Symplectic Fanos and their symmetries

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Acknowledgment

Based on joint papers with **Joel Fine** and **Nick Lindsay**.

Definition. (Algebraic geometry: Fano manifolds)

A smooth complex manifold X is called *Fano* if it has a Kähler metric g satisfying: Take $\omega(u,v) = g(Ju,v)$, then $c_1(X) = [\omega] \in H^2(X, \mathbb{R})$.

Examples of Fanos: $\mathbb{C}P^n$, quadrics, Grassmanians.

Claim. In each dimension there is a finite number of families of Fanos. Manifolds from one family are symplectomorphic to each other.
$\Rightarrow$ This gives us a finite number of symplectic $2n$-manifolds (X, ω) for each n.

Probabilistic Question

Fix dimension $2_\mathbb{R}$, $4_\mathbb{R}$, $6_\mathbb{R}, \ldots$ and pick a random Fano manifold (X, ω). What is the probability that this manifold admits a Hamiltonian S^1-action?

- Dimension $2_\mathbb{R}$. $\mathbb{C}P^1$. Probability$= 1$.
- Dimension $4_\mathbb{R}$. 10 families - del Pezzo surfaces. 5 admit the action: $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$ blown up in ≤ 3 points. Probability$= 1/2$.
- Dimension $6_\mathbb{R}$. Fano 3-folds, 105 families. 62 contain a Fano with a $\mathbb{C}^*$-action. Probability> 0.59.
- Dimension $8_\mathbb{R}$?? Fanos are not classified...

Definition. *Chern classes* of a symplectic manifold (M, ω)

Choose an almost complex structure J *tamed* by ω: $\omega(v, Jv) > 0$ for any $v \neq 0$. The *Chern classes* of (M, ω) are the Chern classes of (TM, J).

Definition. (M, ω) is called a *symplectic Fano* if

$$c_1(M) = [\omega] \in H^2(M, \mathbb{R}).$$

Remark. Symplectic Fanos are often called *monotone* manifolds.

Gromov: Hard vs. Soft in symplectic geometry.

Theorem (Hard: Gromov, Taubes, McDuff, Ohta-Ono)

Every closed 4-dimensional symplectic Fano is symplectomorphic to an algebraic del Pezzo surface. There exist exactly 10 *such manifolds up to a symplectomorphism:* $S^2 \times S^2$ *and* $\mathbb{C}P^2$ *blown up in* ≤ 8 *points.*

Soft

Theorem (Gompf, 1995. Divergence from algebraic geometry)

Let G be any finitely presented group. Then there exists a compact symplectic 4-manifold M^4 with $\pi_1(M^4) = G$.

Conjecture (Eliashberg. Dimension $2n \geq 6$)

Let (M^{2n}, J, h), $n \geq 3$ be an almost complex manifold with a class $h \in H^2(M^{2n})$ such that $h^n \neq 0$. Then there is a symplectic form ω on M^{2n} such that $[\omega] = h$, and a taming $J(\omega)$ is homotopic to J.

Remark. The conjecture is completely open!

- If it holds, any closed smooth oriented 6 manifold X without 2-torsion in $H^3(X, \mathbb{Z})$ and with a 2-class h with $h^3 \neq 0$ is a symplectic Fano.
- However, no non-algebraic Symplectic Fanos are known in dim < 12.

New symplectic Fanos / symplectic domination

Theorem (Fine, P. 2010)

For any $n \geq 6$ there exist symplectic Fanos (M^{2n}, ω) of arbitrary topological complexity.

Proof. Fix n and consider the following matrix in the lie algebra $so(2n,1)$

$$n=1 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, n=2 \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, n=3 \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ldots \begin{pmatrix} J_{2n} & 0 \\ 0 & 0 \end{pmatrix},$$

$J_{2n}^2 = -\mathrm{Id}$, $\langle e_{2n+1} \rangle \perp \langle e_1, \ldots e_{2n}, \rangle$

- The orbit Z_{2n} of this matrix under the action of $SO(2n,1)$ is symplectic.
- **Claim.** This orbit is a *bundle* over the hyperbolic space $\mathbb{H}^{2n}$. The fiber is given by all $J \in SO(2n)$, conjugate to J_{2n}: point, $\mathbb{C}P^1$, $\mathbb{C}P^3$...
- Quotient Z_{2n} by a co-compact torsion free lattice Γ in $SO(2n,1)$.
- For $n=1$ the quotient is a hyperbolic surface. For $n=2$ a 6-manifold with $c_1 = 0$, for $n \geq 3$ a symplectic Fano of dimension $n(n+1)$.
- Z_{2n} is the *Twistor space* of $\mathbb{H}^{2n}$.

