The root system of a multiplicity free manifold

(e.g. $SU(n), U(n)$)

K compact conn. Lie gp.; Lie alg. $\mathfrak{k}$; dual $\mathfrak{k}^*$

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T_K max. torus (e.g. {diag. matrices}) $\quad \mathfrak{t} \quad \mathfrak{t}^*$

Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}^* \equiv (\mathfrak{k}^*)^{T_K}$, $\Lambda \subseteq \mathfrak{t}^*$ weight lattice

$(M, \mu : M \to \mathfrak{k}^*)$ comp. conn. Ham. K-mfd.

THEOREM (Kirwan 1984)

$\mathcal{P}(M) := \mu(M) \cap \mathfrak{t}_+$ is a convex polytope

the momentum polytope of M

Examples: ⓪ Coadjoint orbits $\mathcal{P} = \{pt\}$

① $K = SU(2)$ $\quad T_K = \{\text{diag. matrices in } SU(2)\}$

$\mathfrak{k}^* \cong \mathbb{R}^3, \ \mathfrak{t}^* \cong \mathbb{R}e_3, \ \mathfrak{t}_+ \cong \mathbb{R}_{\geq 0}e_3$

$$M = Ke_3 \times Ke_3 \xrightarrow{\mu} \mathfrak{k}^* : (v,w) \longmapsto v+w$$

$$\cong S^2 \times S^2 \qquad \Longrightarrow \mathcal{P}(M) = [0,2]e_3$$

② $K = SU(n) \circlearrowright V = \{\text{quadratic forms in } n \text{ variables}\}$

$$M = \mathbb{P}(V) \qquad \mathcal{P}(M) = \mathrm{conv}\left(0, 2\omega_1, \frac{2\omega_2}{2}, \dots, \frac{2\omega_{n-1}}{n}\right)$$

$(n = 3)$

③ $K = SU(3) \circlearrowright M = \mathbb{P}(\mathbb{C}^3 \oplus \mathbb{C}^{3*})$

$$\mathcal{P}(M) = \mathrm{conv}(0, \omega_1, \omega_2)$$

Knop's classification of MF manifolds

Invariant momentum map

$$\psi: M \longrightarrow \mathfrak{t}_+ \quad : \{\psi(m)\} = K.\mu(m) \cap \mathfrak{t}_+$$

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\mu"] & \mathfrak{t}_+ \arrow[d, phantom, "\cong"] \\ \mathfrak{k}^* \arrow[r, two heads] & \mathfrak{k}^*/K \end{tikzcd}$$

M is **multiplicity free** if $\psi: M \longrightarrow \mathcal{P}(M)$ is topological quotient for the action of K on M.

Examples: (4) Toric symplectic manifolds
$\left(K = U(1)^n;\ \dim M = 2n;\ \text{action effective} \right)$

. Examples (0) – (3)

Theorem (Delzant 1988) $K = U(1)^n$

(a). A sympl. toric K – mfd M is uniquely determined by $\mathcal{P}(M)$

(b). A polytope $\mathcal{P} \subset \mathfrak{k}_*$ occurs this way iff it is Delzant

Theorem (Knop 2011)

ⓐ (Delzant conjecture) A multiplicity free K-mfd M is uniquely determined by $(\mathcal{P}(M), K_*)$

generic isotropy group; encoded by $\Xi(M) \subseteq \Lambda$ sublattice

ⓑ A pair $(\mathcal{P}, \Xi)$ with $\mathcal{P} \subset \mathfrak{t}_+$ cvx. polytope and Ξ a sublattice of Λ is realized this way IFF for every vertex a of $\mathcal{P}$ there exists a smooth affine spherical $(K_a)^{\mathbb{C}}$-variety X_a such that

$$\mathbb{R}_{\geq 0}(\mathcal{P}-a) = \mathbb{R}_{\geq 0}\Gamma(X_a) \quad \text{and} \quad \Xi = \mathbb{Z}\,\Gamma(X_a),$$

where $\Gamma(X_a)$ is the weight monoid of X_a.

G cx. conn. red. gp. (like $K^{\mathbb{C}}$)
A smooth affine G-variety X is <u>spherical</u> if its coordinate ring $\mathbb{C}[X]$ is multiplicity free as a representation of G.

