On constructing ideals of the Hall algebra of type B

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Abstract

Let $H_v(A_n)$ and $H_v(B_n)$ be the Hall algebras over $\mathbb{Q}(v)$ of the Dynkin quivers A_n and B_n ($n \geq 1$) respectively, where v is an indeterminate and the quivers have linear orientation. By comparing the quantum Serre relations we find a natural algebra epimorphism $\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$. We determine the kernel of π by giving two sets of generators. Let φ be the algebra homomorphism from $H_v(A_n)$ to the quantized Schur algebra $S_v(n+1,r)$ ($r \geqslant 1$) defined in [4] and write $\widetilde{\varphi} : H_{v^2}(A_n) \longrightarrow S_{v^2}(n+1,r)$ for the induced map. We obtain several ideals of $H_v(B_n)$ by lifting the kernel of φ to the kernel of the composition map $\widetilde{\varphi} \circ \pi : H_v(B_n) \longrightarrow S_{v^2}(n+1,r)$.

Key words: Hall algebra, PBW-type basis, generalized quantum Serre relation.

Mathematics Subject Classification (2000): 16G20-16G70-17B37

1 Introduction

Let $H_v(A_n)$ and $H_v(B_n)$ be the Hall algebras over $\mathbb{Q}(v)$ of the Dynkin quivers A_n and B_n ($n \geq 1$) respectively, where v is an indeterminate and the quivers have linear orientation as follows:

$$\begin{array}{ll} A_n & \underset{1}{\cdot} \longrightarrow \cdots \longrightarrow \underset{n-1}{\cdot} \longrightarrow \underset{n}{\cdot} \\ B_n & \underset{1}{\cdot} \longrightarrow \cdots \longrightarrow \underset{n-1}{\cdot} \xrightarrow{(2,1)} \underset{n}{\cdot} \end{array}$$

By Ringel [7], the Hall algebras $H_v(A_n)$ and $H_v(B_n)$ are isomorphic to the positive parts of the corresponding quantum groups, and can be described by quantum Serre relations.

Our main results are the following. By comparing the quantum Serre relations, we find a natural algebra epimorphism $\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$. We determine the kernel of π as an ideal of $H_v(B_n)$ by giving two sets of generators, see Theorem 2.3. Let φ be the algebra homomorphism from $H_v(A_n)$ to the quantized Schur algebra $S_v(n+1,r)$ ($r \in \mathbb{N}$) defined in [4]. Write $\widetilde{\varphi} : H_{v^2}(A_n) \longrightarrow S_{v^2}(n+1,r)$ for the $\mathbb{Q}(v)$-algebra map naturally induced by φ. Let $\psi : H_v(B_n) \longrightarrow S_{v^2}(n+1,r)$ be the composition map of $\widetilde{\varphi}$ and π. We express the kernel of ψ as the sum of two ideals $I_2(B_n)$ and $\mathrm{Ker}(\pi)$ of $H_v(B_n)$, and also as the direct sum of two subspaces $I_1(B_n)$ and $\mathrm{Ker}(\pi)$, in Theorem 2.5. The $\mathbb{Q}(v)$-bases of

[1]The research was partially supported by NSF of China (No.10631010) and NKBRPC (No.2006 CB805905).

$I_2(B_n)$ and $I_1(B_n)$, which are of PBW-type, are obtained in Theorem 2.4 and Theorem 2.5 respectively.

It was first explored in [1] that the quantized Schur algebra $S_v(n+1,r)$ $(r \geqslant 1)$ is closely related to the quantum group, and hence to the Hall algebra, of type A_n. In [4] R.M. Green determined the kernel of $\varphi : H_v(A_n) \longrightarrow S_v(n+1,r)$ explicitly, which has a beautiful basis of PBW-type. Our motivation is to generalize this basis to type B and construct ideals of the Hall algebra $H_v(B_n)$ so that they have representation meaning. It would also be interesting to define the map from $H_v(B_n)$ to the quantized Schur algebra of type B and determine its kernel.

The paper is organized as follows. Section 2 recalls the definition of the Hall algebras $H_v(A_n)$ and $H_v(B_n)$ and states the main theorems. Section 3 proves a so-called generalized quantum Serre relation for type A_n. Section 4 proves our theorems using the results developed in Section 3.

We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of non-negative integers.

2 The Hall algebras $H_v(A_n)$ and $H_v(B_n)$

Root systems and Euler forms: Let $\Phi^+(A_n)$ and $\Phi^+(B_n)$ be the sets of positive roots of the simple Lie algebras of type A_n and B_n $(n \geqslant 1)$ respectively. By Gabriel [3], they are in bijections with the sets of isomorphism classes of the indecomposable representations of the quivers A_n and B_n respectively. For a positive root α, write M_α for an indecomposable representation corresponding to α. We have the following known facts.

1. (Ringel [5]) Write Φ^+ for both $\Phi^+(A_n)$ and $\Phi^+(B_n)$. There exists a 'good' order on Φ^+ such that $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ with $\mathrm{Hom}(M_{\beta_i}, M_{\beta_j}) = 0$ unless $i \leqslant j$, and $\mathrm{Ext}(M_{\beta_i}, M_{\beta_j}) = 0$ unless $j < i$. We define $\beta_i \preccurlyeq \beta_j$ if and only if $i \leqslant j$, and $\beta_i \prec \beta_j$ if and only if $i < j$.

2. $\#\Phi^+(A_n) = \frac{n(n+1)}{2}$, $\#\Phi^+(B_n) = n^2$. Identifying the simple roots provides an embedding of $\Phi^+(A_n)$ into $\Phi^+(B_n)$, which is compatible with their 'good' orders. Let us denote by Φ_1^+ the subset of $\Phi^+(B_n)$ identified with $\Phi^+(A_n)$ and Φ_2^+ the complement, i.e. $\Phi^+(B_n) = \Phi_1^+ \cup \Phi_2^+$.

3. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of both A_n and B_n. Write $0^a1^b0^c$, where $b > 0$ and $a+b+c = n$, for the root $\alpha_{a+1} + \alpha_{a+2} + \cdots + \alpha_{a+b}$ in $\Phi^+(A_n)$ and Φ_1^+. The roots in Φ_2^+ have the form $0^a1^b2^c = \alpha_{a+1} + \cdots + \alpha_{a+b} + 2\alpha_{a+b+1} + \cdots + 2\alpha_{a+b+c}$, where $b, c > 0$ and $a+b+c = n$.

