \\ &= \sum_{s=1}^{p} \lambda_s^{(p)}(\widetilde{Z}) + v_i - v_{i'} - \sum_{s=1}^{p-1} \lambda_s^{(p-1)}(\widetilde{Z}) = (\, pr(\Lambda(\widetilde{Z}))\,)_p + v_i - v_{i'}\end{aligned}$$

$$\begin{aligned}(\, pr(\Lambda(F))\,)_q &= \sum_{s=1}^{q} \lambda_s^{(q)}(F) - \sum_{s=1}^{q-1} \lambda_s^{(q-1)}(F) \\ &= \sum_{s=1}^{q} \lambda_s^{(q)}(\widetilde{Z}) + v_i - v_{i'} - (\sum_{s=1}^{q-1} \lambda_s^{(q-1)}(\widetilde{Z}) + v_i - v_{i'}\,) \\ &= (\, pr(\Lambda(\widetilde{Z}))\,)_q - (v_i - v_{i'})\end{aligned}$$

Thus
$$\overline{\mu(F)\,\mu(\widetilde{Z})} = (v_i - v_{i'})(-e_{pp} + e_{qq}),$$
and the edge e of $\mathcal{P}$ is at least $(v_i - v_{i'})$ multiple of the weight spanning it. Recall from definition of i' that $(v_i - v_{i'}) \geq (v_i - v_{i+1})$.

In case where $v_k > v_i$, $\rho(|z|)$ would be increasing its value from v_i to v_k and we would prove in an analogous way that the edge joining F and F' is at least $(v_{i-1} - v_i)$ multiple of the edge generator.

Notice that different pairs of p and q (such that $F_{pp} \neq F_{qq}$) give different edges. This follows, for example, from the fact that for $j < p$ or $j \geq q$, we have $\lambda_s^{(j)}(F_z) = \lambda_s^{(j)}(F)$. Therefore we found D distinct edges of $\mathcal{P}$ starting from V_F. The Proposition 3.13 gives that these must be all the edges.

Now suppose that m is the index such that the minimum of $\{v_i - v_{i+1} \,|\, i = 1, \ldots, s\}$ is equal to $v_m - v_{m+1}$. There are indices $p < q$ such that $F_{p,p} = v_m$ and $F_{q,q} = v_{m+1}$, or $F_{p,p} = v_{m+1}$ and $F_{q,q} = v_m$. Let F' be the diagonal matrix obtained from F by switching p-th and q-th entry. Then $\widetilde{Z} = F'$, $\widetilde{V} = \Lambda(F')$ and the edge of $\mathcal{P}$ between these two vertices is exactly $(v_m - v_{m+1})$ multiple of the edge generator. □

The above proof used two lemmas that we formulate and prove below.

Lemma 4.2. For z such that $v_i > \frac{(v_i + |z|^2 v_k)}{Z} = \rho(|z|) > v_{i'}$ the point $\Lambda(F_z)$ is in the interior of an edge of $\mathcal{P}$.

Proof. Let m be such that
$$\lambda_m^{(q-1)}(F_z) = v_i > \rho(|z|) = \lambda_{m+1}^{(q-1)}(F_z).$$
We will show that for any $(j,l) \neq (q-1, m)$, $j = 1, \ldots, n-1$, $l = 1, \ldots, j$, we have that
$$\lambda_l^{(j)}(F_z) = \lambda_l^{(j+1)}(F_z) \text{ or } \lambda_l^{(j)}(F_z) = \lambda_{l+1}^{(j+1)}(F_z),$$
and use the Lemma 3.10. The matrix $(F_z)_q := \Phi^q(F_z)$ is diagonal, thus, repeating the proof of Lemma 3.9 for $(F_z)_q$, we can show that the above claim holds for $j < q-1$ and any l. Also, for $j \geq q$ the claim holds, due to equations (4.1) and Lemma 3.9. Thus, for $j \neq q-1$ and any l, the function $\lambda_l^{(j)}$ is equal at F_z to its lower or upper bound.
Now assume $j = q-1$ and notice that
$$\textit{spectrum}((F_z)_q) = \textit{spectrum}((F_z)_{q-1}) \cup \{v_i, v_k\} \setminus \{\rho(|z|)\}.$$
The Figure 4 presents sequences of ordered eigenvalues of $(F_z)_{q-1}$ and $(F_z)_q$. This presentation

