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Schur algebras of classical groups

Qunhua Liu [1]

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China
Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK

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Abstract

Throughout this paper the base field will be $\mathbb{C}$. By Doty's definition [S. Doty, Polynomial representations, algebraic monoids, and Schur algebras of classical type, J. Pure Appl. Algebra 123 (1998) 165–199], a Schur algebra of a classical group G is the image of the representation map $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r})$, where $\mathbb{C}^n$ is the natural representation and r any natural number. These Schur algebras are semisimple over $\mathbb{C}$. Firstly we determine when the Schur algebras are generalized Schur algebras in Donkin's sense (see [S. Donkin, On Schur algebras and related algebras, I, J. Algebra 104 (1986) 310–328]). The main step is to decompose the tensor space $(\mathbb{C}^n)^{\otimes r}$, using path model by Littelmann [P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1994) 329–346]. Secondly we relate Schur algebras with different parameters and form inverse systems from Schur algebras in the same type. We find the inverse limit naturally contains the universal enveloping algebra of the corresponding Lie algebra.

Keywords: Classical groups; Schur algebras; Classical Lie algebras; Path model; Inverse system

E-mail address: lqh98@mails.tsinghua.edu.cn.
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1. Introduction

Throughout this paper the base field will be $\mathbb{C}$. By [3], a Schur algebra of a classical group G is the image of the representation map $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r})$, where $\mathbb{C}^n$ is the natural representation and r any natural number. These Schur algebras are semisimple over $\mathbb{C}$.

There are two aims. Firstly we determine when the Schur algebras are generalized Schur algebras in Donkin's sense (see [2]). For a given Schur algebra, its simple blocks are full matrix algebras labeled by a certain set of dominant weights. The Schur algebra is a generalized Schur algebra if and only if the weight set is saturated in all dominant weights, with respect to the dominance order. We will determine the weight sets for Schur algebras of all classical groups, see Theorem 4.6. The main step is to use Littelmann's path model in [10] to decompose the tensor space $(\mathbb{C}^n)^{\otimes r}$. These weight sets were stated in [11] but in an implicit way (see also the Appendix in [5]). We will prove that over $\mathbb{C}$ Schur algebras are generalized Schur algebras in type A, C and D, and are not in type B, see Theorem 4.8.

Secondly we relate Schur algebras with different parameters. By definition, Schur algebras in types other than A can be embedded naturally into Schur algebras in type A in different (but isomorphic) ways. We will construct some surjections between Schur algebras in the same type. This makes Schur algebras in the same type into an inverse system. Following results from [1], the universal enveloping algebra can be embedded into the inverse limit naturally. The degree difference of Schur algebras in the surjections can be any multiple of n for any types, see Proposition 6.2. In types other than A, it can even be any multiple of 2, see Theorem 6.3.

The paper is organized as follows. In Section 2 we will list the root and weight systems of classical Lie algebras and identify a partition with a certain dominant weight. In Section 3 we will recall Littelmann's path model and theorem on decomposition of tensor products. In Section 4 we will achieve our first aim to determine the structure of Schur algebras, using results from Sections 2 and 3. In Section 5 we will collect some examples of Schur algebras with small parameters. In Section 6 we will discuss maps between different Schur algebras.

Theorems 4.8 and 6.3 hold for Schur algebras over any algebraically closed field of characteristic not 2. These results will appear in a forthcoming paper.

2. Weight systems of classical Lie algebras

We will recall the positive roots and dominant weights for classical Lie algebras in this section (see Humphreys' book [9] for reference).

The Cartan subalgebra $\mathfrak{h}_0$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is a n-dimensional vector space containing all diagonal matrices. For $i = 1, 2, \ldots, n$, set $\varepsilon_i \in \mathfrak{h}_0^*$ sending a diagonal matrix $X = \mathrm{Diag}[x_1, x_2, \ldots, x_n]$ to x_i. Let $\mathfrak{g}$ be a classical Lie algebra over $\mathbb{C}$, and n the size of matrices in $\mathfrak{g}$ in the standard representation. Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{h}_0$.

For a simple root α of $\mathfrak{g}$, let $\check{\alpha}$ be the coroot in $\mathfrak{h}$ such that $\langle \alpha, \check{\alpha} \rangle = 2$. $\lambda \in \mathfrak{h}^*$ is said to be a *dominant weight* if $\langle \lambda, \check{\alpha} \rangle$ is a nonnegative integer for all simple roots α. The *dominance order* is defined as follows. Given two dominant weights λ and μ, $\lambda \trianglerighteq \mu$ if and only if $\lambda - \mu$ is a linear combination of positive roots with positive coefficients. Write $P_{++}(X_m)$ for the set of dominant weights, where X_m is the type of $\mathfrak{g}$. Under dominance order $P_{++}(X_m)$ is a disjoint union of several subsets, such that no order relation exists for weights from different subsets, and for any weight there exists a bigger one in the same subset. We call such a subset a *component* of $P_{++}(X_m)$. We denote by $\mathbb{N}_0$ the set of nonnegative integers, and $\Lambda^+(m,r) = \{(a_1, a_2, \dots, a_m)\colon a_1 + a_2 + \cdots + a_m = r,\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m,\ a_i \in \mathbb{N}_0\}$ the set of partitions. We identify a partition $(a_1, a_2, \dots, a_m)$ with the weight $a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m$.

For type A_m ($m \geqslant 1$), the Lie algebra $\mathfrak{sl}_{m+1}(\mathbb{C}) = \{M \in M_{m+1}(\mathbb{C})\colon \operatorname{tr}(M) = 0\}$, and $n = m+1$. Its Cartan subalgebra $\mathfrak{h}$ is the subspace of $\mathfrak{h}_0$ satisfying $(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)(\mathfrak{h}) = 0$. The simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, m.$$

The positive roots are

$$\Phi^+(A_m) = \{\varepsilon_i - \varepsilon_j\colon 1 \leqslant i < j \leqslant n\}.$$

The dominant weights are

$$\begin{aligned} P_{++}(A_m) &= \{a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m\colon a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m,\ a_i \in \mathbb{N}_0\} \\ &\simeq \bigcup_{r \geqslant 0} \Lambda^+(m,r). \end{aligned}$$

Under the dominance order $P_{++}(A_m)$ is divided into n components: $\bigcup_{k\geqslant 0} \Lambda^+(m, kn+i)$, $i = 0, 1, \dots, m$.

Write $i' = n+1-i$ for $i = 1, 2, \dots, n$. In types not A, $\mathfrak{h}$ is the subspace of $\mathfrak{h}_0$ satisfying $(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)(\mathfrak{h}) = 0$ and $(\varepsilon_i + \varepsilon_{i'})(\mathfrak{h}) = 0$ for $i = 1, 2, \dots, n$. The Lie algebra $\mathfrak{so}_n(\mathbb{C}) = \{M \in M_n(\mathbb{C})\colon M^{\mathrm{tr}}J + JM = 0\}$, where J is any symmetric invertible matrix. We will take J to be the matrix with 1's on all (i, i')-entries and zeroes elsewhere.

For type B_m ($m \geqslant 2$), the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2m+1}(\mathbb{C})$ and $n = 2m+1$. The simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \dots, m-1; \qquad \alpha_m = \varepsilon_m.$$

The positive roots are

$$\Phi^+(B_m) = \{\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i < j \leqslant m\} \cup \{\varepsilon_i\colon i = 1, \dots, m\}.$$

The dominant weights are

$$P_{++}(B_m) = \left\{a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m\colon a_i \in \frac{1}{2}\mathbb{N}_0,\ a_i - a_j \in \mathbb{N}_0,\ \forall i < j\right\}.$$

Under the dominant order $P_{++}(B_m)$ is divided into two components: weights with integral coefficients, $\bigcup_{r\geqslant 0} \Lambda^+(m,r)$, and those with fractional coefficients, $(1/2, 1/2, \dots, 1/2) + \bigcup_{r\geqslant 0} \Lambda^+(m,r)$.

For type C_m ($m \geqslant 2$), $n = 2m$ and the Lie algebra $\mathfrak{sp}_{2m}(\mathbb{C}) = \{M \in M_{2m}(\mathbb{C})\colon\ M^{\mathrm{tr}} J' + J'M = 0\}$, where J' is any anti-symmetric invertible matrix. We will take J' to be the one with 1's on all (i, i')th entries, -1's on (i', i)-entries for $i = 1, 2, \ldots, m$ and zeroes elsewhere. The simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \ldots, m-1; \qquad \alpha_m = 2\varepsilon_m.$$

The positive roots are

$$\Phi^+(C_m) = \{\varepsilon_i \pm \varepsilon_j\colon\ 1 \leqslant i < j \leqslant m\} \cup \{2\varepsilon_i\colon\ i = 1, 2, \ldots, m\}.$$

The dominant weights are

$$\begin{aligned} P_{++}(C_m) &= \{a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m\colon\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m,\ a_i \in \mathbb{N}_0\} \\ &\simeq \bigcup_{r \geqslant 0} \Lambda^+(m, r). \end{aligned}$$

Under the dominance order, $P_{++}(C_m)$ is divided into two components: partitions with odd degrees $\bigcup_{r\ \mathrm{odd}} \Lambda^+(m, r)$ and those with even degrees $\bigcup_{r\ \mathrm{even}} \Lambda^+(m, r)$.

