On the Plateau problem in metric spaces

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The Plateau problem

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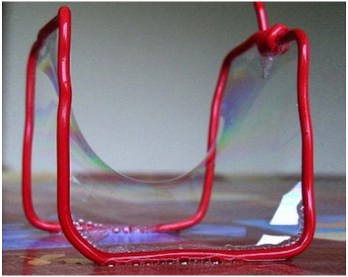

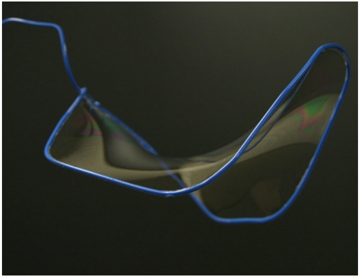

Figures taken from http://images.math.cnrs.fr/

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$$\mathrm{Area}(f) := \int_{\mathbb{D}^2} \sqrt{\det \begin{pmatrix} \langle \partial_1 f, \partial_1 f\rangle & \langle \partial_1 f, \partial_2 f\rangle \\ \langle \partial_2 f, \partial_1 f\rangle & \langle \partial_2 f, \partial_2 f\rangle \end{pmatrix}} = \int_{\mathbb{R}^n} \mathrm{card}\{f^{-1}(y)\}\, \mathrm{d}\mathcal{H}^2(y).$$

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- For $\Gamma \subset \mathbb{R}^n$ a Jordan curve define its set of *fillings* as

$$\Lambda(\Gamma, \mathbb{R}^n) := \{f \in W^{1,2}(\mathbb{D}^2, \mathbb{R}^n) : f|_{\mathbb{S}^1} \text{ parametrizes } \Gamma\}.$$

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Theorem (Douglas, Radó 1930)

Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable Jordan curve. Then there is $f \in \Lambda(\Gamma, \mathbb{R}^n)$ such that

$$\mathrm{Area}(f) \leq \mathrm{Area}(g) \qquad ; \forall g \in \Lambda(\Gamma, \mathbb{R}^n).$$

Moreover, f is *weakly conformal*. I.e. there is $\lambda\colon \mathbb{D}^2 \to [0, \infty)$ such that

$$\langle D_p f(v), D_p f(w) \rangle = \lambda(p) \cdot \langle v, w \rangle \qquad \forall v, w \in \mathbb{R}^2.$$

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Theorem (Lytchak-Wenger 2017)

Let $\Gamma \subset X$ be a Jordan curve. If $\Lambda(\Gamma, X) \neq \emptyset$, then there is $f \in \Lambda(\Gamma, X)$ s.t.

$$\text{Area}(f) \leq \text{Area}(g) \qquad ; \forall g \in \Lambda(\Gamma, X).$$

Moreover, f is *infinitesimally isotropic*.

Previous results

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- Previously known for...
  - ...homogeneously regular Riemannian manifolds X (Morrey '48)
  - ...spaces X which satisfy curvature bounds in the sense of Alexandrov (Nikolaev '79, Mese-Zulkowski 2010)
  - ...certain Finsler manifolds X (Overath-von der Mosel 2014, Pistre-von der Mosel 2017)
- Guo-Wenger (2020) prove the theorem also for many non-proper spaces X.

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Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable closed curve. Then there is $f \in \Lambda(\Gamma, \mathbb{R}^n)$ such that

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Moreover, f is *weakly conformal* on $\mathbb{D}^2 \setminus f^{-1}(\Gamma)$.

- Improves a previous result of Hass from '91.

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- Given $X = \mathbb{R}^n$ regular space and $\Gamma \subset X$ singular curve
- Construct $\widetilde{X}$ singular space and $\widetilde{\Gamma} \subset \widetilde{X}$ regular curve s.t.

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- Apply Lytchak-Wenger to solve the Plateau problem for $(\widetilde{X}, \widetilde{\Gamma})$ □
- Here $\widetilde{X}$ is the mapping cylinder of Γ.

Regularity of solutions

Weyl's lemma (1940)

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- In general infinitesimally isotropic area minimizers $f\colon \mathbb{D}^2 \to X$ need not even be continuous.
- We say that X satisfies a C-*quadratic isoperimetric inequality* if for every Lipschitz curve $\gamma\colon \mathbb{S}^1 \to X$ exists $f \in W^{1,2}(\mathbb{D}^2, X)$ s.t. $f|_{\mathbb{S}^1} = \gamma$ and

$$\mathrm{Area}(f) \leq C \cdot \mathrm{Length}(\gamma)^2.$$

Theorem (Lytchak-Wenger 2017)

If X satisfies a C-quadratic isoperimetric inequality and $f\colon \mathbb{D}^2 \to X$ is an infinitesimally isotropic area minimizer, then f is α-Hölder continuous where $\alpha = \alpha(C)$.