$\Lambda^+ := \Lambda \cap \mathfrak{t}_+$ monoid of dominant wts.
$\longleftrightarrow \mathrm{Irr}(G)$ (highest weight theory)

The <u>weight monoid</u> of X is
$$\Gamma(X) := \{\lambda \in \Lambda^+ \mid V(\lambda) \text{ occurs in } \mathbb{C}[X]\}$$

<u>Theorem</u> (Losev 2009, Knop conjecture)
A smooth aff. sph. G-variety X is uniquely det. by $\Gamma(X)$

Back to the examples

④ toric sympl. mfds:

Knop's sphericity $\Longleftrightarrow$ Delzant condition

b/c the only smooth aff. spherical $T_K^{\mathbb{C}}$-varieties w/ effective action and pointed wt monoid are $\mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_s$ with $x_1, \dots x_s$ basis of Λ

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① $SU(2) \circlearrowright \underbrace{S^2 \times S^2}_{=: M}$ $\quad \mathcal{P}(M) = [\underset{a}{0} \quad \underset{b}{2}]$

$\Xi(M) = 2\Lambda = 2\mathbb{Z}\omega \quad (K_a)^{\mathbb{C}} = SL(2) \quad X_a = SL(2) \Big/ \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}$

$(K_b)^{\mathbb{C}} = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\} \quad X_b = \mathbb{C}_{2\omega} \quad \Gamma(X_a) = 2\Lambda^+ = 2\mathbb{N}\omega$

$\Gamma(X_b) = \mathbb{N}(-2\omega)$

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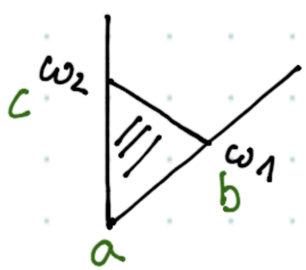
③ $SU(3) \circlearrowright M = \mathbb{P}(\mathbb{C}^3 \oplus \mathbb{C}^{3*})$



$\Xi(M) = \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

$(K_b)^{\mathbb{C}} \cong (K_c)^{\mathbb{C}} \cong GL(2)$

X_b and X_c are repres. of

$(K_a)^{\mathbb{C}} = SL(3)$

$X_a = SL(3) / SL(2)$

$\Gamma(X_a) = \Lambda^+ = \mathbb{N}\omega_1 \oplus \mathbb{N}\omega_2$

- With Pezzini (2019):
  algorithm: INPUT: G, Γ f.g. monoid of dom. wts.
  OUTPUT: YES/No to
  $\exists$? smooth. aff. sph. G-variety with wt. monoid Γ

  (implemented in Sage by W. Kim for $G = SL(n)$ and Γ free and G-saturated)

  applying this algorithm at every vertex of $\mathcal{P}$ yields an algorithm to decide whether $(\mathcal{P}, \Xi)$ is realized by a multiplicity free manifold M.
- There are tables of weight monoids of families of sm. aff. spher. varieties [Krämer 1979, Knop 1997, Avdeev 2010, Paulus-Pezzini-— 2018]

Knop's root system of a MF manifold

Motivating example: (Example 1)

$$\begin{tikzcd} S^2\times S^2 \arrow[r, "\Psi", "\text{inv. mom. map}"'] \arrow[dr, "\text{smooth}"'] & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ & \mathbb{R} \end{tikzcd}$$

$\mathbb{R} : (p,q) \longmapsto \|p+q\|_{\mathbb{R}^3}$

is not smooth.

quotient for the action of $\{\pm 1\}$

Theorem (Knop 2011) M MF K-mfd, $\mathcal{P} = \mathcal{P}(M)$

$\mathfrak{a}^\circ :=$ affine subsp. of $\mathfrak{t}^*$ spanned by $\mathcal{P}$

a) $\exists$ finite subgroup $W_0 \subseteq N_W(\mathfrak{a}^\circ)/C_W(\mathfrak{a}^\circ)$ such that

$\Psi/W_0 : M \longrightarrow \mathfrak{a}^\circ/W_0$ is smooth and

$(\Psi/W_0)^* : \mathcal{C}^\infty(\mathcal{P}/W_0) \xrightarrow{\sim} \mathcal{C}^\infty(M)^K$ iso.

b) There is a unique minimal such W_0, denoted W_M, which is generated by reflections which have a fixed point in $\mathcal{P}$.

c) $\Xi(\Pi)$ is stable under W_M (for action of W_Π on the subspace of t^* parallel to σ^0)

$\Longmapsto$ W_Π is the Weyl group of a root system.

Knop also gives a concrete descr. for a set of simple roots: (in proof)

$$\Sigma(M) := \bigcup_{\substack{a \text{ vertex} \\ \text{of } \mathcal{P}(\Pi)}} \Sigma_{alg}(X_a)$$

$\therefore$ $\Sigma_{alg}(X_a)$ //

primitive elements in $\mathbb{Z}\Gamma(X_a)$ on the extr. rays of

$$\mathbb{R}_{\geq 0}\{\lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Gamma(X_a);\ V(\nu) \subset V(\lambda) \cdot V(\mu)\} \subset t^*$$

(in $\mathbb{C}[X_a]$; $\mathbb{C}[X_a] \underset{G\text{-mod}}{\cong} \bigoplus_{\lambda \in \Gamma(X_a)} V(\lambda)$)

$\Sigma_{alg}(X_a)$ measures "how far" $\mathbb{C}[X_a]$ is from being $\Gamma(X_a)$-graded.

