Asymptotics of ruin probabilities for risk processes under optimal reinsurance policies: the small claim case

Hanspeter Schmidli*

Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

Abstract

We consider a classical risk model with the possibility of reinsurance. Moreover, in one of the models also investment into a risky asset is possible. The insurer follows the optimal strategy. In this paper we find the Cramér-Lundberg approximation in the small claim case and prove that the optimal strategy converges to the asymptotically optimal strategy as the capital increases to infinity.

2000 Mathematical Subject Classification: Primary 60F10; Secondary 60G35, 65K10

Key words: ruin probability, optimal control, Cramér-Lundberg approximation, adjustment coefficient, light tails, martingale methods, change of measure, geometric Brownian motion, Hamilton-Jacobi-Bellman equation

1. Introduction

Recently there was an increased interest in risk models with the possibility of investment into a risky asset. Kalashnikov and Norberg [6] considered a risk model where all surplus was invested into a risky asset modelled as a Black-Scholes model. They found out that the ruin probability in such a model converges to zero like a power law, even if the claim size distribution is light-tailed. The result was generalised by Frolova et. al. [2] to the case where only a (constant) fraction of the surplus is

*Part of the research was done while the author visited the Department 'Matmatica per le Decisioni' at the University of Firenze. The Author thanks for the hospitality at the department.

invested into the risky asset. In a very general setup the result is also obtained by Nyrhinen [7].

If instead of a constant fraction one would invest a constant amount into the risky asset one would get a so-called perturbed risk model. As shown by Dufresne and Gerber [1] the exponential decay of the ruin probability in the small claim case is preserved. The asymptotic behaviour in the subexponential case is not changed, see [15]. This shows that it cannot be optimal to invest the whole surplus into the risky asset. Therefore considered Hipp and Plum [4] a model where the insurer was able to choose dynamically the amount invested into the risky asset. The goal was to minimise the ruin probability. Because it is possible not to invest the ruin probability obtained is smaller than the corresponding ruin probability in the original model.

Another possibility an insurer has to control the business is reinsurance. Schmidli [10] considered the case where the ruin probability is minimised by reinsurance only. In [12] reinsurance as well as investment is allowed. Because theoretically the whole risk can be reinsured the ruin probability always is decaying exponentially. This is because if the insurance risk is a linearly decrasing process and only investment is possible one basically has to control a Brownian motion with drift. And a Brownian motion with (positive) drift has an exponentially decreasing ruin probability. In [10] and [12] proportional reinsurance is considered for convenience. The results are generalised to other types of reinsurance by Vogt [16].

We now introduce the model. Let $S_t = \sum_{i=1}^{N_t} Y_i$ be the aggregate claims process of an insurance portfolio, where $\{N_t\}$ is a Poisson process with rate λ. The claim sizes $\{Y_i\}$ are iid, strictly positive and independent of the claim arrival process. We denote by Y a generic random variable, by $M_Y(r) = \mathbb{E}[\exp\{rY\}]$ its moment generating function and by $G(y)$ its distribution function. All stochastic quantities are defined on a large enough complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The insurer follows a strategy $(A(u), b(u))$ of feedback form, where $(A(u), b(u)) \in$

$\mathcal{A} \subset [0, \infty) \times [0, 1]$. Feedback form is not necessary. We could have allowed all adapted processes $\{(A_t, b_t)\}$ such that (1) below is well defined. But it turns out that an optimal strategy exists and that it is of feedback form. The following cases had been investigated in [4], [10], [12]:

$\mathcal{A} = [0, \infty) \times \{1\}$,	no reinsurance,
$\mathcal{A} = \{0\} \times [0, 1]$,	no investment,
$\mathcal{A} = [0, \infty) \times [0, 1]$,	investment and reinsurance.

$A(u)$ denotes the amount invested into a risky asset, modelled as a geometric Brownian motion

$$\mathrm{d}Z_t = \mu Z_t \,\mathrm{d}t + \sigma Z_t \,\mathrm{d}W_t \,,$$

where $\{W_t\}$ is a standard Brownian motion independent of $\{S_t\}$. We assume here that all economic quantities are discounted. In particular, the claim sizes increase with inflation and the amount "not invested" is put on a bank account or invested in a riskless bond. It is even possible to borrow money at the same rate. The latter can be interpreted that the portfolio under consideration has a debt to the capital resources of the company. The parameters fulfil $\mu, \sigma > 0$.

$b(u)$ is the retention level in proportional reinsurance, i.e. if a claim Y occurs at the time where the surplus is u (prior to the claim payment) then the insurer pays $b(u)Y$ and the reinsurer pays $(1 - b(u))Y$. In order to get this reinsurance cover the insurer has to pay a continuous premium at rate $c(b(u))$. As in [12] we assume that $c(b)$ is strictly decreasing, $c(1) = 0$, and that $c < c(0) < \infty$, where c is the rate at which the insurer gets premiums. We have chosen here proportional reinsurance for simplicity. Other types of reinsurance can be treated similarly, see [16].

In this paper we work with the natural filtration $\{\mathcal{F}_t\}$ of $\{(S_t, W_t)\}$, i.e. the smallest right continuous filtration such that $\{(S_t, W_t)\}$ is adapted. Note that we cannot complete the filtration because we want to change the measure later. The filtration has to be right continuous in order that the ruin time defined below is a stopping time.

