Characteristics of ruin probabilities in classical risk models with and without investment, Cox risk models and perturbed risk models

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Karl Pedersen
Dekan

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Preface

I started to work in risk theory at the beginning of my PhD studies at ETH Zürich. The subject interested me so strongly that I decided to remain in academia for some longer time. A grant from the Swiss National Foundation gave me the opportunity to spend a year at Aarhus University. The earliest papers included in this thesis are from that time. The material included in this thesis was produced during the years 1993 – 1999.

I included ten papers in the present thesis. Together with some papers I wrote with co-authors they build the main part of my publications. The starting point for my interest in risk theory was Paul Embrechts' suggestion to consider general risk models using piecewise deterministic Markov processes. A collaboration with Jan Grandell turned my interests to Cox risk processes. Many of the papers in this thesis are motivated by problems arriving by considering Cox risk models.

The papers were written at the *Department for Theoretical Statistics and Operations Research* at Aarhus University and at the *Department for Actuarial Mathematics and Statistics* at Heriot-Watt University in Edinburgh, where I was employed during the academic year 1994/95. I want to take the opportunity to thank my colleagues at these two departments for the excellent scientific environment and atmosphere they created. Especially I thank Jens Ledet Jensen for his useful comments on a first version of this thesis. And most importantly, I thank Monika and Eliane that they let me pursue an academic career far away from Switzerland.

Dansk sammendrag

I forsikringsmatematik modelleres overskud af en bestemt type af forsikringskontrakter som $\textit{overskud} = \textit{startkapital} + \textit{indtægter} - \textit{udgifter}$. Den første model blev indført af Filip Lundberg [38] i sin PhD-afhandling. Han brugte en sammensat Poissonproces som model for udgifterne og en lineær deterministisk indkomst. Hans arbejde blev generaliseret af Harald Cramér [9] and [10]. Denne model blev grundlag for næsten alle modeller inden for risiko teori.

Startkapitalen er det beløb, et forsikringsselskab vil riskere i et bestemt område. Ruintidspunktet er det tidspunkt, hvor for første gang startkapitalen er tabt. Ruinsandsynligheden er sandsynligheden for, at ruin sker i endelig tid. De klassiske arbejder beskæftiger sig mest med af finde ruinsandsynligheden, grænser for eller approksimationer til ruinsandsynligheden.

I denne afhandling ser vi på den klassiske model og nogle generaliseringer. For den klassiske model løser vi nogle problemer inden for området, som ikke har været løst før. I [I] finder vi fordelingen af overskuddet lige før og lige efter ruin, hvis ruin sker. I [I] ser vi også på tilfældet, hvor den sædvanlige nettoprofit betingelse ikke er opfyldt; det vil sige tilfældet, hvor ruin altid sker i endelig tid. I [X] finder vi optimale genforsikringsstrategier for den klassiske model og for en diffusionaapproksimation til denne.

Det er mange muligheder for at generalisere den klassiske model. Den mest naturlige generalisering er at tillade indkomsten at have en mere generel form. I [II] viser vi, hvordan Markov procesteori kan bruges til at finde information om ruinsandsynligheden. Eftersom det ikke er nemt at finde numeriske metoder til at regne ruinsandsynligheden ud, er man også interesseret i approksimationer. En mulighed er diffusionsapproksimationer. I [III] viser vi, hvordan korrigerede diffusionsapproksimationer kan bruges.

En støjfyldt risikomodel er en model plus en uafhængig proces. For modeller støjfyldt med en Brownsk bevægelse finder vi i [IV] Cramér-Lundberg approksimationen til ruinsandsynligheden, det vil sige, vi finder den eksponentielle rate, med hvilken ruinsandsynligheden kovergerer mod nul, når startkapitalen vokser mod uendelig. I stedet for en Brownsk bevægelse kan man også tage en Lévy proces. I [V] generaliserer vi et arbejde af Furrer [23]. Vi finder fordelingen af trappehøjden for en stationær støjfyldt risikoproces.

En af mine hovedinteresser ind for området har været Cox risiko processer. Man kan forstille sig en Cox risiko proces på følgende måde. Først valges en stokastisk intensitetsfunktion. Denne funktion bruges derefter til at skifte tid i en sammensat Poissonproces. Processen der fremkommer modellerer nu udgifterne i selskabet. Det betyder, at når intensitetsfunktionen stiger stærkt, forventer man mange skader. I [VII] er indført og undersøgt en model, hvor intensitetsfunktionen er stykkevis lineær. Denne model indeholder to vigtige modeller som eksempler: Björk-Grandell modellen [8] og den Markov modulerede Poissonmodel [2]. Cox risikomodeller har

den ulempe, at man ikke kan bruge de klassiske metoder til at finde ruinsandsynligheder eller approximationer til disse. I [VI] udvikler vi et værktøj, som ofte kan bruges til at finde Cramér-Lundberg approksimationer. Eksponenten i Cramér-Lundberg approksimationen er en slags mål for risiko. I praksis kender man jo ikke modellens parametre. Derfor vil man gerne estimere dem. I [VIII] finder vi en stokastisk procedure til at estimere eksponenten.