4. (Crawley-Boevey [2]) By definition, the Euler form $\langle -, - \rangle$ in type A_n is given by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and in type B_n given by

$$\langle \alpha_i,\, \alpha_j\rangle = \begin{cases} 2 & \text{if } \ i = j \in \{1,\dots,n-1\} \\ 1 & \text{if } \ i = j = n \\ -2 & \text{if } \ j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

5. ('Good' order on $\Phi^+(A_n)$ and Φ_1^+) For $\beta = 0^{a_1}1^{b_1}0^{c_1}$ and $\beta' = 0^{a_2}1^{b_2}0^{c_2}$, $\beta \prec \beta'$ if and only if either $a_1 > a_2$ or $(a_1 = a_2,\, b_1 > b_2)$.

Hall algebras and PBW-type bases: Let Q be the linearly oriented quiver A_n or B_n with vertices $\{1, 2, \dots, n\}$. Write S_i for the irreducible representation of Q supported at the vertex i $(1 \leqslant i \leqslant n)$. Let $\Phi^+ = \{\beta_1, \beta_2, \dots, \beta_N\}$ be the set of positive roots with respect to the 'good' order. Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional $\mathbb{F}_q$-representations of Q, where $\mathbb{F}_q$ is a finite field of q elements.

Fix any $[M], [N], [X] \in \mathcal{P}$. By Ringel [6], the Hall number $g^{[X]}_{[M],[N]}$ is a polynomial in $\mathbb{Z}[v]$. When valued at $v = \sqrt{q}$, it counts the number of submodules X_1 of X satisfying $X_1 \cong N$ and $X/X_1 \cong M$. The Hall algebra of type Q, denoted by $H_v(Q)$, is the $\mathbb{Q}(v)$-algebra with basis $\{[M] : [M] \in \mathcal{P}\}$ and product

$$[M] * [N] = v^{\langle \underline{\dim}(M), \underline{\dim}(N)\rangle} [M] \diamond [N],$$

where $[M], [N] \in \mathcal{P}$ and

$$[M] \diamond [N] = \sum_{[X] \in \mathcal{P}} g^{[X]}_{[M],[N]} [X].$$

They are called the star product and the diamond product respectively.

Set $\langle M_\alpha\rangle = v^{\epsilon(\alpha)}[M_\alpha]$ for $\alpha \in \Phi^+$, where $\epsilon(\alpha) = \dim(\mathrm{End}(M_\alpha)) - \dim(M_\alpha)$. Since indecomposable representations have no self-extensions, we have that $\epsilon(\alpha) = \langle \alpha, \alpha\rangle - \dim(M_\alpha)$. By Ringel [7], the Hall algebra $H_v(Q)$ has a PBW-type basis over $\mathbb{Q}(v)$

$$\{\langle M_{\beta_1}\rangle^{(*n_1)} * \langle M_{\beta_2}\rangle^{(*n_2)} * \cdots * \langle M_{\beta_N}\rangle^{(*n_N)} :\ \forall\ n_1, \dots, n_N \in \mathbb{N}_0\},$$

where the divided power

$$\langle M_{\beta_i}\rangle^{(*n_i)} = \frac{\langle M_{\beta_i}\rangle^{*n_i}}{[n_i]_{\langle \beta_i, \beta_i\rangle}!}$$

and $[n]_m! = \prod_{k=1}^{n} [k]_m$ and $[k]_m = \frac{v^{km} - v^{-km}}{v^m - v^{-m}} \in \mathbb{Q}(v)$ for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Furthermore $H_v(Q)$ is isomorphic to the positive part of the corresponding quantum group. Hence the Hall algebra can be described by quantum Serre relations as follows, where the multiplication corresponds to the star product.

The Hall algebra $H_v(A_n)$ is the associative $\mathbb{Q}(v)$-algebra with generators $\{E_i = \langle S_i\rangle :$ $i = 1, 2, \dots, n\}$ and relations

$$(A1) \quad E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad \text{for } |i - j| = 1,$$

$$(A2) \quad E_iE_j = E_jE_i, \quad \text{for } |i-j|>1.$$

Define the root vector $E_\alpha = \langle M_\alpha \rangle$ for $\alpha \in \Phi^+(A_n)$. In particular $E_{\alpha_i} = E_i$ for $1 \leqslant i \leqslant n$. Since all $\langle \alpha, \alpha \rangle = 1$, the divided powers are given by $E_\alpha^{(n_\alpha)} = \frac{E_\alpha^{n_\alpha}}{[n_\alpha]!}$.

The Hall algebra $H_v(B_n)$ is the associative $\mathbb{Q}(v)$-algebra with generators $\{\mathcal{E}_i = \langle S_i \rangle : i = 1, 2, \dots, n\}$ and relations

$$(B1) \quad \mathcal{E}_i^2\mathcal{E}_j - (v^2+v^{-2})\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i + \mathcal{E}_j\mathcal{E}_i^2 = 0, \quad \text{for } |i-j| = 1 \text{ and } i \neq n$$

$$(B2) \quad \mathcal{E}_n^3\mathcal{E}_{n-1} - (v^2+1+v^{-2})\mathcal{E}_n^2\mathcal{E}_{n-1}\mathcal{E}_n + (v^2+1+v^{-2})\mathcal{E}_n\mathcal{E}_{n-1}\mathcal{E}_n^2 - \mathcal{E}_{n-1}\mathcal{E}_n^3 = 0,$$

$$(B3) \quad \mathcal{E}_i\mathcal{E}_j = \mathcal{E}_j\mathcal{E}_i, \quad \text{for } |i-j|>1.$$

Define the root vector $\mathcal{E}_\alpha = \langle M_\alpha \rangle$ for $\alpha \in \Phi^+(B_n) = \Phi_1^+ \cup \Phi_2^+$. In particular $\mathcal{E}_{\alpha_i} = \mathcal{E}_i$ for $1 \leqslant i \leqslant n$.

The Hall algebra and the quantized Schur algebra: We are not going to define the quantized Schur algebra here. Instead we state the work of R.M. Green which is sufficient for our purpose. Let φ be the algebra homomorphism from $H_v(A_n)$ to $S_v(n+1, r)$ defined in [4]. Then the kernel of φ has $\mathbb{Q}(v)$-basis

$$\{\prod_{\alpha \in \Phi^+(A_n)} E_\alpha^{(n_\alpha)} : \sum_{n_\alpha} n_\alpha > r, \; n_\alpha \in \mathbb{N}_0\},$$

where the product respects the 'good' order. We write $I_1(A_n)$ for the ideal $\mathrm{Ker}(\varphi)$ of $H_v(A_n)$.