For type D_m ($m \geqslant 4$), the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2m}$ and $n = 2m$. Simple roots are

$$\begin{aligned} &\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \ldots, m-2; \\ &\alpha_{m-1} = \varepsilon_{m-1} + \varepsilon_m, \qquad \alpha_m = \varepsilon_{m-1} - \varepsilon_m. \end{aligned}$$

Positive roots are

$$\Phi^+(D_m) = \{\varepsilon \pm \varepsilon_j\colon\ 1 \leqslant i < j \leqslant m\}.$$

Dominant weights are

$$\begin{aligned} P_{++}(D_m) = \biggl\{ &\left(a_1 + \cdots + a_{m-2} + \frac{a_{m-1}}{2} + \frac{a_m}{2}\right)\varepsilon_1 + \cdots + \left(\frac{a_{m-1}}{2} + \frac{a_m}{2}\right)\varepsilon_{m-1} \\ &+ \left(\frac{a_{m-1}}{2} - \frac{a_m}{2}\right)\varepsilon_m\colon\ a_i \in \mathbb{N}_0 \biggr\}. \end{aligned}$$

Define

$$\begin{aligned} \Lambda^-(m, r) &= \bigl\{(\lambda_1, \ldots, \lambda_{m-1}, -\lambda_m)\colon\ (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m) \in \Lambda^+(m, r)\bigr\}, \\ \Lambda^\pm(m, r) &= \Lambda^+(m, r) \cup \Lambda^-(m, r). \end{aligned}$$

Then the dominant weights with integral coefficients are $\bigcup_{r \geqslant 0} \Lambda^\pm(m, r)$. Under the dominance order $\bigcup_{r \geqslant 0} \Lambda^\pm(m, r)$ is divided into two parts: those with odd degrees $\bigcup_{r\ \mathrm{odd}} \Lambda^\pm(m, r)$ and those with even degrees $\bigcup_{r\ \mathrm{even}} \Lambda^\pm(m, r)$.

3. Path model

Let us recall what path model is and how it works. The main reference is Littelmann's paper [10] which deals with symmetrizable Kac–Moody Lie algebras.

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra, and $\mathfrak{h}$ the Cartan subalgebra. Write $\mathfrak{h}^*_{\mathbb{R}}$ for the subspace of $\mathfrak{h}^*$ containing linear combinations of simple roots with real coefficients. Then $\dim_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}} = \dim_{\mathbb{C}} \mathfrak{h}^*$. Define a *path* to be a piecewise linear map $\gamma : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$ such that $\gamma(0) = 0$, modulo the equivalence relation $\gamma \sim \gamma'$ if $\gamma = \gamma'$ up to a reparametrization. Let Π be the set of all paths. The product of two paths γ_1 and γ_2 is defined to be

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ \gamma_2(2t-1) + \gamma_1(1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

For a simple root α, let s_α be the linear function on $\mathfrak{h}^*_{\mathbb{R}}$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \check{\alpha}, \lambda \rangle \alpha$$

for any λ in $\mathfrak{h}^*_{\mathbb{R}}$. Define $s_\alpha(\gamma)$ to be the path given by $s_\alpha(\gamma)(t) = s_\alpha(\gamma(t))$.

Given a path γ and a simple root α, consider the map $h_\alpha : [0,1] \to \mathbb{R}$ sending t to $\langle \check{\alpha}, \gamma(t) \rangle$. Set

$$Q = \min\bigl\{\text{image}(h_\alpha) \cap \mathbb{Z}\bigr\}.$$

This is always a negative integer or zero. Let

$$p = \max\bigl\{t \in [0,1]\colon\ h_\alpha(t) = Q\bigr\}.$$

Let P be the integral part of $h_\alpha(1) - Q$ and let $x > p$ such that

$$h_\alpha(x) = Q + 1, \qquad Q < h_\alpha(t) < Q + 1, \quad \forall p < t < x.$$

Define paths γ_1, γ_2 and γ_3 by

$$\gamma_1(t) = \gamma(tp), \qquad \gamma_2(t) = \gamma\bigl(p + t(x-p)\bigr) - \gamma(p),$$
$$\gamma_3(t) = \gamma\bigl(x + t(1-x)\bigr) - \gamma(x).$$

By definition $\gamma = \gamma_1 * \gamma_2 * \gamma_3$. Now define the reflection $f_\alpha(\gamma)$ to be 0 if $P = 0$, and to be $\gamma_1 * s_\alpha(\gamma_2) * \gamma_3$ if $P > 0$.

For any $\mu \in \mathfrak{h}^*_{\mathbb{R}}$, denote by γ_μ the straight-line path sending $t \in [0,1]$ to $t\mu$. Let us look at some examples of reflections on such paths.

Example. Type A_1. $\mathfrak{h}^*_{\mathbb{R}}$ is one-dimensional as in Fig. 1.

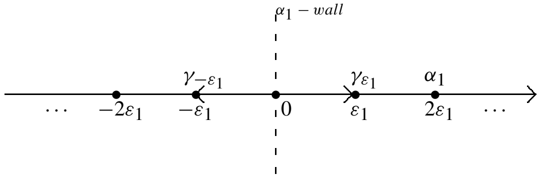

Fig. 1.

Write $\alpha_1 = 2\varepsilon_1$, the unique simple root. Consider the path γ_{ε_1}. The function h_{α_1} sends $t \in [0, 1]$ to $\langle \check{\alpha_1}, \gamma_{\varepsilon_1}(t)\rangle = \langle \check{\alpha_1}, t\varepsilon_1\rangle = t$. Hence $Q = 0$, $P = 1$, $p = 0$, $x = 1$. And

$$f_{\alpha_1}(\gamma_{\varepsilon_1}) = s_{\alpha_1}(\gamma_{\varepsilon_1}) = \gamma_{\varepsilon_1-\alpha_1} = \gamma_{-\varepsilon_1}.$$

Example. Type A_2. $\mathfrak{h}^*_{\mathbb{R}}$ is two-dimensional, see Fig. 2.

For the path γ_{ε_1} and simple root $\alpha_1 = \varepsilon_1 - \varepsilon_2$, the function h_{α_1} sends $t \in [0, 1]$ to t. By a similar argument as in the previous example, we have

$$f_{\alpha_1}(\gamma_{\varepsilon_1}) = s_{\alpha_1}(\gamma_{\varepsilon_1}) = \gamma_{\varepsilon_1-\alpha_1} = \gamma_{\varepsilon_2}.$$

For the simple root $\alpha_2 = \varepsilon_2 - \varepsilon_3$, h_{α_2} sends t to 0. Hence $Q = 0 = P$, and $f_{\alpha_2}(\gamma_{\varepsilon_1}) = 0$. For the positive root $\alpha = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$, h_α sends t to t and

$$f_\alpha(\gamma_{\varepsilon_1}) = s_\alpha(\gamma_{\varepsilon_1}) = \gamma_{\varepsilon_1-\alpha} = \gamma_{\varepsilon_3}.$$

Example. Type B_2. $\mathfrak{h}^*_{\mathbb{R}}$ has the shape shown in Fig. 3.

For the path γ_{ε_1} and simple root $\alpha_1 = \varepsilon_1 - \varepsilon_2$, h_{α_1} sends t to t and

$$f_{\alpha_1}(\gamma_{\varepsilon_1}) = s_{\alpha_1}(\gamma_{\varepsilon_1}) = \gamma_{\varepsilon_1-\alpha_1} = \gamma_{\varepsilon_2}.$$

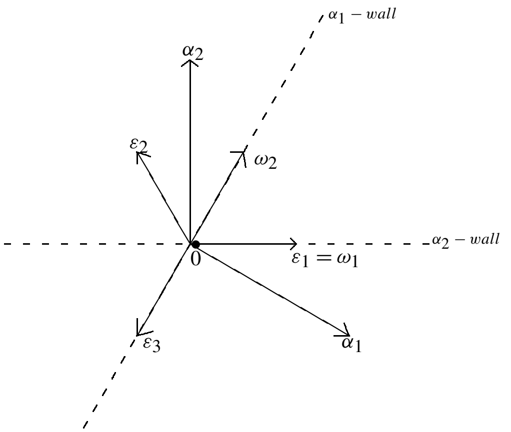

Fig. 2.

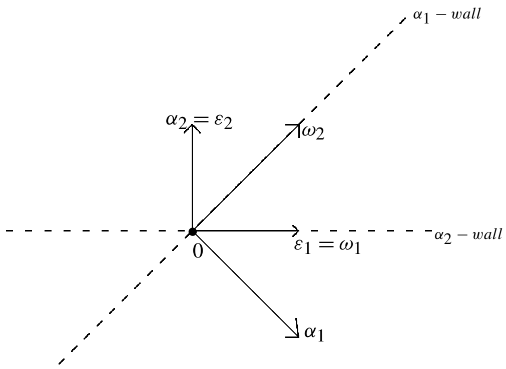

Fig. 3.

