Complexity one Hamiltonian torus actions

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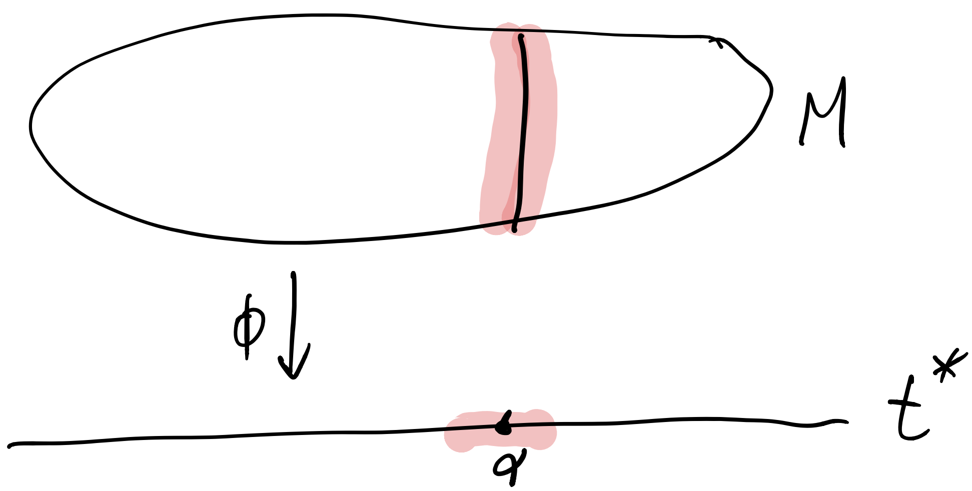
with: Susan Tolman

$$(S^1)^k \cong T \circlearrowright (M,\omega) \xrightarrow{\Phi} \mathfrak{t}^*$$

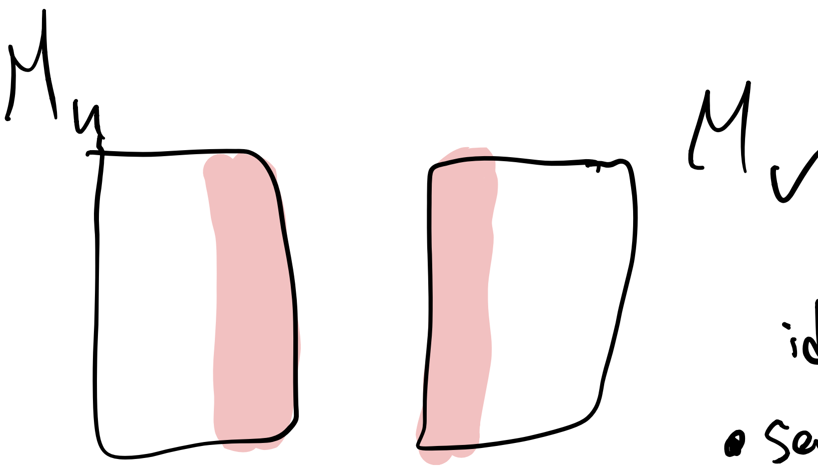
$$\forall \xi \in \mathfrak{t} \qquad d\langle \Phi, \xi \rangle = \xi_M \lrcorner\, \omega$$



Classification ?

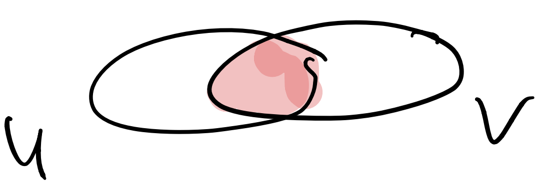
$$T \curvearrowright (M, \omega) \xrightarrow{\phi} t^*$$

- local : nbhd of orbit
- global ? e.g. M compact
- semi-local



idea:

- semi-local classification
- fit them together



$\boxed{\text{Building blocks}}$

Hamiltonian T-models

$$T \supset H \to (S^1)^\ell \circlearrowright \mathbb{C}^\ell \xrightarrow[\Phi_H]{} \mathfrak{h}^*$$

$$z \mapsto \sum \frac{|z_j|^2}{2} \eta_j$$

$$T \times \mathbb{C}^\ell \times \mathfrak{h}^0 \hookrightarrow \underset{\omega_{std}}{(T \times \mathfrak{t}^*) \times \mathbb{C}^\ell}$$

$$\downarrow$$

$$Y = T \times_H \mathbb{C}^\ell \times \mathfrak{h}^0 \, , \quad \omega_Y$$

$$\Phi_Y([t, z, \nu]) = \beta + \underbrace{\Phi_H(z)}_{\text{in } \mathfrak{h}^*} + \underbrace{\nu}_{\text{in } \mathfrak{h}^0}$$

image is convex polyhedral

$\boxed{\text{Local normal form:}}$

$\forall$ Ham. T-mfld M

point $x \in M$

$\exists$ Ham. T-model $Y = T \times_H \mathbb{C}^l \times \mathfrak{h}^0$

s.t. nbhd of x in M $\cong$ nbhd of $[1,0,0]$ in Y

$x \longleftrightarrow [1,0,0]$

stab $x = H$

isotropy $H \circlearrowright T_x M \cong H \circlearrowright \mathbb{C}^l \oplus \text{triv}$

Y is det. up to permutation of coordinates

__proof__: (1) linearize

(2) Moser method

Local properties

properties of Y:

① image is convex polyhedral

② complexity one case: (tall inserted above "one")

$$\begin{tikzcd} Y \arrow[r, "(\Phi_Y, p)"] \arrow[d] & (\text{image } \Phi_Y) \times \mathbb{C} \\ Y/T \arrow[ur, "\text{homeo}"'] & \end{tikzcd}$$

details:

$$1 \longrightarrow H \longrightarrow (S^1)^\ell \longrightarrow S^1 \longrightarrow 1$$

$$(a_1, \dots, a_\ell) \mapsto \prod a_j^{\xi_j}$$

tall: all $\xi_j \geq 0$

$$p([t, z, \nu]) = \prod_{j=1}^{\ell} z_j^{\xi_j}$$

Convexity package

$$T \circlearrowright (M,\omega) \xrightarrow{\ \Phi\ } \mathfrak{t}^*$$

compact, connected

$$\Delta := \Phi(M)$$

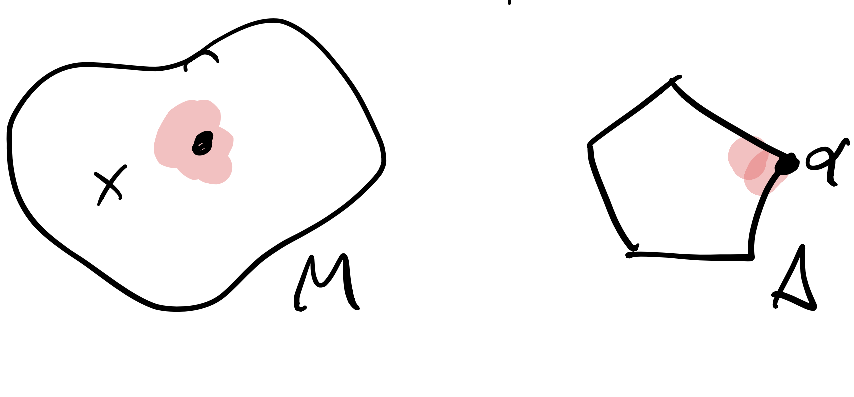
<u>Thm</u>

- Δ is convex
- $\Phi : M \to \Delta$ is open
- $\Phi^{-1}(C)$ is connected $\forall C \subseteq \mathfrak{t}^*$ convex

still true for <u>proper</u> Hamil<u>tonian</u> T-mflds:

$$T \circlearrowright (M,\omega) \xrightarrow[\text{proper}]{\ \Phi\ } \mathcal{T} \subseteq \mathfrak{t}^*$$

convex open



$\boxed{\text{Complexity}}$

$T \circlearrowright (M,\omega) \xrightarrow{\Phi} \mathfrak{t}^*$

connected

$\dim \underbrace{T_{\text{eff}}}_{T/\ker(T \circlearrowright M)} \leq \frac{1}{2} \dim M$