Under the chosen strategy the surplus process is

$$\mathrm{d}X_t = (c - c(b(X_t)) + \mu A(X_t))\,\mathrm{d}t + \sigma A(X_t)\,\mathrm{d}W_t - b(X_{t-})\,\mathrm{d}S_t\,, \qquad X_0 = u\,. \quad (1)$$

The time of ruin is $\tau^{A,b} = \inf\{t \ge 0 : X_t < 0\}$ and the ruin probability is $\psi^{A,b}(u) = \mathbb{P}[\tau^{A,b} < \infty]$. The control function is $\psi(u) = \inf \psi^{A,b}(u)$, where the infimum is taken over all controls such that $\{X_t\}$ is well defined. In order that $\psi(u) < 1$ we have to assume that $c > \lambda \mathbb{E}[Y]$ in the case without investment. If investment is possible the positive safety loading can be achieved by investment.

The following result has been proved in [4], [10] and [12].

Proposition 1. *Suppose there is an increasing function $\delta(u)$ solving the Hamilton-Jacobi-Bellman equation*

$$\sup_{(A,b)\in\mathcal{A}} \tfrac{1}{2}\sigma^2 A^2 \delta''(u) + (c - c(b) + \mu A)\delta'(u) + \lambda(\mathbb{E}[\delta(u - bY)] - \delta(u)) = 0\,, \qquad (2)$$

where $\delta(u) = 0$ for $u < 0$. If investment is possible suppose that $\delta(u)$ is twice continuously differentiable on $(0,\infty)$. Then $\delta(u)$ is bounded and $\delta(u) = \delta(\infty)(1 - \psi(u))$. Moreover, the arguments $(A(u), b(u))$ in (2) maximising the left hand side determine the optimal strategy $\{(A(X_t), b(X_t))\}$. If investment is not possible then there exists an increasing solution to (2). If investment is possible and the claim sizes have a bounded density then there exists a twice continuously differentiable increasing solution to (2). □

Note that if investment is possible then any increasing solution to (2) is concave, yielding that $\psi(u)$ is convex.

We suppose in the rest of this paper that $\psi(u)$ solves

$$\inf_{(A,b)\in\mathcal{A}} \tfrac{1}{2}\sigma^2 A^2 \psi''(u) + (c - c(b) + \mu A)\psi'(u) + \lambda(\mathbb{E}[\psi(u - bY)] - \psi(u)) = 0\,, \qquad (3)$$

where we let $\psi(u) = 1$ for $u < 0$. If investment is possible we also assume that $\psi(u)$ is twice continuously differentiable. In the following sections we investigate the asymptotic behaviour of $\psi(u)$ as $u \to \infty$ as well as the asymptotic behaviour of the strategies $(A(u), b(u))$. Part of this problem has been solved in [5]. There the case without reinsurance had been considered. Several technical problems did not appear in this case that show up in the case with reinsurance. We therefore assume here always that reinsurance is possible. We will in this paper only consider the small claim case, i.e. exponential moments of the claim size distribution exist. The large claim case had been considered in [13] and [14].

2. Lundberg bounds and the change of measure formula

We assume that the distribution tail $1-G(y)$ is decreasing exponentially fast. More specifically, we assume that $M_Y(R) < \infty$ for $R > 0$ defined below and that $c_- > 0$ for c_- defined by (7) below. These are the usual conditions needed for $\psi^{0,1}(u) \sim C\mathrm{e}^{-Ru}$ in the classical risk model, see [8], and in the case without reinsurance, see [5].

We start by defining the Lundberg exponent R. Let $r = R(A, b)$ be the strictly positive solution to (zero if no solution exists)

$$\lambda(M_Y(br) - 1) - (c - c(b) + \mu A)r + \tfrac{1}{2}\sigma^2 A^2 r^2 = 0\,. \tag{4}$$

$R(A, b)$ is the Lundberg exponent in the case of a constant strategy (A, b). The Lundberg exponent for our problem is $R = \sup_{(A,b)\in\mathcal{A}} R(A, b)$. This is, we maximise the Lundberg exponent in order to obtain an asymptotically optimal constant strategy. In the case without investment this problem is discussed in [17] where also (quite weak) conditions are formulated under which $b \mapsto R(A, b)$ is a unimodal function. Note that the function on the left hand side of (4) is strictly convex and zero in $r = 0$ and $r = R(A, b)$. Thus it is positive at $r = R$, and therefore the optimal parameters (A^*, b^*) minimise (4) at $r = R$,

$$\inf_{(A,b)\in\mathcal{A}} \lambda(M_Y(br) - 1) - (c - c(b) + \mu A)r + \tfrac{1}{2}\sigma^2 A^2 r^2 = 0\,.$$

We therefore have $A^* = \mu/(\sigma^2 R)$ and r is the solution to

$$\inf_{b\in[0,1]} \lambda(M_Y(br) - 1) - (c - c(b))r - \frac{\mu^2}{2\sigma^2} = 0\,. \tag{5}$$

