Asymptotische Werte von Crank-Differenzen (Asymptotic values of crank differences)

Diplomarbeit

von

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Einleitung und Zusammenfassung der Ergebnisse

Die Theorie der Partitionen ist ein faszinierendes Beispiel für das Zusammenspiel zwischen analytischen Methoden und Zahlentheorie. Ein schwierige, aber nichtsdestotrotz grundlegende Frage ist, wie viele Partitionen eine ganze Zahl zulässt. Genauer gesagt, wie oft kann man eine ganze positive Zahl inäquivalent in kleinere Summanden zerlegen? Hardy und Ramanujan fanden auf diese Frage eine asymptotische Antwort. Dies bedeutet, sie konnten diese Anzahl für große Zahlen angeben [13]. Die von ihnen benutzte "Hardy-Ramanujan"- Methode, oder auch üblicherweise Kreismethode genannt, konnte von Hans Rademacher weiter verfeinert werden, so dass es ihm möglich war auch für endliche Zahlen eine exakte Antwort auf die Frage zu geben [17]. Das Zusammenspiel zwischen arithmetischen und analytischen Methoden erfolgt durch die Definition einer - sogenannten - erzeugenden Funktion. Dabei nimmt man eine Folge von Zahlen, die z.B. ein arithmetisches Problem kodiert und definiert aus diesen Koeffizienten eine Fourier Reihe, also eine komplexe Funktion. Viele dieser erzeugenden Funktionen sind holomorph. Dies ermöglicht die Analyse solcher Funktionen mit Sätzen aus der Funktionentheorie, wie Cauchys Theorem. Darüber hinaus zeigen viele dieser erzeugenden Funktionen ein interessantes Transformationsverhalten unter Möbiustransformationen. Dies vereinfacht die Analyse dieser Klasse von Funktionen, bzw. ermöglicht viele Aussagen erst. Dieses Transformationsverhalten nennt man *Modularität*. Diese Modularität ermöglicht in vielen Fällen die Berechnung der Koeffizienten einer Fourier Reihe, was bei bestimmten erzeugenden Funktionen gleichbedeutend ist mit der Berechnung arithmetischer Informationen. Diese Eigenschaft ist entscheidend zur Berechnung von $p(n)$, der Anzahl möglicher Partitionen einer ganzen nicht negativen Zahl n. Ramanujan entdeckte, dass $p(n)$ interessante Kongruenzen erfüllt. Er vermutete, dass für jedes $n \in \mathbb{N}_0$ folgende Gleichungen gelten [19]:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Er konnte die ersten beiden Gleichungen sogar beweisen, in dem er zeigte, dass die erzeugende Funktion dieser Werte das 5-fache einer Fourier Reihe ist, die nur ganzzahlige

Koeffizienten hat. Die Kongruenzen sollten aber auch eine kombinatorische Erklärung haben. Das heißt, dass man einer möglichen Partition eine Zahl zuordnen können sollte, genannt der *Rank* (der von Dyson definiert wurde [10]), so dass die Anzahl der Partitionen mit Rank modulo 5 (resp. $7, 11$) immer gleich groß sind. Das sollte also eine Aufteilung in gleichmächtige Gruppen ermöglichen und damit die - so genannten - Ramanujankongruenzen auf kombinatorische Weise erklären [5]. Die Definition des Ranks konnte die ersten beiden Ramanujankongruenzen erklären, aber keine Erklärung für die letzte der drei Kongruenzen liefern. Daher vermutete Dyson [10], dass es noch eine andere - sogenannte - Partitionsstatistik gibt, die auch diese Kongruenz erklärt und sogar alle bis dahin observierten Kongruenzen für $p(n)$. In Analogie zum Rank nannte er diese Funktion den *Crank*. Es stellte sich jedoch heraus, dass die Konstruktion dieser Funktion schwierig ist und erst 40 Jahre später konnte eine Definition gegeben werden, die das Problem löste [3]. In der folgenden Diplomarbeit wollen wir die Fourier Koeffizienten einer unendlichen Familie von "Crank" erzeugenden Funktionen bestimmen und daraus bestimmte Ungleichungen zwischen bestimmten Crank-Funktionen[1] beweisen. Um dorthin zu gelangen geben wir zunächst eine kurze Einführung in die Theorie der Modulformen und in Theorie der Partitionen. Die Beweise der Hauptaussagen erfolgen in Kapitel 4.
Es folgt eine kurze Zusammenfassung der Aussagen, die im Rahmen dieser Diplomarbeit bewiesen wurden. Um die Kreismethode anwenden zu können, ist es nötig ein bestimmtes Transformationsgesetz für die erzeugende Funktion des Cranks zu beweisen.

Proposition 0.1. Sei $C(x;q)$ die erzeugende Funktion des Cranks. Sei $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$, wobei h' eine Lösung der Kongruenzgleichung $hh' \equiv -1 \pmod{k}$ ist. Dann gilt:
Für $c \mid k$ haben wir

$$C\left(e^{\frac{2\pi ia}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = \frac{i\sin(\frac{\pi a}{c})}{z^{\frac{1}{2}}\sin(\frac{\pi ah'}{c})}(-1)^{ak+1}\omega_{h,k}e^{\frac{\pi}{12k}(z^{-1}-z)-\frac{\pi ia^2k_1h'}{c}}C\left(e^{\frac{2\pi iah'}{c}}; e^{\frac{2\pi i}{k}(h'+\frac{i}{z})}\right).$$

Für $c \nmid k$ haben wir

$$\frac{C\left(e^{\frac{2\pi ia}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right)}{(-1)^{ak+l+1}} = \frac{4i\sin(\frac{\pi a}{c})\omega_{h,k}}{z^{\frac{1}{2}}}e^{\frac{-\pi a^2h'k_1}{cc_1}+\frac{2\pi ih'la}{cc_1}}q_1^{\frac{-l^2}{2c_1^2}}e^{\frac{\pi}{12k}(z^{-1}-z)}C\left(ah', \frac{lc}{c_1}, c; q_1\right).$$

Die Funktion $C(a,b,c;q)$ ist definiert wie folgt:

$$\frac{i}{2(q)_\infty}\left(\sum_{m=0}^{\infty}\frac{(-1)^m e^{-\frac{\pi ia}{c}}q^{\frac{m(m+1)}{2}+\frac{b}{2c}}}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}} - \sum_{m=1}^{\infty}\frac{(-1)^m e^{\frac{\pi ia}{c}}q^{\frac{m(m+1)}{2}-\frac{b}{2c}}}{1-e^{\frac{2\pi ia}{c}}q^{m-\frac{b}{c}}}\right).$$

Die $\omega_{h,k}$ sind definiert in (2.5).

[1]Das sind Funktionen, die aus der erzeugenden Funktion des Cranks gebildet werden.

Mit der Kreismethode und dem Transformationsgesetz ist es nun möglich folgendes Theorem zu beweisen:

Theorem 0.2. *Sei* $C(e^{\frac{2\pi j}{c}}; q) = 1 + \sum_{n=1}^{\infty} \tilde{A}\left(\frac{j}{c}; n\right) q^n$. *Seien* $0 < a < c$ *koprime Zahlen,* c *und* n *positive Zahlen, wobei* c *ungerade ist. Dann haben wir*

$$\tilde{A}\left(\frac{j}{c}; n\right) = \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c|k}} \frac{\widetilde{B}_{j,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + \frac{8\sqrt{3}\cdot\sin\left(\frac{\pi j}{c}\right)}{\sqrt{24n-1}}$$
$$\sum_{\substack{1 \le k \le \sqrt{n} \\ c \nmid k \\ r \ge 0 \\ \delta^i_{j,c,k,r} > 0 \\ i \in \{+,-\}}} \frac{D_{j,c,k}(-n, m^i_{j,c,k,r})}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{2\delta^i_{j,c,k,r}(24n-1)}}{\sqrt{3}k}\right) + O\left(n^{\varepsilon}\right).$$

Die $\widetilde{B}_{j,c,k}(n,m)$ sind definiert in (4.5), die $D_{j,c,k}(n,m)$ in (4.6), die $\delta^i_{j,c,k,r}$ in (4.9) und die $m^i_{j,c,k,r}$ in (4.10). Das Theorem ermöglicht nun die Berechnung der asymptotischen Werte von $M(a,c;n)$, wobei $M(a,c;n)$ die Anzahl der Partitionen einer Zahl n mit Crank kongruent zu a modulo c ist.

Theorem 0.3. *Sei* $0 \le a < c$ *mit* c *einer ungeraden Zahl. Dann haben wir:*

$$\begin{aligned} M(a,c;n) =& \frac{2\pi}{c\sqrt{24n-1}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \\ &+ \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} \left(\frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{c|k} \frac{\widetilde{B}_{j,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right. \\ &\left. + \frac{8\sqrt{3}\sin\left(\frac{\pi j}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{a,c,k,r}>0 \\ i\in\{+,-\}}} \frac{D_{j,c,k}(-n, m^i_{j,c,k,r})}{\sqrt{k}} \sinh\left(\sqrt{\frac{2\delta^i_{a,c,k,r}(24n-1)}{3}}\frac{\pi}{k}\right)\right) \\ &+ O(n^{\varepsilon}), \end{aligned}$$

wobei $A_k(n)$ die Kloostermannsumme ist, die in der exakten Formel Radermachers für $p(n)$ aufkommt. Nun sind wir in der Lage die Hauptaussage der Diplomarbeit zu beweisen. Grob gesprochen sagt dieses Theorem aus, dass die Crank-Differenzen $M(a,c,n) - M(b,c,n)$, wenn c groß genug ist, nur noch von den Residuenklassen modulo c abhängen. Exakt bedeutet das:

Theorem 0.4. *Sei* $0 \le a < b \le \frac{c-1}{2}$ *und sei* $c > 11$ *eine ungerade ganze Zahl, dann haben für* $n > N_{a,b,c}$, *wobei* $N_{a,b,c}$ *eine explizite Konstante ist, die folgende Ungleichung:*

$$M(a,c;n) - M(b,c;n) > 0.$$

Aus diesem Theorem lässt sich auch eine Aussage treffen für $c < 13$. Für $c < 13$ ist der Hauptterm, also der Term der für große n den entscheidenden Beitrag für die Fourier Koeffizienten liefert, ein anderer als für $c \geq 13$. In dem Fall $c < 13$ osszilliert der Hauptterm stark und damit wechselt das Vorzeichen der entscheiden Größen je nach Wahl von a, b und somit ergeben sich andere Ungleichungen als im Fall $c \geq 13$. Die verschiedenen Möglichkeiten sind zusammen gefasst im folgenden Theorem.

Theorem 0.5. *Sei $0 \leq a < b \leq \frac{c-1}{2}$. Für $n > \tilde{N}_{a,b,c}$, wobei $\tilde{N}_{a,b,c}$ eine explizite Konstante ist, haben wir:*

1. *Die Crank-Differenzen erfüllen folgende Ungleichung $M(a,5,5n+d) - M(b,5,5n+d)$*

$$\begin{cases} < 0 & \text{wenn } (a,b,d) \in \{(0,b,1),(0,2,2),(1,2,2),(1,2,3)\}, \\ > 0 & \text{wenn } (a,b,d) \in \{(0,b,0),(1,2,1),(0,1,3)\}. \end{cases}$$

2. *Die Crank-Differenzen erfüllen folgende Ungleichung $M(a,7,7n+d) - M(b,7,7n+d)$*

$$\begin{cases} < 0 & \text{wenn } (a,b,d) \in \{(0,1,1),(0,1,6),(0,2,1),(0,2,2),(0,3,1),(0,3,6), \\ & (1,2,2),(1,2,4),(1,3,3),(1,3,4),(2,3,3),(2,3,6)\} \\ > 0 & \text{wenn } (a,b,d) \in \{(0,1,0),(0,1,3),(0,1,4),(0,2,0),(0,2,3),(0,3,0), \\ & (1,2,1),(1,2,6),(1,3,1),(2,3,2)\}. \end{cases}$$

3. *Die Crank-Differenzen erfüllen folgende Ungleichung $M(a,9,3n+d) - M(b,9,3n+d)$*

$$\begin{cases} < 0 & \text{wenn } (a,b,d) \in \{(0,1,1),(0,1,6),(0,1,8),(0,2,1),(0,2,2),(0,2,6)\}, \\ & (0,3,1),(0,3,3),(0,3,6),(0,4,1),(0,4,6),(0,4,8) \\ & (1,2,2),(1,2,4),(1,2,7),(1,3,2),(1,3,3),(1,3,4) \\ & (1,3,5),(1,3,7),(1,4,4),(1,4,7),(2,3,1),(2,3,3) \\ & (2,3,5),(2,3,7),(2,3,8),(2,4,5),(2,4,8),(3,4,0) \\ & (3,4,4),(3,4,6),(3,4,8)\}, \\ > 0 & \text{wenn } (a,b,d) \in \{(0,1,0),(0,1,2),(0,1,3),(0,1,4),(0,1,5),(0,1,7), \\ & (0,2,0),(0,2,3),(0,2,4),(0,2,5),(0,2,7),(0,2,8), \\ & (0,3,0),(0,3,4),(0,3,7),(0,4,0),(0,4,2),(0,4,3), \\ & (0,4,4),(0,4,5),(0,4,7),(1,2,1),(1,2,5),(1,2,8), \\ & (1,3,0),(1,3,1),(1,3,6),(1,3,8),(1,4,1),(2,3,0), \\ & (2,3,2),(2,3,4),(2,3,6),(2,4,2),(3,4,1),(3,4,2), \\ & (3,4,3),(3,4,5),(3,4,7)\}. \end{cases}$$

4. *Die Crank-Differenzen erfüllen folgende Ungleichung* $M(a, 11, 11n+d) - M(b, 11, 11n+d)$

$$\begin{cases} < 0 & \text{wenn } (a,b,d) \in \{(0,1,1),(0,1,7),(0,1,8),(0,1,9),(0,2,1),(0,2,2), \\ & (0,2,9),(0,3,1),(0,3,8),(0,3,9),(0,4,1),(0,4,7), \\ & (0,4,8),(0,5,1),(0,5,9),(1,2,2),(1,2,4),(1,3,3), \\ & (1,4,4),(2,3,3),(2,3,5),(2,3,8),(2,4,8),(3,4,4), \\ & (3,4,7),(3,4,10),(3,5,10),(4,5,5),(4,5,9)\}, \\ > 0 & \text{wenn } (a,b,d) \in \{(0,b,0),(0,1,3),(0,1,4),(0,2,3),(0,2,5),(0,3,4), \\ & (0,3,10),(0,4,3),(0,4,5),(0,5,3),(0,5,4),(1,2,1), \\ & (1,2,5),(1,2,7),(1,2,8),(1,3,1),(1,3,7),(1,3,10), \\ & (1,4,1),(1,4,5),(1,4,9),(1,5,1),(1,5,7),(1,5,8), \\ & (2,3,2),(2,3,4),(2,3,10),(2,4,2),(2,4,9),(2,5,2),(2,5,4), \\ & (3,4,3),(3,4,5),(3,4,9),(3,5,3),(3,5,8),(4,5,4),(4,5,7), \\ & (4,5,8)\}. \end{cases}$$

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Contents

Chapter 1

Introduction

The theory of partitions is an intriguing example for the interplay between number theory and analytic methods. A complicated but nevertheless basic question is: how many partitions does a non-negative integer have. Hardy and Ramanujan found an asymptotic answer to this question [13] using the so-called Circle Method and Hans Rademacher refined the method of Ramanujan and Hardy to give an exact answer [17]. The connection between the analytic and arithmetic methods comes from building up a generating function that admits certain transformation properties when changing the argument of the function. This behavior under linear fractional transformations is called *modularity* and admits a deep analysis of the coefficients of the generating function that might have arithmetical information. Ramanujan found that the partition function fulfills some interesting congruence conditions and gave explanations for his observation by some interesting q-series identities [19]. In fact, there is also a combinatorical explanation for the Ramanujan congruence. In 1947, Dyson conjectured that that these observed properties could be explained by the existence of a function on the set of possible partitions - the so-called rank - which groups the partitions into equally sized congruence classes of rank values [10]. But nevertheless the rank could not explain all mentioned congruences. That was the reason that Dyson conjectured that there is another partition statistic that explain all of the congruences. He called this statistic the crank. It turned out to be difficult to find this partition statistic. Forty years after the conjecture the crank was constructed and it was shown that it explains all Ramanujan congruences simultaneously [3]. In this thesis we want to show certain inequalities of certain functions coming from the crank generating function by computing the asymptotic values of the Fourier coefficients of an infinite family of crank generating functions and by carefully bounding the corresponding error terms occurring in the Circle Method. We will use the theory of modular forms and the transformation rules of certain half-integral weight modular forms to deduce a transformation formula for the crank generating function. In Chapter 2, we will introduce the concept of modularity and explain why it is helpful for arithmetical problems. In Chapter 3, we introduce partitions, and partition statistics. In Chapter 4 we compute the asymptotics of the Fourier coefficients of the crank generating function and especially of the so-called Crank differences. With that we can prove the following main theorem:

Theorem 1.1. *Let $M(a,c;n)$ be the number of partitions of n with crank equal to a modulo c. Let $0 \leq a < b \leq \frac{c-1}{2}$ and let $c > 11$ be an odd integer, then for $n > N_{a,b,c}$, where $N_{a,b,c}$ is an explicit constant, we have the inequality:*

$$M(a,c;n) > M(b,c;n).$$

We outline the tour we have to take to prove this theorem. Firstly, we establish with classical results of modular forms of half-integral weight a transformation formula for the crank generating function. This allows one to detect the asymptotics of the Fourier coefficients of this function by using the Circle Method. From this we can compute the asymptotic value of $M(a,c;n)$ and from that we can prove the main theorem by bounding the error in the circle method explicitly. This theorem is analogous to the main result in [6], where a similar inequality was shown for the rank.

