Realization of GKM fibrations and new examples of Hamiltonian non-Kähler actions

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joint with Oliver Goertsches and Panagiotis Konstantis

Ludwig-Maximilians-Universität München

Mini-workshop on group actions in symplectic and Kähler geometry
Köln, July 29, 2020

Overview

1. A remarkable Hamiltonian torus action
2. GKM graphs and geometric structures
3. GKM fibrations and realization

Tolman's Example

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"Traditional Hamiltonian techniques are not rendered obsolete by more powerful algebraic methods"

Eschenburg's twisted flag

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- E admits a metric of positive sectional curvature (Eschenburg)
- E admits a Kähler structure (Eschenburg, Escher-Ziller)

The Eschenburg flag as a projectivization

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- E is the projectivization of $V := \mathrm{SU}(3) \times_{\mathrm{U}(2)} \mathbb{C}^2 \to \mathrm{SU}(3)/\!\!/\mathrm{U}(2)$.
- the bundles are equivariant w.r.t the action of the diagonal maximal torus $T \subset \mathrm{SU}(3)$ induced by left multiplication on $\mathrm{SU}(3)$.

Symplectic and Kähler structure on E

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On E there is a Kähler form and a T-invariant symplectic form. The two are symplectomorphic.

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ω_K can be built in such a way that $\widetilde{\omega}_K = \int_T t^*\omega_K dt$ is still symplectic.

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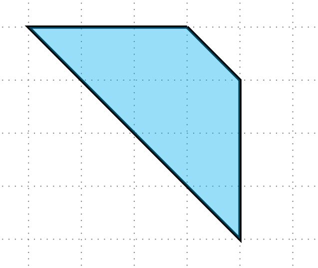
$\mu(E) =$ 

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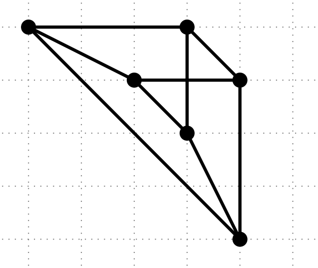
$\mu(E_1) =$ 

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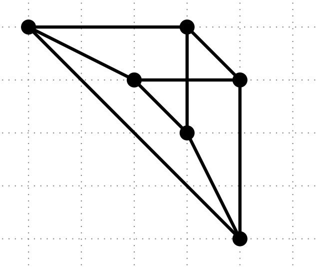
$\mu(E_1) =$ 

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Theorem (Tolman)

A closed Hamiltonian T^2-manifold whose momentum map has the above form, does not admit an invariant Kähler structure.

GKM Manifolds

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Examples: toric manifolds, $T \curvearrowright G/H$ where $T \subset H \subset G$ max. torus

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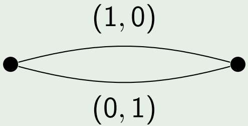

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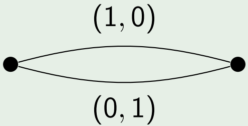

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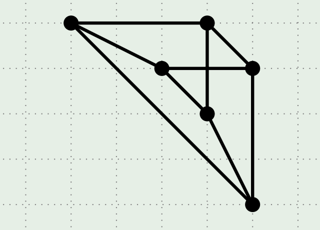

label of an edge =
primitive integral slope

linear realization of the GKM graph

Rigidity in dimension 6

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Is there some correspondence between GKM graphs and GKM T-manifolds?

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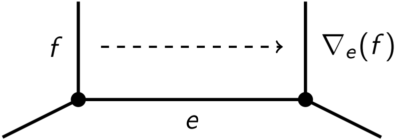

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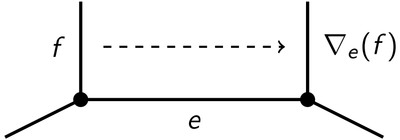

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A graph Γ, together with $\alpha \colon E(\Gamma) \to \mathbb{Z}^k/\pm$ satisfying 1-3 above is called an *abstract GKM graph*.