Example 1. Consider the case where investment and proportional reinsurance is possible. The reinsurer uses an expected value principle, i.e. $c(b) = (1 + \theta)(1 - b)\lambda\mathbb{E}[Y]$. Taking the derivative with respect to b in (5) gives $\lambda M_Y'(bR) = (1 + \theta)\lambda\mathbb{E}[Y]$. Because $M_Y'(r)$ is a continuous increasing function a value for bR can be obtained. Plugging in this values into (5) yields R, and therefore b and A. It is possible that $b > 1$. In this case one has to set $b = 1$ and then to solve

$$\lambda(M_Y(r) - 1) - cr - \tfrac{1}{2}\mu^2/\sigma^2 = 0\,.$$

If the claim sizes are exponentially distributed with parameter α we get $bR = \alpha(1 - \sqrt{1/(1+\theta)})$. This yields

$$R = \frac{\lambda(\sqrt{1+\theta} - 1)^2 + \frac{1}{2}\mu^2/\sigma^2}{\lambda(1+\theta)/\alpha - c}\,,$$

which is positive because $c < c(0)$. The optimal b is now easily obtained. ■

We denote the asymptotically optimal constant strategy by (A^*, b^*), that is $R = R(A^*, b^*)$. From the considerations above it is clear that $A^* = \mu/(R\sigma^2)$ is unique. Waters [17] gives conditions under which also b^* is unique.

We now prove an upper Lundberg bound.

Proposition 2. *There exists a constant* $0 < c_+ < 1$ *such that* $\psi(u) \le c_+ e^{-Ru}$.

Proof. Choose the constant strategy that maximises the Lundberg coefficient. If investment is possible the result follows from [1]. If no investment is possible, the result is the Lundberg inequality for the classical risk process, see [8]. □

For simplicity we suppose that the process is stopped at ruin, i.e. $\mathrm{d}X_t = 0$ for $t > \tau$. Consider now the process

$$M_t = \exp\Bigl\{-R\,(X_{t\wedge\tau} - u) - \int_0^{t\wedge\tau} \theta(X_s)\,\mathrm{d}s\Bigr\}\,,$$

where

$$\theta(u) = \lambda(M_Y(b(u)R) - 1) - (c - c(b(u)) + \mu A(u))R + \tfrac{1}{2}\sigma^2 A^2(u) R^2\,.$$

The function $\theta(u)$ is positive and zero exactly if $(b(u), A(u))$ are the values maximising the Lundberg coefficient. We will later change the measure and readily get the lower bound.

Lemma 1. *The process* $\{M_t\}$ *is a martingale with mean value* 1.

Proof. The result follows in the same way as in [5]. □

The martingale $\{M_t\}$ can be used to change the measure on $\mathcal{F}_t$. We denote the measure by $\mathbb{P}^*$, that is $\mathbb{P}^*[A] = \mathbb{E}[M_t; A]$. It turns out that the measure is independent of t. Indeed, for $A \in \mathcal{F}_t$ and $t < s$,

$$\mathbb{P}^*[A] = \mathbb{E}[M_t; A] = \mathbb{E}[\mathbb{E}[M_s \mid \mathcal{F}_t]; A] = \mathbb{E}[M_s; A]\,.$$

It will become clear from the lemma below that $\mathbb{P}^*$ can be extended to $\mathcal{F}$. However, the two measures are singular on $\mathcal{F}$. Let T be a stopping time. Then for $A \in \mathcal{F}_T \cap \{T < \infty\}$, $\mathbb{P}^*[A] = \mathbb{E}[M_T; A]$. Indeed, by the optional stopping theorem

$$\begin{aligned}\mathbb{P}^*[A \cap \{T \le t\}] &= \mathbb{E}[M_t; A \cap \{T \le t\}] = \mathbb{E}[\mathbb{E}[M_t; A \cap \{T \le t\} \mid \mathcal{F}_T]] \\ &= \mathbb{E}[M_T; A \cap \{T \le t\}]\,.\end{aligned}$$

The claimed formula follows now by the monotone limit theorem. For details see [8] or [9].

Lemma 2. *Under the measure* $\mathbb{P}^*$, *the process* $\{X_t\}$ *is a jump diffusion process with location dependent parameters. The claim intensity is* $\lambda_u^* = \lambda M_Y(b(u)R)$, *the claim size distribution (that is the distribution of* Y, *the jump size is* $b(u)Y$*) is* $\mathrm{d}G_u^*(y) = \mathrm{e}^{b(u)Ry}\,\mathrm{d}G(y)/M_Y(b(u)R)$, *the drift parameter is* $c_u^* = c - c(b(u)) + \mu A(u) - \sigma^2 A^2(u)R$ *and the diffusion parameter is* $\sigma_u^* = \sigma A(u)$. *Moreover,* $\mathbb{P}^*[\tau < \infty] = 1$, *and therefore*

$$\psi(u) = \mathbb{E}^*\Big[\exp\Big\{RX_\tau + \int_0^\tau \theta(X_s)\,\mathrm{d}s\Big\}\Big]\,\mathrm{e}^{-Ru}\,. \tag{6}$$

In particular, the expected value is bounded by c_+.