Theorem (Fine, P. Symplectic domination, 2019)

For any smooth orientable manifold X^{2n} *there exists a symplectic one* (M^{2n}, ω) *that admits a map of positive degree to* X^{2n}*.*

Proof.

Theorem (Ontaneda. Hyperbolisation)

Let X *be a compact oriented manifold and* $\epsilon > 0$*. There exists a degree* 1 *map* $f\colon N \to X$ *from a compact oriented Riemannian manifold* N *of the same dimension, with sectional curvatures in the interval* $[-1-\epsilon, -1]$*.*

Theorem (Reznikov. Twistor spaces of pinched manifolds)

For a small enough $\epsilon(n) > 0$*, the twistor space* Z *of any compact oriented Riemannian manifold* N^{2n} *with sectional curvatures in* $[-1-\epsilon, -1]$ *has a natural integral symplectic form* $\omega \in H^2(Z, \mathbb{Z})$*.*

Theorem (Donaldson. Symplectic hypersufaces)

Let (Z, ω) *be a compact symplectic manifold with* $[\omega] \in H^2(Z, \mathbb{Z})$*. Then there exists a symplectic submanifold* M *of codimension 2, with* $[M]$ *Poincaré dual to* $k[\omega]$ *with integer* $k > 0$*.*

Manifolds with Hamiltonian $\mathbb{T}^k$-symmetries

Theorem (Delzant, 1988)

Any symplectic manifold (M^{2n}, ω) with an effective Hamiltonian $\mathbb{T}^n$-action admits a compatible $\mathbb{T}^n$-invariant Kähler structure. I.e, it's a toric manifold.

The Hamiltonians $(H_1, \ldots, H_n)$ of the $\mathbb{T}^n$-action define the moment map $M^{2n} \to \mathbb{R}^n$ and the image of this map is a *Delzant polytope*. This is a simple polytope whose edges have rational directions. At each vertex the minimal integer vectors along n incoming rays form a basis in $\mathbb{Z}^n \subset \mathbb{R}^n$.

Theorem (Karshon, 1999)

Any symplectic manifold (M^4, ω) with a Hamiltonian S^1-action is S^1-symplectomorphic a toric surface or a blow-up of a ruled surface.

Theorem (Tolman, McDuff, 2009)

Let (M^6, ω) be a Hamiltonian S^1-manifold and suppose $b_2(M^6) = 1$. Then M^6 is S^1-symplectomorphic to one of 4 Fano 3-folds:
1) $\mathbb{C}P^3$, 2) The quadric $Q^3 \subset \mathbb{C}P^3$, 3) The intersection of $G_{\mathbb{C}}(2,5)$ with a plane of codimension 3, 4) X_{22} $(-K^3 = 22)$.

Symplectic Fanos with S^1-action

Conjecture. Fine, P.

Let (M, ω) be a 6-dimensional symplectic Fano manifold with a Hamiltonian S^1-action. Then M is diffeomorphic to a complex projective Fano 3-fold.

Remark. We said *diffeomorphic* to be on the safe side, a stronger version would be to replace diffemorphic by *S^1-symplectomorphic a complex projective Fano with an algebraic S^1-action.*

Theorem (Lindsay, P.)

Let (M, ω) be a symplectic Fano 6-manifold with a Hamiltonian S^1-action. Then M is symplectically birational to $\mathbb{C}P^3$. It has $\pi_1 = 0$ and has $c_1 \cdot c_2 = 24$.

Theorem (Cho)

Let (M, ω) be a 6-dimensional symplectic Fano manifold with a semi-free Hamiltonian S^1-action. Then it has a compatible S^1-invariant Kähler metric.

Remark. The last two results rely on Seiberg-Witten theory.