Toric $:\Longleftrightarrow$ complexity zero

Reduced spaces $M_\alpha := \Phi^{-1}(\alpha)/T$

Φ <u>proper</u>

stratified sympl. spaces

Complexity $= \frac{1}{2} \dim M_\alpha$

$\alpha \in \text{interior}(\Delta)$

$\boxed{\text{Duistermaat - Heckman}}$

$$T^k \circlearrowright (M^{2n}, \omega) \xrightarrow{\Phi} t^*$$

Near regular $\alpha_0 \in t^*$

$$[\omega_\alpha] = [\omega_{\alpha_0}] + \langle \underbrace{\alpha - \alpha_0}_{t^*}, \underbrace{c_1 \begin{pmatrix} \Phi^{-1}(\alpha_0) \\ \downarrow \\ M_{\alpha_0} \end{pmatrix}}_{H^2(M_{\alpha_0}, t)} \rangle$$

DH measure $:= \Phi_* \left| \frac{\omega^n}{n!} \right| = \varphi(\alpha)\, |d\alpha|$

$$\varphi(\alpha) = \int_{M_\alpha} \frac{\omega_\alpha^d}{d!} \, (2\pi)^k \qquad \alpha \in \operatorname{int}(\Delta)$$

$d =$ the complexity

up to constant, determined by the topology of $T \circlearrowright M \xrightarrow{\Phi} t^*$

$\boxed{\text{letting go of the s. form}}$

$(M, \omega, \Phi) \cong (M', \omega', \Phi')$

complexity one

iff

$$\begin{tikzcd} M \arrow[rr, "\exists f"', "\substack{\text{or. preserving} \\ T\text{-eq. diffeo}}"] \arrow[dr, "\Phi"'] & & M' \arrow[dl, "\Phi'"] \\ & \mathfrak{t}^* & \end{tikzcd}$$

Φ-T diffeo

& same DH measure

idea. $\omega_t = (1-t)\omega + t f^* \omega'$ $\quad t \in [0,1]$

- Nondeg
- same momentum map
- same coh. class

$\boxed{\text{Passing to the quotient}}$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & M' \\ M/T \arrow[r, "\bar{f}"] & M'/T \end{tikzcd}$$

ϕ-T diffeo

ϕ-diffeo

$:\Leftrightarrow$ locally lifts to ϕ-T diffeos

$\exists\ \phi$-T diffeo $M \to M'$

iff $\exists\ \phi$-diffeo $M/T \to M'/T$

and same DH measures

$\boxed{\text{Orbit type strata}}$ $T \circlearrowright M$

components of $M_H := \{x \mid \operatorname{stab} x = H\}$

$\mathfrak{X} := \{\text{orbit type strata}\}$

poset wrt $X_1 \subset \overline{X_2}$

$$\underset{\text{component}}{X \subset M_H} \Rightarrow \underset{\text{component}}{\overline{X} \subset M^H}$$

$$T \to T/H \circlearrowright \underset{\substack{\text{smooth}\\\text{sympl.}}}{(\overline{X}, \omega)} \xrightarrow{\Phi} \mathfrak{t}^*$$

$$\text{complexity } \overline{X} \le \text{complexity } M$$

$\boxed{\text{Sympl. toric mflds}}$

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d] & \Delta \subset \mathfrak{t}^* \\ M/T \arrow[ur, "\bar{\Phi}"'] & \end{tikzcd}$$

diffeo
of mflds w/ corners

orbit type strata of M $\longleftrightarrow$ strata of Δ

$\boxed{\text{skeleton}}$ of a cplxty one space:

$$S = \bigsqcup_{\substack{\text{toric} \\ \text{strata in } M_{tall}}} X/T \hookrightarrow M/T$$

"freckles" on reduced spaces

$$T^{n-1} \circlearrowright M^{2n} \xrightarrow{\phi} \Delta \subset \mathfrak{t}^*$$

$$M/T \longrightarrow \Delta$$

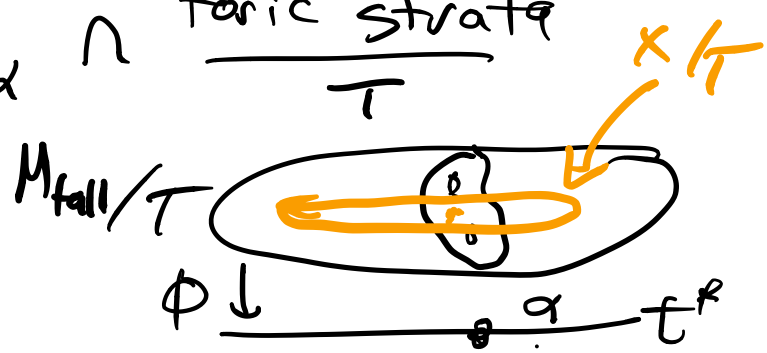
dim = n+1 dim = n−1

$$\Delta = \Delta_{tall} \sqcup \underbrace{\Delta_{short}}_{\text{union of faces of } \Delta}$$