Proof. That $\{X_t\}$ remains a strong Markov process follows by direct verification similarly as in [11]. Calculation of the generator yields the result similarly as in [11]. The infinitesimal drift of the process is

$$c - c(b(u)) + \mu A(u) - \sigma^2 A^2(u)R - \lambda b(u) M_Y'(b(u)R)\,.$$

This is minus the derivative of (4) with respect to r. Because (4) is convex in r, has a zero at zero and $R(A(u), b(u))$ and $R \ge R(A(u), b(u))$ the derivative must be strictly positive. This means that the process $\{X_t\}$ has a negative drift, implying that $\mathbb{P}^*[\tau < \infty] = 1$. □

We can now easily find a lower Lundberg bound. The following result had been proved in [3] and [5] in the case without reinsurance.

Proposition 3. *Suppose that* $c_- > 0$, *where*

$$c_- := \inf_z \frac{1}{\mathbb{E}[\mathrm{e}^{R(Y-z)} \mid Y > z]}\,. \tag{7}$$

Then $\psi(u) \ge c_- \mathrm{e}^{-Ru}$.

Remark. The condition $c_- > 0$ is fulfilled under quite mild conditions. For example, if the hazard rate $G'(y)/(1 - G(y))$ is ultimately bounded away from R and larger than R one has $c_- > 0$. ■

Proof. The expected value in (6) is bounded from below by $\mathbb{E}^*[e^{RX_\tau}]$. Conditioning on $X_{\tau-}$ yields

$$\mathbb{E}^*[e^{RX_\tau} \mid X_{\tau-} = y] = \mathbb{E}^*[e^{R(y-b(y)Y)} \mid Y > y/b(y)] = \frac{e^{Ry} \int_{y/b(y)}^\infty \mathrm{d}G(z)}{\int_{y/b(y)}^\infty e^{Rb(y)z} \,\mathrm{d}G(z)} \,.$$

Taking the infimum over all y (we omit the condition $G(y/b(y)) < 1$) one obtains

$$\mathbb{E}^*[e^{RX_\tau}] \ge \inf_{y\ge 0, b\in(0,1]} \frac{1 - G(y)}{\int_y^\infty e^{Rb(z-y)} \,\mathrm{d}G(z)} \ge \inf_{y\ge 0} \frac{1 - G(y)}{\int_y^\infty e^{R(z-y)} \,\mathrm{d}G(z)} = c_- \,.$$

This is the assertion. □

3. The Cramér-Lundberg approximation

In this section we consider formally the case where investment and reinsurance is possible. If we let $\mu = 0$ in the calculations below then the optimal investment strategy is $A(u) = A^* = 0$ and therefore basically the case with no investment follows. We only need to consider the two cases separately if properties of the derivatives are used because we do not assume existence of the second derivative in the case without investment.

Taking the infimum over A in (3), that is

$$A(u) = -\frac{\mu\psi'(u)}{\sigma^2\psi''(u)} \,,$$

the Hamilton-Jacobi-Bellman equation reads

$$\inf_b -\frac{\mu^2}{2\sigma^2}\frac{\psi'(u)^2}{\psi''(u)} + (c-c(b))\psi'(u) + \lambda\Bigl(\int_0^{u/b} \psi(u-by)\,\mathrm{d}G(y) + 1 - G(u/b) - \psi(u)\Bigr) = 0 \,. \tag{8}$$

b can be replaced by $b(u)$. Using integration by parts the equation can be written as

$$\begin{aligned}\inf_b -\frac{\mu^2}{2\sigma^2}\frac{\psi'(u)^2}{\psi''(u)} &+ (c - c(b))\psi'(u) \\ &+ \lambda\Bigl(\delta(0)(1 - G(u/b)) - \int_0^u (1 - G((u-z)/b))\psi'(z)\,\mathrm{d}z\Bigr) = 0 \,.\end{aligned} \tag{9}$$

Let $f(u) = \psi(u)\mathrm{e}^{Ru}$. Then

$$-\frac{\mu^2}{2\sigma^2}\frac{(Rf(u)-f'(u))^2}{R^2 f(u) - 2Rf'(u) + f''(u)} - (c - c(b(u)))(Rf(u) - f'(u)) \\ + \lambda\Bigl(\int_0^{u/b(u)} f(u - b(u)y)\mathrm{e}^{Rb(u)y}\,\mathrm{d}G(y) + (1 - G(u/b(u)))\mathrm{e}^{Ru} - f(u)\Bigr) = 0\,. \quad (10)$$

Note that $Rf(u) - f'(u) > 0$ and $R^2 f(u) - 2Rf'(u) + f''(u) > 0$ by the corresponding properties of $\psi(u)$.

Because $f(u)$ is bounded we conclude that $f'(u)$ is bounded from above. Because $f(u) > 0$ it is not possible that $f'(u)$ is decreasing to infinity. Thus if $f'(u)$ would be unbounded there must be points where $f'(u)$ is at a local minimum with an arbitrarily small value. But noting that $f(x)$ is bounded and that $f''(u) = 0$ at such a point the left hand side of (10) would be strictly negative. This is not possible, and we conclude that $f'(u)$ is bounded. In the case without investment we will show below that $f'(x)$ is bounded.

