On 3-folds having a holomorphic torus action with 6-fixed points

Nicholas Lindsay

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Motivation

- In a joint work with Dmitri Panov we have proven that a symplectic 6-manifold constructed by Tolman, having a Hamiltonian $\mathbb{T}^2$-action does not have a compatible Kähler metric. Part of the proof used Mori's minimal model program for projective 3-folds.

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- I will discuss a recent work, in which I focused on the category of smooth projective 3-folds. My goal was explore the topology of complex projective 3-folds having a holomorphic $\mathbb{C}^*$-action with 6 fixed points (equivelantly $b_2 = 2$ and finite fixed point set), using Mori's contraction theorem.

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- I will discuss a recent work, in which I focused on the category of smooth projective 3-folds. My goal was explore the topology of complex projective 3-folds having a holomorphic $\mathbb{C}^*$-action with 6 fixed points (equivelantly $b_2 = 2$ and finite fixed point set), using Mori's contraction theorem.
- In particular, I studied an invariant of the underlying 6-manifold, called the Δ-invariant. The Δ-invariant only depends on the integral cohomology ring of the manifold.

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$$\Delta(X) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2).$$

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$$\Delta(X) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2).$$
- One may check that this does not depend on the choice of integral basis, hence is a topological invariant.

An Example (Okonek, Van-de-Ven)

- Suppose E is a rank 2 holomorphic vector bundle over $\mathbb{CP}^2$, then the Δ-invariant of the associated $\mathbb{CP}^1$-bundle is given by a simple formula:

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- These 3-folds are examples of *conic bundles*, i.e. 3-folds having a morphism to an algebraic surface such that all of the fibres are isomorphic to conics in $\mathbb{CP}^2$ (not neccesasarily a topological S^2-bundle) .

First main result

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Theorem

There is a constant K such that any smooth projective 3-fold X having a holomorphic $\mathbb{C}^*$-action with 6-fixed points satisfies one of the following two conditions:

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- We remark that this gives a restriction on the cohomology ring of X, since a conic bundle Y contains a non-zero class $\alpha \in H_2(Y, \mathbb{Z})$, having $\int_Y \alpha^3 = 0$.
- The proof relies on the contraction theorem of Mori (which I will state shortly), and the boundedness of of Fano 3-folds with a certain class of (terminal) singularities.

Mori's contraction theorem

We will need the following Theorem of Mori:

Theorem (Mori)

Let X be a smooth projective 3-fold such that K_X is not nef. Then, there exists a projective variety Y and a morphism $\phi : X \to Y$ associated to a ray in $R \subset H_2(X, \mathbb{R})$. A curve C is sent to a point by $\phi \iff [C] \in R$. Moreover one of the following possibilities occur:

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- $\dim(Y) = 0$ *and X is a Fano 3-fold with* $b_2(X) = 1$.
- $\dim(Y) = 1$, *Y is a smooth, projective curve.*
- $\dim(Y) = 2$, *Y is a smooth, projective surface, ϕ is a conic bundle.*
- $\dim(Y) = 3$, *ϕ is birational, which is an isomorphism away from a smooth divisor $E \subset X$. There is two cases:*
  1. *$\phi(E)$ is a curve. Here, Y is smooth ϕ is the inverse of a blow-up in a smooth curve.*
  2. *$\phi(E)$ is a point. In this case, either $E = \mathbb{CP}^2$ or $E = \mathbb{CP}^1 \times \mathbb{CP}^1$. ϕ is called a divisorial contraction.*

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Theorem (Blanchard)

Suppose that X is a smooth projective variety with an action of a holomorphic torus T and $\phi : X \to Y$ is a Mori extremal contraction. Then there is an action of T on Y, making ϕ equivariant.

Proof of main result 1: Mori fibre spaces

Let X be a smooth projective 3-fold with a $\mathbb{C}^*$-action with 6 fixed points. We note that since X is rational (BB-decomposition), K_X is not nef, hence by Mori's contraction theorem there is an extremal contraction $\phi : X \to Y$. We deal first with the case that $\dim(Y) < \dim(X)$.

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- Firstly we recall that if $\dim(Y) = 0$ then X is a Fano 3-fold with $b_2(X) = 1$. Since X has 6 fixed points, $b_2(X) = 2$, hence this is impossible.

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- If $\dim(Y) = 1$, then the fibre $[F] \in H^2(X, \mathbb{Z})$ satisfies $[F]^2 = 0$, the existence of such an element implies that $\Delta(X) = 0$ (Okonek, Van-de-Ven).

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- If $\dim(Y) = 2$, then ϕ is a conic bundle. By Blanchard's theorem the $\mathbb{C}^*$-action descends to X, making ϕ equivariant. Since the preimage of every fixed point in X contains at least two fixed points, implying that X has exactly 3 fixed points. Hence, $X \cong \mathbb{CP}^2$, as required.

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- There are 4 Fano 3-folds X with $b_2(X) = 1$ having a $\mathbb{C}^*$-action: $\mathbb{CP}^3$, Q the quadric 3-fold, V_5 and V_{22} (Tolman).

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- In the cases Q, and $\mathbb{CP}^3$, then all of the possibilities are exhausted by $Y_n = Bl_{C_n}(\mathbb{CP}^3)$ and a similar family of examples in the quadric 3-fold, $Y'_n = Bl_{C'_n}(Q)$. By direct calculation we may check that

Proof of main result 3: divisorial blow downs with exceptional divisor $E = \mathbb{CP}^1 \times \mathbb{CP}^1$

Here we may show that a smooth projective 3-fold with a $\mathbb{C}^*$-action may not have a extremal contraction with exceptional divisor $E = \mathbb{CP}^1 \times \mathbb{CP}^1$, using a geometric argument involving the Bialynicki-Birula decomposition.

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- This cyclic quotient singularity is terminal, hence the relevant class of Fano 3-folds forms a bounded family (Borisov) (although are not classified to my knowledge).

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- This in turn show that there is finitely many possibilities for Y up to diffeomorphism, in particular the Δ-invariant is bounded above, hence proving the main theorem.