Chapter 2

Theory of modular forms

In the following chapter we want to give some basic definitions around the theory of modular forms and present some transformation formulas which we need to prove our transformation rule for the crank generating function.

2.1 Basic definitions

Let $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ be the *upper half-plane* and $\mathrm{SL}_2(\mathbb{Z})$ be the full *modular group* consisting of 2×2 matrices with integer entries and determinant equal to 1. The *index* of a subgroup Γ is the number of cosets of Γ in $\mathrm{SL}_2(\mathbb{Z})$. For $N \in \mathbb{N}$, we further define:

$$\Gamma_0(4N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \, c \equiv 0 \pmod{4N} \right\}.$$

A *(weakly) holomorphic modular form* is a holomorphic function on the upper half-plane $\mathbb{H}$ that transforms in certain way under the action of the modular group or some subgroup of the modular group with finite index. It is holomorphic (meromorphic) at the cusps of the modular curve. By this curve we mean the quotient of the upper half-plane with the corresponding transformation group. One could postulate that it transforms again to itself, but this condition is too restrictive to obtain interesting general results, so one allows certain correction factors that make the space of all such forms into a finite dimensional $\mathbb{C}$-vector space. Before starting with the formal definitions we want to define certain slash operators. Therefore we need some notation. For d an odd integer we define $\left(\frac{c}{d}\right)$ to be the usual Jacobi symbol. For d a negative odd integer, we define

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if}\, d < 0 \,\text{and}\, c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if}\, d < 0 \,\text{and}\, c < 0. \end{cases}$$

We further define

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

A Dirichlet character modulo $4N$ is a function $\chi : (\mathbb{Z}/4N\mathbb{Z})^* \mapsto \mathbb{C}^*$ such that

$$\chi(xy) = \chi(x)\chi(y).$$

Let f be a complex-valued function on the upper half-plane, λ be any integer and define

$$f(\tau)|_{\lambda+\frac{1}{2}} A := \left(\frac{c}{d}\right)^{-2\lambda-1} \varepsilon_d^{2\lambda+1} (c\tau+d)^{-\lambda-\frac{1}{2}} f\left(\frac{a\tau+b}{c\tau+d}\right) \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N). \quad (2.1)$$

We take the principal branch of the square root. For charcters χ modulo $4N$ we can now give the definition of a half-integral weight modular form.

Definition 2.1 (Modular form of half-integral weight)**.** A *(weakly) modular form of half-integral weight* $\lambda + \frac{1}{2}$ *with Nebentypus* χ is a function $f : \mathbb{H} \to \mathbb{C}$ with the following properties:

1. f is holomorphic on the upper half-plane.
2. $f|_{\lambda+\frac{1}{2}} A = \chi(d) f$ for all $A \in \Gamma_0(4N)$.
3. f is (meromorphic) holomorphic at the cusps.

If in addition f vanishes at all the cusps, then we say that f is a *cusp form*.

Next we want to define a modular form of integral weight. Therefore let g be a complex-valued function on the upper half-plane, let k be any integer and define

$$g(\tau)|_k A = (c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}\right) \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \quad (2.2)$$

Now we can give the following definition

Definition 2.2 (Modular form of integral weight)**.** A *(weakly) holomorphic modular form* g with multiplier $\epsilon : \mathbb{Z}^4 \mapsto \mathbb{C}$ is a function $g : \mathbb{H} \to \mathbb{C}$ with the following properties:

1. g is holomorphic on the upper half-plane.
2. $g|_k A = \epsilon(a,b,c,d) g$ for all $A \in \mathrm{SL}_2(\mathbb{Z})$, where $|\epsilon(a,b,c,d)| = 1$ is called a multiplier system.
3. g is (meromorphic) holomorphic at the cusps.

If in addition g vanishes at all the cusps, then we say that g is a *cusp form*.

Remark 2.3. It is possible to also allow (in condition 2 of both definitions) transformations coming from other finite index subgroups of the modular group or $\Gamma_0(4N)$. That means that condition 2 is only fulfilled for elements of a subgroup and not for all elements of the modular group (resp. $\Gamma_0(4N)$). For every subgroup with finite index we have f $f(\tau+m)=f(\tau)$ for some $m\in\mathbb{N}$, because for every group Γ with $[\mathrm{SL}_2(\mathbb{Z}):\Gamma]<\infty$ we have some $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)\in\Gamma$. So it is possible to expand f in a Fourier series:

$$f(\tau)=\sum_{n=0}^{\infty}a_n e^{\frac{2\pi i n\tau}{m}}=\sum_{n=0}^{\infty}a_n q^{\frac{n}{m}},\quad q:=e^{2\pi i\tau}.$$

It is also possible to expand every half-integral weight modular form in a Fourier series.

The space of all modular forms of a fixed weight is denoted by M_k where k is even, since for k odd all modular forms vanish identically, due to the transformation rule for modular forms. Strictly speaking this is only true for certain finite index subgroups and the full modular group. The space of all cusp forms is denoted by S_k. From the Fourier expansions (also called *q-series*) at every cusp of the transformation group it is easy to see, if a weakly holomorphic modular form $g(\tau)$ is modular or even a cusp form. In order to be modular the Fourier coefficients of $g(\tau)$ have to vanish for negative n and to be a cusp form the extra condition $a_0=0$ should hold at every cusp. Note that $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp. Similar definitions can be made for the half-integral case.

Theorem 2.4. *Let $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N),\chi)$ be the vector space of weight $\lambda+\frac{1}{2}$ modular forms. Then we have:*

$$\dim_{\mathbb{C}} M_{\lambda+\frac{1}{2}}(\Gamma_0(4N),\chi)<\infty$$

PROOF. See [16] Theorem 1.56. The right hand side of the equation is finite. □

The importance of this theorem can not be stressed enough. Let d be the dimension of a certain space of modular forms. If there are $d+1$ modular forms which are elements of this space, then we know that there are linear relations between these functions, because of the vector space structure. That also implies relations among their Fourier coefficients which might be highly non-trivial by direct computations. Next we want to present some examples of modular forms to demonstrate the interplay of arithmetic functions and modular forms. Very important examples are the Eisenstein series:

Example 2.5 (Eisenstein series)**.** First of all, we define series and show that we can gain arithmetic information from their Fourier coefficients.
Let $k>2$ be an integer, $\tau\in\mathbb{H}$ and

$$G_k(\tau):=\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^k}.$$

We call G_k the *Eisenstein series* of weight k. The following theorem explains the significance of these series.

Theorem 2.6. *Let $q := e^{2\pi i\tau}$. Then we have the following facts:*

(i) The Eisenstein series converge absolutely on the upper half-plane and are there analytic functions.

(ii) The Eisenstein series are modular forms.

(iii) We have:

$$\frac{G_k(\tau)(k-1)!}{(2\pi i)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{2.3}$$

where B_k are the Bernoulli numbers, $\sigma_{k-1}(n)$ is the divisor function.

(iv) For $k = 2$ we define

$$G_2(\tau) := B_2\pi^2 - \sum_{n=1}^{\infty} 4\pi^2 \sigma_1(n) q^n$$

. Then the so-called Eisenstein series of weight 2 obeys the following equation:

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \pi i c(c\tau+d).$$

Proof. See [9] (Chapter 1, Proposition 5 and Proposition 6). □

Thus, the Fourier coefficients of the Eisenstein series encode the divisor functions for different weight, giving an example of a modular form having arithmetic information coming from the Fourier expansion at a cusp.

Example 2.7 (discriminant function)**.** We introduce the discriminant function, which is related to Dedekind's η-function. Define $q := e^{2\pi i\tau}$ and

$$\begin{aligned} \eta(\tau) :=& q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \\ \Delta(\tau) :=& q \prod_{n=0}^{\infty} (1-q^n)^{24} = \eta(\tau)^{24}. \end{aligned}$$

We show that this is a cusp form of weight 12 on the full modular group, giving a hint that the function $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$, with a certain multiplier system that is a 24-th root of unity.

Lemma 2.8. *The discriminant function $\Delta(\tau)$ is a cusp form of weight* 12 *on the full modular group.*

Proof. From the product expansion it is clear that $\Delta(\tau) \neq 0$. So we can look at the logarithmic derivative of the function and it is possible to deduce [9]:

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = E_2(\tau).$$

From the transformation rule of E_2 it is possible to see that $(\Delta|_{12}\lambda)(\tau) = C(\lambda)\Delta(\tau)$. Evaluating the constant $C(\lambda)$ on the generators of $\mathrm{Sl}_2(\mathbb{Z})$ shows that the constant is equal to one and this is sufficient because $C : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\star$ is a homomorphism. By the dimension formula we see that $\Delta(\tau)$ is a linear combination of $G_6(\tau)^2$ and $G_4(\tau)^3$. It is obvious that the q-expansion of the function $\Delta(\tau)$ starts with q and hence is a cusp form. This finishes the proof. □

Example 2.9 (Jacobi ϑ-function)**.** The Jacobi theta function is defined in the following way:

$$\vartheta(u;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i\nu^2\tau+2\pi i\nu\left(u+\frac{1}{2}\right)}.$$

Here $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$. We say that τ is the modular variable and u is the elliptic variable. Such a function obeys a modular transformation property and an elliptic transformation property. Functions of this type are called *Jacobi forms* (there are more restrictions, but that is not important here), which were introduced in [11]. One easy observation will be important later on.

Lemma 2.10. *Let B be a positive integer: Then the Jacobi theta function obeys the following equation:*

$$\vartheta(u+B\tau;\tau) = (-1)^B e^{-\pi iB^2\tau-2\pi iBu}\vartheta(u;\tau)$$

Proof. The proof is done by induction. For $B = 1$ we see by the definition of ϑ that

$$\vartheta(u+\tau;\tau) = -e^{-\pi i\tau-2\pi iu}\vartheta(u;\tau)$$

Now assuming that the formula is true for B we can proceed by firstly using the formula for B:

$$\vartheta(u+(B+1)\tau;\tau) = \vartheta(u+\tau+B\tau;\tau) = (-1)^B e^{-\pi iB^2\tau-2\pi iB(u+\tau)}\vartheta(u+\tau;\tau).$$

We now use the formula for $B = 1$ to obtain

$$\begin{aligned}\vartheta(u+(B+1)\tau;\tau) =&(-1)^{B+1}e^{-\pi iB^2\tau-2\pi iB(\tau+u)-\pi i\tau-2\pi iu}\vartheta(u;\tau)\\ =&(-1)^{B+1}e^{-\pi i(B+1)^2-2\pi i(B+1)}\vartheta(u;\theta)\end{aligned}$$

by completing the square. Thus, the formula is true for $B+1$ and so by induction for every $B \in \mathbb{N}$. □

2.2 Transformation formulas for η and ϑ

One key step in the proof of Theorem 1.1 is to deduce the transformation law for the crank generating function under the action of the modular group. One way is to show that the crank generating function is proportional to the quotient of the square of the η-function and Jacobi's ϑ-function. Therefore we need the transformation laws of these two functions to deduce the transformation properties of the crank. We skip the proof of the transformation formula for η and refer to [15] and [4] for the details. We introduce the needed quantities to state the transformation rule. Therefore define

$$\chi(h,h',k) := i^{-\frac{1}{2}}\omega_{h,k}^{-1}e^{-\frac{\pi i}{12k}(h'-h)}. \tag{2.4}$$

Here h' is a solution to $hh' \equiv -1 \pmod{k}$ and

$$\omega_{h,k} := \exp\left(\pi i s\left(h,k\right)\right), \tag{2.5}$$

where the *Dedekind sums* $s(h,k)$ are explicitly given by

$$s(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

In the above, the saw tooth function is defined by

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Theorem 2.11. *For $z \in \mathbb{C}$ with $Re(z) > 0$ we have*

$$\eta\left(\frac{h+iz}{k}\right) = \sqrt{\frac{i}{z}}\chi\left(h,h',k\right)\eta\left(\frac{h'+\frac{i}{z}}{k}\right),$$

where we take the principal branch of the square root. Moreover, η is a modular form of weight $\frac{1}{2}$ with multiplier system.

Next we want to give an important definition that makes notation much easier.

Definition 2.12. The *q-Pochhammer* symbol is given for $n \in \mathbb{N}\cup\{\infty\}$ by:

$$(a)_n := (a,q)_n := \prod_{i=0}^{n-1}\left(1-aq^i\right)$$

Now we can also state the transformation formula for the Jacobi ϑ-function. For completeness, we give the following theorem ([21] Prop 1.3 and [8]);

Theorem 2.13. *Define $x := e^{2\pi i\omega}$ and $q := e^{2\pi i\tau}$ where $\omega \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Let h, k be coprime integer with h' like above. Then ϑ satisfies:*

1. $\vartheta(\omega + 1; \tau) = -\vartheta(\omega; \tau)$.
2. $\vartheta(\omega + \tau; \tau) = -e^{-\pi i\tau - 2\pi i\omega}\vartheta(\omega; \tau)$,
3. *Up to a multiplicative constant, $\omega \mapsto \vartheta(\omega; \tau)$ is the unique holomorphic function satisfying (1), (2).*
4. $\vartheta(-\omega; \tau) = -\vartheta(\omega; \tau)$.
5. *The zeros of ϑ are the points $\omega = n\tau + m$, with $m, n \in \mathbb{Z}$. These are all simple zeros.*
6. $\vartheta(\omega; \tau + 1) = e^{\frac{\pi i}{4}}\vartheta(\omega; \tau)$,
7. $\vartheta\left(\frac{\omega}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{\pi i\omega^2}{\tau}}\vartheta(\omega; \tau)$,
8. *(Jacobi triple product identity)* $\vartheta(\omega; \tau) = -2\sin(\pi\omega)q^{\frac{1}{8}}(q)_\infty(xq)_\infty(x^{-1}q)_\infty$.
9. *If* $Re(z) > 0$, *then* $\vartheta\left(\omega; \frac{h+iz}{k}\right) = \chi^3\sqrt{\frac{i}{z}}e^{-\frac{\pi k\omega^2}{z}}\vartheta\left(\frac{i\omega}{z}; \frac{h' + \frac{i}{z}}{k}\right)$.

Chapter 3

Partitions

3.1 Basic definitions

In the next section we define partitions and we state the Rademacher formula.

Definition 3.1 (Partitions)**.** A *partition* of an non-negative integer n is a finite series of non-increasing positive integers λ_i with $i \in \{1, \dots, k\}$, such that $\sum_{i=1}^{k} \lambda_i = n$. Obviously, such a partition is not unique.

Example 3.2. It is easy to see that the number 3 has three different partitions, which are 3, $2+1$ and $1+1+1$.

After this example we want to ask for the number of partitions of every positive integer. We are hence interested in the following definition.

Definition 3.3 (Partition function)**.** Let n be a non-negative integer and $(\lambda_i)_{i=1,\dots,k}$ a partition of n. Then we let $p(n)$ denote the number of partitions of n. By convention, $p(0) := 1$.

From Example 3.2, we can see that we have $p(3) = 3$. It is easy to see that $p(n)$ is strictly increasing and that $p(n)$ increases rapidly. So it is interesting to see, if there is any closed expression to deduce the value of $p(n)$. To compute $p(n)$ we define the *partition generating function* by

$$P(q) := \sum_{n=0}^{\infty} p(n) q^n,$$

It is possible to show (using $|q| < 1$):

$$P(q) = q^{\frac{1}{24}} \eta(\tau)^{-1} = \prod_{n=0}^{\infty} \frac{1}{1-q^n}.$$

So the generating function is almost a modular form and we may use the transformation rule of $\eta(\tau)$ to get a transformation rule of the generating function. Before stating the theorem, we give the following definition.