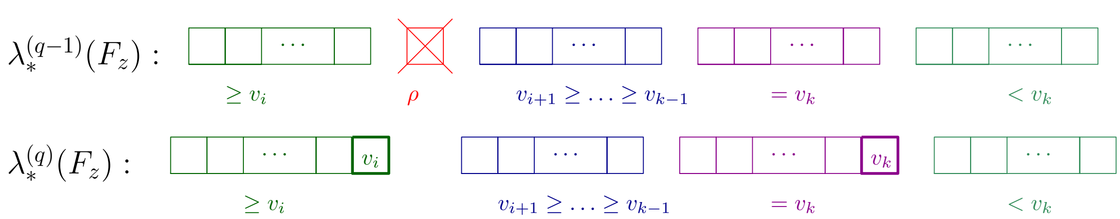


FIGURE 3. Eigenvalues of $(F_z)_{q-1}$ and $(F_z)_q$.

helps to note that
$$\forall_{t \neq m},\ \lambda_t^{(q-1)}(F_z) \geq v_k \ \Rightarrow \lambda_t^{(q-1)}(F_z) = \lambda_t^{(q)}(F_z),$$
$$\forall_{t \neq m},\ \lambda_t^{(q-1)}(F_z) < v_k \ \Rightarrow \lambda_t^{(q-1)}(F_z) = \lambda_{t+1}^{(q)}(F_z).$$

Thus by the Lemma 3.10, $\Lambda(F_z)$ is on the edge of $\mathcal{P}$. All eigenvalues of $(F_z)_q$ are equal to some element of the set $\{v_1, \ldots, v_{n-s}\}$. Therefore $\lambda_m^{(q-1)}(F_z) = \rho(|z|) \in (v_{i'}, v_i)$ is not equal to $\lambda_m^{(q)}(F_z)$ nor $\lambda_{m+1}^{(q)}(F_z)$, so $\Lambda(F_z)$ is not a vertex of $\mathcal{P}$. □

Lemma 4.3. $\Lambda(\,\{F_z \,|\, \rho(|z|) = v_{i'}\}\,)$ is a vertex of $\mathcal{P}$.

Proof. Similarly to the proof of Lemma 4.2, we show that for $(j, l) \neq (q-1, m)$, $j = 1, \ldots, n-1$, $l = 1, \ldots, j$, the function $\lambda_l^{(j)}$ at F_z is equal to its lower or upper bound (again use Figure 4). However this time $\lambda_m^{(q-1)}(F_z) = \rho(|z|) = v_{i'} = \lambda_{m+1}^{(q)}(F_z)$. We use Lemma 3.7 to deduce that $\Lambda(\,\{F_z \,|\, \rho(|z|) = v_{i'}\}\,)$ is a vertex of $\mathcal{P}$. □

5. Low-dimensional examples.

In this section we summarize what is known about Gromov width of $U(n)$ coadjoint orbits. The table below presents low dimensional examples for which it was proved that lower bound of Gromov width is as expected: the minimum of $\lambda_j - \lambda_j$ over $\lambda_i > \lambda_j$. The table also specifies if this fact follows directly from our Main Theorem; if it requires Remark 3.1; or if it was proved using different methods. Generic $U(1)$ orbits, and degenerate $U(2)$ orbits are just points, so their Gromov width is 0. Gromov width of generic orbits satisfying some integrality conditions was already calculated by Zoghi in [Z].

n	λ	Thm 1.1	Rem. 3.1	Other
2	generic $\rightsquigarrow$ sphere degenerate $\rightsquigarrow$points	$\surd$		Delzant Thm; also [Z]
3	any λ	$\surd$		generic - proved in [Z]
4	$(\lambda_1, \lambda_1, \lambda_2, \lambda_2)$ $\rightsquigarrow$ complex Grassmannian of 2-planes in $\mathbb{C}^4$	–	$\surd$	Karshon and Tolman, [KT1, Theorem 1]
4	other λ	$\surd$		generic - proved in [Z]
5	$(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3)$ $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2)$ $(\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2)$	–	$\surd$	
5	other λ	$\surd$		generic - proved in [Z]
6	$(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$	–	–	–

In the case of $n = 6$, there is already an orbit for which we still don't have even the lower bound of the Gromov width. Namely $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$. For all the other orbits, the lower bound or even exact Gromov width is proved in Theorem 1.1 together with Remark 3.1, or in [KT1], or [Z].

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