I praksis modellerer man ofte skader som værende Pareto eller lognormalt fordelt. Disse fordelingsfunktioner tilhører en klasse af funktioner, some man kalder subeksponentielle fordelingsfunktioner. Her kan den klassiske teori, som beskæftiger sig med små skader, ikke anvendes . En approksimation til ruinsandsynligheden i den klassiske model for stor startkapital blev fundet af Embrechts og Veraverbeke [20]. Mere generelle modeller tillader ofte regenereringstider. I mange tilfælde kan man finde en approksimation til ruinsandsynligheden ved kun at betragte processen på disse regenereringstider. Men så skal man vise at tilvæksten mellem to regenereringstider er subexponentiel. Dette lader sig ofte gøre ved at se på en sammensat sum. I [IX] finder vi betingelser for, at en sammensat sum har en subexponentiel fordeling.

List of papers

[I] **Schmidli, H.** (1999). On the distribution of the surplus prior and at ruin. *ASTIN Bull.* **29**, 227 - 244.

[II] **Schmidli, H.** (1994). Risk theory in an economic environment and Markov processes. *Schweiz. Verein. Versicherungsmath. Mitt.* **94**, 51–70.

[III] **Schmidli, H.** (1994). Corrected diffusion approximations for a risk process with the possibility of borrowing and investment. *Schweiz. Verein. Versicherungsmath. Mitt.* **94**, 71–81.

[IV] **Schmidli, H.** (1995). Cramér-Lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance Math. Econom.* **16**, 135–149.

[V] **Schmidli, H.** (1998). Distribution of the first ladder height of a stationary risk process perturbed by α-stable Lévy motion. Research Report No. 394, Dept. Theor. Statist., Aarhus University.

[VI] **Schmidli, H.** (1997). An extension to the renewal theorem and an application to risk theory. *Ann. Appl. Probab.* **7**, 121–133.

[VII] **Schmidli, H.** (1996). Lundberg inequalities for a Cox model with a piecewise constant intensity. *J. Appl. Probab.* **33**, 196–210.

[VIII] **Schmidli, H.** (1997). Estimation of the Lundberg coefficient for a Markov modulated risk model. *Scand. Actuarial J.*, 48–57.

[IX] **Schmidli, H.** (1999). Compound sums and subexponentiality. *Bernoulli* **5**, 999 - 1012.

[X] **Schmidli, H.** (1999). Optimal proportional reinsurance policies in a dynamic setting. Scand. Actuarial J., to appear.

1. Introduction

We start this thesis by introducing some basic terms, explaining what Ruin Theory deals with, and giving an overview over the results collected in this thesis.

The surplus of a certain branch of non-life insurance can be described as

$$\text{Surplus} = \text{Initial capital} + \text{Income} - \text{Outflow}.$$

The first one who considered a model of this type in non-life insurance was Filip Lundberg [38] in his thesis. His work was then generalized by Harald Cramér [9] and [10]. Therefore the model is called *Cramér-Lundberg model* or *classical risk model.* The surplus is modelled as

$$X_t = u + ct - \sum_{k=1}^{N_t} Y_k\,, \tag{1.1}$$

where $u \ge 0$ is the *initial capital*, $c > 0$ is the *premium rate*, N is a *Poisson process* with rate λ, see [41], modelling the number of claims in $(0, t]$ and $(Y_k : k \in \mathbb{N})$ is an iid sequence of positive random variables independent of N, modelling the claim sizes. Here, and in all this thesis, all stochastic objects are assumed to be defined on a complete probability space $(\Omega, \mathcal{F}, P)$. For simplicity we let $Y = Y_1$ be a generic random variable and we denote the distribution function of Y by G. In this model the *time of ruin*

$$\tau = \tau(u) = \inf\{t \ge 0 : X_t < 0\} \tag{1.2}$$

is the first time where the surplus becomes negative. As usual we let $\tau = \infty$ if $\inf\{X_t : t \ge 0\} \ge 0$. Ruin is considered as a technical term. It does not mean that the company becomes bankrupt. The initial capital has interpretation as the capital the company is willing to risk. If ruin occurs, this is interpreted that the company has to take action in order the make the business profitable.

We usually work with the filtration $(\mathcal{F}_t)$, which is assumed to be the smallest right-continuous filtration such that the stochastic process considered, here X, is adapted. Note that we do not assume $(\mathcal{F}_t)$ to be complete. This is important because we later want to change the measure, see [I], [IV], [VI], [VII] and [VIII]. Hence we will define a measure on $\mathcal{F}_t$ for each t and then extend these measures to the whole σ-algebra $\mathcal{F}$. This will only be possible, if the filtration is not completed. Assuming that $(\mathcal{F}_t)$ is right-continuous implies that τ is a stopping-time, see for instance [21]. The martingale approach of [II], [IV], [VI], [VII] and [VIII] can only be applied if τ is a stopping-time.