For the simple root $\alpha_2 = \varepsilon_2$ and the path γ_{ε_2}, h_{α_2} sends t to $2t$. Hence $Q = 0$, $P = 2$, $p = 0$, $x = 1/2$, $\gamma_{\varepsilon_2} = \gamma_2 * \gamma_3$. And

$$f_{\alpha_2}(\gamma_{\varepsilon_2}) = s_{\alpha_2}(\gamma_2) * \gamma_3 = \tilde{\gamma}_0,$$

where $\tilde{\gamma}_0$ means the path

$$\tilde{\gamma}_0(t) = \begin{cases} -t\varepsilon_2 & \text{if } 0 \leqslant t \leqslant 1/2, \\ (t-1)\varepsilon_2 & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Furthermore one can check that $f_{\alpha_2}(\tilde{\gamma}_0(t)) = f_{\alpha_2}^2(\gamma_{\varepsilon_2}) = \gamma_{-\varepsilon_2}$.

Given a dominant weight μ, let

$$\mathcal{P}_\mu = \{f_{\alpha_{i_1}} \cdots f_{\alpha_{i_s}}(\gamma_\mu)\colon\ \alpha_{i_j} \text{ are simple roots, } s \in \mathbf{N}_0\}.$$

Elements in $\mathcal{P}_\mu$ are called *Lakshmibai–Seshadri paths of shape* μ by Littelmann [10], or *LS paths* for short. Given another dominant weight λ (unnecessarily different from μ), an LS path γ of shape μ is said to be λ-*dominant*, if $\lambda + \gamma(t) \in P_{++}(X_m)$ for any $t \in [0, 1]$. Denote by $\mathcal{P}_\mu^\lambda$ for the set of λ-dominant LS paths of shape μ. Then we have the decomposition formula.

Theorem 3.1. (Littelmann [10].) *Given two dominant weights λ and μ, we have a decomposition of $\mathfrak{g}$-representation*

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \mathcal{P}_\mu^\lambda} V\big(\lambda + \nu(1)\big),$$

where $V(\lambda)$ is the irreducible representation with highest weight λ.

For proof of this theorem see [10]. Note that multiplicities in the direct sum are hidden in the paths; indeed different paths in $\mathcal{P}_\mu^\lambda$ can have the same value at $t=1$.

4. Schur algebras of classical groups

Recall that $X = A$, B, C or D by Section 2, and $n = m+1$ in type A_m ($m \geqslant 1$), $2m+1$ in type B_m ($m \geqslant 2$), and $2m$ in type C_m ($m \geqslant 2$) and D_m ($m \geqslant 4$). Let $\mathfrak{g}$ be the simple complex Lie algebra of type X_m, and G the corresponding simply connected Lie group. That is, $G = SL_{m+1}(\mathbb{C})$, $SO_{2m+1}(\mathbb{C})$, $SP_{2m}(\mathbb{C})$ and $SO_{2m}(\mathbb{C})$ in type A_m, B_m, C_m and D_m, respectively. Clearly $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n$, and G a Lie subgroup of GL_n. Consider the natural representation $\mathbb{C}^n$ of G with representation map $\rho: KG \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. For any natural number r, KG acts diagonally on $(\mathbb{C}^n)^{\otimes r}$.

Definition 4.1. (Doty [3].) For $r \geqslant 1$, the Schur algebra of type X_m, denoted by $S^X(n,r)$, is defined to be the image of the representation map

$$\rho^{\otimes r}: KG \to \mathrm{End}_{\mathbb{C}}\big((\mathbb{C}^n)^{\otimes r}\big).$$

When $r = 0$, we define $S^X(n,0) = \mathbb{C}$.

Although we require $m \geqslant 2$ in type B, C, and $m \geqslant 4$ in type D, it is actually unnecessary for the definition. For smaller m, the Lie algebra and Lie group still exist. Hence one can define the corresponding Schur algebra in the same way. The case when $m = 1$ will be explained in Section 6. When $X = A$, $S^A(n,r) = S_{\mathbb{C}}(n,r)$ is just the classical Schur algebra defined by J.A. Green in [7].

The Lie algebra $\mathfrak{g}$ also acts on $\mathbb{C}^n$ by matrix multiplication. It acts on $(\mathbb{C}^n)^{\otimes r}$ by $X(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = \sum_{i=1}^r v_1 \otimes \cdots \otimes v_{i-1} \otimes X v_i \otimes v_{i+1} \otimes \cdots \otimes v_r$, for $X \in \mathfrak{g}$ and v_i in $\mathbb{C}^n$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. This is an associative algebra over $\mathbb{C}$. There is an equivalence between the category of $\mathfrak{g}$-representations and the category of $\mathcal{U}(\mathfrak{g})$-modules. The Schur algebra $S^X(n,r)$ can be defined to be the image of the representation map $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes r}$. This definition enables us to make use of results from Lie theory.

Lemma 4.2. *Let V_1 and V_2 be two finite-dimensional irreducible representations of $\mathfrak{g}$. Then $V_1 \oplus V_2$ is also a representation and the image of the representation map*

$$\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V_1 \oplus V_2)$$

is $\mathrm{End}_{\mathbb{C}}(V_1) \oplus \mathrm{End}_{\mathbb{C}}(V_2)$ if V_1 is nonisomorphic to V_2, and $\mathrm{End}_{\mathbb{C}}(V_1)$ if V_1 is isomorphic to V_2.

Proof. Since V_i is irreducible over $\mathfrak{g}$, the image of representation map $\rho_i : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V_i), \mathcal{U}(\mathfrak{g})/\operatorname{Ker}(\rho)$, is a semisimple algebra. Also V_i is irreducible over $\mathcal{U}(\mathfrak{g})/\operatorname{Ker}(\rho)$. Hence ρ_i is surjective. By definition of direct sums, the representation map

$$\rho_1 \oplus \rho_2 : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V_1 \oplus V_2)$$

factors through $\mathrm{End}_{\mathbb{C}}(V_1) \oplus \mathrm{End}_{\mathbb{C}}(V_2)$. If $V_1 \cong V_2$, we can identify ρ_1 with ρ_2. Hence

$$\operatorname{image}(\rho_1 \oplus \rho_2) \cong \operatorname{image}(\rho_1) = \mathrm{End}_{\mathbb{C}}(V_1).$$

Otherwise, both ρ_1 and ρ_2 are surjective, and

$$\operatorname{image}(\rho_1 \oplus \rho_2) = \operatorname{image}(\rho_1) \oplus \operatorname{image}(\rho_2) = \mathrm{End}_{\mathbb{C}}(V_1) \oplus \mathrm{End}_{\mathbb{C}}(V_2). \qquad \square$$

Corollary 4.3. *Let V_i be pairwise nonisomorphic finite-dimensional irreducible representations of $\mathfrak{g}$, $i = 1, 2, \ldots, s$. Then, for $n_i \in \mathbf{N}$, the image of the representation map $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\bigoplus_{i=1}^{s}(V_i^{\oplus n_i}))$ is $\bigoplus_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(V_i)$.*

Bear in mind that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. So all finite-dimensional representations of $\mathfrak{g}$ are completely reducible with each irreducible direct summand a highest weight module $V(\lambda)$ with $\lambda \in P_{++}(X_m)$, the set of dominant weights. Suppose we have a decomposition

$$\left(\mathbb{C}^n\right)^{\otimes r} = \bigoplus_{\lambda \in \pi^X(n,r)} V(\lambda)^{\oplus l_\lambda},$$

where l_λ is the multiplicity, and $\pi^X(n,r)$ is the subset of $P_{++}(X_m)$ consisting of all highest weights of simple direct summands of $(\mathbb{C}^n)^{\otimes r}$. Now by Corollary 4.3,

$$\begin{aligned} S^X(n,r) &= \operatorname{image}\biggl(\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}\biggl(\bigoplus_{\lambda \in \pi^X(n,r)} V(\lambda)^{\oplus l_\lambda}\biggr)\biggr) \\ &= \bigoplus_{\lambda \in \pi^X(n,r)} \mathrm{End}_{\mathbb{C}}\bigl(V(\lambda)\bigr). \end{aligned}$$

Corollary 4.4. *The Schur algebra is semisimple over $\mathbb{C}$ and*

$$S^X(n,r) \cong \bigoplus_{\lambda \in \pi^X(n,r)} M_{d_\lambda}(\mathbb{C}),$$

where $d_\lambda = \dim_{\mathbb{C}} V(\lambda)$.

Weyl's dimension formula gives the dimension of $V(\lambda)$. To describe the algebra structure of $S^X(n,r)$, we have to determine the set $\pi^X(n,r)$, or equivalently, to decompose the tensor product $(\mathbb{C}^n)^{\otimes r}$ over $\mathfrak{g}$. Note that the natural representation $\mathbb{C}^n$ is irreducible with highest weight ε_1. Let us determine all LS paths of shape ε_1 first.