Definition 3.4. Let h, k be positive integers and $\omega_{h,k}$ be the exponentials of Dedekind sums defined in Chapter 2.2. We then call the following function

$$A_k(n) := \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi i h n}{k}}$$

a *Kloostermann sum.*

Now we can state the famous Rademacher expansion:

Theorem 3.5 (Rademacher)**.** *Let n be a positive integer. The following equality is true:*

$$\begin{aligned} p(n) =& \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right) \\ =& \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6k} \right). \end{aligned}$$

where $I_{\frac{3}{2}}(x)$ is the Bessel function of order $\frac{3}{2}$.

Remark 3.6. The Rademacher formula is astonishing since the left hand side of the equation is an arithmetical function while the right hand side is an analytic expression containing π, square root, Kloostermann sums and Bessel function. However, the importance of the proof does not only rely on the fact that it is one of the biggest achievements in analytic number theory and forms a link between analytic and arithmetic expressions. The theorem is also important because it explains one of the most important techniques in analytic number theory, namely the Circle Method. The Circle Method is a way to obtain statements about Fourier coefficients that encode a generating function which also encodes arithmetical information and will be used later on. We skip the proof of the Rademacher formula and refer for more details to [15] and [4].

3.2 Congruences of $p(n)$, Crank, Rank

The partition function has an interesting congruence property that was observed by Ramanujan [18]. He calculated many values of $p(n)$ and saw that there are certain patterns if the values are sorted in the appropriate ways. More precisely, he noticed:

Theorem 3.7 (Ramanujan)**.** *For every $n \in \mathbb{N}_0$ we have*

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

The first proof of this statement excludes the results for the modulus 11. Ramanujan used some complicated manipulations to show that the generating function for $p(5n+4)$ (resp. $p(7n+5)$) is a q-series with integer coefficients times 5 (resp. 7) [19]. This tells us that the reduction modulo 5 (resp. 7) is zero. However this yields no combinatorial proof. In order to obtain a combinatorial proof, Dyson [10] conjectured that there is a certain partition statistic, a function that gives a value to every partition, explaining the Ramanujan congruences. Unfortunately, the so-called *rank* which was conjectured to be the right function could not explain the congruences for the modulus 11. That was the reason Dyson conjectured that there should be yet another statistic, which he called the *crank*, that simultaneously explains the Ramanujan congruences for all moduli $5, 7$ and 11. We now give the combinatorial definitions of the crank and rank before we investigate their generating functions. Before continuing we fix notation. We mean by λ_1 the largest part of a partition, $o(\lambda)$ the number of ones in a partition and by $\mu(\lambda)$ the number of parts larger than $o(\lambda)$.

Definition 3.8. The *rank* of a partition $\lambda = (\lambda_i)_{i\in\{1,\dots,k\}}$ is defined as:

$$\operatorname{rank} \lambda := \lambda_1 - k.$$

As an example, we show how the rank explains the Ramanujan congruence for the modulus 5 in the case of partitions of 4.

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	0	0
$1+1+2$	-1	4
$1+1+1+1$	-3	2

We denote the number of partitions of a number n with rank m by $N(m,n)$ and the number of partitions of a number n with rank congruent to a modulo c by $N(a,c,n)$.

With this numbers we can define the *rank generating function* as

$$R(x;q) := \sum_{m\in\mathbb{Z}} \sum_{n\geq 0} N(m,n) q^n x^m.$$

Next we define the crank.

Definition 3.9. The *crank* of a partition $\lambda = (\lambda_i)_{i\in\{1,\ldots,k\}}$ is defined as:

$$\operatorname{crank}\lambda := \begin{cases} \lambda_1 & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) \neq 0\ . \end{cases}$$

As an example, we show how the crank explains the Ramanujan congruence for the modulus 11 in the case of partitions of 6.

Partition	Crank	Crank mod 11
6	6	6
$5+1$	$1-1=0$	0
$4+2$	4	4
$3+3$	3	3
$4+1+1$	$1-2=-1$	10
$3+2+1$	$2-1=1$	1
$2+2+1+1$	$0-2=-2$	9
$2+1+1+1+1$	$0-4=-4$	7
$1+1+1+1+1+1$	$0-6=-6$	5
$2+2+2$	2	2
$3+1+1+1$	$0-3=-3$	8

We denote the number of partitions of a number n with crank m by $M(m,n)$ and the number of partitions of a number n with crank congruent to a modulo c by $M(a,c,n)$. We define further the *crank generating function* by summing over all possible partitions and over all possible cranks. To do so, we have to redefine the following crank values of the partition of 1 and the empty partition. We hence set:

$$M(-1,1) = M(0,0) = M(1,1) = 1\,,\ M(0,1) = -1\,,\ M(m,1) = 0 \quad \forall m\,, |m| \geq 2.$$

Hence the crank generating function is

$$C(x;q) := \sum_{m\in\mathbb{Z}} \sum_{n\geq 0} M(m,n) q^n x^m.$$

This is the important technique that allows to translate arithmetical and combinatorial problems into the world of analytic methods, because this function admits certain transformations if we make Möbius transformations on the variable q. To see this behavior, we show that this function has another representation that is useful for the analytic machinery.

3.3 Generating functions of partitions statistics

It is more useful to work with the generating functions of the crank and the rank than with the combinatorial definitions. Next we show:

Theorem 3.10 (Andrews-Garvan)**.** *The generating function of the crank has the following two variable q-series expansion [3], [12]:*

$$C(x;q) = \sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} M(n,m)x^m q^n = \frac{1-x}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-xq^n}. \tag{3.1}$$

PROOF. Firstly, we define $N_v(m,n)$ to be the coefficients of the generating function

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} N_v(m,n)x^m q^n = \frac{(q)_\infty}{(xq)_\infty (x^{-1}q)_\infty}.$$

We show that $N_v(m,n) = M(n,m)$:

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} N_v(m,n)x^m q^n = \frac{(q)_\infty}{(xq)_\infty (x^{-1}q)_\infty} &= \frac{1-q}{(xq)_\infty} \frac{(q^2;q)_\infty}{(q/x)_\infty} \\
&= \frac{1-q}{(xq)_\infty} \sum_{j=0}^{\infty} \frac{(xq)_j (q/x)_j}{(q)_j} \\
&= \frac{1-q}{(xq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j x^{-j}}{(q^2;q)_{j-1}(xq^{j+1})_\infty}.
\end{aligned}$$

In the second step we used the Fine identity (see [2], p. 7). Now using techniques of [2] it is possible to relate the first summand to the partitions with no 1 and the exponent of x counting the largest part λ_1 and for $j > 0$ the second summand generates partitions with $o(\lambda) = j$ and with the exponent of x equal to $\mu(\lambda) - o(\lambda)$. To complete the proof, we have to show the following identity:

$$\frac{(q)_\infty}{(xq)_\infty (x^{-1}q)_\infty} = \frac{1-x}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-xq^n}.$$

This can be deduced directly from Lemma 3.1 of [14] using hypergeometric series. □

For completeness, we now quote the analog of this result for the rank.

Theorem 3.11. *The generating function of the rank has the following two variable q-series expansion [10]*

$$R(x;q) = \sum_{m\in\mathbb{Z}} \sum_{n\geq 0} N(m,n)x^m q^n = \frac{1-x}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-xq^n}.$$

Remark 3.12. It is impressive to see that the rank and crank generating function are related. In the next chapter we will see that although they have the same shape and are motivated by the same fact, it will become clear that in the world of modular forms they are completely different objects.

Chapter 4

Asymptotic formula for crank differences

4.1 Transformation of the crank generating function

In this section, we prove how the crank generating function transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$. As noted above, we only need the transformation formulas of η and ϑ. Throughout, let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $0 \leq h < k$ with $(h, k) = 1$. Let $x = e^{2\pi i u}$ and $q = e^{-2\pi z}$. Let h' be a solution to the congruence $hh' \equiv -1 \pmod{k}$ if k is odd and let h' be a solution to the congruence $hh' \equiv -1 \pmod{2k}$ if k is even. Let $0 < l < c_1$, be the unique solution to $l \equiv ak_1 \pmod{c_1}$, where $k_1 := \frac{k}{(k,c)}$ and $c_1 := \frac{c}{(c,k)}$. Finally let $0 < a < c$ be coprime integers with c odd.

Remark 4.1. Firstly, we want to explain why it is possible to deduce the transformation properties of the crank generating functions with the transformation formulas of η and ϑ. Therefore we use (3.1):

$$C(x; q) = \frac{(q)_\infty^2}{(xq)_\infty (x^{-1}q)_\infty (q)_\infty}.$$

The Jacobi theta function obeys the Jacobi triple product identity (see Theorem 2.13)

$$\vartheta(u; iz) = -2\sin(\pi u) q^{\frac{1}{8}} (q)_\infty (xq)_\infty (x^{-1}q)_\infty,$$

and the Dedekind η-function can be expressed in the following way

$$\eta(iz) = q^{\frac{1}{24}} (q)_\infty.$$

Plugging in the η^2 and the Jacobi tripel product identity we arrive at:

$$C\left(e^{2\pi i u}; e^{-2\pi z}\right) = \frac{-2\sin(\pi u) q^{\frac{1}{24}} \eta^2(iz)}{\vartheta(u; iz)}. \tag{4.1}$$

From this we notice that it is sufficient to know the transformation properties of η and ϑ. Using these transformation formulas and properties of the Jacobi ϑ function to shift arguments we obtain:

Proposition 4.2. We define $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$. Then the following is true:

(1) For $c \mid k$ we have

$$C\left(e^{\frac{2\pi ia}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = \frac{i\sin\left(\frac{\pi a}{c}\right)}{z^{\frac{1}{2}}\sin\left(\frac{\pi ah'}{c}\right)}(-1)^{ak+1}\omega_{h,k}e^{\frac{\pi}{12k}\left(z^{-1}-z\right)-\frac{\pi ia^2k_1h'}{c}} \times C\left(e^{\frac{2\pi iah'}{c}}; e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}\right).$$

(2) For $c \nmid k$ we have

$$C\left(e^{\frac{2\pi ia}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = \frac{4i\sin(\frac{\pi a}{c})\omega_{h,k}(-1)^{ak+l+1}}{z^{\frac{1}{2}}}e^{-\frac{\pi a^2h'k_1}{cc_1}+\frac{2\pi ih'la}{cc_1}} \times q_1^{-\frac{l^2}{2c_1^2}}e^{\frac{\pi}{12k}(z^{-1}-z)}C\left(ah', \frac{lc}{c_1}, c; q_1\right),$$

where the function $C(a,b,c;q)$ is defined as follows

$$\frac{i}{2(q)_\infty}\left(\sum_{m=0}^{\infty}\frac{(-1)^m e^{-\frac{\pi ia}{c}}q^{\frac{m(m+1)}{2}+\frac{b}{2c}}}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}} - \sum_{m=1}^{\infty}\frac{(-1)^m e^{\frac{\pi ia}{c}}q^{\frac{m(m+1)}{2}-\frac{b}{2c}}}{1-e^{\frac{2\pi ia}{c}}q^{m-\frac{b}{c}}}\right).$$

Proof. (1). Using Theorem 2.11 and Theorem 2.13, we obtain

$$C\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{2\sin(\pi u)i}{z^{\frac{1}{2}}}\omega_{h,k}e^{\frac{\pi i}{12k}(h'-h)}e^{\frac{\pi i}{12k}(h+iz)}\frac{\eta^2\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}{\vartheta\left(\frac{iu}{z};\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}e^{\frac{\pi ku^2}{z}},$$

where the $\omega_{h,k}$ were defined in (2.5). We now assume that $c \mid k$, define $A := \frac{ak}{c} \in \mathbb{Z}$ and write $u = \frac{a}{c}$. First of all we replace η^2 in the numerator by rewriting (4.1) such that it is possible to identify the crank at two different positions in $\mathbb{H}\times\mathbb{C}$:

$$C\left(e^{2\pi i\frac{a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = i\frac{\sin(\frac{\pi a}{c})}{\sin(\frac{\pi ah'}{c})}\omega_{h,k}e^{\frac{\pi}{12k}(z^{-1}-z)+\frac{\pi a^2k}{zc^2}}C\left(e^{\frac{2\pi iah'}{c}}; e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}\right) \times \frac{\vartheta\left(\frac{ah'}{c};\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}{\vartheta\left(\frac{ia}{zc};\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}. \quad (4.2)$$

The fraction of theta functions can be simplified by noting that the elliptic variable of the numerator function can be seen as a shift of the elliptic variable of denominator function by $A\cdot\tau$ where τ is the modular variable defined as $\tau := \frac{1}{k}(h' + \frac{i}{z})$. This allows to simplify the quotient of ϑ-functions:

$$\frac{\vartheta(\frac{ah'}{c};\frac{1}{k}(h'+\frac{i}{z}))}{\vartheta(\frac{ia}{zc};\frac{1}{k}(h'+\frac{i}{z}))} = -\frac{\vartheta(\frac{ah'}{c};\frac{1}{k}(h'+\frac{i}{z}))}{\vartheta(-\frac{ia}{zc};\frac{1}{k}(h'+\frac{i}{z}))} = -\frac{\vartheta(-\frac{ia}{zc}+\frac{ak}{c}(\frac{1}{k}(h'+\frac{i}{z}));\frac{1}{k}(h'+\frac{i}{z}))}{\vartheta(-\frac{ia}{zc};\frac{1}{k}(h'+\frac{i}{z}))}$$
$$= \frac{\vartheta(-\frac{ia}{zc};\frac{1}{k}(h'+\frac{i}{z}))}{\vartheta(-\frac{ia}{zc};\frac{1}{k}(h'+\frac{i}{z}))}(-1)^{\frac{ak}{c}} e^{-\frac{\pi i a^2k^2}{c^2}(\frac{1}{k}(h'+\frac{i}{z}))+\frac{-2\pi a^2 k}{zc^2}}$$
$$= (-1)^{ak+1} e^{-\frac{\pi i a^2 k h'}{c^2}} e^{\frac{\pi a^2 k}{zc^2}} e^{-\frac{2\pi a^2 k}{zc^2}}.$$

Inserting this expression into the equation (4.2) yields the transformation formula for the case $c \mid k$.
(2). The case $c \nmid k$ is more difficult, because in general we have $\frac{ak}{c} \notin \mathbb{Z}$. Again using the transformation rules of ϑ and η, we obtain:

$$C\left(e^{\frac{2\pi i a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{2\sin(\pi u)i}{z^{\frac{1}{2}}}\omega_{h,k} e^{\frac{\pi i}{12k}(h'+iz)} \frac{\eta^2\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}{\vartheta\left(\frac{ia}{zc};\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)} e^{\frac{\pi k a^2}{zc^2}}. \tag{4.3}$$

Since we defined l to be the solution to the congruence condition $l \equiv ak_1 \pmod{c_1}$ by definition it is clear that $B := \frac{l-ak_1}{c_1} \in \mathbb{Z}$. We may shift the theta function in the elliptic variable by the modular variable multiplied by B, where the modular variable is given above. We compute the shifted theta function using Lemma 2.10 and obtain

$$\vartheta\left(\frac{-ah'}{c}+\frac{l}{c_1}\tau;\tau\right) = \vartheta\left(\frac{ia}{zc}+\frac{B}{k}\left(h'+\frac{i}{z}\right);\tau\right)$$
$$= (-1)^{ak+l} e^{-\pi i B^2\tau} e^{-2\pi i B\frac{ia}{zc}} \vartheta\left(\frac{ia}{zc};\tau\right).$$

This is equivalent to the following equation:

$$\vartheta\left(\frac{ia}{zc};\tau\right) = (-1)^{ak+l} e^{\frac{\pi i (l-ak_1)^2}{c_1^2}\tau} e^{2\pi i\left(\frac{l-ak_1}{c_1}\right)\frac{ia}{zc}} \vartheta\left(\frac{-ah'}{c}+\frac{l}{c_1}\tau;\tau\right).$$

Inserting this expression for theta function into (4.3) yields:

$$\frac{C\left(e^{2\pi i\frac{a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right)}{(-1)^{ak+l+1}} = 2\frac{\sin(\frac{\pi a}{c})}{z^{\frac{1}{2}}} i\omega_{h,k} e^{\frac{\pi}{12k}\left(h'+\frac{i}{z}\right)} q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi i l a h'}{cc_1}-\frac{\pi i a^2 k_1 h'}{cc_1}} \frac{\eta^2(\tau)}{\vartheta\left(-\frac{ah'}{c}+\frac{l}{c_1}\tau;\tau\right)}.$$