Now from the $\mathbb{Q}(v)$-algebra homomorphism $\varphi : H_v(A_n) \longrightarrow S_v(n+1, r)$, we get naturally a $\mathbb{Q}(v)$-algebra homomorphism $\widetilde{\varphi} : H_{v^2}(A_n) \longrightarrow S_{v^2}(n+1, r)$, where $H_{v^2}(A_n)$ and $S_{v^2}(n+1, r)$ are obtained from $H_v(A_n)$ and $S_v(n+1, r)$ respectively with v replaced by v^2 everywhere. We write $I_2(A_n)$ for the ideal $\mathrm{Ker}(\widetilde{\varphi})$ of $H_{v^2}(A_n)$. It is clear that $I_2(A_n)$ has a $\mathbb{Q}(v)$-basis of the same form as $I_1(A_n)$, with v replaced by v^2 everywhere.

Main results: We start with two lemmas about the positive roots of A_n and B_n.

Lemma 2.1. *(Type A_n) For a positive and non-simple root $\alpha \in \Phi^+(A_n)$, there exist (unnecessarily unique) $\gamma_1, \gamma_2 \in \Phi^+(A_n)$ and a short exact sequence:*

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\alpha \longrightarrow M_{\gamma_2} \longrightarrow 0.$$

Proof. Write $\alpha = 0^a1^b0^c$ with $a+b+c = n$ and $b \geqslant 2$. Take b_1, $b_2 \in \mathbb{N}$ such that $b_1+b_2 = b$. Set

$$\gamma_1 = 0^{a+b_1}1^{b_2}0^c, \quad \gamma_2 = 0^a1^{b_1}0^{b_2+c}.$$

Then $\alpha = \gamma_1 + \gamma_2$, $\langle \gamma_1, \gamma_2 \rangle = 0$, $\langle \gamma_2, \gamma_1 \rangle = -1$, and $\gamma_1 \prec \gamma_2$. Now the lemma follows from the Auslander-Reiten quiver of the linearly oriented A_n. □

Lemma 2.2. *(Type B_n)* (1) *For a positive and non-simple root $\alpha \in \Phi_1^+$, there exist (unnecessarily unique) γ_1, $\gamma_2 \in \Phi_1^+$ and a short exact sequence*

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\alpha \longrightarrow M_{\gamma_2} \longrightarrow 0.$$

(2) *For a positive root $\beta \in \Phi_2^+$, there exist uniquely γ_1, and $\gamma_2 \in \Phi_1^+$ such that there is an Auslander-Reiten sequence*

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\alpha \longrightarrow M_{\gamma_2} \longrightarrow 0.$$

Proof. (1) Suppose $\alpha = 0^a 1^b 0^c$ with $a+b+c = n$ and $b \geqslant 2$. Then γ_1, and γ_2 of the same form as in Lemma 2.1 will play the role. Note that they have different Euler form now: $\langle \gamma_1, \gamma_2 \rangle = 0$, $\langle \gamma_2, \gamma_1 \rangle = -2$.

(2) Suppose $\beta = 0^a 1^b 2^c$ with $a+b+c = n$ and $b, c > 0$. Take $\gamma_1 = 0^{a+b} 1^c$ and $\gamma_2 = 0^a 1^{b+c}$ in Φ_1^+. Then $\beta = \gamma_1 + \gamma_2$, $\gamma_1 \prec \gamma_2$ and

$$\langle \gamma_1, \gamma_2 \rangle = 1, \quad \langle \gamma_2, \gamma_1 \rangle = -1.$$

It is clear that such pair (γ_1, γ_2) is unique. The existence of the Auslander-Reiten sequence follows from the Auslander-Reiten quiver of B_n. $\square$

Our main results are the following and will be proved in Section 4.

Theorem 2.3. *There exists a $\mathbb{Q}(v)$-algebra epimorphism*

$$\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$$

sending $\mathcal{E}_i$ to E_i. The kernel of π is an ideal of $H_v(B_n)$ generated by

$$\mathcal{E}_n^2 \mathcal{E}_{n-1} - (v^2 + v^{-2}) \mathcal{E}_n \mathcal{E}_{n-1} \mathcal{E}_n + \mathcal{E}_{n-1} \mathcal{E}_n^2,$$

and also generated by

$$\{\mathcal{E}_\beta - \frac{v^2 - 1}{v + v^{-1}} \mathcal{E}_{\gamma_1} \mathcal{E}_{\gamma_2} : \ \forall \, \beta \in \Phi_2^+\},$$

where γ_1 and $\gamma_2 \in \Phi_1^+$ are determined uniquely by β by Lemma 2.2 (2).

Fix any positive integer $r \in \mathbb{N}$.

Theorem 2.4. *Write $I_2(B_n)$ for the ideal of the Hall algebra $H_v(B_n)$ generated by*

$$\{ \prod_{\alpha \in \Phi_1^+, \, \beta \in \Phi_2^+} \mathcal{E}_\alpha^{(n_\alpha)} \mathcal{E}_\beta^{(n_\beta)} : \sum_{n_\alpha} n_\alpha + 2 \sum_{n_\beta} n_\beta > r, \ n_\alpha, n_\beta \in \mathbb{N}_0 \},$$

where the product respects the 'good' order. Then the set of generators is actually a $\mathbb{Q}(v)$-basis of $I_2(B_n)$.

Consider the composition map $\psi : H_v(B_n) \longrightarrow S_{v^2}(n+1, r)$ of $\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$ and $\widetilde{\varphi} : H_{v^2}(A_n) \longrightarrow S_{v^2}(n+1, r)$.

Theorem 2.5. *Write $I_1(B_n)$ for the $\mathbb{Q}(v)$-subspace of $H_v(B_n)$ with basis (the product respects the 'good' order)*

$$\{\prod_{\alpha\in\Phi_1^+}\mathcal{E}_\alpha^{(n_\alpha)}:\sum_{n_\alpha} n_\alpha > r,\ n_\alpha\in\mathbb{N}_0\}.$$

The kernel of the composition map ψ is the sum $I_2(B_n)+Ker(\pi)$ as an ideal of $H_v(B_n)$, and is the direst sum $I_1(B_n)\oplus Ker(\pi)$ as a $\mathbb{Q}(v)$-subspace of $H_v(B_n)$.