Let $g(u) = Rf(u) - f'(u) = -\psi'(u)\mathrm{e}^{Ru}$. Then $g(u) > 0$ and $g'(u) < Rg(u)$. Moreover, $g(u)$ is bounded. Equation (9) then reads

$$-\frac{\mu^2}{2\sigma^2}\frac{g(u)^2}{Rg(u) - g'(u)} - (c - c(b(u)))g(u) \\ + \lambda\Bigl(\delta(0)(1 - G(u/b(u))\mathrm{e}^{Ru} + \int_0^u (1 - G(y/b(u)))\mathrm{e}^{Ry} g(u-y)\,\mathrm{d}y\Bigr) = 0\,. \quad (11)$$

From (5) we have

$$\lambda\frac{M_Y(b^*R) - 1}{R} g(u) - (c - c(b^*))g(u) - \frac{\mu^2}{2R\sigma^2} g(u) = 0\,.$$

Taking the difference to (11) yields

$$-\frac{\mu^2}{2\sigma^2 R}\frac{g(u)g'(u)}{Rg(u) - g'(u)} + (c(b(u)) - c(b^*))g(u) + \lambda\Bigl(\delta(0)(1 - G(u/b(u))\mathrm{e}^{Ru} \\ + \int_0^u (1 - G(y/b(u)))\mathrm{e}^{Ry} g(u-y)\,\mathrm{d}y - \frac{M_Y(b^*R) - 1}{R} g(u)\Bigr) = 0\,.$$

$b(u)$ is the value at which the minimum of the left had side is taken. If we therefore replace $b(u)$ by b^* we obtain

$$-\frac{\mu^2}{2\sigma^2 R}\frac{g(u)g'(u)}{Rg(u)-g'(u)} + \lambda\int_0^u (g(u-y)-g(u))\mathrm{e}^{Ry}(1-G(y/b^*))\,\mathrm{d}y \\ - \lambda g(u)\int_u^\infty \mathrm{e}^{Ry}(1-G(y/b^*))\,\mathrm{d}y + \lambda\delta(0)\mathrm{e}^{Ru}(1-G(u/b^*)) \geq 0\,, \quad (12)$$

where we used that

$$M_Y(b^*R)-1 = \int_0^\infty\int_0^{b^*y} R\mathrm{e}^{Rz}\,\mathrm{d}z\,\mathrm{d}G(y) = R\int_0^\infty (1-G(z/b^*))\mathrm{e}^{Rz}\,\mathrm{d}z\;.$$

Note that the last two terms in (12) tend to zero as $u\to\infty$ provided $g(u)$ is bounded. We will now show that $g(u)$, and therefore also $f'(u)$ is bounded in the case without investment.

Suppose that $g(u)$ is unbounded. Let $u_n := \inf\{u : g(u) = 2nR\}$. Suppose first that $G(x_0) = 1$ for some finite x_0. For n large enough $b^*u_n > x_0$. That means that the left hand side in (12) is strictly negative because $g(u_n - y) < g(u_n)$. Hence we can assume $G(x) < 1$ for all $x > 0$. For n large enough we have $\delta(0)\mathrm{e}^{Ru_n}(1-G(u_n/b^*)) < \varepsilon$. Let $u' = \sup\{u : (1-G(u/b^*))\mathrm{e}^{Ru} \geq \varepsilon\}$. For n large enough $u' \leq u_n$. This implies that

$$nR\varepsilon\int_0^{u'} 1\!\!1_{g(u_n-z)\leq Rn}\,\mathrm{d}z \leq \int_0^{u_n}(g(u_n)-g(u_n-y))\mathrm{e}^{Ry}(1-G(y/b^*))\,\mathrm{d}y < \varepsilon\;.$$

Thus $\int_0^{u'} 1\!\!1_{g(u_n-z)\leq Rn}\,\mathrm{d}z \leq 1/(nR)$. Recall that $f(u) < 1$. If $g(u) > nR$ then $f'(u) = Rf(u) - g(u) < -(n-1)R$. Thus

$$\begin{aligned} f(u_n) &= f(u_n-u') + \int_0^{u'} f'(u_n-z)\,\mathrm{d}z \\ &< 1 + R\int_0^{u'} 1\!\!1_{g(u_n-z)\leq Rn}\,\mathrm{d}z - (n-1)R\int_0^{u'} 1\!\!1_{g(u_n-z)>Rn}\,\mathrm{d}z \\ &\leq 1 + 1/n - (n-1)R(u' - 1/(nR)) = 2-(n-1)Ru'\;. \end{aligned}$$

For n large enough one therefore would have $f(u_n) < 0$. Thus $g(u)$ is bounded.

Let $\zeta = \overline{\lim}_{u\to\infty} g(u)/R$. Then $\zeta > 0$. Indeed, if $\lim_{u\to\infty} g(u) = 0$ then for x large enough $g(u) < c\_R/2$. Then $f'(u) = Rf(u) - g(u) > c\_R/2$. This contradicts that $f(u)$ is bounded.

We now have to consider the cases with and without investment separately.