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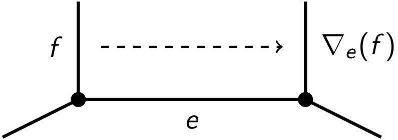

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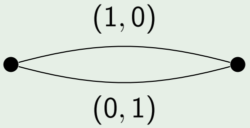
If $\alpha \colon E(\Gamma) \to \mathbb{Z}^k$ such that $\alpha(e) = -\alpha(\bar{e})$ and 3 holds with "$+$", then (Γ, α) is a *signed GKM graph*.

Examples (GKM graphs and geometric structures)



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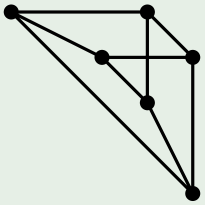

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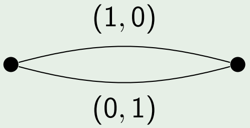

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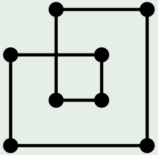
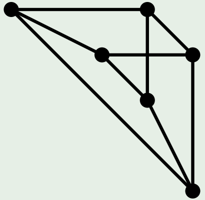


almost complex



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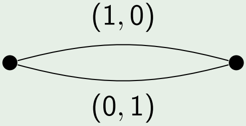

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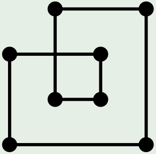

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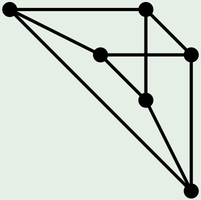

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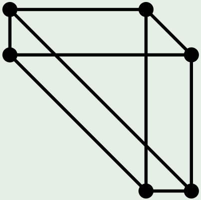

Kähler ($T^2 \curvearrowright \mathrm{SU}(3)/T^2$)

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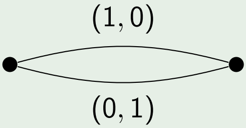

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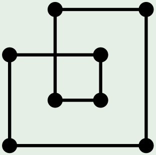

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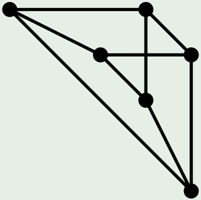

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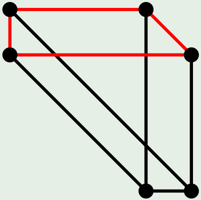

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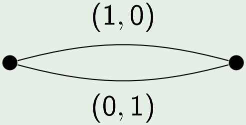

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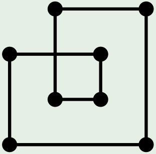

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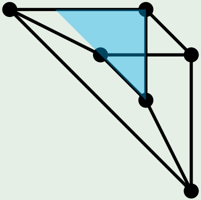

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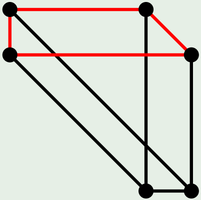

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A closer look at the Eschenburg flag

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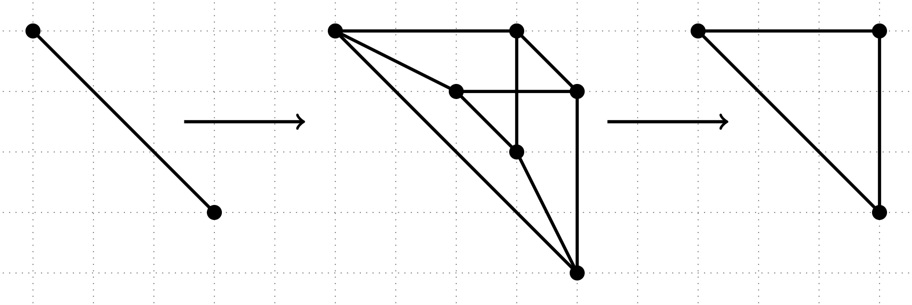
$$\mathbb{C}P^1 \longrightarrow E \longrightarrow \mathbb{C}P^2$$

yields a GKM fibration (Guillemin–Sabatini–Zara)