The quantities of interest in ruin theory are the ruin probabilities

$$\psi(u) = P[\tau < \infty], \qquad \psi(u, t) = P[\tau \le t]\,. \tag{1.3}$$

In order that $\psi(u) \ne 1$ one has to assume that $c > \lambda\mu$ where $\mu = E[Y]$. The *safety loading* $(c - \lambda\mu)/(\lambda\mu)$ is the risk premium per unit time. In the classical work, a light tail condition (*small claims*) on the distribution tail of Y is assumed. Suppose there is a solution $R > 0$, called the *adjustment coefficient*, to the equation

$$\lambda(M_Y(r) - 1) - cr = 0 \tag{1.4}$$

where $M_Y(r) = E[\exp\{rY\}]$ is the moment generating function of Y. Then

$$\psi(u) < e^{-Ru}\,. \tag{1.5}$$

Equation (1.5) is called *Lundberg's inequality.* It can be sharpened to

$$a_- e^{-Ru} \leq \psi(u) \leq a_+ e^{-Ru}$$

with

$$a_- = \inf_{0 \leq x < r_G} \frac{e^{Rx} \int_x^\infty \overline{G}(y)\,dy}{\int_x^\infty e^{Ry} \overline{G}(y)\,dy}\,, \qquad a_+ = \sup_{0 \leq x < r_G} \frac{e^{Rx} \int_x^\infty \overline{G}(y)\,dy}{\int_x^\infty e^{Ry} \overline{G}(y)\,dy}$$

where $\overline{G}(x) = 1 - G(x)$ denotes the distribution tail of Y and $r_G = \sup\{x : G(x) < 1\}$ is the right end point of the support of G, see [41]. Moreover, the asymptotic behaviour of $\psi(u)$ is found to be

$$\lim_{u \to \infty} \psi(u) e^{Ru} = \frac{c - \lambda\mu}{\lambda M_Y'(R) - c} =: C \tag{1.6}$$

where the right-hand side has to be interpreted as zero if $M_Y'(R) = \infty$. If $C \neq 0$ this gives rise to an approximation to the ruin probability, $\psi(u) \approx Ce^{-Ru}$. This approximation is called the *Cramér-Lundberg approximation.*

In actuarial mathematics the small claim condition often is not fulfilled. Many claim size distributions of interest do not have exponential moments, such as the *Pareto distribution* ($G(x) = 1 - (1 + x/\beta)^{-\alpha}$) or the *lognormal distribution* ($G(x) = \Phi((logx - m)/s)$). The latter two distributions are popular in insurance, for instance for industrial fire insurance or third liability car insurance. Most heavy tailed distributions (*large claims*) of interest belong to the class of *subexponential distributions.* A distribution is called subexponential if

$$\lim_{x \to \infty} \frac{\overline{G^{*2}}(x)}{\overline{G}(x)} = 2\,. \tag{1.7}$$

Here $G^{*n}(x)$ denotes the n-fold convolution of G. Because $P[\max\{Y_1, Y_2\} > x] = 1 - (1 - \overline{G}(x))^2 \sim 2\overline{G}(x)$ where $f(x) \sim g(x)$ means $\lim_{x \to \infty} f(x)/g(x) = 1$ it follows that for large x with large probability the sum $Y_1 + Y_2$ exceeds the level x only if one of the variables Y_1 and Y_2 exceeds the level x. Moreover, to say that G is subexponential is for each integer $n \geq 2$ equivalent to

$$\lim_{x \to \infty} \frac{\overline{G^{*n}}(x)}{\overline{G}(x)} = n\,.$$

This indicates that with large probability the sum of n random variables can only exceed the level x if one of the n variables exceeds the level x. This is indeed often observed in actuarial applications. The aggregate loss is determined by the largest loss. Another property of a subexponential distribution is

$$\lim_{x\to\infty} \frac{\overline{G}(x+z)}{\overline{G}(x)} = 1$$

for all $z \in \mathbb{R}$. This means that for all z, given that $Y > x$ for a large level x, then also $Y > x + z$ with a large probability.

The asymptotic behaviour of $\psi(u)$ in the large claim case was found by Embrechts and Veraverbeke [20]. Assume that the distribution $G_I(x) = \mu^{-1} \int_0^x \overline{G}(y)\, dy$, also called the *integrated tail distribution*, is subexponential. Then

$$\lim_{u\to\infty} \frac{\psi(u)}{\int_u^\infty \overline{G}(x)\, dx} = \frac{\lambda}{c - \lambda\mu}\,. \tag{1.8}$$

For large initial capital u this gives an approximation in the heavy tailed case.

An interesting question in this model is then, what happens if ruin occurs, and how ruin does occur. Segerdahl [57] considers the question: when does ruin occur provided ruin occurs. Dufresne and Gerber [16] and Dickson [15] consider the question: what is the capital just prior to and at ruin, if ruin occurs. In [I] their results are generalized and also the cases of negative safety loading and of no safety loading are considered. Explicit expressions in terms of ruin probabilities are obtained for the joint distribution of the surplus prior and after ruin. From that approximations for both small and large initial capital can be obtained for all cases of interest. No safety loading or even negative safety loading can occur in a free market, where one insurance type, for example motor insurance, is used to attract customers, who then also will sign contracts for other types of insurance, even though their premiums are "too" high. Note also that the premiums considered here do not include administration costs. Administration costs have to be added to the premium later.

The classical risk model serves now as a skeleton for more realistic risk models. We consider here mainly two types of generalizations. The first type includes investment and borrowing into the model. The classical risk model considers the case where the interest rate and the inflation rate cancel, and where the premium rate increases with inflation. For a discussion of this fact see [44]. In reality, however, the return from the investment of an insurance company is larger than the loss by inflation. In [II] and [III] a risk model with a constant difference between interest and inflation is considered. Moreover, borrowing is allowed. The latter may be considered as borrowing from another branch of the same insurance company. This means that "ruin" has to be replaced by "*absolute ruin*", the first time where the outgo for interest becomes larger than the premium income. Methods from Markov process theory that allow an analytical treatment of the model are described in [II]. An invariance principle to get a *diffusion approximation* to the model is obtained in [45]. That is, one considers a sequence of risk processes converging weakly to a diffusion process. One can show that the ruin probabilities in finite time then converge to the ruin probability in finite time of the diffusion process. Considering classical risk processes, also the ruin probabilities in infinite time converge to the ruin probability in infinite time of the diffusion process. The diffusion process is then considered as a approximation to the original risk model. In [III] a method going back to Siegmund [58], called *corrected diffusion approximation*, gives a refinement of the classical invariance principle introduced by Iglehart [35].