Now replacing the quotient $\frac{\eta^2}{\vartheta}$ by the Crank generating function we get:

$$\frac{C\left(e^{2\pi i\frac{a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right)}{(-1)^{ak+l}} = \frac{\sin(\frac{\pi a}{c})}{z^{\frac{1}{2}}} i\omega_{h,k} e^{\frac{\pi}{12k}\left(z^{-1}-z\right)} q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi i l a h'}{cc_1}-\frac{\pi i a^2 k_1 h'}{cc_1}} \frac{C\left(e^{-\frac{2\pi i a h'}{c}+\frac{2\pi i l}{c_1}\tau}; e^{2\pi i\tau}\right)}{\sin\left(-\frac{\pi a h'}{c}+\frac{\pi l}{c_1}\tau\right)}. \tag{4.4}$$

We define $x := e^{-\frac{2\pi i a h'}{c}+\frac{2l\pi i}{c_1}\tau}$, and use the exponential representation of the sine to deduce

$$\frac{C\left(e^{-\frac{2\pi i a h'}{c}+\frac{2\pi i l}{c_1}\tau}; e^{2\pi i\tau}\right)}{\sin\left(-\frac{\pi a h'}{c}+\frac{\pi l}{c_1}\tau\right)} = \frac{2i\,(1-x)}{(q_1)_\infty\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{1-xq_1^m}$$
$$= \frac{-2ix^{\frac{1}{2}}}{(q_1)_\infty}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{1-xq_1^m}.$$

Inserting this expression into (4.4), we arrive at

$$C\left(e^{2\pi i\frac{a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) = \frac{(-1)^{ak+l+1}\sin\left(\frac{\pi a}{c}\right)}{z^{\frac{1}{2}}} i\omega_{h,k} e^{\frac{\pi}{12k}\left(z^{-1}-z\right)} q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi i l a h'}{c_1}-\frac{\pi i a^2 k_1 h'}{c_1 c}}$$
$$\times \frac{2i}{(q_1)_\infty x^{-\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{1-xq_1^m}$$
$$= \frac{(-1)^{ak+l+1}\sin\left(\frac{\pi a}{c}\right)}{z^{\frac{1}{2}}} i\omega_{h,k} e^{\frac{\pi}{12k}\left(z^{-1}-z\right)} q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi i l a h'}{c_1}-\frac{\pi i a^2 k_1 h'}{c_1 c}}$$
$$\times \frac{2i}{(q_1)_\infty}\sum_{m\in\mathbb{Z}}\frac{(-1)^m e^{-\frac{\pi i a h'}{c}} q_1^{\frac{l}{2c_1}+\frac{m(m+1)}{2}}}{1-e^{-\frac{2\pi i a h'}{c}} q_1^{m+\frac{l}{2c_1}}}$$
$$= \frac{(-1)^{ak+l+1}\sin\left(\frac{\pi a}{c}\right)}{z^{\frac{1}{2}}} i\omega_{h,k} e^{\frac{\pi}{12k}\left(z^{-1}-z\right)} q_1^{-\frac{l^2}{2c_1^2}} e^{\frac{2\pi i l a h'}{c_1}-\frac{\pi i a^2 k_1 h'}{c_1 c}} C\left(ah', \frac{lc}{c_1}, c; q_1\right).$$

This completes the proof of the transformation formula. □

Remark 4.3. It is interesting to see that the transformation formula is similar to the rank case. The main and important difference is that we have no mock part and that that the step function s does not appear (see [7]). This reveals the fact that the rank and the crank generating functions look very similar but have completely different behavior under the action of the modular group.

4.2 Circle method and asymptotic formula

In this section we give an asymptotic formula for the coefficients of the crank generating function. To state the theorem, we have to fix notation and define the following sum for $m, n \in \mathbb{Z}$:

$$\widetilde{B}_{a,c,k}(n,m) := (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{h \pmod{k}^*} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{\pi i a^2 k_1 h'}{c}} \cdot e^{\frac{2\pi i}{k}(nh+mh')}. \quad (4.5)$$

Here the sum runs over all primitive residue classes modulo k and this summation is denoted by $h \pmod{k}^*$. For the case $c \nmid k$ we define

$$D_{a,c,k}(m,n) = (-1)^{ak+l} \sum_{h \pmod{k}^*} \omega_{h,k} e^{\frac{2\pi i}{k}(nh+mh')}, \quad (4.6)$$

where l is defined above. Firstly, an important lemma (compare Lemma 3.2 in [7]) is established that is needed to bound certain terms:

Lemma 4.4. *Let $n, m, k, D \in \mathbb{Z}$ with $(D,k) = 1$, $0 \leq \sigma_1 < \sigma_2 \leq k$. Then there are constants C_1 and C_2 such that:*

(1) We have

$$\left| \sum_{\substack{h (\mathrm{mod}\, k)^* \\ \sigma_1 \leq Dh' \leq \sigma_2}} \omega_{h,k} e^{\frac{2\pi i}{k}(hn+h'm)} \right| \leq C_1 \cdot \gcd(24n+1,k)^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon}; \quad (4.7)$$

(2) We have

$$\left| \frac{\sin\left(\frac{\pi a}{c}\right)}{(-1)^{ak+1}} \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \leq Dh' \leq \sigma_2}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{\pi i a^2 k_1 h'}{c}} e^{\frac{2\pi i}{k}(hn+h'm)} \right| \leq C_2 \cdot \gcd(24n+1,k)^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon}. \quad (4.8)$$

The constants C_1 and C_2 are independent of a and k.

Proof. In [2] part one is proven and part two follows from part one and the proof of Lemma 3.2 in [7] after defining $\tilde{c} := c$ if k is odd and $\tilde{c} = 2c$ if k is even and checking that $e^{-\frac{\pi i a^2 k_1 h'}{c}} \sin^{-1}\left(\frac{\pi a h'}{c}\right)$ only depends on h' modulo $\tilde{c}$. To show this, we insert an explicit representative of the equivalence class and show that all of the terms that do not depend on h' cancel. This establishes Lemma 4.4. □

In the following theorem, we investigate the main contributions to the Fourier coefficients of the crank generating function using the circle method and the proven transformation formula. In addition, we give a rough bound of the error term. To state the theorem we need some notation. We define:

$$\delta^i_{a,c,k,r} := \begin{cases} -\left(\frac{1}{2}+r\right)\frac{l}{c_1}+\frac{1}{2}\left(\frac{l}{c_1}\right)^2+\frac{1}{24} & \text{if } i=+, \\ \frac{l}{2c_1}+\frac{1}{2}\left(\frac{l}{c_1}\right)^2-\frac{23}{24}-r\left(1-\frac{l}{c_1}\right) & \text{if } i=-, \end{cases} \tag{4.9}$$

and

$$m^+_{a,c,k,r} := \frac{1}{2c_1^2}\left(-a^2k_1^2+2lak_1-ak_1c_1-l^2+lc_1-2ark_1c_1+2lc_1r\right), \tag{4.10}$$

$$m^-_{a,c,k,r} := \frac{1}{2c_1^2}\left(-a^2k_1^2+2lak_1-ak_1c_1-l^2+2c_1^2r-2lrc_1+2ark_1c_1+2lc_1+2c_1^2-ak_1c_1\right).$$

Then we have the following theorem.

Theorem 4.5. *Let* $C\left(e^{\frac{2\pi ia}{c}};q\right) =: 1+\sum_{n=1}^{\infty}\widetilde{A}\left(\frac{a}{c};n\right)q^n$. *If* $0<a<c$ *are co-prime integers,* c *is odd and* n *is a positive integer, then we have*

$$\widetilde{A}\left(\frac{a}{c};n\right) = \frac{4\sqrt{3}i}{\sqrt{24n-1}}\sum_{\substack{1\le k\le\sqrt{n}\\ c|k}}\frac{\widetilde{B}_{a,c,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)+\frac{8\sqrt{3}\cdot\sin\left(\frac{\pi a}{c}\right)}{\sqrt{24n-1}}$$

$$\times\sum_{\substack{1\le k\le\sqrt{n}\\ c\nmid k\\ r\ge 0\\ \delta^i_{a,c,k,r}>0\\ i\in\{+,-\}}}\frac{D_{a,c,k}(-n,m^i_{a,c,k,r})}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{2\delta^i_{a,c,k,r}(24n-1)}}{\sqrt{3}k}\right)+O\left(n^{\varepsilon}\right).$$

Proof. To prove our asymptotic formula for the crank coefficients we use the Hardy-Ramanujan method(also called Circle Method): By Cauchy's theorem we have for $n>0$

$$\widetilde{A}\left(\frac{a}{c};n\right) = \frac{1}{2\pi i}\int_C\frac{C\left(e^{\frac{2\pi ia}{c}};q\right)}{q^{n+1}}dq,$$

where C is an arbitrary path inside the unit circle surrounding 0 counterclockwise. Choosing a circle with radius $e^{-\frac{2\pi}{n}}$ and as a parametrisation $q=e^{-\frac{2\pi}{n}+2\pi it}$ with $0\le t\le 1$ gives

$$\widetilde{A}\left(\frac{a}{c};n\right) = \int_0^1 C\left(e^{\frac{2\pi ia}{c}};e^{-\frac{2\pi}{n}+2\pi it}\right)e^{2\pi-2\pi int}dt.$$

We define

$$\vartheta'_{h,k} := \frac{1}{k\left(\widetilde{k}_1 + k\right)}, \qquad \vartheta''_{h,k} := \frac{1}{k\left(\widetilde{k}_2 + k\right)},$$

where $\frac{h_1}{\widetilde{k}_1} < \frac{h}{k} < \frac{h_2}{\widetilde{k}_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor n^{1/2} \rfloor$. For more on Farey fractions see [4]. We know that

$$\frac{1}{k + \widetilde{k}_j} \leq \frac{1}{N+1} \qquad (j = 1, 2).$$

Now we decompose the path of integration along Farey arcs $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$, where $\Phi = t - \frac{h}{k}$ and $0 \leq h < k \leq N$ with $(h,k) = 1$. From this decomposition of the path we can rewrite the integral along these arcs:

$$\widetilde{A}\left(\frac{a}{c}; n\right) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} C\left(e^{2\pi i \frac{a}{c}}; e^{\frac{2\pi i}{k}(h+iz)}\right) e^{\frac{2\pi n z}{k}} d\Phi,$$

where $z = \frac{k}{n} - k\Phi i$. We insert our transformation formula into the integral and obtain

$$\begin{aligned}
\widetilde{A}\left(\frac{a}{c}; n\right) &= i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c|k}} \omega_{h,k} \frac{(-1)^{ak+1}}{\sin(\frac{\pi a h'}{c})} e^{-\frac{\pi i a^2 k_1 h'}{c} - \frac{2\pi i h n}{k}} \\
&\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}} C\left(e^{\frac{2\pi i a h'}{c}}; q_1\right) d\Phi \\
&- 4i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c\nmid k}} \omega_{h,k} (-1)^{ak+l} e^{-\frac{\pi i a^2 h' k_1}{c c_1} + \frac{2\pi i h' l a}{c c_1} - \frac{2\pi i h n}{k}} \\
&\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}} q_1^{-\frac{l^2}{2c_1^2}} C\left(ah', \frac{lc}{c_1}, c; q_1\right) d\Phi \\
&=: \Sigma_1 + \Sigma_2.
\end{aligned}$$

To deduce the main contribution of Σ_1 we note that the principal part of $C\left(e^{\frac{2\pi i a h'}{c}}; q_1\right)$ in the q_1 variable in the limit $z \to 0$ is 1 and from that it is possible to write

$$C\left(e^{\frac{2\pi i a h'}{c}}; q_1\right) =: 1 + \sum_{r \in \mathbb{N}} \sum_{s \pmod{c}} a(r,s) e^{\frac{2\pi i h'}{k} m_{r,s}} q_1^r$$

where $m_{r,s}$ takes values in $\mathbb{Z}$ and $\sum_{s \pmod{c}} a(r,s) = p(r)$ for $r > 1$. Only the constant term will contribute to the main term while the other terms will contribute to the error, because for large n these terms are suppressed exponentially. So from that the Σ_1 part can be written as:

$$\Sigma_1 = S_1 + S_2.$$

Here

$$S_1 := i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c|k}} \omega_{h,k}\frac{(-1)^{ak+1}}{\sin(\frac{\pi a h'}{c})} e^{-\frac{\pi i a^2 k_1 h'}{c}-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} d\Phi$$

and

$$S_2 := i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c|k}} \omega_{h,k}\frac{(-1)^{ak+1}}{\sin(\frac{\pi a h'}{c})} e^{-\frac{\pi i a^2 k_1 h'}{c}} e^{-\frac{2\pi i h n}{k}}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} \sum_{r\in\mathbb{N}} \sum_{s \pmod c} a(r,s) e^{\frac{2\pi i h'}{k} m_{r,s}} q_1^r d\Phi.$$

To bound the error term S_2 it is helpful to recall some easy facts:

(i) $z = \frac{k}{n} - i\Phi k$;

(ii) $-\vartheta'_{h,k} \le \Phi \le \vartheta''_{h,k}$;

(iii) $Re(z) = \frac{k}{n}$;

(iv) $|z|^2 = \frac{k^2}{n^2} + k^2\Phi^2 \ge \frac{k^2}{n^2}$;

(v) $|z|^{-\frac{1}{2}} \le k^{-\frac{1}{2}} n^{\frac{1}{2}}$;

(vi) $|z|^2 \le \frac{k^2}{n^2} + \frac{k^2}{k^2(k+\widetilde{k}_2)^2} \le \frac{2}{n}$;

(vii) $\mathrm{Re}(z^{-1}) = \frac{\mathrm{Re}(z)}{|z|^2} \ge \frac{k}{2}$;

(viii) $\vartheta'_{h,k} + \vartheta''_{h,k} \le \frac{2}{k\sqrt{n}}$.