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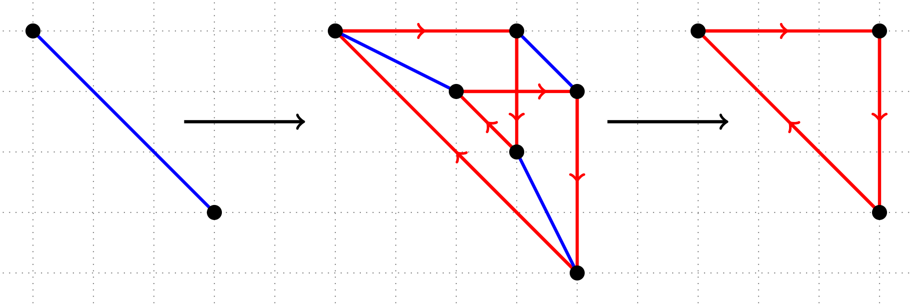
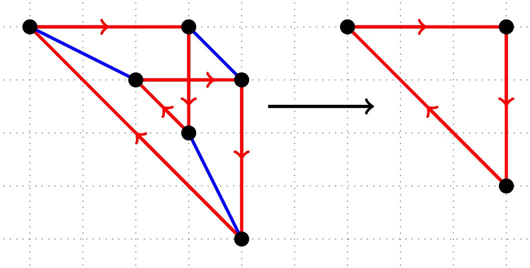


GKM fibrations



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- a map $\varphi\colon V(\Gamma) \to V(B)$

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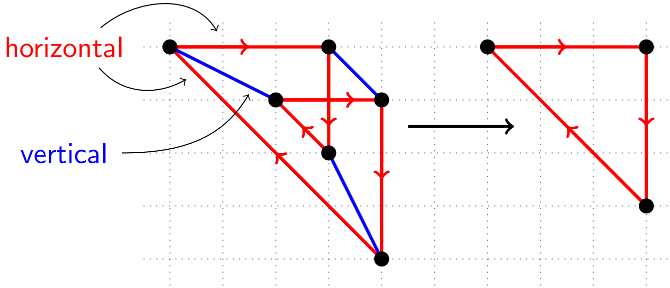

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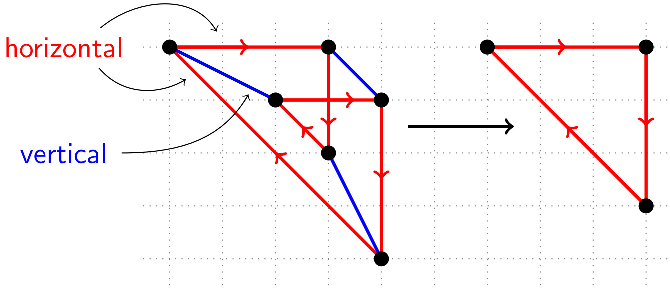

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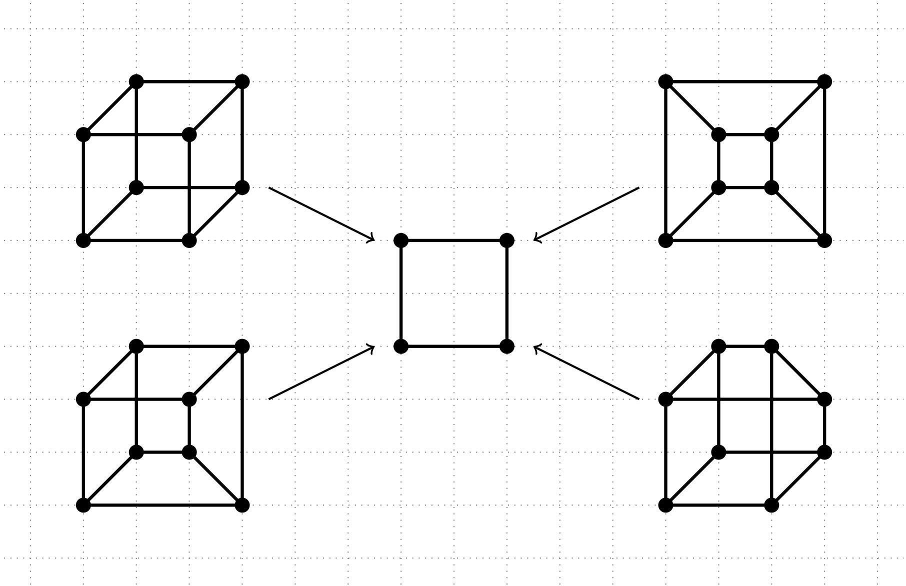
- for every $p \in V(\Gamma)$ a bijection $\varphi_p\colon H_p \to E(B)_{\varphi(p)}$ satisfying $\alpha(e) = \alpha_B(\varphi_p(e))$