Investigation of real data shows that for some branches of insurance the classical risk model only does fit if the number of individual contracts is very large. In many branches, statistical testing shows that the Poisson distribution does not fit. The actuaries therefore started to use a *negative binomial distribution* ($P(N = n) = \binom{-\alpha}{n} p^{-\alpha}(p-1)^n$) for the number of claims in a certain time interval. The main reason for the good fit is that there are two parameters in this model. We therefore have to construct a point process with negative binomially distributed increments, or increments that approximately are negative binomially distributed. An observation is then that a negative binomial distribution can be obtained by choosing a parameter λ from a gamma distribution, and then, conditioned on λ, the number of claims is Poisson distributed with parameter λ. This is a special case of a *mixed Poisson distribution*. As a generalization, any distribution could be used for mixing. This indicates that a *mixed Poisson process*, also called *Pólya process*, has the right properties. That is, the Poisson parameter λ is stochastic. A comprehensive treatment of mixed Poisson processes can be found in the recent book [31].

Unfortunately, the mixed Poisson process is useless for our purposes. Indeed, we have that N_t/t converges almost surely to λ, which means that after some time t, $(N_{t+s} - N_t : s \geq 0)$ behaves almost like a Poisson process. What is needed is some variability that the Poisson process does not have. If we now let (λ_t) be a stationary process, and N conditioned on $(\lambda_t : 0 \leq t < \infty)$ be an inhomogeneous Poisson process, see [41], we get increments that are nearly negative binomially distributed, but the variability does not vanish.

Such a process was first considered by *Ammeter* [1] as early as 1948. He let the intensity process be constant over one year and be Γ distributed. The level of the intensity was assumed to be independent in different years. This allowed him to get a negative binomial distribution for the annual number of claims. This model was then generalized by *Björk and Grandell* [8]. They let the time in which the intensity is constant have an arbitrary distribution. The pair (level, duration) was assumed to build an iid sequence of vectors. They obtained Lundberg's inequality. The Cramér-Lundberg approximation was obtained in [VI]. A mathematical definition of the model will be given in Section 4.1.

Janssen [36] considered a semi-Markovian model, i.e. the time till the next claim and the claim size depend on an environmental Markov chain in discrete time with a finite state space, i.e. the time between the $j-1$-st and the j-th claim and the j-th claim size given the environmental Markov chain are conditionally independent of the other inter-arrival times and the other claim sizes. And the claim size distribution depends on the Markov chain via the chain at time j only. Asmussen [2] formulated the process as a *Markov modulated risk model*, i.e. a Cox model where the intensity is a Markov chain in continuous time. The intensity then also works as an environmental process, and the claim sizes are dependent on the present level of the intensity. The model will be defined in Section 4.1.

A generalization containing both the Björk-Grandell model and the Markov modulated model was constructed in [VII]. Here the intensity process is a Markov chain in continuous time with state space $[0,\infty)$. As in the Markov modulated risk model the claim sizes can depend on the level of the intensity. Lundberg bounds are obtained under some regularity conditions similar to the ones used in [8]. With the tools from [VI] one may also obtain a Cramér-Lundberg approximation in this model. In order that this is possible, regeneration points have to exist. *Regeneration points* are time instants, after which the process is dependent on the past via the state at regeneration point only, and follows the same law between regeneration points. For the construction of regeneration points the notion of *petite sets* may be useful, see for instance [39].

Alternatively, reality can be seen as the Cramér-Lundberg model plus an error. Because X is a *Lévy process* (a process with independent and stationary increments), the natural way to describe such a *perturbation*, is to add an independent Lévy process B to X. Gerber [25] uses a Brownian motion to perturb the risk process. Furrer and Schmidli [24] generalize this to other risk models. They obtain exponential bounds for the ruin probability. In [IV] also Cramér-Lundberg approximations are obtained. Furrer [23] uses an α-stable Lévy motion as a perturbation. A nice Pollaczek-Khinchin type formula is obtained. An attempt to understand what is behind this formula is given in [V]. Moreover, it is shown that the exponentially distributed time between claims is crucial for obtaining the Pollaczek-Khinchin formula.

In praxis, the claim size distributions and the intensity process have to be estimated. A small error in the estimation of the claim size distribution may yield a large error in the adjustment coefficient. Thus methods to estimate the adjustment coefficient are called for. Grandell [30] used the empirical distributions of a Cramér-Lundberg model to estimate the adjustment coefficient. Csörgő and Steinebach [11] estimated the adjustment coefficient of a Cramér-Lundberg model via order statistics of certain cycles. In the case of a Markov modulated risk model a similar method is successful, see [VIII]. Here time is reversed, which yields a storage model, see [5]. Then the maxima between two times where the content is empty is considered. The tail of the distribution of these maxima decreases then exponentially with the adjustment coefficient of the risk model as decay rate. Thus the problem is similar to the estimation of the coefficient of regular variation. In fact, *Hill's estimator* (see Chapter 6) turns out to be strongly consistent in this case. Moreover, it seems as though the estimator can also be used in the Cox model with a piecewise constant

	small claims	large claims	properties at ruin	reinsurance	estimation
classical	[I]	[I]	[I]	[X]	
interest	[II], [III]	[II]			
diffusion approximation	[III]			[X]	
perturbed	[IV], [V]	[V]			
Cox models	[VI], [VII]	[IX]			[VIII]

Table 1: *Overview of the subjects of the papers*

intensity of [VII].