Lemma 4.5. *For type A_m, $\mathcal{P}_{\varepsilon_1} = \{\gamma_{\varepsilon_1}, \gamma_{\varepsilon_2}, \dots, \gamma_{\varepsilon_{m+1}}\}$. For type B_m,*

$$\mathcal{P}_{\varepsilon_1} = \{\gamma_{\varepsilon_1}, \gamma_{\varepsilon_2}, \dots, \gamma_{\varepsilon_m}, \ \tilde{\gamma}_0, \gamma_{-\varepsilon_m}, \dots, \gamma_{-\varepsilon_1}\},$$

where $\tilde{\gamma}_0$ is the path

$$\tilde{\gamma}_0(t) = \begin{cases} -t\varepsilon_m & \text{if } 0 \leqslant t \leqslant 1/2, \\ (t-1)\varepsilon_m & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

For type C_m and D_m, $\mathcal{P}_{\varepsilon_1} = \{\gamma_{\varepsilon_1}, \gamma_{\varepsilon_2}, \dots, \gamma_{\varepsilon_m}, \ \gamma_{-\varepsilon_m}, \dots, \gamma_{-\varepsilon_1}\}$.

Proof. This lemma follows from the construction of reflections. Let us prove it case by case.

Type A_m: For a simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2, \dots, m)$ and a path γ_{ε_j} $(j = 1, 2, \dots, n)$, $f_{\alpha_i}(\gamma_{\varepsilon_j})$ is not zero if and only if $j = i$. In this case h_{α_i} sends $t \in [0, 1]$ to t and hence $Q = 0$, $P = 1$, $p = 0$, $s = 1$. It follows that $f_{\alpha_i}(\gamma_{\varepsilon_i}) = s_{\alpha_i}(\gamma_{\varepsilon_i}) = \gamma_{\varepsilon_{i+1}}$. Therefore

$$\mathcal{P}_{\varepsilon_i} = \big\{ f_{\alpha_{i_1}} \cdots f_{\alpha_{i_s}}(\gamma_\mu)\colon \alpha_{i_j} \text{ simple}, \ s \in \mathbf{N}_0 \big\} = \{\gamma_{\varepsilon_1}, \gamma_{\varepsilon_2}, \dots, \gamma_{\varepsilon_{m+1}}\}.$$

Type B_m: For a simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, \dots, m-1)$ and a path γ_{ε_j}, $f_{\alpha_i}(\gamma_{\varepsilon_j})$ is not zero if and only if $j = i$. In this case, similar to the situation in type A, we have $f_{\alpha_i}(\gamma_{\varepsilon_i}) = \gamma_{\varepsilon_{i+1}}$, $i = 1, \dots, m-1$. For a simple root α_i $(i = 1, \dots, m-1)$ and the path $\gamma_{-\varepsilon_j}$, $f_{\alpha_i}(\gamma_{-\varepsilon_j})$ is not zero if and only if $j = i+1$. In this case $f_{\alpha_i}(\gamma_{-\varepsilon_{i+1}}) = \gamma_{-\varepsilon_i}$, $i = 1, \dots, m-1$. When $i = m$, the simple root $\alpha_m = \varepsilon_m$, and similar to the example B_2, we have $f_{\alpha_m}(\gamma_{\varepsilon_m}) = \tilde{\gamma}_0$ and $f_{\alpha_m}(\tilde{\gamma}_0) = \gamma_{-\varepsilon_m}$.

For type C_m and D_m, notice that the only cases when $f_{\alpha_i}(\pm\gamma_{\varepsilon_j})$ is not zero are

$$\begin{gathered} f_{\alpha_i}(\gamma_{\varepsilon_i}) = \gamma_{\varepsilon_{i+1}}, \quad i = 1, \dots, m-1, \\ f_{\alpha_i}(\gamma_{-\varepsilon_{i+1}}) = \gamma_{-\varepsilon_i}, \quad i = 1, \dots, m-1, \\ f_{\alpha_m}(\gamma_{\varepsilon_m}) = \gamma_{-\varepsilon_m}, \end{gathered}$$

and the LS paths of shape ε_1 follow. □

For $0 \leqslant i \leqslant n$, we introduce the following notation:

$$\Lambda_i^+(n, r) = \big\{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)\colon \lambda_1, \dots, \lambda_{n-i} \neq 0\big\},$$

which contains partitions in $\Lambda^+(n, r)$ with the first $n - i$ positions nonzero and the later ones free. In particular $\Lambda_n^+(n, r) = \Lambda^+(n, r)$. In Section 2 we have defined the following sets for type D:

$$\begin{gathered} \Lambda^-(n, r) = \big\{\lambda = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)\colon (\lambda_1, \dots, \lambda_{n-1}, \lambda_n) \in \Lambda^+(n, r)\big\}, \\ \Lambda^\pm(n, r) = \Lambda^+(n, r) \cup \Lambda^-(n, r). \end{gathered}$$

Assume $\Lambda^+(n,r)$, $\Lambda^\pm(n,r)$, $\Lambda^-(n,r)$, $\Lambda_i^+(n,r) = \emptyset$ if $r < 0$. Recall that we identify a partition $(a_1, a_2, \ldots, a_m)$ with a dominant weight $a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m$ in Section 2. The following theorem gives a complete description of $\pi^X(n,r)$ for all types.

Theorem 4.6. *The weight sets which describe Schur algebras of classical groups are*:

$$\pi^A(n,r) = \bigcup_{i\geqslant 0} \Lambda^+(m, r-ni),$$

$$\pi^B(n,r) = \bigcup_{i\geqslant 0} \Lambda^+(m, r-2i) \cup \bigcup_{i\geqslant 0} \Lambda_i^+\bigl(m, r-(2i+1)\bigr),$$

$$\pi^C(n,r) = \bigcup_{i\geqslant 0} \Lambda^+(m, r-2i),$$

$$\pi^D(n,r) = \bigcup_{i\geqslant 0} \Lambda^\pm(m, r-2i),$$

where the second term in $\pi^B(n,r)$ involves of Λ_i^+, and $\pi^D(n,r)$ involves of $\Lambda^\pm$ as defined above.

Proof. We will prove it case by case by induction on r. For any type X_m, when $r=1$, $\mathbb{C}^n = V(\varepsilon_1)$ is the irreducible representation with highest weight ε_1. Hence by identifying a dominant weight with a partition, $\pi^X(n,1) = \{\varepsilon_1\} = \{(1,0,\ldots,0)\} = \Lambda^+(m,1)$ fits every type. In general, assume that $\pi^X(n,r)$ has the required form. We have a decomposition

$$\bigl(\mathbb{C}^n\bigr)^{\otimes r} = V(\varepsilon_1)^{\otimes r} = \bigoplus_{\lambda\in\pi^X(n,r)} V(\lambda)^{\oplus l_\lambda},$$

where l_λ is the multiplicity of the simple module $V(\lambda)$ in $(\mathbb{C}^n)^{\otimes r}$. Therefore by Theorem 3.1

$$\begin{aligned}\bigl(\mathbb{C}^n\bigr)^{\otimes r+1} &= \bigoplus_{\lambda\in\pi^X(n,r)} \bigl(V(\lambda)\otimes V(\varepsilon_1)\bigr)^{\oplus l_\lambda}\\ &= \bigoplus_{\lambda\in\pi^X(n,r)} \bigoplus_{\gamma\in\mathcal{P}_{\varepsilon_1}^\lambda} V\bigl(\lambda+\gamma(1)\bigr)^{\oplus l'_\lambda},\end{aligned}$$

where l'_λ is the multiplicity of the simple module $V(\lambda+\gamma(1))$ in $(\mathbb{C}^n)^{\otimes r+1}$. Hence by definition $\pi^X(n,r+1) = \{\lambda+\gamma(1)\colon \lambda\in\pi^X(n,r),\ \gamma\in\mathcal{P}_{\varepsilon_1}^\lambda\}$, where $\mathcal{P}_{\varepsilon_1}^\lambda = \{\gamma\in\mathcal{P}_{\varepsilon_1}\colon \gamma(t)+\lambda\in P_{++}(A_m),\ \forall 0\leqslant t\leqslant 1\}$. Hence $\pi^X(n,r+1) = P_{++}(A_m)\cap\{\lambda+\gamma(1)\colon \lambda\in\pi^X(n,r),\ \gamma\in\mathcal{P}_{\varepsilon_1}\}$. It remains to show this set coincides with that in the theorem.