Some fibrations over a square



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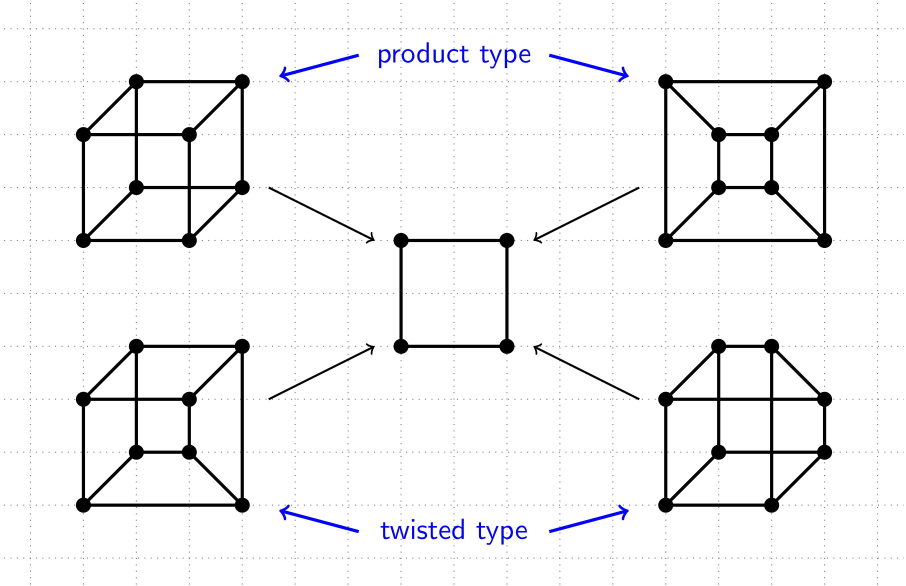

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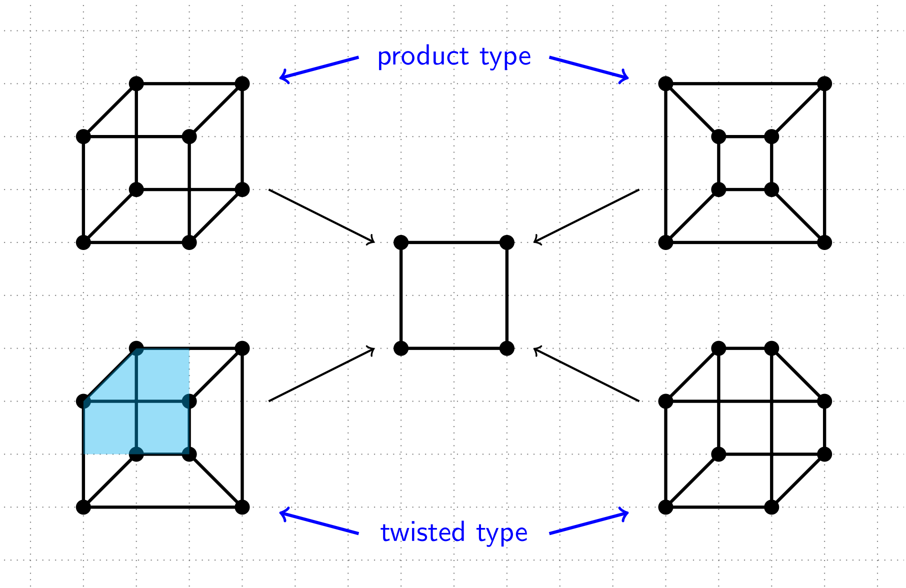

More non-Kähler graphs

Theorem (Goertsches, Konstantis, Z.)