For the Cox models described above the small claim case is more or less solved. Thus the question arises, what happens in the large claim case. This question was solved under some regularity conditions in [6]. Consider the aggregate claim between two regeneration points. In [6] conditions are given that assure that this distribution determines the tail of the increment between two regeneration points. Assume that both the aggregate claim size distribution and its integrated tail distribution are subexponential. Then the ruin probability behaves for large initial capital as the ruin probability of the discrete version of the model obtained by only observing the process at the regeneration times. Thus it is important to know whether the distribution of a a compound sum, its integrated tail distribution, respectively, is subexponential. If one of the distributions of the summands or the number of the summands is subexponential and the other is light tailed, it is shown in [IX] that then the compound distribution is subexponential.

Recently, methods from stochastic control theory were applied to actuarial problems, see Asmussen and Taksar [7], Højgaard and Taksar [33], [34], and Hipp and Taksar [32]. A decision actuaries have to take is to determine a retention level in reinsurance. Preferably, an optimal level should be determined. Waters [61] considers the asymptotically best strategy if the ruin probability has to be minimized. Højgaard and Taksar [33] maximize the mean discounted future surplus. In [X] the ruin probability is minimized where the reinsurance strategy for a proportional reinsurance treaty can be adapted continuously. This yields a risk process where the premium income depends on the present surplus.

A summary on the models and the subjects found in the papers [I] – [X] is given in Table 1.

Let us end this introduction by an overview of the remaining parts of the thesis. In Chapter 2 we will consider a classical risk model and investigate the surplus prior and after ruin. Chapter 3 deals with the classical risk model where interest and borrowing is included. First, the Markov process method is explained (Section 3.1), and then, the model with interest and borrowing is investigated (Section 3.2). Finally, corrected diffusion approximations are discussed (Section 3.3). In Chapter 4 we consider perturbed risk models. First, we find Cramér-Lundberg approximations for risk models perturbed by Brownian motion in the small claim case (Section 4.1). Then we generalize results by Dufresne and Gerber [17] and by Furrer [23] on the distribution of the modified "ladder heights" in perturbed risk models (Section 4.2). The Cox risk models are discussed in Chapter 5. We start by giving an extension to the renewal theorem, that can be used for obtaining Cramér-Lundberg approximations in Cox models (Section 5.1), and then we introduce a quite general Cox risk model (Section 5.2). An estimation procedure for inference on the adjustment coefficient is given in Chapter 6. Criteria for subexponentiality of compound sums are derived in Chapter 7. Finally, in Chapter 8 the optimal reinsurance strategies for a diffusion approximation (Section 8.1) and for the classical risk model (Section 8.2) are found.

2. Classical risk models

Consider now a classical risk model (1.1), where N is a Poisson process with rate λ and $(Y_k : k \in \mathbb{N})$ is an iid sequence of positive random variables independent of N. We use the notation introduced in Section 1. In [I] we are not only interested in the ruin probabilities, but in the joint distribution that ruin occurs, that the capital after ruin is below some level $-x$ and that just prior to ruin, the capital was above the level y. If $X_{\tau-}$ is small, one may recognize ruin before it occurs, and thus take action to prevent ruin. If ruin usually will happen from a high surplus $X_{\tau-}$, there is no way to react before ruin has occurred, except by underwriting reinsurance. If $-X_\tau$ will be small then ruin is not a severe event, but if $-X_\tau$ is large the whole insurance company may become bankrupt. Even a reinsurer could be affected.

The distributions considered here were introduced by Dufresne and Gerber [16] and also investigated by Dickson [15]. In their work they assumed absolute continuity of the claim size distribution and positive safety loading. In [I] the claim size distribution can be arbitrary and no positive safety loading condition is assumed.

By Markov process theory or directly, as in [I], it follows that $f(u;x,y) = P[\tau < \infty, X_\tau < -x, X_{\tau-} > y]$ is absolutely continuous with respect to u and its density fulfils the equation

$$cf'(u;x,y) + \lambda\Big(\int_0^u f(u-z;x,y)\,dG(y) + \mathbb{1}_{u\geq y}\overline{G}(u+x) - f(u;x,y)\Big) = 0 \qquad (2.1)$$

where the derivative is taken with respect to u. From this the following two equations follow

$$\hat{f}(s;x,y) = \int_0^\infty e^{-su} f(u;x,y)\,du = \frac{cf(0;x,y) - \lambda\int_y^\infty \overline{G}(z+x)e^{-sz}\,dz}{cs - \lambda(1-M_Y(-s))}\,, \qquad (2.2)$$

where as before $M_Y(r)$ is the moment generating function, and

$$c(f(u;x,y) - f(0;x,y)) = \lambda\int_0^u f(u-z,x,y)\overline{G}(z)\,dz - \mathbb{1}_{u>y}\lambda\int_y^u \overline{G}(z+x)\,dz\,. \quad (2.3)$$

Equation (2.2) is obtained by multiplying (2.1) by e^{-su} and then integrating over $(0,\infty)$, and equation (2.3) is just obtained by integration over $(0,u]$. The equations above constitute the key point in analysing the function $f(u;x,y)$.