3 Generalized quantum Serre relations

In this section we prove some equalities in the Hall algebra $H_v(A_n)$. Recall that the root vector $E_\alpha = \langle M_\alpha\rangle = v^{\epsilon(\alpha)}[M_\alpha]$ for a positive root $\alpha\in\Phi^+(A_n)$, where $\epsilon(\alpha) = \dim(\mathrm{End}(M_\alpha)) - \dim(M_\alpha) = \langle\alpha,\alpha\rangle - \dim(M_\alpha)$. The Euler form $\langle -,-\rangle$ and the symmetric Euler form $(-,-)$ on the root lattice are defined by

$$\langle\alpha,\beta\rangle = \langle\underline{\dim}(M_\alpha),\underline{\dim}(M_\beta)\rangle = \dim(\mathrm{Hom}(M_\alpha,M_\beta)) - \dim(\mathrm{Ext}(M_\alpha,M_\beta)),$$

$$(\alpha,\,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle,$$

for any positive roots $\alpha,\beta\in\Phi^+(A_n)$, see [2].

Lemma 3.1. *For $\alpha,\,\beta\in\Phi^+(A_n)$, the following are equivalent:*

(1) *The symmetric Euler form $(\alpha,\,\beta) = -1$.*

(2) *There exists a short exact sequence of one of the following forms*

$$0\longrightarrow M_\alpha\longrightarrow M_{\alpha+\beta}\longrightarrow M_\beta\longrightarrow 0,$$

$$0\longrightarrow M_\beta\longrightarrow M_{\alpha+\beta}\longrightarrow M_\alpha\longrightarrow 0.$$

(3) *The sum $\alpha+\beta$ is again a positive root in $\Phi^+(A_n)$*

Proof. (1) $\Rightarrow$ (2): Note that for any $\gamma_1,\gamma_2\in\Phi^+(A_n)$, the Euler form $\langle\gamma_1,\gamma_2\rangle\in\{1,0,-1\}$. Hence $(\alpha,\,\beta)=-1$ if and only if

$$(\langle\alpha,\beta\rangle = 0,\ \langle\beta,\alpha\rangle = -1)\quad\text{or}\quad(\langle\alpha,\beta\rangle = -1,\ \langle\beta,\alpha\rangle = 0).$$

In the first case, from the properties of 'good' order on $\Phi^+(A_n)$ it follows that $\mathrm{Ext}(M_\alpha,M_\beta) = 0$, $\mathrm{Hom}(M_\alpha,M_\beta)=0$, $\mathrm{Hom}(M_\beta,M_\alpha)=0$ and $\dim(\mathrm{Ext}(M_\beta,M_\alpha))=1$. We have a non-split short exact sequence of the form

$$0\longrightarrow M_\alpha\longrightarrow X\longrightarrow M_\beta\longrightarrow 0.$$

Assume the middle term X is decomposable. Then there exists a nonzero proper direct summand X_1 such that the composition map $M_\alpha\longrightarrow X_1\longrightarrow M_\beta$ is nonzero. This is a contradiction with $\mathrm{Hom}(M_\alpha,M_\beta)=0$. Hence X is indecomposable and in particular the dimension vector $\underline{\dim}(X) = \alpha + \beta$. Similarly the second case gives rise to a short exact sequence of the form

$$0 \longrightarrow M_\beta \longrightarrow M_{\alpha+\beta} \longrightarrow M_\alpha \longrightarrow 0.$$

Proof of the existence of such an X_1: Write $X = \oplus_{i=1}^s X_i$, where $s \geqslant 2$ and each direct summand X_i is indecomposable. Write $f = (f_1, \ldots, f_s)^{tr}$ and $g = (g_1, \ldots, g_s)$, where $f_i : M_\alpha \longrightarrow X_i$ and $g_i : X_i \longrightarrow M_\beta$. Suppose $g_i \circ f_i \neq 0$ for any i. Then $\mathrm{Im}(f_i) \subseteq \mathrm{Ker}(g_i)$ for any i and

$$\mathrm{Im}(f) \subseteq \oplus_{i=1}^s \mathrm{Im}(f_i) \subseteq \oplus_{i=1}^s \mathrm{Ker}(g_i) \subseteq \mathrm{Ker}(g).$$

But $\mathrm{Im}(f) = \mathrm{Ker}(g)$ by the exactness of the short exact sequence. So it holds for any i that $\mathrm{Im}(f_i) = \mathrm{Ker}(g_i)$ and

$$\mathrm{Im}(f) = \oplus_{i=1}^s \mathrm{Im}(f_i) = \oplus_{i=1}^s \mathrm{Ker}(g_i) = \mathrm{Ker}(g).$$

Note that $\mathrm{Im}(f) \cong M_\alpha$ is indecomposable. Hence there exists a unique j such that f_j is nonzero and $\mathrm{Im}(f) = \mathrm{Im}(f_j)$. On the other hand

$$M_\beta \cong \mathrm{Cok}(g) = \oplus_{i=1}^s X_i/\mathrm{Ker}(g_i) = X_j/\mathrm{Ker}(g_j) \oplus \bigoplus_{i:i\neq j} X_i/\mathrm{Ker}(g_i)$$

is indecomposable. Hence there exists a unique k such that $X_k/\mathrm{Ker}(g_k) \neq 0$. If $k \neq j$, then the short exact sequence is split. So we have $k = j$. Then $X = X_j$ is indecomposable, a contradiction!

Another method: show that in type A_n, let $\alpha = 0^{a_1}1^{b_1}0^{c_1}$ and $\beta = 0^{a_2}1^{b_2}0^{c_2}$. Then $(\alpha, \beta) = -1$ if and only if $a_1 = a_2 + b_2$ or $a_2 = a_1 + b_1$. Prove case by case, not hard. (2007.07.16)

(2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (1): Since $\alpha + \beta$ is a positive root in $\Phi^+(A_n)$, we have that

$$1 = \langle \alpha + \beta, \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + (\alpha, \beta) + \langle \beta, \beta \rangle.$$

Also $1 = \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$. Hence $(\alpha, \beta) = -1$. $\square$

Proposition 3.2. *(Generalized quantum Serre Relations) In $H_v(A_n)$, we have the following generalized quantum Serre relations:*

$$(S) \qquad E_\alpha^2 E_\beta - (v + v^{-1}) E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 = 0,$$

for all positive roots α, β satisfying $(\alpha, \beta) = -1$.