We split the integral in the following way (this is possible because $\widetilde{k}_1, \widetilde{k}_2 \le N$):

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(\widetilde{k}_2+k)}} + \int_{-\frac{1}{k(\widetilde{k}_1+k)}}^{-\frac{1}{k(N+k)}}. \tag{4.11}$$

Then S_2 can be rewritten into three sums each sum corresponding to one of the three integrations (4.11):

$$S_2 = S_{21} + S_{22} + S_{23}.$$

For example

$$S_{21} = i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c|k}}\omega_{h,k}\frac{(-1)^{ak+1}}{\sin(\frac{\pi a h'}{c})}e^{-\frac{\pi i a^2 k_1 h'}{c}-\frac{2\pi i h n}{k}}$$
$$\times\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{1}{2}}e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}}\sum_{r\in\mathbb{N}}\sum_{s \pmod c} a(r,s)e^{\frac{2\pi i h' m_{r,s}}{k}}q_1^r d\Phi.$$

Taking the absolute value of this it is possible to bound the term. Before doing that we define

$$a(r) := \sum_{s \pmod c}|a(r,s)|,$$

where the $a(r)$ are exactly $p(r)$ except from some constant term ambiguity. We proceed:

$$|S_{21}| \le \sum_{r=1}^{\infty}\sum_{c|k}\sum_{s \pmod c}|a(r,s)|\left|(-1)^{ak+1}\sin\left(\frac{\pi a}{c}\right)\sum_{h}\frac{\omega_{h,k}}{\sin(\frac{\pi a h'}{c})}e^{-\frac{\pi i a^2 k_1 h'}{c}-\frac{2\pi i h n}{k}+\frac{2\pi i m_{r,s}h'}{k}}\right|$$
$$\times k^{-\frac{1}{2}}n^{\frac{1}{2}}\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}}\left|e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}}q_1^r\right| d\Phi$$
$$\le \sum_{r=1}^{\infty}\sum_{c|k}\sum_{s \pmod c}|a(r,s)|\left|(-1)^{ak+1}\sin\left(\frac{\pi a}{c}\right)\sum_{h}\frac{\omega_{h,k}}{\sin(\frac{\pi a h'}{c})}e^{-\frac{\pi i a^2 k_1 h'}{c}-\frac{2\pi i h n}{k}+\frac{2\pi i m_{r,s}h'}{k}}\right|$$
$$\times k^{-\frac{1}{2}}n^{\frac{1}{2}}e^{2\pi+\frac{\pi}{12n}}e^{-\pi r}\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} d\Phi$$

Using Lemma 4.4 (2) we may bound this by

$$C\sum_{r=1}^{\infty}\sum_{k}\sum_{s \pmod c}|a(r,s)|e^{-\pi r}\,(24n-1,k)^{\frac{1}{2}}\,n^{\frac{1}{2}}k^{-\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}k^{-1}n^{-\frac{1}{2}}$$
$$= C\sum_{r=1}^{\infty}|a(r)|e^{-\pi r}\sum_{k}k^{-1+\varepsilon}\,(24n-1,k)^{\frac{1}{2}} \le C_1\sum_{k}k^{-1+\varepsilon}\,(24n-1,k)^{\frac{1}{2}}$$
$$\le C_1\sum_{k\le N}k^{-1+\varepsilon}\sum_{\substack{d|k\\ d|24n-1}}d^{\frac{1}{2}} \le C_1\sum_{\substack{d|24n-1\\ d\le N}}d^{\frac{1}{2}}\sum_{k\le N/d}(kd)^{-1+\varepsilon}$$
$$\le C_1\sum_{\substack{d|24n-1\\ d\le N}}d^{-\frac{1}{2}}\sum_{k\le N/d}k^{-1}N^{\varepsilon} \le C_2 n^{\varepsilon}\sum_{\substack{d|24n-1\\ d\le N}}d^{-\frac{1}{2}} \le C_3 n^{\varepsilon}.$$

where C_1, C_2 and C_3 are constants. We conclude that $S_{21} = O(n^{\varepsilon})$. S_{22} and S_{23} are bounded in the same way and so we just consider S_{22}. We can rewrite the integral in the following way

$$\int_{-\frac{1}{k(\tilde{k}_1+k)}}^{-\frac{1}{k(N+k)}} = \sum_{\ell=\tilde{k}_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} . \tag{4.12}$$

Plugging in this splitting of the integral we obtain the bound

$$|S_{22}| \le \Bigg| \sum_{r=0}^{\infty} \sum_{c|k} \sum_{s \pmod c} a(r,s) \sum_{\ell=\tilde{k}_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}} q_1^r e^{\frac{2\pi}{12kz}+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)} d\Phi$$
$$(-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_h \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{\pi i a^2 k_1 h'}{c}} e^{-\frac{2\pi i h n}{k}} e^{\frac{2\pi i m_{r,s} h'}{k}} \Bigg| =: A.$$

We use the condition $N < k + \tilde{k}_1 \le \ell$ and so we can rearrange the summation from $\sum_{\ell=\tilde{k}_1+k}^{N+k-1}$ to $\sum_{\ell=N+1}^{N+k-1}$, but we also have to rewrite the sum over h to count all the terms that contribute:

$$A = \Bigg| \sum_{r=0}^{\infty} \sum_{c|k} \sum_{s \pmod c} a(r,s) \sum_{\ell=N+1}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}} q_1^r e^{\frac{2\pi}{12kz}+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)} d\Phi$$
$$(-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \\ N<k+\tilde{k}_1\le \ell}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{\pi i a^2 k_1 h'}{c}} e^{-\frac{2\pi i h n}{k}} e^{\frac{2\pi i m_{r,s} h'}{k}} \Bigg| . \tag{4.13}$$

Now by the theory of Farey fractions we have

$$\tilde{k}_1 \equiv -h' \pmod k, \; \tilde{k}_2 \equiv h' \pmod k, \; N-k \le \tilde{k}_i \le N,$$

for $i = 1, 2$. This can be seen by [4], Theorem 5.4 where it is proven that adjacent Farey fractions fulfill some unimodular relations that are equivalent to the above statement. We see that it is possible to use Lemma 4.4(2) to bound contributions of (4.13) and with that also S_{22}. This is done like in the S_{21} case by using the facts listed above and using the same bounds. The only difference is that we need to be careful about the bound of the sum over the different integrals. An easy calculation shows that the following bound can be obtained:

$$\sum_{\ell=N+1}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} d\Phi \le \frac{2}{k\sqrt{n}}$$

So all the terms can be bounded the same way. Thus, we obtain the same result:

$$S_{21} = O(n^{\varepsilon});\; S_{22} = O(n^{\varepsilon});\; S_{23} = O(n^{\varepsilon}).$$

So Σ_1 is equal to:

$$i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c|k}}\omega_{h,k}\frac{(-1)^{ak+1}}{\sin(\frac{\pi ah'}{c})}e^{-\frac{\pi i a^2 k_1 h'}{c}-\frac{2\pi i hn}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}}e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}}d\Phi+O(n^{\varepsilon}).$$

Next we want to analyze S_1. Therefore we use a similar trick like (4.12) to split the integral:

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}=\int_{-\frac{1}{kN}}^{\frac{1}{kN}}-\int_{-\frac{1}{kN}}^{-\frac{1}{k(k+\widetilde{k}_1)}}-\int_{\frac{1}{k(k+\widetilde{k}_2)}}^{\frac{1}{kN}}.$$

and denote by S_{11}, S_{12}, S_{13} the corresponding sums. It is possible to show that S_{12} and S_{13} contribute to the error term. We begin with S_{12}. Similar to the analysis of the error terms of S_2 we write for the integral:

$$\int_{-\frac{1}{kN}}^{-\frac{1}{k(k+\widetilde{k}_1)}}=\sum_{\ell=N}^{k+\widetilde{k}_1-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}.$$

Plugging into S_{12} gives:

$$S_{12}=\sum_{c|k}\sum_{\ell=N}^{k+\widetilde{k}_1-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}}e^{\frac{\pi}{12kz}+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)}d\Phi\frac{\sin\left(\frac{\pi a}{c}\right)}{(-1)^{ak+1}}\sum_{h}\frac{\omega_{h,k}}{\sin\left(\frac{\pi ah'}{c}\right)}e^{-\frac{\pi i a^2 h' k_1}{c}-\frac{2\pi hn}{k}}.$$

Now due to the condition $\widetilde{k}_1\le N$ we have that $\ell\le k+\widetilde{k}_1-1\le N+k-1$ which restricts the summation over h. We can now bound S_{12} by summing over more integrals:

$$\begin{aligned}|S_{12}|\le\sum_{c|k}\sum_{\ell=N}^{k+N-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}&\left|z^{-\frac{1}{2}}e^{\frac{\pi}{12kz}+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)}\right|d\Phi\\ &\times\left|\sin\left(\frac{\pi a}{c}\right)(-1)^{ak+1}\sum_{\substack{h\\ \ell\le k+\tilde{k}_1-1\le N-k-1}}\frac{\omega_{h,k}}{\sin\left(\frac{\pi ah'}{c}\right)}e^{-\frac{\pi i a^2 h' k_1}{c}}e^{\frac{2\pi hn}{k}}\right|=O(n^{\varepsilon}),\end{aligned}$$

using again Lemma 4.4 and the facts listed at the beginning of the proof. In the next step we detect the main contributions from the second sum Σ_2 in the Circle Method. We rewrite Σ_2 in such a way that it is easy to see if certain terms contribute to the main part using geometric series, which is possible because $|q_1| < 1$:

$$\begin{aligned}C\left(ah', \frac{lc}{c_1}, c; q_1\right) &= \frac{i}{2(q_1)_\infty}\left(\sum_{m=0} \frac{(-1)^m e^{-\frac{\pi iah'}{c}} q_1^{\frac{m^2+m}{2}+\frac{l}{2c_1}}}{1-e^{-\frac{2\pi iah'}{c}} q_1^{m+\frac{l}{c_1}}} - \sum_{m=1} \frac{(-1)^m e^{\frac{\pi iah'}{c}} q_1^{\frac{m^2+m}{2}-\frac{l}{2c_1}}}{1-e^{\frac{2\pi iah'}{c}} q_1^{m-\frac{l}{c_1}}}\right)\\
&= \frac{i}{2(q_1)_\infty}\left(\sum_{m=0}(-1)^m \sum_{r=0} e^{-\frac{\pi iah'}{c}-\frac{2\pi irah'}{c}} q_1^{\frac{m}{2}(m+1)+\frac{l}{2c_1}+rm+\frac{rl}{c_1}}\right.\\
&\qquad\left. - \sum_{m=1}(-1)^m \sum_{r=0} e^{\frac{\pi iah'}{c}+\frac{2\pi iarh'}{c}} q_1^{\frac{m}{2}(m+1)-\frac{l}{2c_1}+rm-\frac{rl}{c_1}}\right).\end{aligned}$$

From this expression and the following explanation it is possible to see that we can write

$$e^{-\frac{\pi ia^2h'k_1}{cc_1}+\frac{2\pi ih'la}{cc_1}+\frac{\pi}{12kz}} q_1^{\frac{-l^2}{2c_1^2}} C\left(ah', \frac{lc}{c_1}, c; q_1\right) =: \sum_{r\geq r_0} \sum_{s \pmod c} b(r,s) e^{\frac{2\pi i m_{r,s} h'}{k}} q_1^r. \tag{4.14}$$

We next explain that $m_{r,s} \in \mathbb{Z}$ and r_0 is possibly negative. The part with negative r contributes to the main part. We rewrite (4.14) further by using $1/(q_1)_\infty = 1 + O(q_1)$ inside of $C(ah', \frac{lc}{c_1}, c; q_1)$. So, the main contribution of

$$e^{-\frac{\pi ia^2h'k_1}{cc_1}+\frac{2\pi ih'la}{cc_1}+\frac{\pi}{12kz}} q_1^{\frac{-l^2}{2c_1^2}} C\left(ah', \frac{lc}{c_1}, c; q_1\right)$$

comes from the following expression:

$$\pm\frac{i}{2} e^{-\frac{\pi ia^2h'k_1}{cc_1}+\frac{2\pi ih'la}{cc_1}+\frac{\pi}{12kz}} q_1^{\frac{-l^2}{2c_1^2}} (-1)^m q_1^{\frac{m}{2}(m+1)\pm\frac{l}{2c_1}+rm\pm\frac{rl}{c_1}} e^{\mp\frac{\pi iah'}{c}\mp\frac{2\pi iah'r}{c}}. \tag{4.15}$$

From this is possible to split the expression into the roots of unity and to the part that depends on the variable z. The roots of unity look like:

$$\exp\left(\frac{2\pi ih'}{k}\left(-\frac{a^2k_1k}{2cc_1} + \frac{lak}{cc_1} - \frac{l^2}{2c_1^2} + rm \pm \frac{rl}{c_1} \mp \frac{kra}{c} + \frac{m(m+1)}{2} \pm \frac{l}{2c_1} \mp \frac{ak}{2c}\right)\right).$$

Rewriting the expression in the second bracket, using the congruence condition $l \equiv ak_1 \pmod{c_1}$, $l^2 \pm l$ is always even and rearranging the sum it is possible to show that the contribution of the roots of unity looks like $\exp\left(\frac{2\pi ih'm_{r,s}}{k}\right)$ where $m_{r,s}$ is a sequence in $\mathbb{Z}$. The interesting part happens for $\exp\left(\frac{\pi}{kz}T\right)$, where T is defined in the following way:

$$T := \frac{l^2}{c_1^2} + \frac{1}{12} - 2rm \mp 2r\frac{l}{c_1} - m(m+1) \mp \frac{l}{c_1}.$$

This part contributes to the circle method exactly if $T > 0$ which is equivalent to $-T < 0$. Firstly we treat the case with the plus sign in (4.15). By multiplying by (-1) and assuming $m > 0$ it is possible to show

$$-T = -\frac{l^2}{c_1^2} - \frac{1}{12} + 2rm + 2r\frac{l}{c_1} + m(m+1) + \frac{l}{c_1} > -1 - \frac{1}{12} + 2 + 1 > 1 > 0.$$

So, $-T > 0$ and this gives for all r no contribution to the Circle Method. For $m = 0$ define r to be a solution to the following inequality:

$$-\frac{l^2}{c_1^2} - \frac{1}{12} + 2r\frac{l}{c_1} + \frac{l}{c_1} < 0.$$

This is equivalent to $T > 0$ and so this contributes to the main part in the Circle Method. Now choosing the minus sign in the equation (4.15) that becomes

$$T = -\frac{l^2}{c_1^2} - \frac{1}{12} + 2rm - 2r\frac{l}{c_1} + m(m+1) - \frac{l}{c_1}.$$

Assuming that $m \geq 2$, it is possible to show that $-T > 3 > 0$ and this gives no contribution. For $m = 1$ we define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := -x^2 - x(1+2r) - \frac{1}{12} + 2 + 2r.$$

Calculating the maximum and computing the values of the function we see that on the boundary the function is negative, i.e., $f(1) = -\frac{1}{12} < 0$. Thus this contributes to the main part in the Circle Method. So there are two contributions coming from each of the two terms of $C(ah', \frac{lc}{c_1}, c; q_1)$. The first one comes from the first sum, if $m = 0$, and this contributes with

$$\frac{i}{2} e^{-\frac{\pi i a^2 h' k_1}{cc_1} + \frac{2\pi i h' la}{cc_1} - \frac{\pi i h' a}{c} + \frac{\pi}{12kz}} q_1^{\frac{-l^2}{2c_1^2} + \frac{l}{2c_1}} \sum_{\substack{r \geq 0 \\ \delta^+_{a,c,k,r} > 0}} e^{-\frac{2\pi i h' ar}{c}} q_1^{\frac{rl}{c_1}},$$

where $\delta^+_{a,c,k,r} = \frac{l^2}{2c_1^2} + \frac{1}{24} - (r + 1/2)\frac{l}{c_1}$. The second contribution comes from the second sum, if $m = 1$, and this contributes with

$$e^{-\frac{\pi i a^2 h' k_1}{cc_1} + \frac{2\pi i h' la}{cc_1} + \frac{\pi i h' a}{c} + \frac{\pi}{12kz}} q_1^{\frac{-l^2}{2c_1^2} + \frac{l}{2c_1} + 1} \sum_{\substack{r \geq 0 \\ \delta^-_{a,c,k,r} > 0}} e^{\frac{2\pi i h' ar}{c}} q_1^{r\left(1 - \frac{l}{c_1}\right)},$$

where $\delta^-_{a,c,k,r} = \frac{l^2}{2c_1^2} - \frac{23}{24} - r\left(1 - \frac{l}{c_1}\right) + \frac{l}{2c_1}$. Thus we have for the leading order of Σ_2 the following expression:

$$2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c \nmid k \\ i \in \{-,+\}}} (-1)^{ak+l} \sum_h \omega_{h,k} e^{\frac{2\pi i}{k}\left(-nh + m^i_{a,c,k,r} h'\right)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{2\pi}{kz}\delta^i_{a,c,k,r}} d\Phi.$$

Now it is possible to rewrite the sum over k into the sum where the k's have the same values for c_1 and l and thus the $\delta^i_{a,c,k,r}$ are constant in each class and the condition $\delta^i_{a,c,k,r} > 0$ is independent of k in each class. Moreover it is clear as c_1 and l are finite numbers and for arbitrary large r there do not exist any solutions to $\delta^i_{a,c,k,r} > 0$, so that there are only finitely many solution to the inequality. That means it is possible to split the sum over r into positive $\delta^i_{a,c,k,r}$, which by the above argument is a finite sum and into negative $\delta^i_{a,c,k,r}$, where the part with negative $\delta^i_{a,c,k,r}$ contributes to the error. By symmetrizing the integral and now using Lemma 4.4 (1) it is possible to bound all the terms exactly the same way we did for Σ_1:

$$\Sigma_2 = 2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{a,c,k,r}>0 \\ i\in\{-,+\}}} (-1)^{ak+l} \sum_h \omega_{h,k} e^{\frac{2\pi i}{k}\left(-nh+m^i_{a,c,k,r}h'\right)} \times \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{2\pi}{kz}\delta^i_{a,c,k,r}} d\Phi + O(n^\varepsilon).$$

Another way to argue is to plug in the expansion (4.14) directly and split the sum over r into positive and non-positive powers. Then by our analysis above we see that the coefficients of the expansion do not depend on a and k, because the roots of unity are all expressions in l/c_1. So we can bound all the terms with k by using Lemma 4.4 and as the $b(r,s)$ grow exactly like the partition function with r and so smaller than $\exp(-\frac{\pi r}{12kz})$ the product of theses two quantities can also be bounded by a constant. So at the end we have

$$\Sigma_2 = 2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{a,c,k,r}>0 \\ i\in\{+,-\}}} D_{a,c,k}(-n, m^i_{a,c,k,r}) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{2\pi}{kz}\delta^i_{a,c,k,r}} d\Phi + O(n^\varepsilon) \tag{4.16}$$

and by the analysis before

$$\Sigma_1 = i\sum_{c\mid k} \tilde{B}_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} d\Phi + O(n^\varepsilon). \tag{4.17}$$