$\alpha \in \Delta_{short}$: $M_\alpha = \{\text{point}\}$

$\alpha \in \Delta_{tall}$: $M_\alpha =$ 

"freckles" = $M_\alpha \cap \dfrac{\text{toric strata}}{T}$



Trivialization of the quotient

$$
\begin{tikzcd}
M_{tall}/T \arrow[r, "\overset{\text{homeo}}{\cong}"] \arrow[d, "\phi"'] & \Delta_{tall} \times \quad \arrow[dl] \\
\Delta_{tall} &
\end{tikzcd}
$$

$S = \bigsqcup X/T$

$X \subset M_{tall}$

toric stratum

skeleton: $S \hookrightarrow M_{tall}/T \cong \Delta_{tall} \times$

(Φ, f) "painting"

$\boxed{\text{Theorem}}$ K + Tolman

$\Delta = \Delta_{tall}$: 2014

$\Delta \neq \Delta_{tall}$: in progress

Complexity one $T \circlearrowright (M,\omega) \xrightarrow{\Phi} t^*$ are classified by:

- $\Delta = \Phi(M)$
- DH measure
- genus
- toric skeleton $S \xrightarrow{\bar{\Phi}} t^*$ + isotropy label
- painting

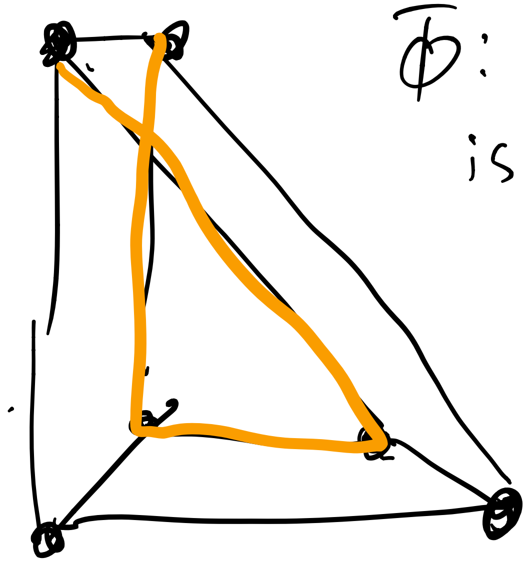
$$S \xrightarrow[(\bar{\Phi}, f)]{} \Delta_{tall} \times \Sigma$$

up to:

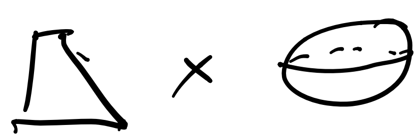
$$\begin{array}{ccc} S & \xrightarrow{f} & \Sigma \\ \cong \downarrow & & \downarrow \text{homeo} \\ S' & \xrightarrow{f'} & \Sigma \end{array}$$

+ homotopy

Kähler $\Rightarrow$ painting is trivial.

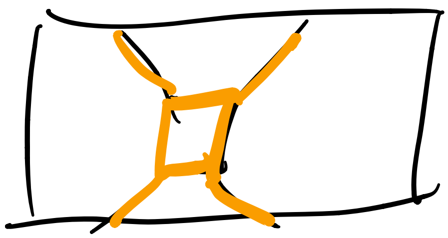


component of $\Phi : S \to \mathfrak{t}^*$ is not 1-1

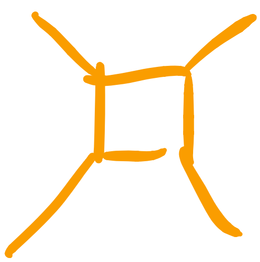
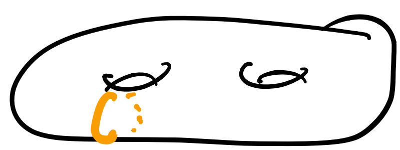
$S \xrightarrow{(\Phi, f)}$ 

is 1-1

$\Rightarrow$ f cannot be loc. const.



fall

 $\xrightarrow{f}$ 

Thank you!