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- By Reshetnyak's majorization theorem

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- This is sharp since $C(\ell^\infty) = \frac{1}{2\pi}$ (Ivanov 2011).
- However, if $\dim(X) < \infty$, then $C(X) < \frac{1}{2\pi}$ (C. 2020)

Branch points

Theorem (Ossermann '70, Gulliver-Ossermann-Royden '73)

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- Let $f: \mathbb{D}^2 \to \mathbb{C}^2 \cong \mathbb{R}^4$ be given by $f(z) = (z^3, z^2)$. Then f is a weakly conformal area minimizer, but $\partial_1 f(0) = \partial_2 f(0) = 0$ (Federer '65).

Theorem (Gulliver-Lesley '73)

If $f: \mathbb{D}^2 \to \mathbb{R}^n$ is a weakly conformal area minimizer, then there are $p_1, \ldots, p_m \in \mathbb{D}^2$ s.t. the restriction of f to $\mathbb{D}^2 \setminus \{p_1, \ldots, p_m\}$ is an immersion.

Branch points in the metric setting

Theorem (Lytchak-Wenger 2018, Stadler 2018)

If $C(X) \leq \frac{1}{4\pi}$ and $f\colon \mathbb{D}^2 \to X$ is an infinitesimally isotropic area minimizer, then there are $p_1, \ldots, p_m \in \mathbb{D}^2$ s.t. the restriction of f to $\mathbb{D}^2 \setminus \{p_1, \ldots, p_m\}$ is locally injective.

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- Let $B \subset\subset \mathbb{D}^2$ be a ball. Then $X := \mathbb{D}^2/B$ satisfies a quadratic isoperimetric inequality with $C(X) = \frac{1}{2\pi}$. The quotient map $f\colon \mathbb{D}^2 \to X$ is an infinitesimally isotropic area minimizer, but constant on B.

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Question (Lytchak-Wenger 2018)

Can the branch set be controlled if $C(X) < \frac{1}{2\pi}$?

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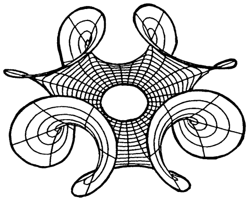
- Any positive result of this type would apply if X is a finite dimensional Banach space (or more generally a Finsler manifold).
- However, the answer is negative (C.-Romney, 2020)

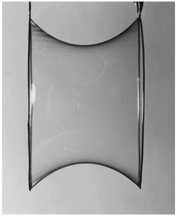
Generalization: The Plateau-Douglas problem

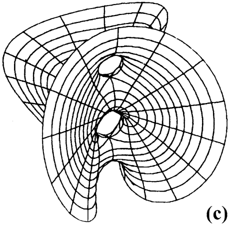
- Let $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ be a configuration disjoint Jordan curves in $\mathbb{R}^n$.
- Let S be a compact, connected, orientable, smooth surface with k boundary components.

Plateau-Douglas problem

Does Γ span a surface of least area which has the shape of S?







(c)

Figures taken from Dierkes-Hildebrandt-Sauvigny "Minimal surfaces"

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Theorem (Douglas '39)

If (Γ, S) satisfy the *Douglas condition*, then there is $f \in \Lambda(S, \Gamma, \mathbb{R}^n)$ s.t.

$$\text{Area}(f) \leq \text{Area}(g) \qquad ; \forall g \in \Lambda(S, \Gamma, \mathbb{R}^n).$$

Moreover, there is a Riemannian metric h on S such that $f : (S, h) \to \mathbb{R}^n$ is weakly conformal.

The Plateau-Douglas problem in metric spaces

- Let $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ be a configuration disjoint Jordan curves in a proper metric space X.
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- Previously known for...
  - ...homogeneously regular Riemannian manifolds X (Jost '85)
  - ...spaces X which satisfy a local quadratic isoperimetric inequality (Fitzi-Wenger 2020)
- New for general Riemannian manifolds X.

Proof sketch

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$$\text{Area}(f) \leq \text{Area}(g) \qquad ; \forall g \in \Lambda(S, \Gamma, X).$$

Moreover, there is a Riemannian metric h on S such that $f : (S, h) \to X$ is weakly conformal.

Proof:

- There are proper metric spaces $(X_n)_{n \in \mathbb{N}}$ such that
    - X_n satisfies a local quadratic isoperimetric inequality,
    - $X \subset X_n$ isometrically, and
    - $d_H(X_n, X) \to 0$.

Proof sketch

- Let $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ be a configuration disjoint Jordan curves in a proper metric space X.

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$\rightsquigarrow$ $f := \lim_{n\to\infty} f_n$ is solution to PD problem (Γ, X) $\square$

Thank You!