To finish the proof we have to evaluate integrals of the following form:

$$I_{k,t} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{t}{z}\right)} d\Phi.$$

Substituting $z = k/n - ik\Phi$ gives

$$I_{k,t} = \frac{1}{ki}\int_{k/n-\frac{i}{N}}^{k/n+\frac{i}{N}} z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{t}{z}\right)} dz.$$

We introduce the circle through the complex conjugated points $k/n \pm i/N$ which is tangent to the imaginary axis at 0 and denote this circle by Γ. Writing a complex number on the circle by $z = x + iy$ we have as a circle equation $x^2 + y^2 = \alpha x$ with $\alpha = \frac{k}{n} + \frac{n}{N^2 k}$. On the smaller arc that is the arc going from the two complex conjugated points through zero we clearly have $\mathrm{Re}(z) \leq \frac{k}{n}$, $\mathrm{Re}\left(z^{-1}\right) < k$ and $2 > \alpha > \frac{1}{k}$. From evaluating the integral on the smaller arc we get that the integral is bounded by $O(n^{-\frac{1}{8}})$[1]. So it possible to change the path of integration to the larger arc because we have no singularities enclosed by the larger arc anymore. So by Cauchys Theorem we obtain:

$$I_{k,t} = \int_\Gamma z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{t}{z}\right)} dz + O(n^{-\frac{1}{8}}).$$

Transforming the circle to a straight line by $s = \frac{2\pi r}{kz}$ gives:

$$I_{k,t} = \frac{2\pi}{k}\left(\frac{2\pi t}{k}\right)^{1/2} \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma+i\infty} s^{-\frac{3}{2}} e^{s+\frac{\beta}{s}} ds + O\left(n^{-\frac{1}{8}}\right),$$

where $\gamma \in \mathbb{R}$ and $\beta = \frac{\pi^2 t}{6k^2}(24n-1)$. By the Hankel integral formula [7] we get

$$I_{k,t} = \frac{4\sqrt{3}}{\sqrt{k(24n-1)}} \sinh\left(\sqrt{\frac{2t(24n-1)}{3}}\frac{\pi}{k}\right) + O\left(n^{-\frac{1}{8}}\right).$$

Now at the end we have

$$\Sigma_2 + \Sigma_1 = 2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{a,c,k,r}>0 \\ i\in\{+,-\}}} D_{a,c,k}\left(-n, m^i_{a,c,k,r}\right) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{2\pi}{kz}\delta^i_{a,c,k,r}} d\Phi$$
$$+ i\sum_{c|k} \widetilde{B}_{a,c,k}\left(-n,0\right) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} d\Phi + O(n^{\varepsilon})$$

finishing the proof of Theorem 4.5 after inserting the expressions for $I_{k,t}$. □

Now let $M(a, c; n)$ be the number of partitions of n with crank equal to a modulo c. From the Theorem 4.5 it is now easy to give asymptotics for the functions $M(a, c; n)$:

[1] we will make this statement more precise in the next section, see (4.24)

Corollary 4.6. *Let $0 \leq a < c$ with c and odd integer. Then we have:*

$$\begin{aligned}M(a,c;n) =& \frac{2\pi}{c\sqrt{24n-1}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6k}\right)\\ &+ \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} \left(\frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{c|k} \frac{\tilde{B}_{j,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right.\\ &\left.+ \frac{8\sqrt{3}\sin\left(\frac{\pi j}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{j,c,k,r}>0 \\ i\in\{+,-\}}} \frac{D_{j,c,k}(-n, m^i_{j,c,k,r})}{\sqrt{k}} \sinh\left(\sqrt{\frac{2\delta^i_{j,c,k,r}(24n-1)}{3}}\frac{\pi}{k}\right)\right)\\ &+ O(n^{\varepsilon}).\end{aligned}$$

Proof. The proof follows easily from the following identity:

$$\sum_{n=0}^{\infty} M(a,c;n)q^n = \frac{1}{c}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} C(\zeta_c^j;q). \tag{4.18}$$

Plugging in the coefficients for $C(\zeta_c^j;q)$, the Rademacher formula and comparing termwise in the q-expansion shows the corollary. To prove (4.18) we notice that for the right hand side of (4.18) the following holds

$$\begin{aligned}\frac{1}{c}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} C(\zeta_c^j;q) =& \frac{1}{c}\sum_{j \pmod c} \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)\zeta_c^{mj}\zeta_c^{-aj}q^n\\ =& \frac{1}{c}\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)\left(\sum_{j \pmod c} \zeta_c^{(m-a)j}\right) q^n,\end{aligned}$$

where the first term on the left hand side of the first equation corresponds to the case $j=0$ on the right hand side. Using the orthogonality of the roots of unity

$$\sum_{j \pmod c} \zeta_c^{rj} = \begin{cases} 0 & r \not\equiv 0 \pmod c, \\ c & r\equiv 0 \pmod c,\end{cases}$$

we obtain

$$\begin{aligned}\frac{1}{c}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} C(\zeta_c^j;q) =& \sum_{n\geq 0} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv a \pmod c}} M(m,n)q^n\\ =& \sum_{n\geq 0} M(a,c;n)q^n,\end{aligned}$$

and this finishes the proof of (4.18). □

4.3 Inequalities of crank differences

Now we can prove our Theorem 1.1 by bounding all the error terms that occur in the circle method explicitly. To aid the reader, we repeat the statement.

Theorem 4.7. *Let $M(j,c;n)$ be the number of partitions of n with crank congruent to j modulo c. Let $0 \leq a < b \leq \frac{c-1}{2}$ and let $c > 11$ be an odd integer, then we have for $n > N_{a,b,c}$, where $N_{a,b,c}$ is an explicit constant, the inequality:*

$$M(a,c;n) > M(b,c;n).$$

We use Theorem 4.5, with a change of variables that does not effect the theorem. For completeness we repeat the needed quantities. If $c \nmid k$ we redefine $0 < l < c$ to be the unique solution to the congruence $l \equiv jk \pmod{c}$ and let h, k be coprime integers. Define h' by $hh' \equiv -1 \pmod{k}$ if k is odd and by $hh' \equiv -1 \pmod{2k}$ if k is even. Let $\omega_{h,k}$ be the multiplier occurring in the transformation law of the partition function $p(n)$ which satisfies $|\omega_{h,k}| = 1$, see (2.5). We make a technical assumption throughout the proof, that c is prime. The bounds for c non-prime would differ slightly, but for simplicity we restrict to that case. Moreover, we have, for $n, m \in \mathbb{Z}$, the following sums of Kloosterman type

$$\widetilde{B}_{j,c,k}(n,m) = (-1)^{jk+1} \sin\left(\frac{\pi j}{c}\right) \sum_{h \pmod{k}^*} \frac{\omega_{h,k}}{\sin\left(\frac{\pi j h'}{c}\right)} \cdot e^{-\frac{\pi i j^2 h'}{c}} \cdot e^{\frac{2\pi i}{k}(nh+mh')}$$

if $c|k$, and

$$D_{j,c,k}(n,m) = (-1)^{jk+l} \sum_{h \pmod{k}^*} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(nh+mh')}.$$

Here the sums run through all primitive residue classes modulo k. Moreover, for $c \nmid k$, let

$$\delta^i_{j,c,k,r} = \begin{cases} -\left(\frac{1}{2}+r\right)\frac{l}{c} + \frac{1}{2}\left(\frac{l}{c}\right)^2 + \frac{1}{24} & \text{if } i = +, \\ \frac{l}{2c} + \frac{1}{2}\left(\frac{l}{c}\right)^2 - \frac{23}{24} - r\left(1 - \frac{l}{c}\right) & \text{if } i = -, \end{cases}$$

and

$$m^i_{j,c,k,r} = \begin{cases} \frac{1}{2c^2}\left(-j^2k^2 + 2ljk - jkc - l^2 + lc - 2jrkc + 2lcr\right) & \text{if } i = +, \\ \frac{1}{2c^2}\left(-j^2k^2 + 2ljk - jkc - l^2 + 2c^2 r \right. & \text{if } i = -. \\ \qquad \left. -2lrc + 2jrkc + 2lc + 2c^2 - jkc\right) & \end{cases}$$

Note that due to the redefinition the quantities have changed. So now we can give the Proof of Theorem 4.7.

Proof. Firstly, we define

$$\rho_j(a,b,c) := \left(\cos\left(\frac{2\pi a j}{c}\right) - \cos\left(\frac{2\pi b j}{c}\right)\right).$$

It is possible to write the crank differences as (see (4.18))

$$\sum_{n} \left(M(a,c;n) - M(b,c;n)\right) q^n = \frac{2}{c} \sum_{j=1}^{\frac{c-1}{2}} \rho_j(a,b,c) C\left(\zeta_c^j; q\right), \tag{4.19}$$

where we defined $\zeta_c = e^{\frac{2\pi i}{c}}$. We deduce the asymptotic behavior of (4.19) using Theorem 4.5. So we insert Theorem 4.5 into the equation (4.19) and get directly

$$M(a,c;n) - M(b,c;n) = \sum_{j=1}^{\frac{c-1}{2}} \left(S_j(a,b,c;n) + \sum_{i\in\{-,+\}} T_j^i(a,b,c;n) + O\left(n^{\varepsilon}\right) \right) \tag{4.20}$$

where we have

$$S_j(a,b,c;n) := \rho_j(a,b,c) \frac{8\sqrt{3}i}{c\sqrt{24n-1}} \sum_{\substack{1\le k\le\sqrt{n} \\ c|k}} \frac{\widetilde{B}_{j,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right), \tag{4.21}$$

$$\begin{aligned} T_j^i(a,b,c;n) := \rho_j(a,b,c) &\frac{16\sqrt{3}\cdot \sin\left(\frac{\pi j}{c}\right)}{c\sqrt{24n-1}} \\ &\times \sum_{\substack{1\le k\le\sqrt{n} \\ c\nmid k \\ r\ge 0 \\ \delta^i_{j,c,k,r}>0}} \frac{D_{j,c,k}(-n, m^i_{j,c,k,r})}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{2\delta^i_{j,c,k,r}(24n-1)}}{\sqrt{3}k}\right). \end{aligned} \tag{4.22}$$

This looks similar to the rank case treated in [6]. Firstly, we detect the main contribution coming from sinh. It is a strictly increasing function and so we have to detect the largest argument. In S_j the condition $c \mid k$ has to be fulfilled and so the largest argument occurs if $k = c$. We show using $\delta^+_{j,c,k,r}$ with $k = 1, r = 0$ and $j = 1$ that the argument of the hyperbolic sine of S_j is always smaller then the argument of the hyperbolic sine of T_j^i. As $c \mid k$ we have to show that:

$$\sqrt{\frac{1}{3c^2} - \frac{1}{3c} + \frac{1}{36}} > \frac{1}{6c}.$$

By squaring both sides and solving a polynomial equation we see that this is equivalent to $c > 11$. So, for $c > 11$ the main contribution to the crank differences comes from T_j^i, so we have to detect the largest argument occurring in the T_j^i. To see what is the largest argument we compare $\delta^+_{j,c,k,r}$ and $\delta^-_{j,c,k,r}$. First of all it is clear that the largest argument occurs if $r = 0$ for fixed j, k. So we set $r = 0$ and see that $\delta^-_{j,c,k,0} < \delta^+_{j,c,k,0}$, because

$0 < l < c$. Assuming $\frac{l}{c} < \frac{1}{2}$, which we may do by the symmetry of the parabola in the argument l/c we see that $\delta^{+}_{j,c,k,0} \leq \delta_0 := \frac{1}{2c^2} + \frac{1}{24} - \frac{1}{2c}$. For $k = 1$ we get $l = j$ and so if $j \neq 1$ we have $\delta^{i}_{j,c,1,0} < \delta_0$ if $j \neq 1$. This implies that the biggest argument occurs for $k = 1$, $r = 0$ and $j = 1$. So the main contribution is

$$T_1^{+}(a,b,c,n) = \frac{2}{c}\rho_1(a,b;c)\frac{8\sqrt{3}\sin\left(\frac{\pi}{c}\right)}{\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{2\delta_0(24n-1)}}{\sqrt{3}}\right).$$

From this it is already possible to deduce the theorem, because for sufficiently large n the main contribution comes from T_1^{+}. The sign of T_1^{+} is determined by the sign of the ρ_1 which is positive since we have $0 < \frac{\pi}{c} < \frac{\pi}{11} < \frac{\pi}{2}$ which implies that in this range the $\cos(\pi x/c)$ is decreasing. Thus for $0 < a < b < \frac{c-1}{2}$ we have that $\cos\left(\frac{\pi a}{c}\right) > \cos\left(\frac{\pi b}{c}\right)$ and that explains why ρ_1 is positive. So for sufficiently large n we have $N(a,c;n) > N(b,c;n)$. The next step is to clarify what sufficiently large exactly means by bounding all the error terms explicitly in terms of c and n, beginning with the contributions of S_j, T_j^{i} for $j > 1$ and T_1^{-}.

Bounding the contributions of S_j:

For S_j it is easily seen:

$$\begin{aligned}|S_j(a,b,c)| \leq& \frac{8|\rho_j(a,b,c)|\sqrt{3}}{c\sqrt{24n-1}}\sum_{\substack{1\leq k\leq\sqrt{n}\\ c|k}}\frac{|\tilde{B}_{j,c,k}(-n,0)|}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)\\ \leq& \frac{8|\rho_j(a,b,c)|\sqrt{3}}{c\sqrt{24n-1}}\left|\sin\left(\frac{\pi j}{c}\right)\right|\sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right)\sum_{\substack{1\leq k\leq\sqrt{n}\\ c|k}}\frac{1}{\sqrt{k}}\sum_{\substack{h=1\\ (h,k)=1}}^{k}\frac{1}{|\sin(\frac{\pi h}{c})|}.\end{aligned}$$

Here we used that the biggest argument in the hyperbolic sine occurs if $c = k$ and that h and h' run over the same primitive residue classes modulo k and so we changed in the summation the argument of the sine from $jh' \to h$ and with that to another representative of the equivalence class. Here it is important to note that we are using that c is prime. We further used $|\exp(\pi i x)| = 1$ for $x \in \mathbb{R}$. The inner sum can be further estimated by

$$\sum_{\substack{h=1\\ (h,k)=1}}^{k}\frac{1}{|\sin(\frac{\pi h}{c})|} \leq \frac{2k}{c}\sum_{h=1}^{\frac{c-1}{2}}\frac{1}{|\sin(\frac{\pi h}{c})|} \leq \frac{2k}{\pi}\sum_{h=1}^{\frac{c-1}{2}}\frac{1}{h\left(1-\pi^2/24\right)} \leq \frac{2k\left(1+\log\left(\frac{c-1}{2}\right)\right)}{(1-\pi^2/24)}. \quad (4.23)$$

In the first inequality it used that the absolute value of the sine is not bigger than 1 and that c is odd. In the second inequality it is used that $\sin(x) > x - x^3/6$ for $x < 1$ and we used that the summation runs to $(c-1)/2$ by bounding in the x^3-term h by $c/2$. In the last step we have used $\sum_{h=1}^{\frac{c-1}{2}} h^{-1} = 1 + \sum_{h=2}^{\frac{c-1}{2}} h^{-1}$ and estimated the sum by an integral.