Type A_m, $n = m+1$, and $\varepsilon_n = -(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_m)$. Induction hypothesis says that $\pi^A(n,r) = \bigcup_{i\geqslant 0}\Lambda^+(m,r-ni)$. For any path $\gamma_{\varepsilon_j}\in\mathcal{P}_{\varepsilon_1}$ $(j=1,2,\ldots,n)$, $\gamma_{\varepsilon_j}(1) = \varepsilon_j$. Therefore by Lemma 4.5, $\pi^A(n,r+1) = P_{++}(A_m)\cap\{\lambda+\varepsilon_j\colon \lambda\in\bigcup_{i\geqslant 0}\Lambda^+(m,r-ni),$

$j = 1, 2, \ldots, n\}$ is inside $\bigcup_{i \geqslant 0} \Lambda^+(m, r+1-ni)$. On the other hand, for any $i \geqslant 0$, $P_{++}(A_m) \cap \{\lambda + \varepsilon_j \colon \lambda \in \Lambda^+(m, r-ni),\ j = 1, 2, \ldots, m\} = \Lambda^+(m, r+1-ni)$, $P_{++}(A_m) \cap \{\lambda + \varepsilon_n \colon \lambda \in \Lambda^+(m, r-ni)\} = \Lambda^+(m, r+1-n(i-1))$. Hence $\pi^A(n, r+1)$ has the desired form.

Type B_m, $n = 2m+1$. By hypothesis

$$\pi^B(n,r) = \bigcup_{i \geqslant 0} \Lambda^+(m, r-2i) \cup \bigcup_{i \geqslant 0} \Lambda_i^+\big(m, r-(2i+1)\big).$$

If $\lambda \in \Lambda^+(m, r-2i)$, then $P_{++}(B_m) \cap \{\lambda + \gamma(1) \colon \gamma \in \mathcal{P}_{\varepsilon_1}^{\lambda}\}$ is contained in $\Lambda^+(m, r+1-2i) \cup \Lambda^+(m, r+1-2(i-1)) \cup \Lambda_0^+(m, r-2i)$. If $\lambda \in \Lambda_i^+(m, r-(2i+1))$, then $P_{++}(B_m) \cap \{\lambda + \gamma(1) \colon \gamma \in \mathcal{P}_{\varepsilon_1}^{\lambda}\}$ is contained in $\Lambda_{i+1}^+(m, r+1-(2(i+1)+1)) \cup \Lambda_i^+(m, r+1-(2i+1))$. Therefore weights in $\pi^B(n, r+1)$ have the required form. Conversely, all weights in $\bigcup_{i \geqslant 0} \Lambda^+(m, r+1-2i)$ come from $\bigcup_{i \geqslant 0} \Lambda^+(m, r-2i)$. All weights in $\Lambda_i^+(m, r+1-(2i+1))$ come from $\Lambda_i^+(m, r-(2i+1)) \cup \Lambda_{i-1}^+(m, r-(2i-1))$.

Type C_m, $n = 2m$. By hypothesis $\pi^C(n,r) = \bigcup_{i \geqslant 0} \Lambda^+(m, r-2i)$. Notice that for $i \geqslant 0$, $P_{++}(C_m) \cap \{\lambda + \varepsilon_j \colon \lambda \in \Lambda^+(m, r-2i),\ j = 1, 2, \ldots, m\} = \Lambda^+(m, r+1-2i)$, and $P_{++}(C_m) \cap \{\lambda - \varepsilon_j \colon \lambda \in \Lambda^+(m, r-2i),\ j = 1, 2, \ldots, m\} = \Lambda^+(m, r-1-2i) = \Lambda^+(m, r+1-2(i-1))$. Therefore by Lemma 4.5, $\pi^C(n, r+1)$ has the desired form.

Type D_m, $n = 2m$. By hypothesis $\pi^D(n,r) = \bigcup_{i \geqslant 0} \Lambda^{\pm}(m, r-2i)$. By Lemma 4.5, $\pi^D(n, r+1) = P_{++}(D_m) \cap \{\lambda \pm \varepsilon_j \colon \lambda \in \Lambda^{\pm}(m, r-2i),\ j = 1, 2, \ldots, m\}$, which is inside $\bigcup_{i \geqslant 0} \Lambda^{\pm}(m, r+1-2i)$. Conversely, all weights in $\bigcup_{i \geqslant 0} \Lambda^+(m, r+1-2i)$ come from $\bigcup_{i \geqslant 0} \Lambda^{\pm}(m, r+1-2i)$. All weights in $\Lambda^-(m, r+1-2i)$ come from $\Lambda^-(m, r-2i)$. It follows that $\pi^D(n, r+1)$ has the desired form. □

Following Donkin, we say a subset π of $P_{++}(X_m)$ is *saturated*, if for $\lambda \in \pi$, $\mu \in P_{++}(X_m)$, $\lambda \trianglerighteq \mu$ (under the dominance order) implies $\mu \in \pi$. For a dominant weight $\lambda = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m$, we define the *degree* of λ to be $a_1 + a_2 + \cdots + a_m$. See Section 2 for the list of positive roots of classical Lie algebras.

Corollary 4.7. *Let $r \geqslant 1$ be any natural number. In type B_m with $m \geqslant 2$, $\pi^B(n,r)$ is not saturated in $P_{++}(B_m)$. In types not B, $\pi^X(n,r)$ is saturated in $P_{++}(X_m)$.*

Proof. Type A_m: Positive roots of $\mathfrak{sl}_n$ are $\varepsilon_i - \varepsilon_j$ for $1 \leqslant i < j \leqslant n$. They are of degree either 0 or n. Hence by definition of dominance order, there is no order between weights of degree difference not 0 or n. It follows that $\bigcup_{i \geqslant 0} \Lambda^+(m, r-ni)$ is a saturated subset in $P_{++}(A_m)$.

Types C_m and D_m: Positive roots are of degree 0 or 2. Hence there is no order between weights of degree difference not 0 or 2. It follows that $\pi^C(n,r)$ and $\pi^D(n,r)$ are saturated.

Type B_m ($m \geqslant 2$): Positive roots are $\{\varepsilon_i \pm \varepsilon_j \colon 1 \leqslant i < j \leqslant m\} \cup \{\varepsilon_i \colon 1 \leqslant i \leqslant m\}$. Hence the dominant weight $(r-1, 0, \ldots, 0)$ is smaller than $(r, 0, \ldots, 0)$, which is in $\pi^B(n,r)$ by Theorem 4.6. But by definition of $\Lambda_0^+(m, r-1)$, $(r-1, 0, \ldots, 0)$ is not in $\Lambda_0^+(m, r-1)$, and hence not in $\pi^B(n,r)$. Therefore $\pi^B(n,r)$ is not saturated. □

For a saturated subset π of $P_{++}(X_m)$, Donkin defined an algebra $S(\pi)$ in [2], called the generalized Schur algebra. In characteristic zero, $S(\pi)$ is semisimple and has the form $\bigoplus_{\lambda\in\pi} M_{d_\lambda}(\mathbb{C})$, where d_λ is the dimension of the irreducible representation with the highest weight λ. So by Corollary 4.4, whenever the weight set $\pi^X(n,r)$ is saturated the Schur algebra $S^X(n,r)$ is a generalized Schur algebra. We have deduced the following result.

Theorem 4.8. *Let $r \geqslant 1$ be any natural number. In type B_m $(m \geqslant 2)$, $S^B(n,r)$ is not a generalized Schur algebra. In types not B, $S^X(n,r)$ is a generalized Schur algebra.*

In particular when $r = 0$, $\pi^X(n,0) = \{0\}$ which is always saturated. Hence $S^X(n,0) = \mathbb{C}$ is a generalized Schur algebra in any type.

5. Examples

In this section we will collect some examples of Schur algebras with small parameters.

Example. In type A_1, by Theorem 4.6,

$$\pi^A(2,r) = \bigcup_{i\geqslant 0} \Lambda^+(1, r-2i) = \left\{r\varepsilon_1,\ (r-2)\varepsilon_1, \ldots, \left(r - 2\left\lfloor \frac{r}{2} \right\rfloor\right)\varepsilon_1\right\},$$

where $\lfloor r/2 \rfloor$ is the largest integer which is less than or equal to $r/2$. In this case $d_{l\varepsilon_1} = l+1$. Hence the Schur algebra is semisimple of the form

$$S^A(2,r) \cong \bigoplus_{i\geqslant 0} M_{r+1-2i}(\mathbb{C}).$$

Example. The Lie algebra $\mathfrak{sp}_2$ can be viewed as of type C_1. It is defined by $J' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. And $XJ' + J'X = 0$ if and only if $\operatorname{tr}(X) = 0$. Hence we have a Lie algebra isomorphism $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$. It follows that $S^A(2,r) \cong S^C(2,r)$, for any nonnegative integer r.

Example. The Lie algebra $\mathfrak{so}_3$ can be viewed as of type B_1. It has only one positive root ε_1. When $r = 1$, $\pi^B(3,1) = \{\varepsilon_1\}$, which is not saturated because the dominant weight 0 is smaller than ε_1 and 0 is not in the set. When $r \geqslant 2$, we will show that $S^B(3,r)$ is isomorphic to $S^A(2,2r)$, the Schur algebra of type A_1. In particular it is a generalized Schur algebra.