Let $(\Gamma, \alpha) \to (B, \alpha_B)$ be a signed GKM fibration of twisted type, in which Γ is 3-regular and (B, α_B) is 2-regular, effective, and of polytope type.

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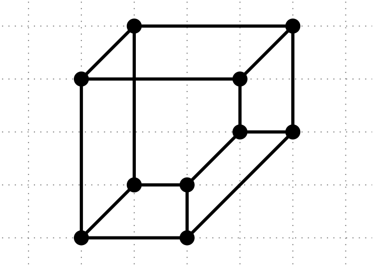

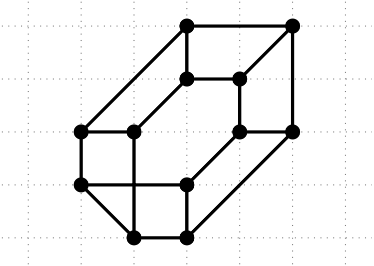

Realization of fibrations

Theorem (Goertsches, Konstantis, Z.)

Let $(\Gamma, \alpha) \to (B, \alpha_B)$ be a fiberwise signed GKM fibration, with Γ 3-regular and (B, α_B) 2-regular and effektive.

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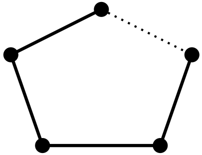
- *If the fibration is signed, then realizations admit T^2-invariant almost complex structures.*
- *If (B, α_B) is of polytope type, then the realizing actions are Hamiltonian. In this case there also exists a Kähler structure on $\mathbb{P}(V)$ which is symplectomorphic to a T^2-invariant symplectic form.*

Proof of the realization theorem

Step 1: find realization X for the base

start with the suitable n-gon of 2-spheres which will be X_1 and glue inside a free 2-cell $D^2 \times T^2$.



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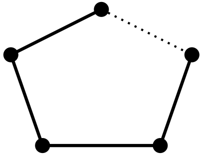
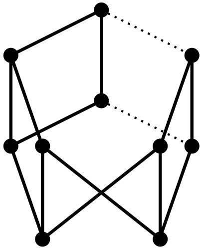


Step 2: construct V over X_1

Construct V over every 2-sphere in X_1 separately and glue those bundles together over the fixed points such that $\mathbb{P}(V|_{X_1})$ has the desired 1-skeleton.



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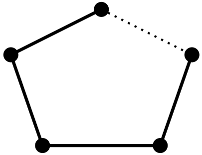

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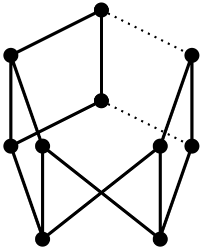

Step 3: extend V over all of X

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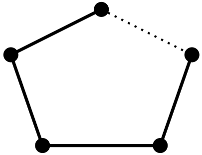

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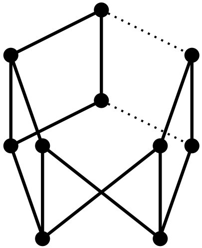

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Step 4: construct geometric structures on $\mathbb{P}(V)$

Classification of GKM fibrations

In the situation of the main theorem: for a fixed base graph (B, α_B) with n vertices

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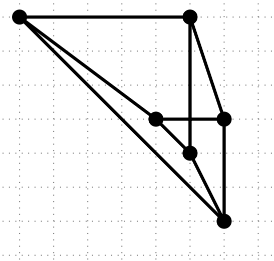
Even over $\mathbb{C}P^2$, this produces infinitely many Tolman-type examples with 6 fixed points (pairwise not homotopy equivalent).



Thank you for your attention!