2.1. Positive safety loading

First we have to find $f(0;x,y)$ in the case $c > \lambda\mu$. Because $f(u;x,y) \leq \psi(u) \to 0$ as $u \to \infty$ we obtain

$$f(0;x,y) = \frac{\lambda}{c}\int_y^\infty \overline{G}(z+x)\,dz$$

by letting $u \to \infty$ in (2.3). Using $f(u;0,0) = \psi(u)$ the Laplace transform (2.2) can be inverted and yields

$$f(u;x,y) = \frac{\lambda}{c-\lambda\mu}\big(\bar\psi(u)\int_y^\infty \overline{G}(z+x)\,dz - \mathbb{1}_{u>y}\int_y^u \bar\psi(u-z)\overline{G}(z+x)\,dz\big) \quad (2.4)$$

where $\bar\psi(u) = 1-\psi(u)$. Equation (2.4) does not give an explicit expression for $f(u;x,y)$. However, there is a large literature on the calculation of $\psi(u)$, which can then also be used to calculate $f(u;x,y)$.

Often, one is interested in $f(u;x,y)$ for large u. A limit can be found if for each $z \in \mathbb{R}$ the limit

$$\gamma(z) = \lim_{u\to\infty}\frac{\psi(u+z)}{\psi(u)}$$

exists. In this case

$$\lim_{u\to\infty}\frac{f(u;x,y)}{\psi(u)} = \frac{1}{c-\lambda\mu}\big(c\gamma(x) - \lambda\int_0^{y+x}\gamma(x-z)\overline{G}(z)\,dz - \lambda\int_{x+y}^\infty \overline{G}(z)\,dz\big). \quad (2.5)$$

The cases where the limit $\gamma(z)$ is known are described in [20]. Namely, in the small claim case $\gamma(z) = e^{-Rz}$ for some $R > 0$, where R is the adjustment coefficient in the Cramér case. The most interesting case is the subexponential case. Assume the distribution function $G_I(u) = \mu^{-1}\int_0^u \overline{G}(z)\,dz$ is subexponential. Then $\gamma(z) = 1$ for all $z \in \mathbb{R}$. This gives immediately that $f(u;x,y) \sim \psi(u)$, so for fixed x and y we do not get any information on $(X_{\tau-}, -X_\tau)$. Let us try with $x = 0$ and $y = u$. Then

$$f(u;0,u) = \frac{\lambda}{c-\lambda\mu}\bar\psi(u)\int_u^\infty \overline{G}(z)\,dz \sim \psi(u)$$

as $u \to \infty$. Thus for large u we have $X_{\tau-} > u$ with a large probability. Trying functions $x(u)$ and $y(u) \ge u$ we get

$$f(u;x(u),y(u)) = \frac{\lambda}{c-\lambda\mu}\bar\psi(u)\int_{x(u)+y(u)}^\infty \overline{G}(z)\,dz \sim \psi(x(u)+y(u))\,.$$

The asymptotic behaviour of $f(u;x,y)$ can now be found for the two main classes of subexponential distributions.

Regularly varying tail Assume that $\overline{G}(z) = L(z)z^{-\alpha}$ for some $\alpha \ge 1$ and some *slowly varying function* $L(z)$, i.e. $L(tz)/L(z)$ converges to one as $z \to \infty$. Then $\psi(u) \sim CL(z)u^{-(\alpha-1)}$ for some constant C and

$$\lim_{u\to\infty}\frac{f(u;au,bu)}{\psi(u)} = (a+b)^{-(\alpha-1)}$$

provided $a \ge 0$ and $b \ge 1$.

Maximum domain of attraction of the Gumbel distribution A distribution function G is said to belong to the *maximum domain of attraction* of a extremal distribution H if there exist numbers a_n and b_n such that $(G(a_n x + b_n))^n$ converges pointwise to $H(x)$. For an introduction to extremal theory see [18]. If the tail of G is not regularly varying, then under mild assumptions, see [29], we find that G is in the maximum domain of attraction of the *Gumbel distribution* ($H(x) = \exp(-e^{-x})$). In this case, with $a(z) = E[Y - u \mid Y > u]$, we have

$$\lim_{u\to\infty} \frac{f(u; x(u), u + za(u) - x(u))}{\psi(u)} = e^{-z}$$

provided $x(u) \leq za(u)$.