Lemma 3. *Suppose investment is allowed. Then*

i) *For any $\varepsilon, \beta, x_0 > 0$ there exists $x \ge x_0$ such that $g(u) > R\zeta - \varepsilon$ for all $u \in [x - \beta, x]$.*

ii) *We have $\overline{\lim}_{u\to\infty} f(u) = \zeta$.*

iii) *For any $\varepsilon, \beta, x_0 > 0$ there exists $x \ge x_0$ such that $f(u) > \zeta - \varepsilon$ for all $u \in [x-\beta, x]$.*

Proof. i) If $g(u)$ is ultimately monotone then $g(u)$ is converging and the assertion is trivial. We therefore assume that $g(u)$ is not ultimately monotone. Suppose first $G(\beta/b^*) < 1$. Choose $0 < \chi < \varepsilon/3$ to be determined later. Let $\eta = \inf_{y\le\beta} \mathrm{e}^{Ry}(1 - G(y/b^*))$. We can then find $x \ge x_0$ such that $g(x) > R\zeta - \chi$, $g'(x) \ge 0$, $g(z) < R\zeta + \chi$ for all $z \ge x - \beta$, and

$$\phi(x) := 2R\zeta\lambda \int_x^\infty \mathrm{e}^{Ry}(1 - G(y/b^*))\,\mathrm{d}y + \lambda\delta(0)\mathrm{e}^{Rx}(1 - G(x/b^*)) < \chi\,.$$

Then

$$\lambda \int_0^x (g(x-y) - g(x))\mathrm{e}^{Ry}(1 - G(y/b^*))\,\mathrm{d}y > -\chi\;.$$

Suppose now $g(x - y) \le R\zeta - \varepsilon$ for some $y \le \beta$. We have that $g'(z) \le Rg(z) \le R(R\zeta + \chi)$. Thus $g(z) < R\zeta - 2\varepsilon/3$ for all $z \in [y, y + \varepsilon/(3R(R\zeta + \chi))]$. That means we can estimate the integral above

$$\lambda 2\chi \int_0^\infty \mathrm{e}^{Ry}(1 - G(y/b^*))\,\mathrm{d}y - \frac{\lambda\eta\varepsilon^2}{9R(R\zeta + \chi)} > -\chi\;.$$

If one chooses χ small enough one obtains a contradiction. This proves the result if $G(\beta/b^*) < 1$.

Suppose the result is proven in the sense of the construction above for $\tilde{\beta} = 2\beta/3$. There exits $\tilde{\chi}$ such that whenever $g(u) > R\zeta - \tilde{\chi}$, $g'(u) \geq 0$, $g(z) < R\zeta + \tilde{\chi}$ for all $z \geq u - 2\beta$ and $\phi(u) < \tilde{\chi}$ then $g(z) > R\zeta - \varepsilon$ for all $z \in [u - 2\beta/3, u]$. Choose now $x \geq x_0 + \beta$ such that $g'(x) \geq 0$, $g(z) > R\zeta - \tilde{\chi}$ for all $z \in [x - 2\beta/3, x]$, $g(z) < R\zeta + \tilde{\chi}$ for $z \geq x - 3\beta$, and $\phi(z) < \tilde{\chi}$ for $z > x - \beta$. If there is $\tilde{x} \in [x - 2\beta/3, x - \beta/3)$ such that $g'(\tilde{x}) \geq 0$ the result follows from the proof above. If there is no such $\tilde{x}$ then $g(z)$ is decreasing on $[x - 2\beta/3, x - \beta/3]$. If $g(z)$ is also decreasing on $[x - \beta, x - 2\beta/3]$ the result follows. Otherwise $z_0 := \sup\{z \leq x - 2\beta/3 : g(z) = 0\}$ exists, and the result follows from the considerations above. This proves the assertion if $G(2\beta/(3b^*)) < 1$. In the same way the result follows if $G((2/3)^n \beta/b^*) < 1$ by induction.

ii) Denote by $\eta = \overline{\lim}_{u\to\infty} f(u)$. Note there must be points where $f'(u)$ is arbitrarily close to zero and $f(u)$ is close to η. Because $g(u) = Rf(u) - f'(u)$ we must have $\eta \leq \zeta$. Suppose $\eta < \zeta$. Recall that $g'(z) < Rg(z)$. By i) there exits x such that $g(z) > (\eta + 2\zeta)R/3$ for $z \in [x - 2/(R(\zeta - \eta)), x]$. On this interval $f'(z) = Rf(z) - g(z) < -R(\zeta - \eta)/2$. This implies that $f(x) < 0$, which is a contradiction.

iii) There is $x \geq x_0 + 2\zeta/(\varepsilon R)$ such that $g(z) > R(\zeta - \varepsilon/2)$ for all $z \in [x - \beta - 2\zeta/(\varepsilon R), x]$. If for some $y \in [x - \beta - 2\zeta/(\varepsilon R), x - 2\zeta/(\varepsilon R)]$ we had $f(y) \leq \zeta - \varepsilon$ then $f'(y) = Rf(y) - g(y) < -R\varepsilon/2$ would be decreasing. Thus $f(z) \leq \zeta - \varepsilon$ for $z \in [y, x]$. This would imply that $f(x) < 0$. Therefore $f(y) > \zeta - \varepsilon$ for all $y \in [x - \beta - 2\zeta/(\varepsilon R), x - 2\zeta/(\varepsilon R)]$. □

In a similar way one can prove the corresponding result for the case without investment.