In fact as a Lie algebra, $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$. $\mathfrak{so}_3$ has a unique simple root ε_1, and the fundamental weight $\varepsilon_1/2$. $\mathbb{C}^3$ is the irreducible $\mathfrak{so}_3$-representation with highest weight ε_1. When $r \geqslant 2$, by the same calculation as in Theorem 4.6, we have

$$\pi^B(3,r) = \{r\varepsilon_1, (r-1)\varepsilon_1, \ldots, \varepsilon_1, 0\}.$$

On the other hand, $\mathfrak{sl}_2$ has the unique simple root $2\varepsilon_1$, and the fundamental weight ε_1. $\mathbb{C}^3$ is the irreducible $\mathfrak{sl}_2$-representation with highest weight $2\varepsilon_1$. When $r \geqslant 2$, by Theorem 4.6,

$$\pi^A(2, 2r) = \{2r\varepsilon_1, 2(r-1)\varepsilon_1, \dots, 2\varepsilon_1, 0\}.$$

This gives rise to an isomorphism of $\mathbb{C}$-algebras $S^B(3, r) \cong S^A(2, 2r)$ when $r \geqslant 2$.

By Theorem 4.6, when $m \geqslant 2$ (and $m \geqslant 4$ in type D), $\pi^A(n, 2) = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2\}$, and $\pi^X(n, 2) = \{2\varepsilon_1, \varepsilon_2 + \varepsilon_2, 0\}$ when $X \neq A$. The following lemma tells us the dimensions d_λ of irreducible representations with highest weights λ in these sets.

Lemma 5.1. *For type A_m ($m \geqslant 2$),*

$$d_{2\varepsilon_1} = \frac{n^2+n}{2}, \qquad d_{\varepsilon_1+\varepsilon_2} = \frac{n^2-n}{2}.$$

For type B_m ($m \geqslant 2$) and D_m ($m \geqslant 4$),

$$d_{2\varepsilon_1} = \frac{n^2+n}{2} - 1, \qquad d_{\varepsilon_1+\varepsilon_2} = \frac{n^2-n}{2}, \qquad d_0 = 1.$$

For type C_m ($m \geqslant 2$),

$$d_{2\varepsilon_1} = \frac{n^2+n}{2}, \qquad d_{\varepsilon_1+\varepsilon_2} = \frac{n^2-n}{2} - 1, \qquad d_0 = 1.$$

The next proposition follows from Corollary 4.4.

Proposition 5.2. *Under the same assumption of m as in the previous lemma, the Schur algebras are*

$$S^A(n,2) \cong M_{\frac{n^2+n}{2}}(\mathbb{C}) \oplus M_{\frac{n^2-n}{2}}(\mathbb{C}),$$
$$S^X(n,2) \cong M_{\frac{n^2+n}{2}-1}(\mathbb{C}) \oplus M_{\frac{n^2-n}{2}}(\mathbb{C}) \oplus \mathbb{C}, \quad X = B \text{ or } D,$$
$$S^C(n,2) \cong M_{\frac{n^2+n}{2}}(\mathbb{C}) \oplus M_{\frac{n^2-n}{2}-1}(\mathbb{C}) \oplus \mathbb{C}.$$

In particular we have the dimension formula

$$\dim S^A(n,2) = \frac{n^2(n^2+1)}{2},$$
$$\dim S^X(n,2) = \frac{n^2(n^2+1)}{2} - (n+2)(n-1), \quad X = B \text{ or } D,$$
$$\dim S^C(n,2) = \frac{n^2(n^2+1)}{2} - (n-2)(n+1).$$

Proof of Lemma 5.1. For a dominant weight λ, Weyl's dimension formula says that

$$d_\lambda = \dim\bigl(V(\lambda)\bigr) = \frac{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, \lambda+\rho\rangle}{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, \rho\rangle},$$

where Φ^+ is the set of positive roots, $\rho = (\sum_{\alpha\in\Phi^+}\alpha)/2$ and $\langle h, \lambda\rangle$ is the valuation of a weight λ on an element h in the Cartan subalgebra. Also note that when $r = 2$ the multiplicity of simple direct summands in $(\mathbb{C}^n)^{\otimes r}$ is always 1. Hence $n^2 = \sum_{\lambda\in\pi^X(n,2)} d_\lambda$. Let us check the formula now case by case.

Type A_m, $\Phi^+ = \{\varepsilon_i - \varepsilon_j\colon\ 1 \leqslant i < j \leqslant m+1\}$, $\rho = (m\varepsilon_1 + (m-2)\varepsilon_2 + \cdots + (-m)\varepsilon_{m+1})/2$. We have that $\langle\rho, \varepsilon_i \check{-} \varepsilon_j\rangle = j - i$ for $i < j$. Denote by $j - i$ the length of the positive root $\varepsilon_i - \varepsilon_j$. For $1 \leqslant i \leqslant n-1$, there are $(n-i)$'s positive roots of length i. And $\langle\varepsilon_1, (\varepsilon_i \check{-} \varepsilon_j)\rangle \neq 0$ if and only if $i = 1$. In such cases $\langle\varepsilon_1, (\varepsilon_1 \check{-} \varepsilon_j)\rangle = 1$. Hence

$$\begin{aligned} d_{2\varepsilon_1} &= \frac{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, 2\varepsilon_1+\rho\rangle}{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, \rho\rangle} \\ &= \frac{3\cdot 1^{n-2}\cdot 4\cdot 2^{n-3}\cdot 5\cdot 3^{n-4}\cdots n\cdot (n-2)^1\cdot (n+1)^1}{1^{n-1}\cdot 2^{n-2}\cdot 3^{n-3}\cdots (n-2)^2\cdot (n-1)^1} \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

Note that $d_{2\varepsilon_1} + d_{\varepsilon_1+\varepsilon_2} = n^2$. It follows that

$$d_{\varepsilon_1+\varepsilon_2} = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Type B_m, $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j\colon\ 1 \leqslant i < j \leqslant m\} \cup \{\varepsilon_i\colon\ 1 \leqslant i \leqslant m\}$, $\rho = ((2m-1)\varepsilon_1 + (2m-3)\varepsilon_2 + \cdots + 3\varepsilon_{m-1} + \varepsilon_m)/2$. We have that for a positive root α, $\langle\varepsilon_1, \check{\alpha}\rangle \neq 0$ if and only if $\alpha = \varepsilon_1$ or $\varepsilon_1 \pm \varepsilon_j$ for $j = 2, 3, \ldots, m$. And $\langle\rho, \check{\varepsilon_1}\rangle = (2m-1)/2$, $\langle\rho, \varepsilon_1 \check{-} \varepsilon_j\rangle = j - 1$, $\langle\rho, \varepsilon_1 \check{-} \varepsilon_j\rangle = j + m - 2$. Hence

$$\begin{aligned} d_{2\varepsilon_1} &= \frac{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, 2\varepsilon_1+\rho\rangle}{\prod_{\alpha\in\Phi^+}\langle\check{\alpha}, \rho\rangle} \\ &= \frac{3\cdot 4\cdot 5\cdots (2m-1)\cdot (2m)\cdot (2m+3)/2}{1\cdot 2\cdot 3\cdots (2m-3)\cdot (2m-2)\cdot (2m-1)/2} \\ &= m(2m+3) = \frac{n(n+1)}{2} - 1. \end{aligned}$$

Since the irreducible module with highest weight 0 is the trivial module $\mathbb{C}$, it has the dimension $d_0 = 1$. By $d_{2\varepsilon_1} + d_{\varepsilon_1+\varepsilon_2} + d_0 = n^2$, we have that

$$d_{\varepsilon_1+\varepsilon_2} = n^2 - 1 - \left(\frac{n(n+1)}{2} - 1\right) = \frac{n(n-1)}{2}.$$

The results for type C and D follow from similar calculations. $\square$

We can define Schur algebras for any subgroup of general linear groups in the same way as for classical groups. Suppose G is a subgroup of GL_n. Consider the natural representation $\mathbb{C}^n$ over G. For a natural number r, the Schur algebra $S^G(n,r)$ is defined to be the image of the representation map $KG \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r})$.

Consider the special orthogonal group SO_2 and orthogonal group O_2. They are subgroups of GL_2. Note that SO_2 can be viewed of type D_1. For a natural number r, write $S^D(2,r)$ and $S^{D'}(2,r)$ for the Schur algebras associated to SO_2 and O_2, respectively. By definition both $S^D(2,r)$ and $S^{D'}(2,r)$ are subalgebras of $S^A(2,r)$. And by Proposition 6.1 we can choose the symmetric matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to define $\mathfrak{so}_2$ and $\mathfrak{o}_2$. In this way we will have a nice embedding of Schur algebras $S^D(2,r)$ and $S^{D'}(2,r)$ into $S^A(2,r)$. Following J.A. Green's notation [7], $S^A(2,r)$ has the $\mathbb{C}$-basis $\{\xi_{\underline{i}\,\underline{j}}\colon (\underline{i},\underline{j}) \in \Omega^2(2,r)\}$, where $\Omega^2(2,r)$ is a set of representatives of the symmetric group Σ_r-orbits on $I(2,r) \times I(2,r)$, and $I(2,r) = \{(i_1, i_2, \ldots, i_r)\colon i_1, \ldots, i_r = 1 \text{ or } 2\}$.