Proof. By definition, it suffices to prove for the star product that

$$[M_\alpha]^{*2} * [M_\beta] - (v + v^{-1})[M_\alpha] * [M_\beta] * [M_\alpha] + [M_\beta] * [M_\alpha]^{*2} = 0.$$

Since $(\alpha, \beta) = -1$, we have either ($\langle \alpha, \beta \rangle = 0$, $\langle \beta, \alpha \rangle = -1$) or ($\langle \alpha, \beta \rangle = -1$, $\langle \beta, \alpha \rangle = 0$).

In the first case, by Lemma 3.1 we have a short exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M_{\alpha+\beta} \longrightarrow M_\beta \longrightarrow 0.$$

Hence

$$\begin{aligned}
[M_\alpha]^{*2} * [M_\beta] &= (v(v^2+1)[M_\alpha \oplus M_\alpha]) \diamond [M_\beta] \\
&= v(v^2+1)[M_\alpha \oplus M_\alpha \oplus M_\beta], \\
[M_\alpha] * [M_\beta] * [M_\alpha] &= ([M_\alpha] \diamond [M_\beta]) * [M_\alpha] = [M_\alpha \oplus M_\beta] * [M_\alpha] \\
&= [M_\alpha \oplus M_\beta] \diamond [M_\alpha] \\
&= (v^2+1)[M_\alpha \oplus M_\beta \oplus M_\alpha] + [M_\alpha \oplus M_{\beta+\alpha}], \\
[M_\beta] * [M_\alpha]^{*2} &= v(v^2+1)[M_\beta] * [M_\alpha \oplus M_\alpha] \\
&= v(v^2+1)v^{-2}[M_\beta] \diamond [M_\alpha \oplus M_\alpha] \\
&= (v+v^{-1})([M_\beta \oplus M_\alpha \oplus M_\alpha] + [M_{\alpha+\beta} \oplus M_\alpha]).
\end{aligned}$$

The relation (S) follows.

Now assume $\langle \alpha, \beta \rangle = -1$, $\langle \beta, \alpha \rangle = 0$. By Lemma 3.1 we have a short exact sequence

$$0 \longrightarrow M_\beta \longrightarrow M_{\alpha+\beta} \longrightarrow M_\alpha \longrightarrow 0.$$

Hence

$$\begin{aligned}
[M_\alpha]^{*2} * [M_\beta] &= (v(v^2+1)v^{-2}[M_\alpha \oplus M_\alpha]) \diamond [M_\beta] \\
&= (v+v^{-1})([M_\alpha \oplus M_\alpha \oplus M_\beta] + [M_\alpha \oplus M_{\alpha+\beta}]), \\
[M_\alpha] * [M_\beta] * [M_\alpha] &= v^{-1}([M_\alpha] \diamond [M_\beta]) * [M_\alpha] \\
&= v^{-1}([M_\alpha \oplus M_\beta] + [M_{\alpha+\beta}]) * [M_\alpha] \\
&= v^{-1}v([M_\alpha \oplus M_\beta] + [M_{\alpha+\beta}]) \diamond [M_\alpha] \\
&= ((v^2+1)[M_\alpha \oplus M_\beta \oplus M_\alpha] + [M_{\alpha+\beta} \oplus M_\alpha]), \\
[M_\beta] * [M_\alpha]^{*2} &= v(v^2+1)[M_\beta] * [M_\alpha \oplus M_\alpha] \\
&= v(v^2+1)[M_\beta] \diamond [M_\alpha \oplus M_\alpha] \\
&= v(v^2+1)[M_\beta \oplus M_\alpha \oplus M_\alpha].
\end{aligned}$$

The relation (S) follows similarly. $\square$

Note that the quantum Serre relation $(A1)$ in Section 2 is a special case of the generalized quantum Serre relation (S).

Proposition 3.3. *Let α, β, $\gamma \in \Phi^+(A_n)$ be three positive roots satisfying $\alpha = \beta + \gamma$ and $\beta \prec \gamma$. Then it holds in $H_v(A_n)$ that*

$$E_\alpha = E_\gamma E_\beta - v^{-1} E_\beta E_\gamma, \quad E_\alpha E_\beta = v E_\beta E_\alpha.$$

Proof. By Lemma 3.1, there exist a short exact sequence

$$0 \longrightarrow M_\beta \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow 0$$

and $\langle \beta, \gamma \rangle = 0$, $\langle \gamma, \beta \rangle = -1$. By definition of the product in the Hall algebra $H_v(A_n)$, we have that

$$[M_\beta] * [M_\gamma] = [M_\beta] \diamond [M_\gamma] = [M_\beta \oplus M_\gamma],$$

$$[M_\gamma] * [M_\beta] = v^{-1}[M_\gamma] \diamond [M_\beta] = v^{-1}([M_\beta \oplus M_\gamma] + [M_\alpha]).$$

Hence

$$[M_\alpha] = v[M_\gamma] * [M_\beta] - [M_\beta] * [M_\gamma].$$

Notice that $\dim(M_\alpha) = \dim(M_\beta) + \dim(M_\gamma)$ and $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 1$. Since the root vector $E_\alpha = \langle M_\alpha \rangle = v^{\epsilon(\alpha)}[M_\alpha]$ and $\epsilon(\alpha) = \langle \alpha, \alpha \rangle - \dim(M_\alpha)$, it holds that $E_\alpha = E_\gamma E_\beta - v^{-1} E_\beta E_\gamma$.

Now from Proposition 3.2, we have the generalized quantum Serre relation

$$E_\gamma E_\beta^2 - (v + v^{-1}) E_\beta E_\gamma E_\beta + E_\beta^2 E_\gamma = 0.$$

Hence

$$\begin{aligned} E_\alpha E_\beta &= (E_\gamma E_\beta - v^{-1} E_\beta E_\gamma) E_\beta = E_\gamma E_\beta^2 - v^{-1} E_\beta E_\gamma E_\beta \\ &= v E_\beta E_\gamma E_\beta - E_\beta^2 E_\gamma = v E_\beta (E_\gamma E_\beta - v^{-1} E_\beta E_\gamma) \\ &= v E_\beta E_\alpha. \end{aligned}$$

□

4 Proof of main results

The root systems $\Phi^+(A_n)$ and $\Phi^+(B_n) = \Phi_1^+ \cup \Phi_2^+$ are described in Section 2. Recall that $\Phi_1^+ = \Phi^+(A_n)$ by identifying the simple roots.