Lemma 4. *Consider the case without investment. Then*

i) *For any $\varepsilon, \delta, \beta, x_0 > 0$ there exists $x \geq x_0$ such that $\int_0^\beta \mathbb{1}_{g(x-z) \leq R\zeta - \varepsilon}\, \mathrm{d}z < \delta$.*

ii) *We have $\overline{\lim}_{u\to\infty} f(u) = \zeta$.*

iii) *For any* $\varepsilon, \beta, x_0 > 0$ *there exists* $x \geq x_0$ *such that* $f(u) > \zeta - \varepsilon$ *for all* $u \in [x-\beta, x]$.

□

This enables to prove our main result.

Theorem 1. *There exists a constant* $\zeta \in (0,1)$ *such that* $\lim_{u\to\infty} \psi(u)e^{Ru} = \zeta$.

Proof. Choose $\varepsilon, \beta > 0$. Then there is $x_0 \geq \beta$ such that $f(u) > \zeta - \varepsilon$ for $u \in [x_0 - \beta, x_0]$. Suppose $x \geq 2x_0$ and let $T = \inf\{t > 0 : X_t < x_0\}$. Note that $\mathbb{P}^*[T < \infty] = 1$. From (6) it follows that

$$f(u) = \mathbb{E}^*\Bigl[\exp\Bigl\{RX_\tau + \int_0^\tau \theta(X_s)\,\mathrm{d}s\Bigr\}\Bigr] = \mathbb{E}^*\Bigl[f(X_T)\exp\Bigl\{\int_0^T \theta(X_s)\,\mathrm{d}s\Bigr\}\Bigr]\ .$$

In particular,

$$f(u) \geq (\zeta - \varepsilon)\mathbb{P}^*[x_0 - X_T \leq \beta]\ .$$

By choosing β appropriately one can obtain $\mathbb{P}^*[x_0 - X_T \leq \beta] > 1 - \varepsilon$, see also the proof of Proposition 3. Thus $f(u) \geq (\zeta - \varepsilon)(1 - \varepsilon)$. Because ε is arbitrary we have $\underline{\lim}_{u\to\infty} f(u) \geq \zeta = \overline{\lim}_{u\to\infty} f(u)$. That $\zeta \in (0,1)$ follows from Propositions 2 and 3. □

4. Convergence of the strategies

From the proof of Theorem 1 we see that $\int_0^T \theta(X_s)\,\mathrm{d}s$ has to be close to zero. That means that $\{(A(X_s), b(X_s))\}$ should most of the time be close to (A^*, b^*). We therefore expect the functions $A(u)$ and $b(u)$ to converge to A^* and b^*, respectively.

We start proving convergence of the derivatives of $f(u)$.

Lemma 5. *It holds that* $\lim_{u\to\infty} f'(u) = 0$. *If investment is possible then also* $\lim_{u\to\infty} f''(u) = 0$.

Remark. The above lemma means that derivatives and limit can be interchanged, i.e. $\psi'(u) \sim -\zeta R e^{-Ru}$ and $\psi''(u) \sim \zeta R^2 e^{-Ru}$. ■

Proof. From Lemmata 3 and 4 and Theorem 1 we find

$$\varliminf_{u\to\infty} f'(u) = \lim_{u\to\infty} Rf(u) - \varlimsup_{u\to\infty} g(u) = 0\,.$$

We first consider the case with investment. Suppose $\varlimsup_{u\to\infty} f'(u) = \varepsilon > 0$. Then $\varliminf_{u\to\infty} g(u) = R\zeta - \varepsilon$. Because $g'(u) < Rg(u)$ we have $g'(u) < R\zeta\varepsilon/2$ for infinitely many intervals of length $\delta = \varepsilon/(3R\sup g(z))$. Thus $f'(u) > \varepsilon/2$ for infinitely many intervals of length δ. But this contradicts that $f(u)$ is converging. Note that we obtain $\lim_{u\to\infty} g(u) = R\zeta$.

Choose a sequence $\{u_n\}$ tending to infinity such that $f''(u_n)$ converges to some value κ. Note that $\lim_{n\to\infty} g'(u_n) = -\kappa$. By restricting to a subsequence we can also assume that $b(u_n)$ converges to a value b_0. The limit of (10) is

$$-\frac{\mu^2}{2\sigma^2}\frac{R^2\zeta^2}{R^2\zeta+\kappa} - (c - c(b_0))R\zeta + \lambda(M_Y(b_0R) - 1)\zeta = 0\,.$$

Replacing b_0 by b^* yields, see (5),

$$-\frac{\mu^2}{2\sigma^2}\frac{R^2\zeta}{R^2\zeta+\kappa} - (c - c(b^*))R + \lambda(M_Y(b^*R) - 1) \le 0\,.$$

By the definition of the Lundberg exponent the latter inequality is

$$\frac{\mu^2}{2\sigma^2} \le \frac{\mu^2}{2\sigma^2}\frac{R^2\zeta}{R^2\zeta+\kappa}\,.$$

We conclude that $-R^2\zeta \le \kappa \le 0$. Letting $n \to \infty$ in (12) gives

$$\frac{\mu^2}{2\sigma^2 R}\frac{R\zeta\kappa}{R\zeta+\kappa} \ge 0\,.$$

Thus $\kappa \ge 0$. This shows that $\kappa = 0$, i.e. $\lim f''(u) = 0$.