Proposition 5.3. (1) *As a subalgebra of $S^A(2,r)$, $S^D(2,r)$ has a $\mathbb{C}$-basis* $\{\xi_{1^{r-i}2^i,1^{r-i}2^i}\colon i = 0, 1, \ldots, r\}$. *As a $\mathbb{C}$-algebra, $S^D(2,r)$ is isomorphic to $\mathbb{C}^{\oplus(r+1)}$.*

(2) *As a subalgebra of $S^A(2,r)$, $S^{D'}(2,r)$ has a basis* $\{\xi_{1^{r-i}2^i,1^{r-i}2^i},\ \xi_{1^{r-i}2^i,2^{r-i}1^i}\colon i = 0, 1, \ldots, r\}$. *As a $\mathbb{C}$-algebra,*

$$S^{D'}(2,r) \cong \begin{cases} (M_2(\mathbb{C}))^{\oplus \frac{r+1}{2}} & \text{if } r \text{ is odd,} \\ (M_2(\mathbb{C}))^{\oplus \frac{r}{2}} \oplus \mathbb{C}^{\oplus 2} & \text{if } r \text{ is even.} \end{cases}$$

In particular, $\dim(S^{D'}(2,r)) = 2(r+1)$.

Proof. By definition $\mathfrak{so}_2 = \{X \in M_2(\mathbb{C})\colon X^{\mathrm{tr}}J + JX = 0\}$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as above. This is a one-dimensional Lie algebra with a basis $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence the universal enveloping algebra $\mathcal{U}(\mathfrak{so}_2)$ is isomorphic to the polynomial algebra $\mathbb{C}[X]$ with X the variable. Let

$$\tau : \mathcal{U}(\mathfrak{so}_2) \to \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}^2\big)$$

be the natural representation map. It sends the matrix X to itself. For $r \geqslant 1$, the representation map

$$\tau^{\otimes r} : \mathcal{U}(\mathfrak{so}_2) \to \mathrm{End}_{\mathbb{C}}\big(\big(\mathbb{C}^2\big)^{\otimes r}\big)$$

sends X to the r-fold tensor product of X. It factors through $S^A(2,r)$. And the image of $\tau^{\otimes r}$ is $S^D(2,r)$, generated by the image of X.

Write $\{v_1, v_2\}$ for the natural basis of $\mathbb{C}^2$. For $\underline{i} = (i_1, i_2, \ldots, i_r)$ in $I(2,r)$, write $v_{\underline{i}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$. Then $\{v_{\underline{i}}\colon \underline{i} \in I(2,r)\}$ provides a basis of $(\mathbb{C}^2)^{\otimes r}$. Write $E_{\underline{i},\underline{j}}$,

$\underline{i}, \underline{j} \in I(2, r)$, for the matrix in $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^2)^{\otimes r})$ sending $v_{\underline{k}}$ to $v_{\underline{i}}$ if $\underline{k} = \underline{j}$ and zero otherwise. Inside $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^2)^{\otimes r})$, an element $\xi_{\underline{i},\underline{j}}$ in $S^A(2, r)$ equals $\sum_{\sigma \in \Sigma_r} E_{\underline{i},\underline{j}\sigma}$, where σ acts on $\underline{j}$ by place permutation, and the sum is taken over those σ's which produce different $(\underline{i}, \underline{j}\sigma)$'s.

By definition $\tau(X) = E_{11} - E_{22} = \xi_{11} - \xi_{22}$ in $M_2(\mathbb{C})$. Inductively one finds that $\tau^{\otimes r}(X) = X^{\otimes r} = \sum_{i=0}^{r}(r - 2i)\xi_{1^{r-i}2^i,1^{r-i}2^i}$ in $S^A(2, r)$. Elements $\xi_{1^{r-i}2^i,1^{r-i}2^i}$ ($i = 0, 1, \ldots, r$) in the sum are idempotents and they are orthogonal to each other. Keeping multiplying $X^{\otimes r}$ with itself in $S^A(2, r)$, one obtains $(X^{\otimes r})^s = \sum_{i=0}^{r}(r - 2i)^s \xi_{1^{r-i}2^i,1^{r-i}2^i}$, for $s \geqslant 1$. $S^D(2, r)$ is the subalgebra of $S^A(2, r)$ generated by $\tau^{\otimes r}(X)$. Hence for $s \geqslant 1$, $(X^{\otimes r})^s$ are in $S^D(2, r)$. Notice that for different i's in $\{0, 1, \ldots, r\}$, $(r - 2i)$'s are different. By solving a Vandermonde matrix one obtains that for each i, $\xi_{1^{r-i}2^i,1^{r-i}2^i}$ is a linear combination of $(X^{\otimes r})^s$ with $s \in \{0, 1, \ldots, r\}$. Hence $\{\xi_{1^{r-i}2^i,1^{r-i}2^i}\colon i = 0, 1, \ldots, r\}$ is contained in $S^D(2, r)$. On the other hand, the subspace with basis $\{\xi_{1^{r-i}2^i,1^{r-i}2^i}\colon 0 \leqslant i \leqslant r\}$ is in fact a subalgebra of $S^A(2, r)$. Hence it is $S^D(2, r)$ already.

To get the second part of the proposition, one applies similar calculation to $\mathfrak{o}_2$. $\mathfrak{o}_2$ is two dimensional with a basis $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The representation map $\tau^{\otimes r}\colon \mathcal{U}(\mathfrak{so}_2) \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^2)^{\otimes r})$ factors through $S^A(2, r)$. It sends X to $\sum_{i=0}^{r}(r - 2i)\xi_{1^{r-i}2^i,1^{r-i}2^i}$, and Y to $\sum_{i=0}^{r}(r - 2i)\xi_{1^{r-i}2^i,2^{r-i}1^i}$. $S^{D'}(2, r)$ is the subalgebra of $S^A(2, r)$ generated by the image of X and Y. By similar calculation for $0 \leqslant i \leqslant r$, $\xi_{1^{r-i}2^i,1^{r-i}2^i}$ is in the image, and $\xi_{1^{r-i}2^i,2^{r-i}1^i}$, the product of $\xi_{1^{r-i}2^i,1^{r-i}2^i}$ and $\tau^{\otimes r}(Y)$, is also in the image. The subspace with basis $\{\xi_{1^{r-i}2^i,1^{r-i}2^i},\ \xi_{1^{r-i}2^i,2^{r-i}1^i}\colon i = 0, 1, \ldots, r\}$ is closed under the multiplication hence it is just the algebra $S^{D'}(2, r)$.

The algebraic structure follows easily from knowing the basis. $\square$

6. Maps between Schur algebras

This section collects some maps between Schur algebras $S^X(n, r)$ with different parameters. The main result is Theorem 6.3.

In Section 2 we defined the special orthogonal (symplectic, respectively) Lie algebras by using a particular symmetric (anti-symmetric, respectively) matrix. In fact the Lie algebras defined by different symmetric (anti-symmetric, respectively) matrices are isomorphic. The corresponding different Schur algebras are isomorphic as well.

Proposition 6.1. *Let $X = B$, C or D, and $\mathfrak{g}_1$, $\mathfrak{g}_2$ the classical Lie algebras of type X_m defined by two matrices J_1, J_2. The isomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ gives rise to an isomorphism of different embedding of Schur algebras of type X into Schur algebras of type A. In particular, we have the commutative diagram, see Fig.* 4.

The proof is straightforward by inducing the isomorphism between Lie algebras to the isomorphisms between the universal enveloping algebras, and between the Schur algebras.

Consider the polynomial algebra $\mathbb{C}[c_{ij}]$ in n^2 indeterminates. It is a bialgebra with comultiplication and counit given by $\Delta(c_{ij}) = \sum_{k=1}^{n} c_{ik} \otimes c_{kj}$ and $\epsilon(c_{ij}) = \delta_{ij}$ on gen-

$$\begin{tikzcd}
 & & & \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r}) \\
 & \mathcal{U}(\mathfrak{sl}_n) \arrow[rru] \arrow[r, two heads] & S^A(n,r) \arrow[ru] & \\
\mathcal{U}(\mathfrak{g}_1) \arrow[ru] \arrow[rr, dashed, two heads] \arrow[rd, "\simeq"] & & S_1^X(n,r) \arrow[u, dashed] \arrow[rd, dashed, "\simeq"] & \\
 & \mathcal{U}(\mathfrak{g}_2) \arrow[uu] \arrow[rr, two heads] & & S_2^X(n,r) \arrow[luu]
\end{tikzcd}$$

Fig. 4.

erators. Each homogeneous part $\mathbb{C}[c_{ij}]_r$ is a coalgebra. By [7] $S^A(n,r)$, the Schur algebra of type A, is isomorphic to the linear dual of $\mathbb{C}[c_{ij}]_r$. The determinant of the $n \times n$ matrix (c_{ij}) is a homogeneous polynomial of degree n. Multiplying by this polynomial gives rise to an injection from $\mathbb{C}[c_{ij}]_r$ to $\mathbb{C}[c_{ij}]_{n+r}$. Taking the dual of this injection, one obtains a surjection from $S^A(n,n+r)$ to $S^A(n,r)$. For each $i = 0,\dots,n-1$, we have an inverse system $\{S^A(n,kn+i)\colon k \geqslant 0\}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_n)$ can be embedded into the direct sum of these inverse limits. This was developed by Beilinson, Lusztig and MacPherson [1] in the quantum case. See also [4,6,8]. By the natural embedding of Schur algebras into $S^A(n,r)$, we can give the analogue of the above for other classical types. And since we are working over $\mathbb{C}$, the inverse limit has an explicit expression.