We now have:

$$|S_j(a,b,c)| \leq \frac{16|\rho_j(a,b,c)|\sqrt{3}}{c\sqrt{24n-1}} \frac{\left|\sin\left(\frac{\pi j}{c}\right)\right| \left(1+\log\left(\frac{c-1}{2}\right)\right)}{\pi\left(1-\frac{\pi^2}{24}\right)} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right) \sum_{\substack{1\leq k\leq\sqrt{n} \\ c|k}} k^{1/2}$$
$$\leq \frac{64n^{3/4}\left(1+\log\left(\frac{c-1}{2}\right)\right)}{\sqrt{24n-1}c^2\sqrt{3}\pi\left(1-\frac{\pi^2}{24}\right)} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right).$$

Here it is used that $|\rho_j(a,b,c)| \leq 2$ and the following estimation of the sum:

$$\sum_{\substack{1\leq k\leq\sqrt{n} \\ c|k}} k^{\frac{1}{2}} \leq c^{\frac{1}{2}} \sum_{1\leq j\leq\lfloor\frac{N}{c}\rfloor} j^{\frac{1}{2}} \leq c^{\frac{1}{2}} \int_1^{\lfloor\frac{N}{c}\rfloor} x^{\frac{1}{2}}dx \leq \frac{2}{3c}n^{\frac{3}{4}}.$$

Next we want to bound the T_j^i for $j \geq 2$ and T_1^-. Firstly notice that is possible to bound $D_{j,c,k}(-n, m_{j,c,k,r}^i)$ trivially by k. The reason is that we sum over roots of unity and the sum runs over all primitive residue classes modulo k. That explains the bound. Moreover we can bound the hyperbolic sine by the positive part because $\sinh(x) = (e^x - e^{-x})/2$.
Bounding the contributions of T_j^i for $j \geq 2$
Using the exponential function we can bound the terms in the sum of T_j^i in the following way (here for $k \geq 2$):

$$\frac{D_{j,c,k}(-n, m_{j,c,k,r}^i)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{2\delta_{j,c,k,r}^i(24n-1)}}{\sqrt{3}k}\right) \leq \frac{k^{\frac{1}{2}}}{2} e^{\frac{\pi\sqrt{2\delta_0(24n-1)}}{2\sqrt{3}}}.$$

The number of r satisfying the condition $\delta_{j,c,k,r}^i > 0$ can be bounded in terms of c: First of all we find the number of solutions to the equation as a function of l for fixed c. Now define the function $g_c : [1, c-1] \to \mathbb{R}$ by $f(l) = \frac{l}{2c} + \frac{1}{2} + \frac{c}{24l}$. We added one to the equation to afterwards take the Gauss bracket. The largest values occur on the boundary of the interval, namely $l = 1$ and $l = c-1$, as the function has its minimum in the interior of the interval and is a continuous function. For $c > 11$ the function take its maximum for $l = 1$ and may be bounded by $\frac{(c+18)}{24}$. For the other cases we checked by hand that the number of solutions to the equation $\delta > 0$, which is $\lfloor\frac{l}{2c} + \frac{1}{2} + \frac{c}{24l}\rfloor$, can be bound by $\frac{(c+18)}{24}$, where we inserted the maximizing $l = c-1$. Thus we can bound T_j^i for $k \geq 2$ by the following expression :

$$\frac{4(c+18)}{3\sqrt{3}c\sqrt{24n-1}} n^{3/4} e^{\pi\frac{\sqrt{2\delta_0(24n-1)}}{2\sqrt{3}}}.$$

Since $\delta_{j,c,,1,0} < \delta_0$ is decreasing in j, for $j > 1$ we bound the $k = 1$ contribution by the argument of $j = 2$

$$\frac{2(c+18)}{\sqrt{3}c\sqrt{24n-1}} e^{\pi\frac{\sqrt{2\delta_{2,c,1,0}(24n-1)}}{\sqrt{3}}}.$$

Before coming to the error terms of the Circle Method we have to bound the contribution of T_1^-.

Contribution of T_1^-:

By the same analysis, it is possible to bound this term by a similar expression as the ones before. By bounding the sinh by the exponential function, using $\rho \leq 2$ and showing $\delta_{1,c,k,r}^-$ is smaller then $\delta_{2,c,1,0}^+$ and so choosing the right argument in the exponential function T_1^- can be bounded by

$$\frac{2(c-1)}{\sqrt{3}c\sqrt{24n-1}} e^{\pi \frac{\sqrt{2\delta_{2,c,1,0}^+(24n-1)}}{\sqrt{3}}}.$$

Here we bounded the number of solutions to $\delta_{j,c,k,r}^- > 0$ by $\frac{c-1}{24}$ which is a rough bound, but makes sense for all odd c (We could bound the number of solutions stricter, but we would have had to put an extra condition on c or introduce a heavyside function that reflects the fact the there are no solutions for $c < 23$). Now we want to make the $O(n^\varepsilon)$-term in the Theorem 4.5 explicit. We had $\tilde{A}\left(\frac{j}{c}; n\right) = \Sigma_1 + \Sigma_2$ with $\Sigma_1 = S_1 + S_2$, where $S_2 =: S_{err}$ contributes to the error in the circle method. See Theorem 4.5 for details.

Contribution of the error of Σ_1

We again use the bounds $|z| \geq \frac{k}{n}$, $\text{Re}(z) = \frac{k}{n}$, and $C(\zeta_c^h, q_1) = 1 + C(\zeta_c^h, q_1) - 1$ to obtain:

$$S_{err} \leq 2 \left| \sin\left(\frac{\pi j}{c}\right) \right| e^{2\pi} \sum_{\substack{k \leq N \\ c|k}} k^{-\frac{3}{2}} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} \frac{1}{\sin\left(\frac{\pi h}{c}\right)} \max_z \left| e^{\frac{\pi}{12kz}} \left(C(\zeta_c^h, q_1) - 1\right) \right|.$$

Here we used again a change of variables $jh' \to h$. The next step is to estimate $|e^{\frac{\pi}{12kz}}(C(\zeta_c^h, q_1) - 1)|$. Remember that (3.1):

$$\begin{aligned} C(\zeta_c^h, q_1) &= \frac{1}{(q_1)_\infty} + \frac{(1-\zeta_c^h)}{(q_1)_\infty} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{1-\zeta_c^h q_1^m} \\ &= \frac{1}{(q_1)_\infty} + \frac{(1-\zeta_c^h)}{(q_1)_\infty} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{(1-\zeta_c^h q_1^m} + \frac{(1-\zeta_c^{-h})}{(q_1)_\infty} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{1-\zeta_c^{-h} q_1^m}. \end{aligned}$$

From this it is easily seen that $|e^{\frac{\pi}{12kz}}(C(\zeta_c^h, q_1) - 1)|$ may be bounded by

$$e^{\frac{\pi}{24}} \sum_{n=1}^{\infty} p(n) e^{-\pi n} + e^{\frac{\pi}{24}} \sum_{n=0}^{\infty} p(n) e^{-\pi n} \sum_{m=1}^{\infty} e^{-\pi m(m+1)/2} \left| \frac{1-\zeta_c^h}{1-\zeta_c^h q_1^m} + \frac{1-\zeta_c^{-h}}{1-\zeta_c^{-h} q_1^m} \right|.$$

Note that the summation index n on the first term starts with 1 where one the second sum with 0, that means we absorbed the -1 into the sum. We bound the term further by noting that

$$\left| \frac{1-\zeta_c^h}{1-\zeta_c^h q_1^m} + \frac{1-\zeta_c^{-h}}{1-\zeta_c^{-h} q_1^m} \right| \leq 2 \frac{1 + |\cos\left(\frac{\pi}{c}\right)|}{1-e^{-\pi m}}.$$

Defining $c_2 := \sum_{n=1}^{\infty} p(n)e^{-\pi n}$ and $c_1 := \sum_{m=1}^{\infty} \frac{e^{-\frac{\pi m(m+1)}{2}}}{1-e^{-\pi m}}$ it is possible to bound S_{err} by:

$$2e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| e^{\frac{\pi}{24}} \left(c_2 + 2\left(1 + \left|\cos\left(\frac{\pi}{c}\right)\right|\right) c_1(1+c_2)\right) \sum_{\substack{k\leq N\\ c|k}} k^{-\frac{3}{2}} \sum_{\substack{h=1\\ (h,k)=1}}^{k-1} \frac{1}{|\sin\left(\frac{\pi h}{c}\right)|}.$$

Using (4.23) and estimating the sum over k by an integral expression we obtain after evaluating the integral the upper bound for S_{err}:

$$\frac{2e^{2\pi+\frac{\pi}{24}} \left|\sin\left(\frac{\pi j}{c}\right)\right| \left(c_2 + 2\left(1 + |\cos\left(\frac{\pi}{c}\right)|\right) c_1(1+c_2)\right) n^{\frac{1}{4}} \left(1 + \log\left(\frac{c-1}{2}\right)\right)}{\pi\left(1 - \frac{\pi^2}{24}\right) c}.$$

We continue by repeating this procedure for the $O(n^{\varepsilon})$-term of Σ_2:
Contribution of the error of Σ_2
The error correspond to the terms where $\delta^i_{j,c,k,r}$ is not positive. Therefore we define $\tilde{M}(jh', l, c; q_1)$ to be the terms with positive exponents in the q_1-expansion of

$$e^{\frac{\pi}{12kz}} q_1^{-\frac{l^2}{2c^2}} C(jh', l, c; q_1)$$

and bound $\tilde{M}$. Writing for the entire sum T_{err}, using the usual bounds of $|z|$ and doing the usual change of variable $jh' \to h$ the following bound is easily obtained:

$$T_{err} \leq 8e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_{\substack{h,k\\ c\nmid k}} k^{-\frac{3}{2}} \max_z \tilde{M}(h, l, c; q_1).$$

The difficult bounds come from the function $\tilde{M}$. Remember that $C(h, l, c; q_1)$ has the following expansion

$$C(h,l,c,q_1) = \frac{i\zeta_{2c}^{-h} q_1^{\frac{l}{2c}}}{2(q_1)_\infty \left(1 - \zeta_c^{-h} q_1^{\frac{l}{c}}\right)} + \frac{i\zeta_{2c}^{h} q_1^{-\frac{l}{2c}+1}}{2(q_1)_\infty \left(1 - \zeta_c^{h} q_1^{1-\frac{l}{c}}\right)}$$
$$-\frac{i\zeta_{2c}^{h} q_1^{-\frac{l}{2c}}}{2(q_1)_\infty} \sum_{m=2}^{\infty} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{\left(1 - \zeta_c^{h} q_1^{m-\frac{l}{c}}\right)} + \frac{i\zeta_{2c}^{-h} q_1^{\frac{l}{2c}}}{2(q_1)_\infty} \sum_{m=1}^{\infty} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{\left(1 - \zeta_c^{-h} q_1^{m-\frac{l}{c}}\right)}.$$

We bound the contributions of $\tilde{M}$ term by term beginning with the first one. We rewrite the denominator by a geometrical sum and by $\frac{1}{(q_1)_\infty} = \sum_{m=0} p(m) q_1^m$. So we have

$$\frac{i\zeta_{2c}^{-h} q_1^{\frac{l}{2c}}}{2(q_1)_\infty \left(1 - \zeta_c^{-h} q_1^{\frac{l}{c}}\right)} = \frac{i}{2} \zeta_{2c}^{-h} q_1^{\frac{l}{2c}} \sum_{m=0} p(m) q_1^m \sum_{r=0} \zeta_c^{-hr} q_1^{\frac{rl}{c}}.$$

Maximizing $|z|$ and taking the absolute value of this expression, noting that for $m=0$ not all the terms correspond to the error but for all higher m they do, and using that $\mathrm{Re}(z^{-1}) \geq \frac{k}{2}$, we gain the following contribution to $\tilde{M}$

$$\frac{1}{2}e^{-\frac{\pi l}{2c}+\frac{l^2}{2c^2}+\frac{\pi}{24}}\left(\sum_{r\geq r_0} e^{-\frac{\pi l r}{c}} + \sum_{r=0} e^{-\frac{\pi l r}{c}} \sum_{m=1} p(m)e^{-\pi m}\right),$$

where $r_0 := \lceil -\frac{1}{2} + \frac{l}{2c} + \frac{c}{24l} \rceil$. Using

$$\sum_{r\geq r_0} e^{-\frac{\pi l r}{c}} = \frac{e^{-\frac{\pi r_0 l}{c}}}{\left(1-e^{\frac{-\pi l}{c}}\right)} \quad , \quad c_2 = \sum_{m=1} p(m)e^{-\pi m}$$

and the usual geometrical series we can bound the term further by

$$\frac{e^{-\frac{\pi l}{2c}+\frac{\pi l^2}{2c^2}+\frac{\pi}{24}-\frac{\pi r_0 l}{c}}}{2\left(1-e^{\frac{-\pi l}{c}}\right)} + \frac{e^{-\frac{\pi l}{2c}+\frac{l^2}{2c^2}+\frac{\pi}{24}}c_2}{2\left(1-e^{\frac{-\pi l}{c}}\right)} = \frac{e^{\frac{\pi l}{c}\left(-\frac{1}{2}+\frac{l}{2c}+\frac{c}{24l}-r_0\right)}\left(1+c_2 e^{\frac{\pi r_0 l}{c}}\right)}{2\left(1-e^{\frac{-\pi l}{c}}\right)}$$
$$\leq \frac{\left(1+c_2 e^{\pi\delta_0}\right)}{2\left(1-e^{-\frac{\pi}{c}}\right)}.$$

The second sum can be bounded exactly the same way. In the third and fourth summand all the terms will contribute to the error as was shown in the Theorem 4.5. We obtain

$$\left| -\frac{i\zeta_{2c}^h q_1^{-\frac{l}{2c}+\frac{l^2}{2c^2}} e^{\frac{\pi}{12kz}}}{2(q_1)_\infty} \sum_{m=2}^{\infty} \frac{(-1)^m q_1^{\frac{m(m+1)}{2}}}{\left(1-\zeta_c^h q_1^{m-\frac{l}{c}}\right)} \right| \leq \frac{1}{2}e^{-\frac{\pi l}{2c}+\frac{\pi}{24}+\frac{\pi l^2}{2c^2}}(c_2+1)\sum_{m=2} \frac{e^{-\frac{\pi m(m+1)}{2}}}{1-e^{-\pi m+\pi\frac{l}{c}}}$$
$$\leq \frac{1}{2}e^{\pi\delta_0}(c_2+1)c_3,$$

where

$$c_3 := \sum_{m=2} \frac{e^{-\frac{\pi m(m+1)}{2}}}{1-e^{-\pi m+\pi}}.$$

We proceed by repeating this step for the fourth sum in the $C(h,l,c;q_1)$ expansion. As there is nothing new we omit the step and just give the bound for $\tilde{M}$. It is

$$e^{\pi\delta_0}c_1(1+c_2).$$

So at the end the error terms coming from the function $\tilde{M}$ can be bounded by the following function $f(c)$ that just depends on c

$$f(c) := \frac{1+c_2 e^{\pi\delta_0}}{\left(1-e^{-\frac{\pi}{c}}\right)} + e^{\pi\delta_0}c_1(1+c_2) + \frac{1}{2}e^{\pi\delta_0}(c_2+1)c_3.$$

So the error can be bounded by

$$T_{err} \leq 8e^{2\pi} f(c) \sum_{\substack{h,k \\ h \nmid k}} k^{-\frac{3}{2}} \leq 16e^{2\pi} f(c) n^{\frac{1}{4}} \left| \sin\left(\frac{\pi j}{c}\right) \right| .$$

Here it used that the sum over h can be bounded trivially by k because the sum runs over the residue class modulo k and we estimated the sum over k by an integral expression. As a next step we want to bound the contributions that come from symmetrizing the integral. In Theorem 4.5 we have shown that it is possible to split the integration over the Farey arcs by making the integral bounds symmetric and showed that the needed integral to correct this symmetrization will contribute to the error. These terms have to be made explicit.