Consider now the case without investment. Because $b(u)$ is bounded away from zero and $f(u)$ converges to ζ we conclude from (10) that

$$(c - c(b(u)))(f'(u) - R\zeta) + \lambda\zeta(M_Y(Rb(u)) - 1) < \varepsilon$$

for u large enough. We conclude that

$$(c - c(b(u)))f'(u) < -\zeta\theta(u) + \varepsilon \,.$$

Because $\underline{\lim}_{u\to\infty}(c-c(b(u)))f'(u) = 0$ we conclude that $\lim_{u\to\infty}(c-c(b(u)))f'(u) = 0$. Moreover, $\zeta\theta(u) < 2\varepsilon$ for u large enough. Choosing ε small enough this proves that $c - c(b(u))$ is bounded away from zero. Therefore $f'(u)$ converges to zero. □

In [5] it is shown that the strategy $A(u)$ converges in the case without reinsurance. In [10, 12] it was conjectured that the strategy $b(u)$ converges to the asymptotically optimal b^*. This is motivated by the optimal rate e^{-Ru} which corresponds to the strategy (A^*, b^*). It is clear that, if $b(u)$ converges, the limit must be b^*. We now prove convergence of the strategy $(A(u), b(u))$.

Theorem 2. *Suppose investment is possible. Then* $\lim_{u\to\infty} A(u) = A^*$. *Suppose that* b^* *is uniquely defined. Then* $\lim_{u\to\infty} b(u) = b^*$.

Proof. By the definition of the function $A(u)$ we find

$$\lim_{u\to\infty} A(u) = -\lim_{u\to\infty} \frac{\mu\psi'(u)}{\sigma^2\psi''(u)} = \lim_{u\to\infty} \frac{\mu}{\sigma^2} \frac{Rf(u) - f'(u)}{R^2 f(u) - 2Rf'(u) + f''(u)} = \frac{\mu}{R\sigma^2} = A^* \,.$$

Let $\{u_n\}$ be a sequence such that $\lim_{n\to\infty} b(u_n) = b_0$ for some $b_0 \in [0,1]$. By Lemma 5 and Theorem 1 the limit of (10) is

$$-\frac{\mu^2\zeta}{2\sigma^2} - (c - c(b_0))R\zeta + \lambda(M_Y(b_0 R) - 1)\zeta = 0 \,.$$

By (5) and because b_0 is unique we have $b_0 = b^*$. □

Remark. If b^* is not unique b_0 can be any of the points where $R(A^*, b)$ is maximal. In order to find the limit of $b(u)$ (if a limit exists) one needs to determine close to which point of maximisation $b(u)$ lies for large u. ■

References

[1] **Dufresne, F. and Gerber, H.U.** (1991). Risk theory for the compound Poisson process that is perturbed by diffusion. *Insurance Math. Econom.* **10**, 51–59.

[2] **Frolova, A., Kabanov, Y. and Pergamenshchikov, S.** (2002). In the insurance business risky investments are dangerous. *Finance Stoch.* **6**, 227–235.

[3] **Gaier, J., Grandits, P. and Schachermeyer, W.** (2003). Asymptotic ruin probabilities and optimal investment. *Ann. Appl. Probab.* **13**, 1054–1076.

[4] **Hipp, C. and Plum, M.** (2000). Optimal investment for insurers. *Insurance Math. Econom.* **27**, 215–228.

[5] **Hipp, C. and Schmidli, H.** (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scand. Actuarial J.*, to appear.

[6] **Kalashnikov, V.V and Norberg, R.** (2002). Power tailed ruin probabilities in the presence of risky investments.. *Stochastic Process. Appl.* **98**, 211–228.

[7] **Nyrhinen, H.** (2001). Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Process. Appl.* **92**, 265–285.

[8] **Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J.L.** (1999). *Stochastic Processes for Insurance and Finance*. Wiley, Chichester.

[9] **Schmidli, H.** (1995). Cramér-Lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance Math. Econom.* **16**, 135–149.

[10] **Schmidli, H.** (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J.*, 55–68.

[11] **Schmidli, H.** (2001). Risk processes conditioned on ruin. *Working paper* **176**, *Laboratory of Actuarial Mathematics, University of Copenhagen.*

[12] **Schmidli, H.** (2002). On minimising the ruin probability by investment and reinsurance. *Ann. Appl. Probab.* **12**, 890–907.

[13] **Schmidli, H.** (2004). Asymptotics of ruin probabilities for risk processes under optimal reinsurance and investment policies: the large claim case. *Queueing Syst. Theory Appl.* **46**, 149–157.

[14] **Schmidli, H.** (2004). On Cramér-Lundberg approximations for ruin probabilities under optimal excess of loss reinsurance. *Working paper* **193**, *Laboratory of Actuarial Mathematics, University of Copenhagen.*

[15] **Veraverbeke, N.** (1993). Asymptotic estimates for the probability of ruin in a Poisson model with diffusion. *Insurance Math. Econom.* **13**, 57–62.

[16] **Vogt, M.** (2004). *Optimale Dynamische Rückversicherung — Ein Kontrolltheoretischer Ansatz*. Verlag Versicherungswirtschaft GmbH, Karlsruhe.

[17] **Waters, H.R.** (1983). Some mathematical aspects of reinsurance. *Insurance Math. Econom.* **2**, 17–26.