Proposition 6.2. *We have the commutative diagram, see Fig.* 5. *The map ρ^X is injective from the universal enveloping algebra to the inverse limit.*

$$\begin{tikzcd}
 & & \mathcal{U}(\mathfrak{sl}_n) \arrow[lld, "\rho^A"'] \arrow[ld] \arrow[rd] & \\
\bigoplus_{i=0}^{n-1} \varprojlim_k S^A(n,kn+i) & S^A(n,r) & & S^A(n,n+r) \arrow[ll, two heads] \\
 & & \mathcal{U}(\mathfrak{g}) \arrow[uu, dashed] \arrow[lld, dashed, "\rho^X"'] \arrow[ld, dashed] \arrow[rd, dashed] & \\
\bigoplus_{i=0}^{n-1} \varprojlim_k S^X(n,kn+i) \arrow[uu] & S^X(n,r) \arrow[uu] & & S^X(n,n+r) \arrow[ll, two heads] \arrow[uu]
\end{tikzcd}$$

Fig. 5.

Proof. By Corollary 4.4 we have algebra isomorphisms

$$S^X(n,r) \cong \bigoplus_{\lambda \in \pi^X(n,r)} M_{d_\lambda}(\mathbb{C}),$$

$$S^X(n,n+r) \cong \bigoplus_{\lambda \in \pi^X(n,n+r)} M_{d_\lambda}(\mathbb{C}).$$

Notice that by Theorem 4.6, $\pi^X(n,r) \subseteq \pi^X(n,n+r)$ for each X and each r, n. It follows that $S^X(n,r)$ is a direct summand, as well as a quotient, of $S^X(n,n+r)$. The restriction of the quotient map to simple blocks labeled by $\pi^X(n,r)$ is the identity.

Therefore for each $i = 0, 1, \ldots, n-1$, $\{S^X(n,kn+i)\colon k \geqslant 0\}$ forms an inverse system. The inverse limit $\varprojlim_{k \geqslant 0} S^X(n,kn+i)$ is the direct product of the full matrix algebras of sizes d_λ with λ in $\bigcup_{k \geqslant 0} \pi^X(n,kn+i)$. Taking the direct sum of the inverse limits over $i = 0, 1, \ldots, n-1$, we obtain:

$$\bigoplus_{i=0}^{n-1} \varprojlim_{k \geqslant 0} S^X(n,kn+i) = \bigoplus_{i=0}^{n-1} \prod_{\lambda \in \bigcup_{k \geqslant 0} \pi^X(n,kn+i)} M_{d_\lambda}(\mathbb{C}).$$

It equals $\prod_{\lambda \in P_{++}(A_m)} M_{d_\lambda}(\mathbb{C})$ in type A, and has $\prod_{\lambda \in P'_{++}(A_m)} M_{d_\lambda}(\mathbb{C})$ as a subalgebra in other types, where $P'_{++}(X_m) = \bigcup_{r \geqslant 0} \pi^X(n,r)$ the set of dominant weights with integral coefficients. In type C, all dominant weights are of integral coefficients. Hence $P'_{++}(C_m) = P_{++}(C_m)$. Notice that a dominant weight may occur in $\bigcup_{k \geqslant 0} \pi^X(n,kn+i)$ (for different $i = 0, \ldots, n-1$) for more than once and at most n times.

The surjection from $S^X(n,n+r)$ to $S^X(n,r)$ is compatible with the representation map. By property of inverse systems, there is a map from $\mathcal{U}(\mathfrak{g})$ to the inverse limit

$$\rho^X : \mathcal{U}(\mathfrak{g}) \to \bigoplus_{i=0}^{n-1} \varprojlim_{k} S^X(n,kn+i).$$

By [1], the map in type A

$$\rho^A : \mathcal{U}(\mathfrak{sl}_n) \to \prod_{\lambda \in P_{++}(A_m)} M_{d_\lambda}(\mathbb{C})$$

is injective. That is, for any element U in $\mathcal{U}(\mathfrak{sl}_n)$ there exists an r such that the representation map $\mathcal{U}(\mathfrak{sl}_n) \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r})$ does not send U to zero. Now that $\mathcal{U}(\mathfrak{g})$ is a subalgebra of $\mathcal{U}(\mathfrak{sl}_n)$, for any element U' in $\mathcal{U}(\mathfrak{g})$, there exists an r' such that the representation map $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r'})$does not send U' to zero. Namely, the image of U' in $S^X(n,r')$ is nonzero. Consequently the map ρ^X is injective. $\square$

Furthermore notice that when $X \neq A$, $\pi^X(n,r) \subseteq \pi^X(n,r+2)$. We have in fact a stronger relation between Schur algebras of types not A.

Theorem 6.3. *For $X \neq A$, we have two inverse systems $\{S^X(n, 2k+i)\colon k \geqslant 0\}$ for $i = 0, 1$. The universal enveloping algebra is contained in the inverse limit $\prod_{\lambda \in P'_{++}(X_m)} M_{d_\lambda}(\mathbb{C})$, where $P'_{++}(X_m)$ is the set of dominant weights with integral coefficients. In type B, for any $i = 0, 1$ and any $j = 0, 1, \ldots, n-1$,*

$$\varprojlim_{k \geqslant 0} S^X(n, 2k+i) = \varprojlim_{k \geqslant 0} S^X(n, kn+j) = \prod_{\lambda \in P'_{++}(B_m)} M_{d_\lambda}(\mathbb{C}).$$

In types C and D, $\varprojlim_{k \geqslant 0} S^X(n, 2k+i) = \varprojlim_{k \geqslant 0} S^X(n, kn+j)$ for $i \equiv j \pmod 2$, and

$$\bigoplus_{i=0}^{1} \varprojlim_{k \geqslant 0} S^X(n, 2k+i) = \prod_{\lambda \in P'_{++}(X_m)} M_{d_\lambda}(\mathbb{C}).$$

Proof. By a similar argument as above, the embedding between weight sets $\pi^X(n, r) \subseteq \pi^X(n, r+2)$ induces a quotient map between Schur algebras $S^X(n, r+2) \to S^X(n, r)$. Hence $\{S^X(n, 2k)\colon k \geqslant 0\}$ and $\{S^X(n, 2k+1)\colon k \geqslant 0\}$ are both inverse systems. The inverse limits are

$$\varprojlim_{k \geqslant 0} S^X(n, 2k) = \prod_{\pi^{\text{even}}} M_{d_\lambda}(\mathbb{C})$$

and

$$\varprojlim_{k \geqslant 0} S^X(n, 2k+1) = \prod_{\pi^{\text{odd}}} M_{d_\lambda}(\mathbb{C}),$$

where $\pi^{\text{even}} = \bigcup_{k \geqslant 0} \pi^X(n, 2k)$, $\pi^{\text{odd}} = \bigcup_{k \geqslant 0} \pi^X(n, 2k+1)$.

Write φ_2 for the surjection from $S^X(n, r+2)$ to $S^X(n, r)$. Write ϕ_n for the surjections from $S^X(n, n+r)$ to $S^X(n, r)$ in Proposition 6.2. Since the surjections are induced by the inclusions of weight sets, they are compatible. In type B_m, $n = 2m+1$ and $(\phi_n)^2 = (\varphi_2)^n$ from $S^B(n, 2n+r)$ to $S^B(n, r)$ where multiplication means composition. In types $X_m = C_m$ or D_m, $n = 2m$ and $\phi_n = (\varphi_2)^m$ from $S^X(n, n+r)$ to $S^X(n, r)$.

In type B, $n = 2m+1$ is odd, and $\pi^B(n, r)$ is a subset of both $\pi^B(n, r+2)$ and $\pi^B(n, n+r)$. It follows that $\pi^{\text{odd}} = \pi^{\text{even}} = \bigcup_{k \geqslant 0} \pi^B(n, kn+j) = P'_{++}(B_m)$ for any $j = 0, 1, \ldots, n-1$. In types C and D, $n = 2m$ is even, and $\pi^{\text{odd}} \cap \pi^{\text{even}} = \emptyset$. Hence the inverse limit of Schur algebras are as stated in the theorem. The universal enveloping algebra in contained in the inverse limit by the same argument as in Proposition 6.2. □

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