Lemma 4.1. *Let* $\alpha = 0^a 1^b 0^c$, $\gamma_1 = 0^{a+b_1} 1^{b_2} 0^c$, $\gamma_2 = 0^a 1^{b_1} 0^{b_2+c} \in \Phi_1^+$, *where* $a, c \in \mathbb{N}_0$, $b, b_1, b_2 \in \mathbb{N}$ *such that* $a + b + c = n$ *and* $b_1 + b_2 = b$. *We have in the Hall algebra* $H_v(A_n)$

$$E_\alpha = E_{\gamma_2} E_{\gamma_1} - v^{-1} E_{\gamma_1} E_{\gamma_2}$$

and in the Hall algebra $H_v(B_n)$

$$\mathcal{E}_\alpha = \mathcal{E}_{\gamma_2} \mathcal{E}_{\gamma_1} - v^{-2} \mathcal{E}_{\gamma_1} \mathcal{E}_{\gamma_2}.$$

Proof. Clearly $\alpha = \gamma_1 + \gamma_2$, and $\gamma_1 \prec \gamma_2$ with respect to the 'good' order. The first equality in $H_v(A_n)$ now follows from Proposition 3.3.

Consider the Euler form on $\Phi^+(B_n)$. We have that

$$\langle \gamma_1, \gamma_2 \rangle = 0, \quad \langle \gamma_2, \gamma_1 \rangle = -2, \quad \langle \gamma_2, \gamma_2 \rangle = 2,$$

and $\langle \gamma_1, \gamma_1 \rangle = \langle \alpha, \alpha \rangle = 2$ when $c = 0$, and $\langle \gamma_1, \gamma_1 \rangle = \langle \alpha, \alpha \rangle = 1$ when $c \neq 0$. Also $\dim(M_\alpha) = \dim(M_{\gamma_1}) + \dim(M_{\gamma_2})$. Hence $\epsilon(\alpha) + 2 = \epsilon(\gamma_1) + \epsilon(\gamma_2)$.

By Lemma 2.2 (1) there is a short exact sequence

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\alpha \longrightarrow M_{\gamma_2} \longrightarrow 0.$$

We have in the Hall algebra $H_v(B_n)$ that

$$[M_{\gamma_1}] * [M_{\gamma_2}] = [M_{\gamma_1}] \diamond [M_{\gamma_2}] = [M_{\gamma_1} \oplus M_{\gamma_2}],$$

$$[M_{\gamma_2}] * [M_{\gamma_1}] = v^{-2}[M_{\gamma_2}] \diamond [M_{\gamma_1}] = v^{-2}([M_{\gamma_2} \oplus M_{\gamma_1}] + [M_\alpha]).$$

Hence

$$[M_\alpha] = v^2[M_\gamma] * [M_\beta] - [M_\beta] * [M_\gamma].$$

The equality $\mathcal{E}_\alpha = \mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - v^{-2}\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}$ follows from the definition that $\mathcal{E}_\alpha = v^{\epsilon(\alpha)}[M_\alpha]$. □

Lemma 4.2. *Let $\beta = 0^a 1^b 2^c \in \Phi_2^+$, and $\gamma_1 = 0^{a+b}1^c$, $\gamma_2 = 0^a 1^{b+c} \in \Phi_1^+$. We have in the Hall algebra $H_v(A_n)$*

$$E_{\gamma_2}E_{\gamma_1} = vE_{\gamma_1}E_{\gamma_2}$$

and in the Hall algebra $H_v(B_n)$

$$\mathcal{E}_\beta = \frac{1}{v+v^{-1}}(\mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - \mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}).$$

Proof. Let $\gamma_3 = 0^a 1^b 0^c \in \Phi^+(A_n)$. Then $\gamma_2 = \gamma_1 + \gamma_3$ and $\gamma_1 \prec \gamma_3$. The relation in $H_v(A_n)$ follows from Proposition 3.3.

For the relation in $H_v(B_n)$, note that $\beta = \gamma_1 + \gamma_2$ and $\dim(\mathrm{End}(M_\beta)) = 2, \dim(\mathrm{End}(M_{\gamma_1})) = \dim(\mathrm{End}(M_{\gamma_2})) = 1$, $\dim(M_\beta) = \dim(M_{\gamma_1}) + \dim(M_{\gamma_2})$. Thus $\epsilon_\beta = \epsilon_{\gamma_1} + \epsilon_{\gamma_2}$. From $\mathcal{E}_\alpha = \langle M_\alpha \rangle = v^{\epsilon_\alpha}[M_\alpha]$ for any positive root α, it suffices to prove that

$$(R) \qquad [M_\beta] = \frac{1}{v+v^{-1}}([M_{\gamma_2}] * [M_{\gamma_1}] - [M_{\gamma_1}] * [M_{\gamma_2}]).$$

By Lemma 2.2 (2), we have the Auslander-Reiten sequence

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\beta \longrightarrow M_{\gamma_2} \longrightarrow 0,$$

and $\langle \gamma_1, \gamma_2 \rangle = 1$, $\langle \gamma_2, \gamma_1 \rangle = -1$. Hence $\mathrm{Hom}(M_{\gamma_2}, M_{\gamma_1}) = 0$, $\mathrm{Ext}(M_{\gamma_1}, M_{\gamma_2}) = 0$, $\dim(\mathrm{Ext}(M_{\gamma_2}, M_{\gamma_1})) = 1$, $\dim(\mathrm{Hom}(M_{\gamma_1}, M_{\gamma_2})) = 1$ and M_{γ_1} is a submodule of M_{γ_2}. We have

$$[M_{\gamma_2}] * [M_{\gamma_1}] = v^{-1}[M_{\gamma_2}] \diamond [M_{\gamma_1}] = v^{-1}(v^2[M_{\gamma_2} \oplus M_{\gamma_1}] + (v^2+1)[M_\beta]),$$

$$[M_{\gamma_1}] * [M_{\gamma_2}] = v^1[M_{\gamma_1}] \diamond [M_{\gamma_2}] = v[M_{\gamma_1} \oplus M_{\gamma_2}].$$

The relation (R) follows now. □

From the description of the Hall algebras $H_{v^2}(A_n)$ and $H_v(B_n)$ by quantum Serre relations in Section 2, one sees directly that sending $\mathcal{E}_i$ to E_i $(i = 1, 2, \ldots, n)$ provides an algebra epimorphism $\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$.