2.2. Negative safety loading

Assume now $c < \lambda\mu$, with the possibility that $\mu = \infty$. This situation may occur in practise. An insurance company does not know the exact claim size distribution nor the claim arrival intensity. To estimate the claim arrival intensity is no problem. The estimator N_t/t converges exponentially fast. The situation is completely different for the mean value μ. The mean value may be determined by the distribution function far out in the tail, a region where one usually not does have any observations. Therefore, the premium estimate of the insurance company might give a premium that yields negative safety loading. In this case we do not know beforehand whether $f(u; x, y)$ converges as $u \to \infty$. Thus $f(0; x, y)$ cannot be found from (2.3) by simply considering the limit as $u \to \infty$. Note that the numerator of (2.2) has a strictly positive root R, i.e. $cR - \lambda(1 - M_Y(-R)) = 0$. Because $\hat{f}(R; x, y) \leq R^{-1}$ we must have that also the denominator is zero. This yields

$$f(0; x, y) = \frac{\lambda}{c} \int_y^\infty \overline{G}(x + z) e^{-Rz}\, dz\,. \tag{2.6}$$

Introducing the distribution $G_Q(z) = (M_Y(-R))^{-1} \int_0^z e^{-Rv}\, dG(v)$ and the parameter $\lambda_Q = M_Y(-R)\lambda$ we get a risk model $\tilde{X}$ with claim intensity λ_Q and claim size distribution $G_Q(z)$. This model may also be obtained by a change of measure argument. In this model $c > \lambda_Q \mu_Q$, and thus its ruin probability $\psi_Q(u) < 1$. Inversion of (2.2) yields then

$$\begin{aligned} f(u; x, y) = \frac{\lambda e^{Ru}}{c - \lambda M_Y'(-R)} \Big(&(1 - \psi_Q(u)) \int_y^\infty \overline{G}(z + x) e^{-Rz}\, dz \\ &- 1\!\!1_{u>y} \int_y^u (1 - \psi_Q(u - z)) \overline{G}(x + z) e^{-Rz}\, dz \Big). \end{aligned} \tag{2.7}$$

Besides the value at zero we also can find $f(u; x, y)$ for large initial capital, namely

$$\lim_{u\to\infty} f(u; x, y) = \frac{\lambda}{\lambda\mu - c} \int_y^\infty (1 - e^{-Rz}) \overline{G}(x + z)\, dz$$

if $\mu < \infty$ and $\lim_{u\to\infty} f(u; x, y) = 1$ if $\mu = \infty$.

2.3. No safety loading

If $c = \lambda\mu$ then $0 \leq s\hat{f}(s; x, y) \leq 1$, from which we find

$$f(0; x, y) = \frac{1}{\mu} \int_y^\infty \overline{G}(z + x)\, dz\,. \tag{2.8}$$

Plugging this into (2.3) gives

$$f(u; x, y) = \frac{1}{\mu} \int_0^u f(u - z; x, y)\overline{G}(z)\, dz + \frac{1}{\mu} \int_{u \vee y}^\infty \overline{G}(x + z)\, dz\,. \tag{2.9}$$

The latter is an ordinary *renewal equation*. For an introduction to *renewal theory* see for instance [22] or [41]. In this case $G_I(z)$ is the ladder-height distribution. We find

$$f(u; x, y) = \frac{1}{\mu}\Big(\int_{y+x}^\infty \overline{G}(z)\, dz\, U(u) - 1\!\!1_{u>y} \int_y^u U(u - z)\overline{G}(z + x)\, dz \Big) \tag{2.10}$$

where $U(z) = \sum_{k=0}^\infty G_I^{*k}(z)$ is the renewal measure.

The value of $f(u; x, y)$ for large u follows then from the key renewal theorem. If $E[Y^2] < \infty$, i.e. if $G_I(z)$ is a distribution with finite mean we have

$$\lim_{u\to\infty} f(u; x, y) = \frac{\int_y^\infty z\overline{G}(z + x)\, dz}{\int_0^\infty z\overline{G}(z)\, dz}\,.$$

If $E[Y^2] = \infty$ then $\lim_{u\to\infty} f(u; x, y) = 1$. Note that the behaviour is similar to the negative safety loading case. However, finite second moment is needed in order to obtain a non-trivial limit, whereas in the negative safety loading case only finite first moment was required. This has to do with the *ladder-height distributions* (2.8) and (2.6). The former has finite mean iff the claim size distribution has finite second mean, the latter iff the claim size distribution has finite mean.

3. Risk models with interest and borrowing

3.1. The Markov process method

In [II] and [III] the classical risk model is enlarged to allow for interest and borrowing. The tool used for the analysis are the *piecewise deterministic Markov processes* (PDMP) introduced in [13] and [14]. Let us first recall some facts from Markov process theory. Let X be a Markov process with (full) generator $\mathfrak{A}$. The *full generator* $\mathfrak{A}$ is the set of functions (f, g) such that

$$\big(f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds\big)$$

is a martingale. The *domain of the generator* is the set $\mathcal{D}(\mathfrak{A}) = \{f : \exists g, (f, g) \in \mathcal{A}\}$. We often write $\mathfrak{A}f$ for a version g of all the functions g such that $(f, g) \in \mathfrak{A}$. Let $f(x) \geq 0$ be an increasing function with $\mathfrak{A}f(x) = 0$ and $f(x) = 0$ for $x < -b$, for some value $b \in \mathbb{R}$. In particular, we assume that $f \in \mathcal{D}(\mathfrak{A})$. For a PDMP this means that f must be absolutely continuous along the deterministic paths, has to fulfil some boundary condition and an integration condition, see [13]. In this case, the equation $\mathfrak{A}f(x) = 0$ is just an integro-differential equation, as for example (2.1) with $x = y = 0$. We now have that $(f(X_t)1\!\!1_{\sup\{X_s : 0 \leq s \leq t\} \geq -b})$ is a martingale.

