Equivariant Gompf gluing (and its applications)

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Definition

The **x-ray** is the set

$$K = \bigcup_{H \subset T} K^H$$

together with the moment polytopes $\mu(X)$ for $X \in K$.

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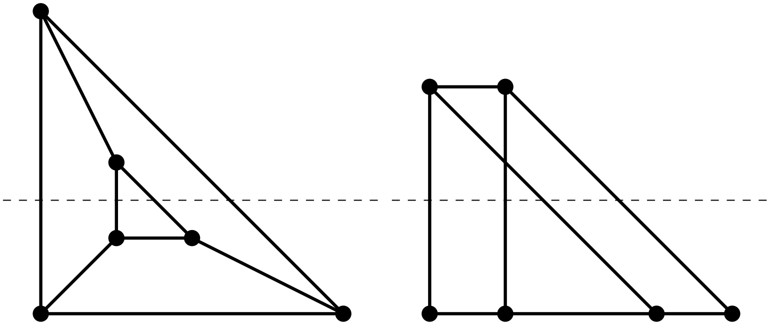

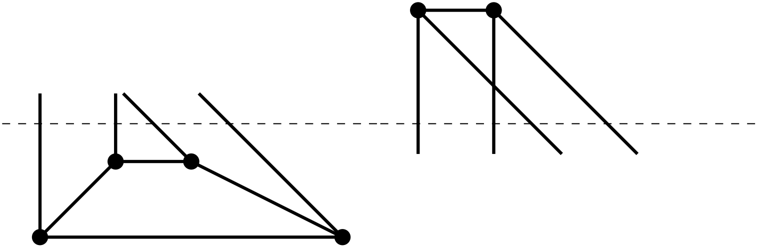

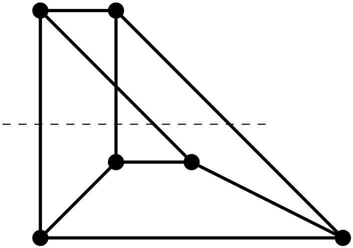

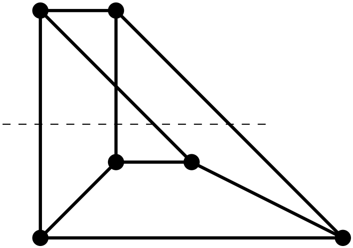

This construction does not depend on the choice of the gluing map.

Let M_1, M_2 be smooth oriented manifolds of dimension $2n$ and N a compact, connected, smooth, oriented manifold of dimension $2(n-1)$. Suppose moreover that there exist orientation preserving embeddings $i_1, i_2 \colon N \to M_1, M_2$.

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The normal bundles ν_1, ν_2 of N in M_1, M_2 are then oriented and their structure group can be assumed to be S^1 using some fixed metric.

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We define an orientation reversing isomorphism

$$i \colon E_2 \setminus 0_{E_2} \to E_2 \setminus 0_{E_2}, \quad i(x) = \frac{x}{||x||^2}.$$

Theorem

Let V_1, V_2 be tubular neighborhoods of N in M_1, M_2. For any orientation reversing isomorphism $\psi\colon E_1 \to E_2$, the isomorphism $i \circ \psi\colon E_1 \setminus 0_{E_1} \to E_2 \setminus 0_{E_2}$ is orientation preserving.

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Any two closed k-forms ω_1, ω_2 on M_1, M_2 with $i_1^* \omega_1 = i_2^* \omega_2$ induce a canonical cohomology class Ω on $M_1 \#_\psi M_2$.

Definition

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Theorem

Let M be a G-manifold. Then any equivariant isotopy ω_t with compact support is induced by an isotopy of equivariant diffeomorphisms $f_t \colon M \to M$, i.e. $f_t^*\omega_t = \omega_0$.

Theorem

Suppose that M_1 and M_2 admit symplectic forms ω_1 and ω_2 satisfying $i_1^*\omega_1 = i_2^*\omega_2$. Then $M_1 \#_\psi M_2$ admits a canonical isotopy class of symplectic forms of class Ω independent of isotopies of the embeddings or ψ.

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If furthermore M_1, M_2 and N are G-Hamiltonian and i_1, i_2, ψ are equivariant, then $M_1 \#_\psi M_2$ is G-Hamiltonian with respect to some equivariant isotopy class.

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- *There is an open dense subset $M_{\mu<t} \subset M_{cut}$ which is G-equivariantly symplectomorphic to $\mu^{-1}((-\infty, t))$.*

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- *The complement of $M_{\mu<t}$ in M_{cut} is a $2(n-1)$-dimensional Hamiltonian G-manifold, equivariantly symplectomorphic to $\mu^{-1}(t)/S^1$.*
- *The S^1-moment image of M_{cut} is $\mu(M) \cap (-\infty, t]$.*

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Since t is a regular value of μ, it is one of H, and so $M_{cut} = H^{-1}(t)/S^1$ is a symplectic manifold (with the canonical G-action).

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Theorem

The first Chern classes of E_1 and E_2 have opposite sign and there is a canonical orientation reversing isomorphism $\psi_{cut}\colon E_1 \to E_2$. Thus, if we assume N to be compact, the assumptions for equivariant Gompf gluing are fulfilled and we have $M_1 \#_\psi M_2 = M$ as Hamiltonian G-manifold.

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$$\mu_T(M) \cap \{\xi \in \mathfrak{t}^* \mid \xi(X) \leq \alpha(X)\}.$$

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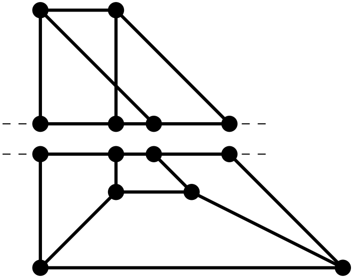
Lemma

Any choice of f induces canonical embeddings $j_1, j_2 \colon N \to M_1, M_2$ and an orientation reversing isomorphism $\psi \colon E_1 \to E_2$, such that $M_1 \#_\psi M_2 = M$ as Hamiltonian T^2-manifolds.



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In our case N is equivariantly symplectomorphic to $S^2 \times S^2$ with the diagonal action and symplectic form

$$C_1\omega_{S^2} \oplus C_2\omega_{S^2}, \quad C_1 \neq C_2,$$

where C_1, C_2 are certain positive constants and ω_{S^2} the standard form.

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where C_1, C_2 are certain positive constants and ω_{S^2} the standard form. In particular $H^1(N, \mathbb{Z}) = 0$.

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Lemma

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Thus, the construction does not depend on the choice of f.

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