Lemma 4.3. *The image of the root vectors $\mathcal{E}_\alpha$ of $H_v(B_n)$ under π are*

$$\pi(\mathcal{E}_\alpha) = E_\alpha, \quad if\ \alpha \in \Phi_1^+,$$

$$\pi(\mathcal{E}_\beta) = \frac{(v^2-1)}{(v+v^{-1})} E_{\gamma_1} E_{\gamma_2}, \quad if\ \beta \in \Phi_2^+,$$

where $\gamma_1, \gamma_2 \in \Phi_1^+$ are uniquely determined by $\beta \in \Phi_2^+$ by Lemma 2.2 (2).

Proof. Suppose $\alpha \in \Phi_1^+$. By Lemma 4.1 and induction on α, it is clear that the expression of $\mathcal{E}_\alpha \in H_v(B_n)$ into $\mathcal{E}_i$ is the same as the expression of $E_\alpha \in H_{v^2}(A_n)$ into E_i. Then $\pi(\mathcal{E}_\alpha) = E_\alpha$ follows from that $\pi(\mathcal{E}_i) = E_i$.

Suppose $\beta \in \Phi_2^+$. By Lemma 4.2 and the relation we obtained just now, we have

$$\begin{aligned}\pi(\mathcal{E}_\beta) &= \pi\Big(\frac{1}{v+v^{-1}}(\mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - \mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2})\Big) = \frac{1}{v+v^{-1}}(E_{\gamma_2}E_{\gamma_1} - E_{\gamma_1}E_{\gamma_2}) \\ &= \frac{(v^2-1)}{(v+v^{-1})} E_{\gamma_1} E_{\gamma_2}.\end{aligned}$$

□

We are now prepared for the proof of our main results.

Proof. (**Proof of Theorem 2.3**) Note that

$$\begin{aligned}&E_n^3 E_{n-1} - (v^2+1+v^{-2})E_n^2 E_{n-1} E_n + (v^2+1+v^{-2}) E_n E_{n-1} E_n^2 - E_{n-1} E_n^3 \\ &\quad = E_n(E_n^2 E_{n-1} - (v^2+v^{-2}) E_n E_{n-1} E_n + E_{n-1} E_n^2) \\ &\quad\quad - (E_n^2 E_{n-1} - (v^2+v^{-2}) E_n E_{n-1} E_n + E_{n-1} E_n^2) E_n.\end{aligned}$$

So the kernel of π is generated by

$$\mathcal{E}_n^2 \mathcal{E}_{n-1} - (v^2+v^{-2}) \mathcal{E}_n \mathcal{E}_{n-1} \mathcal{E}_n + \mathcal{E}_{n-1} \mathcal{E}_n^2.$$

We write $I_3(B_n)$ for the ideal of $H_v(B_n)$ generated by

$$\{\mathcal{E}_\beta - \frac{v^2-1}{v+v^{-1}} \mathcal{E}_{\gamma_1} \mathcal{E}_{\gamma_2} :\ \forall\, \beta \in \Phi_2^+\},$$

where $\gamma_1, \gamma_2 \in \Phi_1^+$ are determined by β by Lemma 2.2 (2). By Lemma 4.3, the ideal $I_3(B_n)$ is contained in the kernel of π.

Consider $\beta = 0^{n-2}1^1 2^1 = \alpha_{n-1} + 2\alpha_n \in \Phi_2^+$. It uniquely determines $\gamma_1 = \alpha_n$, $\gamma_2 = \alpha_{n-1} + \alpha_n \in \Phi_1^+$. We have in $H_v(B_n)$

$$\mathcal{E}_\beta = \frac{1}{v+v^{-1}}(\mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - \mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2})$$

by Lemma 4.2, and

$$\mathcal{E}_{\gamma_2} = \mathcal{E}_{n-1}\mathcal{E}_n - v^{-2}\mathcal{E}_n\mathcal{E}_{n-1}$$

by Lemma 4.1. Therefore in $I_3(B_n)$

$$\begin{aligned}
\mathcal{E}_\beta - \frac{v^2-1}{v+v^{-1}}\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2} &= \frac{1}{v+v^{-1}}(\mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - \mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}) - \frac{v^2-1}{v+v^{-1}}\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2} \\
&= \frac{\mathcal{E}_{\gamma_2}\mathcal{E}_{\gamma_1} - v^2\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}}{v+v^{-1}} = \frac{\mathcal{E}_{\gamma_2}\mathcal{E}_n - v^2\mathcal{E}_n\mathcal{E}_{\gamma_2}}{v+v^{-1}} \\
&= \frac{\mathcal{E}_{n-1}\mathcal{E}_n^2 - (v^2+v^{-2})\mathcal{E}_n\mathcal{E}_{n-1}\mathcal{E}_n + \mathcal{E}_n^2\mathcal{E}_{n-1}}{v+v^{-1}}.
\end{aligned}$$

It follows that $\mathcal{E}_{n-1}\mathcal{E}_n^2 - (v^2+v^{-2})\mathcal{E}_n\mathcal{E}_{n-1}\mathcal{E}_n + \mathcal{E}_n^2\mathcal{E}_{n-1}$, and hence $\mathrm{Ker}(\pi)$, lie in $I_3(B_n)$. Hence $\mathrm{Ker}(\pi) = I_3(B_n)$. $\square$

Proof. (**Proof of Theorem 2.4**) Define the degree function on the root vectors and on the PBW-basis of $H_v(B_n)$ as follows:

$$deg(\mathcal{E}_\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Phi_1^+ \\ 2 & \text{if } \gamma \in \Phi_2^+ \end{cases}$$

and for $\gamma_1, \gamma_2 \in \Phi^+(B_n)$ with $\gamma_1 \preccurlyeq \gamma_2$

$$deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}) = deg(\mathcal{E}_{\gamma_1}) + deg(\mathcal{E}_{\gamma_2}).$$

For any element $\mathcal{E} \in H_v(B_n)$, define the degree $deg(\mathcal{E})$ to be the minimal degree of the PBW-basis elements which have non-zero coefficients in $\mathcal{E}$.