Tolman's manifold

Theorem (Tolman 1998)

There exists a symplectic 6 manifold $M_{\mathcal{T}}$ with $b_2(M_{\mathcal{T}}) = 2$ and with a family of symplectic structures $\omega_{\lambda_1,\lambda_2}$, $0 < \lambda_1 < \lambda_2$, that admits a Hamiltonian $\mathbb{T}^2$-action but doesn't admit a compatible $\mathbb{T}^2$-invariant Kähler form.

Questions about Tolman's manifolds open till 2019.

1. Does the manifold $M_{\mathcal{T}}$ have any Kähler metric?
2. Does $(M_{\mathcal{T}}, \omega)$ have a compatible Kähler metric?

Theorem (Goertsches, Konstantis, Zoller. 2019)

Tolman's manifold is diffeomorphic to a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^2$. Furthermore $M_{\mathcal{T}}$ is a projectivisation of a rank 2 bundle E with $c_1(E) = -1$, $c_2(E) = -1$.

Theorem (Lindsay, P. 2019)

For $2\lambda_1 \geq \lambda_2$ the symplectic form $\omega_{\lambda_1,\lambda_2}$ doesn't admit a compatible Kähler metric.

Construction of Tolman's manifold

(1) Start with two toric 3-folds $\hat{M}$ and $\tilde{M}$.
(2) $\hat{M}$ is $\mathbb{C}P^1 \times \mathbb{C}P^2$.
(3) $\tilde{M}$ is the projectivisation of the bundle $\mathcal{O} \oplus \mathcal{O}(-3)$ over $\mathbb{C}P^2$.
(4) Choose $\mathbb{T}^2$-subactions and symplectic forms, so that the moment images are as on the figure ($0 < \lambda_1 < \lambda_2$):
(5) Glue the gray halves

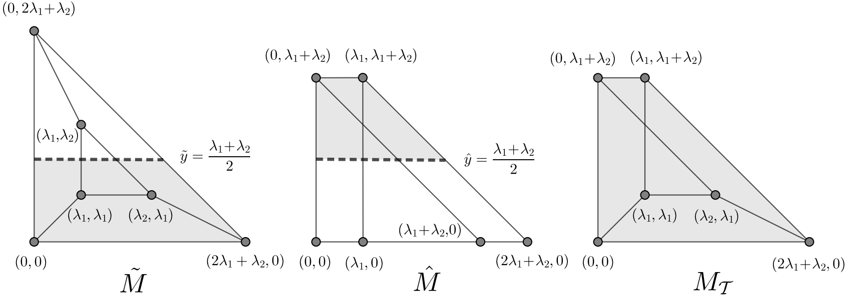

Figure 1: Tolman's sum construction

Remark. If λ_1 is tiny, all three pictures look like a triangle $= \mathbb{C}P^2$!

Idea of proof of non-Kählerness for $2\lambda_1 \geq \lambda_2$

1. Localisation: $c_1^3(M_{\mathcal{T}}) = 64$, $c_1(M_{\mathcal{T}})$ is divisible by 2. Ring structure on $H^*(M_{\mathcal{T}})$ in terms of classes $[\omega_{\lambda_1,\lambda_2}]$.
2. Since $b_2 = 2$, if $M_{\mathcal{T}}$ is Kähler then is $M_{\mathcal{T}}$ projective.
3. Apply minimal model programme. We have tree possibilities
   - (a) One can blow down $\mathbb{C}P^2$ on $M_{\mathcal{T}}$ to get a 3-Fano. But then c_1^3 increases. However 3-Fanos have $c_1^3 \leq 64$.
   - (c) $M_{\mathcal{T}}$ is a quadric fibration over $\mathbb{C}P^1$. Impossible for topological reasons.
   - (b) $M_{\mathcal{T}}$ is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^2$.
4. Analyse Kähler cones of $\mathbb{C}P^1$-bundles over $\mathbb{C}P^2$ that are projectivisations of E with $c_1(E) = -1$, $c_2(E) = -1$.

Remark/Question

This is the first known symplectic manifold admitting a Hamiltonian S^1-action with isolated fixed points, but without a compatible Kähler structure.
Question. Can one always deform such an example to an algebraic one?

Remark. Tolman's manifold is almost a Fano, i.e., c_1" $\geq$ "0!

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