Back to examples

(4) M toric $\Rightarrow \Sigma(M) = \emptyset$

(in fact, for general K, a in interior of $\mathfrak{t}_+ \Rightarrow \Sigma_{alg}(X_a) = \emptyset$)

(1) $SU(2) \curvearrowright M = S^2 \times S^2$ $\quad \Sigma_{alg}\left(SL(2)/T \right) = \{\alpha\}$

$\Rightarrow \Sigma(M) = \{\alpha\}$

(3) $SU(3) \curvearrowright M = \mathbb{P}(\mathbb{C}^3 \oplus \mathbb{C}^{3*})$ $\quad \Sigma_{alg}\left(SL(3)/SL(2) \right) = \{\alpha_1 + \alpha_2\}$

and $\Sigma(M) = \{\alpha_1 + \alpha_2\}$

Remarks: a) up to changes in root lengths, the root system controls the automorphisms of M [Knop 2011]

b) $\exists$ Tables of $\Sigma_{alg}(X)$ for X smooth aff. sph.

c) aforementioned algorithm w/ Pezzini computes $\Sigma_{alg}(X)$ as a byproduct

"Kählerizable"

<u>Theorem</u> (Pezzini, –):
M admits an invariant compatible complex structure
$\Longleftrightarrow \left(\Xi(M), \mathcal{P}(M), \Sigma(M)\right)$ is an $\mathbb{R}$-momentum triple as def. in [Cupit-Foutou, Pezzini, –]
$\Longleftrightarrow \left(\Xi(M), \mathcal{P}(M), \Sigma(M)\right)$ satisfies 5 (elementary) "Luna" axioms

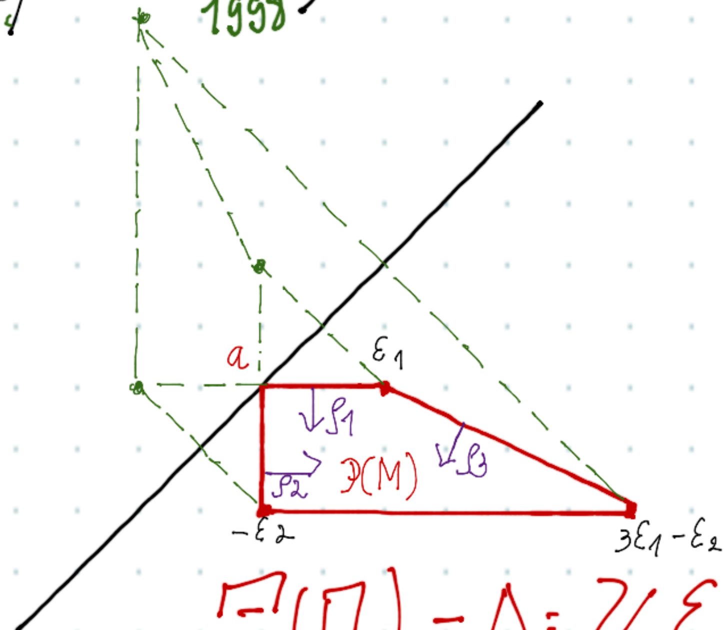
<u>Remarks:</u> (a) logically, these are conditions on $\left(\Xi(M), \mathcal{P}(M)\right)$
(b) simplification of an earlier Kähler. criterion in [CF, P, –]
(c) important ingredient for $\boxed{\Leftarrow}$ is the combinatorial classif. of spherical varieties (formerly known as Luna conjecture)
(d) toric case: $\Sigma(M) = \emptyset$ and one recovers Delzant's result that all toric symplectic mfds. are Kählerizable

Corollary: $\dim \mathcal{P}(\Pi) = 1 \Longrightarrow M$ is Kählerizable

Examples: ⓪ - ④ are Kählerizable

⑤ (Woodward 1998 / Tolman 1998)

$$K = U(2)$$



$$(K_a)^{\mathbb{C}} = GL(2)$$

$$X_a = GL(2) \Big/ \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} : t \in \mathbb{C}^\times \right\}$$

$$\Xi(\Pi) = \Lambda = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$\Sigma_{alg}(X_a) = \{\alpha\} = \Sigma(\Pi)$$

one of the "Luna" axioms says: if $\alpha \in \Sigma(\Pi)$ then $\mathcal{P}(\Pi)$ has at most 2 inward pointing facet normals pairing positively with α; $\langle \rho_i, \alpha \rangle > 0$ for $i \in \{1,2,3\}$ $\Longrightarrow M$ not Kählerizable

(as Woodward had told us)