For a positive integer r, let V_r be the subspace of $H_v(B_n)$ with a $\mathbb{Q}(v)$-basis

$$\{\prod_{\alpha\in\Phi_1^+,\,\beta\in\Phi_2^+} \mathcal{E}_\alpha^{(n_\alpha)}\mathcal{E}_\beta^{(n_\beta)} : \sum_{n_\alpha} n_\alpha + 2\sum_{n_\beta} n_\beta > r\}.$$

To see that V_r is an ideal, it suffices to show that for any positive root $\alpha \in \Phi^+(B_n)$ and any word $\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}$ in the PBW-basis of $H_v(B_n)$, i.e. $\gamma_1 \preccurlyeq \gamma_2 \preccurlyeq \ldots \preccurlyeq \gamma_m$, we have

$$deg(\mathcal{E}_\alpha\mathcal{E}_{\gamma_1}\cdots\mathcal{E}_{\gamma_m}) \geqslant deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}),$$

$$deg(\mathcal{E}_{\gamma_1}\cdots\mathcal{E}_{\gamma_m}) \leqslant deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}\mathcal{E}_\alpha).$$

We shall prove the first inequality only.

If $\alpha \preccurlyeq \gamma_1$, it is well ordered already and

$$\begin{aligned}
deg(\mathcal{E}_\alpha\mathcal{E}_{\gamma_1}\cdots\mathcal{E}_{\gamma_m}) &= deg(\mathcal{E}_\alpha) + deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}) \\
&\leqslant deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}).
\end{aligned}$$

If $\gamma_1 \prec \alpha$ and $\langle\alpha,\gamma_1\rangle = 0$, then $\mathrm{Ext}(M_\alpha, M_{\gamma_1}) = 0$ and $\mathcal{E}_\alpha\mathcal{E}_{\gamma_1} = v^{-\langle\gamma_1,\alpha\rangle}\mathcal{E}_{\gamma_1}\mathcal{E}_\alpha$. Degree is preserved in this order-changing.

If $\gamma_1 \prec \alpha$ and $\langle \alpha, \gamma_1 \rangle \neq 0$, then $\mathrm{Ext}(M_\alpha, M_{\gamma_1}) \neq 0$ and there exists a short exact sequence of the form

$$0 \longrightarrow M_{\gamma_1} \longrightarrow X \longrightarrow M_\alpha \longrightarrow 0$$

where X is of dimension vector $\gamma_1 + \alpha$.

If X is decomposable, say $X = \bigoplus_i M_{\beta_i}$ with $\beta_i \in \Phi^+(B_n)$ satisfying that $\sum_i \beta_i = \gamma_1 + \alpha$ and $\gamma_1 \prec \beta_i \prec \alpha$. We have

$$\mathcal{E}_\alpha \mathcal{E}_{\gamma_1} = c_1 \mathcal{E}_{\gamma_1} \mathcal{E}_\alpha + c_2 \prod_i \mathcal{E}_{\beta_i}$$

with the coefficients $c_1, c_2 \in \mathbb{Q}(v)$. From the Auslander-Reiten quiver of B_n, one sees that $deg(\prod_i \mathcal{E}_{\beta_i}) = deg(\mathcal{E}_\alpha \mathcal{E}_{\gamma_1})$, and that for any β_i, there is no $\gamma \in \Phi^+(B_n)$ with $\gamma_1 \precneq \gamma$ and $\mathrm{Ext}(M_{\beta_i}, M_\gamma) \neq 0$.

If X is indecomposable, suppose $X = M_\beta$ with $\beta = \gamma_1 + \alpha$. We have

$$\mathcal{E}_\alpha \mathcal{E}_{\gamma_1} = d_1 \mathcal{E}_{\gamma_1} \mathcal{E}_\alpha + d_2 \mathcal{E}_\beta$$

with the coefficients $d_1, d_2 \in \mathbb{Q}(v)$. Since M_{γ_1} is a submodule of M_β, we have $deg(\mathcal{E}_\beta) \geq \mathcal{E}_{\gamma_1}$. There is at most one positive root $\gamma \in \Phi^+(B_n)$ with $\gamma_1 \precneq \gamma$ such that $\mathrm{Ext}(M_\beta, M_\gamma) \neq 0$, which happens only when $\gamma = \gamma_1$ and $\beta = 0^a 1^b \in \Phi_1^+$ with $a > 0$, $a + b = n$. In this case we have short exact sequences

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_\beta \longrightarrow M_\alpha \longrightarrow 0,$$

$$0 \longrightarrow M_{\gamma_1} \longrightarrow M_{\beta+\gamma_1} \longrightarrow M_\beta \longrightarrow 0.$$

And

$$deg(\mathcal{E}_\beta) = deg(\mathcal{E}_{\gamma_1}) = 1,$$

$$deg(\mathcal{E}_{\beta+\gamma_1}) = 2 = deg(\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}) = deg(\mathcal{E}_{\gamma_1}^2).$$

With these facts, doing induction on the length m of the word $\mathcal{E}_{\gamma_1}\mathcal{E}_{\gamma_2}\cdots\mathcal{E}_{\gamma_m}$ completes the proof. □

Now Theorem 2.5 is just a direct corollary.

Proof. (**Proof of Theorem 2.5**) Recall that the algebra map $\psi : H_v(B_n) \longrightarrow S_{v^2}(n+1, r)$ is the composition of $\pi : H_v(B_n) \longrightarrow H_{v^2}(A_n)$ and $\widetilde{\varphi} : H_{v^2}(A_n) \longrightarrow S_{v^2}(n+1, r)$. We know from Section 2 that $\mathrm{Ker}(\widetilde{\varphi}) = I_2(A_n)$. By Lemma 4.3 the map π sends the ideal $I_2(B_n)$ of $H_v(B_n)$ to the ideal $I_2(A_n)$ of $H_{v^2}(A_n)$. Therefore

$$\begin{aligned}\mathrm{Ker}(\psi) &= \mathrm{Ker}(\widetilde{\varphi} \circ \pi) = \pi^{-1}(\mathrm{Ker}(\widetilde{\varphi})) \\ &= \pi^{-1}(I_2(A_n)) = I_2(B_n) + \mathrm{Ker}(\pi).\end{aligned}$$

Note that although $I_1(B_n)$ is not an ideal of the Hall algebra $H_v(B_n)$, the image of $I_1(B_n)$ under π is exactly $I_2(A_n)$ and they have the same dimension over $\mathbb{Q}(v)$. Hence $\mathrm{Ker}(\psi)$ decomposes into the direct sum of $I_1(B_n)$ with $\mathrm{Ker}(\pi)$ as a $\mathbb{Q}(v)$-vector space. □

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