Plateau's problem for singular curves

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October 15, 2019

1. Classical Plateau problem

Classical Plateau problem

- Let $\Gamma \subset \mathbb{R}^3$ be a closed, rectifiable Jordan curve.
- Set $\Lambda\left(\Gamma, \mathbb{R}^3\right) := \left\{u \in W^{1,2}\left(D, \mathbb{R}^3\right) \middle| u_{|S^1} \text{ parametrizes } \Gamma\right\}$.

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- Define

$$\mathrm{Fill}(\Gamma) := \inf_{u \in \Lambda(\Gamma, \mathbb{R}^3)} \mathrm{Area}(u).$$

- Here

$$\mathrm{Area}(u) := \int_D J(d_p u) dp = \int_{\mathbb{R}^3} \sharp\{u^{-1}(y)\} \, d\mathcal{H}^2(y)$$

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Question (Plateau's problem)

Is there $u_{\min} \in \Lambda(\Gamma, \mathbb{R}^3)$ satisfying

$$\mathrm{Area}(u_{\min}) = \mathrm{Fill}(\Gamma)?$$

If yes: Regularity of obtained $u_{\min}$?

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- Yes (Douglas, Radó, '30).
- Regularity:
  - $u_{\min}$ is smooth and conformal in the interior of D.
  - $u_{\min}$ is continuous on $\overline{D}$.
  - Additional regularity of $\Gamma \rightsquigarrow$ Additional boundary regularity of $u_{\min}$.
  - E. g. Γ bi-Lipschitz to $S^1 \rightsquigarrow u_{\min}$ Hoelder continuous on $\overline{D}$.

2. Plateau's problem for self-intersecting curves

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- (Hass, '91) "Yes". Setting: continuous maps and Lebesgue notion of surface area (not $W^{1,2}$ and Area).

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- (Hass, '91) "Yes". Setting: continuous maps and Lebesgue notion of surface area (not $W^{1,2}$ and Area).
- (C., 2019) Yes. Regularity: u_{min} Hoelder continuous on $\overline{D}$.

3. Plateau's problem in metric spaces.

Plateau's problem in metric spaces

- Let X be a proper metric space and $\Gamma \subset X$ be a closed, rectifiable Jordan curve.
- Let $\Lambda(\Gamma, X) := \left\{ u \in W^{1,2}(D, X) \,\middle|\, u_{|S^1} \text{ parametrizes } \Gamma \right\}$.
- Define

$$\mathrm{Fill}(\Gamma) := \inf_{u \in \Lambda(\Gamma, X)} \mathrm{Area}(u).$$

- Here

$$\mathrm{Area}(u) := \int_D J(\mathrm{apmd}_p u) dp = \int_X \sharp\{u^{-1}(y)\} \, d\mathcal{H}^2(y)$$

Question (Plateau's problem)

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Classical Plateau problem in metric spaces

- X proper metric space, $\Gamma \subset X$ a closed, rectifiable Jordan curve.

Question (Plateau's problem)

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- X proper metric space, $\Gamma \subset X$ a closed, rectifiable Jordan curve.

Question (Plateau's problem)

Is there $u_{\text{min}} \in \Lambda(\Gamma, X)$ satisfying

$$\text{Area}(u_{\text{min}}) = \text{Fill}(\Gamma)?$$

If yes: Regularity of obtained u_{min}?

- Yes, if $\Lambda(\Gamma, X) \neq \emptyset$ (Lytchak-Wenger, 2017).
- Regularity, if X satisfies a quadratic isoperimetric inequality:
  - u_{min} is locally Hoelder continuous (~~smooth~~) and quasi-conformal in the interior of D.
  - u_{min} is continuous on $\overline{D}$.
  - Γ bi-Lipschitz to $S^1 \rightsquigarrow u_{\text{min}}$ Hoelder continuous on $\overline{D}$.

Main result

- $\Gamma \subset \mathbb{R}^3$ a closed, rectifiable curve.

Theorem (C., 2019)

There exists $u_{\min} \in \Lambda(\Gamma, \mathbb{R}^3)$ *which is Hoelder continuous on* $\overline{D}$ *and satisfies*

$$\text{Area}(u_{\min}) = \text{Fill}(\Gamma).$$

Main result

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Theorem (C., 2019)

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$$\text{Area}(u_{\min}) = \text{Fill}(\Gamma).$$

- The theorem holds more generally in proper metric spaces satisfying a quadratic isoperimetric inequality.
- E.g. $\mathbb{R}^n$, finite dimensional normed spaces, compact Riemannian and Finsler manifolds, higher dimensional Heisenberg groups, proper CAT(0) spaces, compact Alexandrov spaces,...

Thank You!