Symmetrizing

We had used

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{-\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} - \int_{\frac{1}{k(k+k_1)}}^{\frac{1}{kN}} .$$

Plugging into the first term of the main contribution we are left with the following error term:

$$S_{1err} = -i \sin\left(\frac{\pi j}{c}\right) \sum_{\substack{h,k \\ c|k}} \frac{\omega_{h,k}(-1)^{ak+1}}{\sin\left(\frac{\pi j h'}{c}\right)} e^{-\frac{\pi i j^2 k h'}{c^2} - \frac{2\pi i h n}{k}}$$
$$\times \left(\int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} + \int_{\frac{1}{k(k+k_1)}}^{\frac{1}{kN}} \right) z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}} d\Phi.$$

Taking the absolute value and the usual bound of $|z|$ we can bound the term by

$$|S_{1err}| \leq \left| \sin\left(\frac{\pi j}{c}\right) \right| e^{2\pi + \frac{\pi}{12}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ c|k}} \frac{n^{\frac{1}{2}}}{k^{\frac{1}{2}}} \sum_{\substack{h \\ (h,k)=1}} \frac{1}{\left| \sin\left(\frac{\pi h}{c}\right) \right|} \frac{2}{kN}$$
$$\leq \frac{4 \left| \sin\left(\frac{\pi j}{c}\right) \right| e^{2\pi + \frac{\pi}{12}} \left(1 + \log\left(\frac{c-1}{2}\right)\right)}{\pi \left(1 - \frac{\pi^2}{24}\right)} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ c|k}} k^{-\frac{1}{2}}$$
$$\leq \frac{8e^{2\pi + \pi/12} \left(1 + \log\left(\frac{c-1}{2}\right)\right) n^{\frac{1}{4}}}{\pi \left(1 - \frac{\pi^2}{24}\right) c} .$$

Now we do the same for the second main contribution. Remember that we have to bound the following term:

$$S_{2err} = 2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta^i_{a,c,k,r}>0 \\ i\in\{-,+\}}} (-1)^{ak+l} \sum_h \omega_{h,k} e^{\frac{2\pi i}{k}(-nh+m^i_{a,c,k,r}h')}$$

$$\times \left(\int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} + \int_{\frac{1}{k(k+k_1)}}^{\frac{1}{kN}} \right) z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{2\pi}{kz}\delta^i_{a,c,k,r}} d\Phi$$

Completely analogous to S_{1err} it is possible to show:

$$S_{2err} \leq 8e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_{\substack{r,k \\ \delta^i_{j,c,k,r}>0 \\ i\in\{-,+\}}} k^{-\frac{1}{2}} e^{2\pi\delta^i_{j,c,k,r}}.$$

As a next step we evaluate the sum over r with $i=+$ and bound it in terms of δ_0. As it is the biggest argument, we can also bound the term with $i=-$ analogously. So we restrict to the case $i=+$. The sum over r gives

$$\begin{aligned}
S_{2err} \leq & 16e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_{\substack{r,k \\ \delta^+_{j,c,k,r}>0 \\ i\in\{-,+\}}} k^{-\frac{1}{2}} e^{2\pi\delta^+_{j,c,k,r}} \\
= & 16e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_k k^{-\frac{1}{2}} \sum_{r\leq r_0-1} e^{-\frac{\pi l}{c}+\frac{\pi l^2}{c^2}+\frac{\pi}{12}-\frac{2\pi l r}{c}} \\
= & 16e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_k k^{-\frac{1}{2}} \frac{e^{-\frac{\pi l}{c}+\frac{\pi l^2}{c^2}+\frac{\pi}{12}} \left(e^{-\frac{2\pi l}{c}r_0} - 1\right)}{e^{-\frac{2\pi l}{c}} - 1} \\
\leq & 16e^{2\pi} \left|\sin\left(\frac{\pi j}{c}\right)\right| \sum_k k^{-\frac{1}{2}} \frac{e^{2\pi\delta_0}}{1-e^{-\frac{2\pi}{c}}} \\
\leq & 32e^{2\pi} n^{\frac{1}{4}} \left|\sin\left(\frac{\pi j}{c}\right)\right| \frac{e^{2\pi\delta_0}}{1-e^{-\frac{2\pi}{c}}}.
\end{aligned}$$

Two facts should be explained. We had used that the summation over r is an error term if it starts with r_0, so here we have to sum over all the r-terms where $r \leq r_0 - 1$. As a next step we calculated the geometric series in r and bounded the term similar to the Σ_2-error. Finally an estimation of the k-sum gives the final expression. The last contribution we have to bound is coming from the evaluation of the integral. There it is used that it is possible to change the path of integration if one accounts the integral over the smaller arc. This term has to be made explicit.

Integrating along the smaller arc
Remember that we had to compute integrals of the following form

$$I_{k,t} = \frac{1}{ki} \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{t}{z}\right)} dz.$$

Now we denote the circle through $\frac{k}{n} \pm \frac{i}{N}$ and tangent to the imaginary axis at 0 by Γ. For $z = x + iy$, Γ is given by $x^2 + y^2 = \frac{k}{n} + \frac{n}{N^2 k} x =: \alpha x$. The path of integration can be changed into the larger arc, while on the smaller arc we have the following bounds: $\frac{1}{k} < \alpha < 2$, $\mathrm{Re}(z) \leq \frac{k}{n}$ and $\mathrm{Re}(z^{-1}) < k$. This can be used to bound the integral over the smaller arc which we denoted by Γ_S. Splitting $I_{k,t} = I_{k,t}^{main} + I_{k,t}^{err}$ we can bound $I_{k,t}^{err}$

$$\begin{aligned}
I_{k,t}^{err} \leq & \frac{2}{k} e^{2\pi + 2\pi t} \int_{\Gamma_S} |z|^{-\frac{1}{2}} dz \leq \frac{2}{k} e^{2\pi + \frac{\pi}{12}} \left| \int_0^{\frac{k}{n}} (x^2 + y^2)^{-\frac{1}{4}} (dx + idy) \right| \\
= & \frac{2}{k} e^{2\pi + 2\pi t} \left| \int_0^{\frac{k}{n}} (\alpha x)^{-\frac{1}{4}} (dx + idy) \right| = \frac{2}{k} e^{2\pi + 2\pi t} \left| \int_0^{\frac{k}{n}} (\alpha x)^{-\frac{1}{4}} dx + i \int_0^{\frac{k}{n}} (\alpha x)^{-\frac{1}{4}} dy \right| \\
= & \frac{2}{k} e^{2\pi + 2\pi t} \alpha^{-\frac{1}{4}} \left| \int_0^{\frac{k}{n}} x^{-\frac{1}{4}} dx + i \int_0^{\frac{k}{n}} x^{-\frac{1}{4}} dy \right| \\
= & \frac{2}{k} e^{2\pi + 2\pi t} \alpha^{-\frac{1}{4}} \left| \frac{4}{3} \left(\frac{k}{n} \right)^{\frac{3}{4}} + i \int_0^{\frac{k}{n}} x^{-\frac{1}{4}} \frac{dy}{dx} dx \right| \\
= & \frac{2}{k} e^{2\pi + 2\pi t} \alpha^{-\frac{1}{4}} \left| \frac{4}{3} \left(\frac{k}{n} \right)^{\frac{3}{4}} + i \int_0^{\frac{k}{n}} x^{-\frac{3}{4}} \frac{\alpha - 2x}{2\sqrt{\alpha - x}} dx \right|.
\end{aligned}$$

Next we compute the derivative of $f : [0, \alpha] \to \mathbb{R}$ defined by $x \to \frac{\alpha - 2x}{2\sqrt{\alpha - x}}$ and see that the derivative is negative for $x < \frac{3}{2}\alpha$ and so for all $x \in [0, \alpha]$. That means that the function $f(x)$ has its maximum at $x = 0$ and so we can bound further:

$$\begin{aligned}
I_{k,t}^{err} \leq & \frac{2}{k} e^{2\pi + 2\pi t} \alpha^{-\frac{1}{4}} \left| \frac{4}{3} \left(\frac{k}{n} \right)^{\frac{3}{4}} + i\alpha^{\frac{1}{2}} \int_0^{\frac{k}{n}} x^{-\frac{3}{4}} dx \right| \\
\leq & \frac{2}{k} e^{2\pi + 2\pi t} \left(\frac{4}{3} \left(\frac{k}{n} \right)^{\frac{3}{4}} \alpha^{-\frac{1}{4}} + 2\alpha^{\frac{1}{4}} \left(\frac{k}{n} \right)^{\frac{1}{4}} \right) \\
\leq & \frac{2}{k} e^{2\pi + 2\pi t} \left(\frac{4}{3} + 2^{\frac{5}{4}} \right) n^{-\frac{1}{8}}.
\end{aligned} \tag{4.24}$$

Here it is used that $k \leq \sqrt{n}$ and so $\frac{k}{n} \leq n^{-1/2}$, $\alpha < 2$. Combining the contributions from Σ_1, using usual formulas like (4.23), and estimation of the sum over k, we can bound the whole contribution by ($t = 1/24$):

$$\frac{4\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{c}\right)\right|\left(1+\log\left(\frac{c-1}{2}\right)\right)e^{2\pi+\frac{\pi}{12}}n^{\frac{3}{8}}}{\pi c\left(1-\frac{\pi^2}{24}\right)}.$$

The same can be done for Σ_2,

$$\begin{aligned} &4\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{c}\right)\right|\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{c}}}n^{-\frac{1}{8}}\sum_{k\leq\sqrt{n}}\frac{1}{k} \\ \leq &4\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{c}\right)\right|\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{c}}}\left(1+\log(\sqrt{n})\right)n^{-\frac{1}{8}} \\ \leq &8\left(\frac{4}{3}+2^{\frac{5}{4}}\right)\left|\sin\left(\frac{\pi j}{c}\right)\right|\frac{e^{2\pi\delta_0+2\pi}}{1-e^{-\frac{2\pi}{c}}}, \end{aligned}$$

where it is used that the the log grows smaller then every root function, that means that we can bound the contribution of n. Explicitly that means that we calculated the maximum of $f(x) = (1 + \log(\sqrt{x}))x^{-\frac{1}{8}}$ and the maximal value can be bounded by 2.
The explicit constant
Denoting the different error terms by $\tilde{\Sigma}_{errj}$ and the main part by T_1^+ we can conclude that

$$N_{a,b,c} = \min\left\{n \in \mathbb{N} \,\middle|\, T_1^+(a,b,c,n) - \sum_j \tilde{\Sigma}_{errj}(c,n) > 0\right\}.$$

This finishes the proof of the theorem. □

For $c < 13$ the S_j will give the main contributions to the circle method as noticed in the last theorem. As the sign of S_j depend on the sign of $\tilde{B}_{j,c,k}$ and the $\tilde{B}_{j,c,k}$ oscillate we will have the following

Corollary 4.8. *For $n > \tilde{N}_{a,b,c}$ where $\tilde{N}_{a,b,c}$ is an explicit constant we have*

1. *If $0 \leq a < b \leq 2$, then the difference $M(a,5,5n+d) - M(b,5,5n+d)$ is*

$$\begin{cases} < 0 & \text{if } (a,b,d) \in \{(0,b,1),(0,2,2),(1,2,2),(1,2,3)\}, \\ > 0 & \text{if } (a,b,d) \in \{(0,b,0),(1,2,1),(0,1,3)\}. \end{cases}$$

2. *If $0 \leq a < b \leq 3$, then the difference $M(a,7,7n+d) - M(b,7,7n+d)$ is*

$$\begin{cases} < 0 & \text{if } (a,b,d) \in \{(0,1,1),(0,1,6),(0,2,1),(0,2,2),(0,3,1),(0,3,6), \\ & (1,2,2),(1,2,4),(1,3,3),(1,3,4),(2,3,3),(2,3,6)\} \\ > 0 & \text{if } (a,b,d) \in \{(0,1,0),(0,1,3),(0,1,4),(0,2,0),(0,2,3),(0,3,0), \\ & (1,2,1),(1,2,6),(1,3,1),(2,3,2)\}. \end{cases}$$

3. If $0 \leq a < b \leq 4$*, then the difference* $M(a, 9, 3n+d) - M(b, 9, 3n+d)$ *is*

$$\begin{cases} < 0 & \text{if } (a,b,d) \in \{(0,1,1),(0,1,6),(0,1,8),(0,2,1),(0,2,2),(0,2,6)\}, \\ & (0,3,1),(0,3,3),(0,3,6),(0,4,1),(0,4,6),(0,4,8) \\ & (1,2,2),(1,2,4),(1,2,7),(1,3,2),(1,3,3),(1,3,4) \\ & (1,3,5),(1,3,7),(1,4,4),(1,4,7),(2,3,1),(2,3,3) \\ & (2,3,5),(2,3,7),(2,3,8),(2,4,5),(2,4,8),(3,4,0) \\ & (3,4,4),(3,4,6),(3,4,8)\}, \\ > 0 & \text{if } (a,b,d) \in \{(0,1,0),(0,1,2),(0,1,3),(0,1,4),(0,1,5),(0,1,7), \\ & (0,2,0),(0,2,3),(0,2,4),(0,2,5),(0,2,7),(0,2,8), \\ & (0,3,0),(0,3,4),(0,3,7),(0,4,0),(0,4,2),(0,4,3), \\ & (0,4,4),(0,4,5),(0,4,7),(1,2,1),(1,2,5),(1,2,8), \\ & (1,3,0),(1,3,1),(1,3,6),(1,3,8),(1,4,1),(2,3,0), \\ & (2,3,2),(2,3,4),(2,3,6),(2,4,2),(3,4,1),(3,4,2), \\ & (3,4,3),(3,4,5),(3,4,7)\}. \end{cases}$$

4. If $0 \leq a < b \leq 5$*, then the difference* $M(a, 11, 11n+d) - M(b, 11, 11n+d)$ *is*

$$\begin{cases} < 0 & \text{if } (a,b,d) \in \{(0,1,1),(0,1,7),(0,1,8),(0,1,9),(0,2,1),(0,2,2), \\ & (0,2,9),(0,3,1),(0,3,8),(0,3,9),(0,4,1),(0,4,7), \\ & (0,4,8),(0,5,1),(0,5,9),(1,2,2),(1,2,4),(1,3,3), \\ & (1,4,4),(2,3,3),(2,3,5),(2,3,8),(2,4,8),(3,4,4), \\ & (3,4,7),(3,4,10),(3,5,10),(4,5,5),(4,5,9)\}, \\ > 0 & \text{if } (a,b,d) \in \{(0,b,0),(0,1,3),(0,1,4),(0,2,3),(0,2,5),(0,3,4), \\ & (0,3,10),(0,4,3),(0,4,5),(0,5,3),(0,5,4),(1,2,1), \\ & (1,2,5),(1,2,7),(1,2,8),(1,3,1),(1,3,7),(1,3,10), \\ & (1,4,1),(1,4,5),(1,4,9),(1,5,1),(1,5,7),(1,5,8), \\ & (2,3,2),(2,3,4),(2,3,10),(2,4,2),(2,4,9),(2,5,2),(2,5,4), \\ & (3,4,3),(3,4,5),(3,4,9),(3,5,3),(3,5,8),(4,5,4),(4,5,7), \\ & (4,5,8)\}. \end{cases}$$

PROOF. The proof uses computer techniques. As c is odd and less than 13, the inequalities are easily checked by hand using MAPLE. That is done by assuming $c < 11$ and $k = c$, because this yields the largest argument in the hyperbolic sine in S_j and from that we only have to compute which sign $\sum_j \rho_j(a,b,c)\widetilde{B}_{j,c,c}(-n,0)$ has to see which inequality the crank differences obey. For $c = 11$ the arguments of the hyperbolic sines could match and cancellation between S_j and T_j can occur. So we have to add to $\sum_j \rho_j(a,b,c)\widetilde{B}_{j,c,c}(-n,0)$ also $\rho_1(a,b,11)\sin\left(\frac{\pi}{11}\right)$ corresponding to the maximal argument in the hyperbolic sine coming from the combination $k = 1$, $j = 1$, $r = 0$ to see which inequality the crank differences obey. Computing all the signs gives the complete list. Two things should be mentioned. The largest argument occur if $c = k$ and that avoids problems in the computation of the $\widetilde{B}_{j,c,c}(-n,0)$ because $c = 9$ is not prime. The other important fact is that we have to modify the constant. As the S_j are no error terms for $c < 11$ the $T_1^+(a,b,c,n)$ can be bounded by the error term $\Sigma_{err}^{new}(a,b,c,n) := \frac{2}{\rho(a,b,c)}T_1^+(a,b,c,n)$. So we obtain

$$\tilde{N}_{a,b,c} := \min\left\{n \in \mathbb{N} \,\middle|\, \sum_j S_j(a,b,c,n) - \sum_j \Sigma_{errj} - \Sigma_{err}^{new} > 0\right\}.$$

where the Σ_{errj} are all the error terms of Theorem 4.20 except for the S_j-terms. □

Chapter 5

Results

In this diploma thesis we have computed asymptotic values of the Fourier coefficients of an infinite family of crank generating functions. To do so we had to prove a statement about the transformation of the crank generating functions under Möbiustransformation. This allowed us to use the Hardy-Ramanujan-Rademacher method to compute these coefficients asymptotically (see Theorem 4.5). From that theorem we could extract the asymptotic value of $M(a, c; n)$. Again using Theorem 4.5 we could compute certain crank differences asymptotically and see that the crank differences obey certain inequalities by detecting the main contribution of the main part of theorem 4.5. Instead of bounding the error in the asymptotic expansion by $O(n^{\varepsilon})$ we bounded the all error term by term explicit by a contribution depending on n and c. From that we could see that for $c > 11$ and all $n > N_{a,b,c}$ the sum of all error is less then the main contribution. This fact is quite surprising in the sense that for $c < 13$ the behavior is different. For that case the main contributions highly oscillate and there is no general inequality that the crank differences obey. We computed all inequalities for the cases $c < 13$ by using computer algebra programs. This thesis is an analog of [6] and [7], where the same analysis was done for the Rank. Due to mock terms the rank is more involved, but needs for the analog statement about the Rank differences only $c > 9$. Note that the restriction $c > 11$ could be seen heuristically as a consequence that the crank explains more Ramanujan congruences simultaneously as the rank.

Eigenständigkeitserklärung Hiermit bestätige ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

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