Let $\tau = \inf\{t \geq 0 : X_t < -b\}$ denote the first passage time of the boundary $-b$. Because the martingale $(f(X_{\tau\wedge t}))$ is positive, it follows by the martingale convergence theorem that $f(X_\tau) = \lim_{t\to\infty} f(X_{\tau\wedge t})$ exists and is integrable. For the processes considered here one has that $X_t \to \infty$ on $\{\tau = \infty\}$. This implies that either $f(\infty) < \infty$ or $P[\tau < \infty] = 1$. If $f(\infty) < \infty$ then the martingale stopping theorem yields

$$f(X_0) = E[f(X_\tau)] = f(\infty)P[\tau = \infty]$$

because by the choice of f, $f(X_\tau) = 0$ on $\{\tau < \infty\}$. This shows that any solution to $\mathfrak{A}f(x) = 0$ is a multiple of $P[\tau = \infty]$. So the problem left is just to determine the solution to $\mathfrak{A}f(x) = 0$ and to verify that $f \in \mathcal{D}(\mathfrak{A})$. Note that $f \in \mathcal{D}(\mathfrak{A})$ will automatically follow if the solution f is bounded and fulfils the boundary conditions, see [12]. Martingale methods were introduced to actuarial mathematics by Gerber [27]. An approach similar to the one presented above can also be found in [28].

3.2. A model with interest and borrowing

Let (C_t) be a classical risk model (1.1). We consider the process (X_t) fulfilling

$$dX_t = 1\!\!1_{X_t \geq \Delta}\beta_1(X_t - \Delta)\,dt + 1\!\!1_{X_t < 0}\beta_2 X_t\,dt + dC_t$$

where β_1 is the interest short rate paid for surplus above the level $\Delta \geq 0$ and β_2 is the interest short rate that has to be paid for borrowed money. For surplus in $[0, \Delta]$

no interest is paid. The level Δ can be interpreted as the amount the company wants to keep as a liquid reserve, and therefore has to be invested at a lower return.

As time of ruin we consider here the time $\tau = \inf\{t : \beta_2 X_t \le -c\}$, called *absolute ruin time*, where the outgo for interest becomes larger than the premium income. Note that almost surely $\{\tau < \infty\} = \{\lim_{t\to\infty} X_t = -\infty\}$.

A special case is the model considered by *Gerber* [26]. He let $\Delta = 0$ and $\beta_1 = \beta_2$. Another special case is the model considered by *Dassios and Embrechts* [12]. They let $\Delta = \infty$.

The model was considered in [44] and [19]. It was observed there, that the solution can be found in three steps. First, find the solution to Gerber's model, yielding the solution for negative x. Second, find the solution to the Dassios-Embrechts model, yielding the solution for $x \le \Delta$. Third, find the solution to the model considered here. The method is illustrated for exponentially distributed claim sizes. For general claim size distributions the Laplace transforms of the desired functions

$$f(x) = 1\!\!1_{-c/\beta_2 \le x < 0} f_3(x) + 1\!\!1_{0 \le x < \Delta} f_2(x) + 1\!\!1_{x \ge \Delta} f_1(x)$$

are found in [19],

$$\hat{f}_3(s) = K\frac{1}{s}\exp\{\frac{cs}{\beta_2} - \frac{\lambda}{\beta_2}\int_0^s \frac{1 - M_Y(-\xi)}{\xi}\,d\xi\}\,, \tag{3.1}$$

$$\hat{f}_2(s) = \frac{cf_3(0) - h_2(s)}{cs - \lambda(1 - M_Y(-s))}\,, \qquad s > s_0,$$

where $h_2(s) = \int_0^\infty \int_x^{x+c/\beta_2} f_3(x-y)\,dG(y)e^{-sx}\,dx$ and $s_0 = \sup\{s \ge 0 : cs - \lambda(1 - M_Y(-s)) = 0\}$,

$$\hat{f}_1(s) = \frac{\exp\{\beta_1^{-1}(cs - \lambda\int_0^s \frac{1-M_Y(-\xi)}{\xi}\,d\xi)\}}{\beta_1 s e^{\Delta s}} \int_s^\infty \frac{cf_2(\Delta) - h_1(\eta)e^{\Delta\eta}}{\exp\{\beta_1^{-1}(c\eta - \lambda\int_0^\eta \frac{1-M_Y(-\xi)}{\xi}\,d\xi)\}}\,d\eta\,,$$

where

$$h_1(s) = \lambda\int_\Delta^\infty \Big[\int_{x-\Delta}^x f_2(x-y)\,dG(y) + \int_x^{x+c/\beta_2} f_3(x-y)\,dG(y)\Big]e^{-sx}\,dx\,.$$

Here the functions are considered to be defined on the intervals $[-c/\beta_2, \infty)$, $[0, \infty)$ and $[\Delta, \infty)$, respectively.

In [II] we find finally an explicit inversion formula for the Laplace transform for Gerber's model, which is the function f_3. Moreover, a dual *shot-noise process* is found. We observe that

$$\{\tau \le t\} = \{\sum_{k=1}^{N_t} Y_k e^{-\beta_2 T_k} > u + \frac{c}{\beta_2}\}$$

where (T_k) are the claim times. The quantity

$$S_t = \sum_{k=1}^{N_t} Y_k e^{-\beta_2 T_k}$$

has the same distribution as a shot-noise process at time t, starting at zero. Thus the finite time ruin probability can be found via shot-noise theory. In particular, $S = S_\infty$ has the stationary distribution of a shot noise process. From (3.1) we obtain the well-known characteristic function of the stationary distribution of a shot-noise process

$$\psi_S(\vartheta) = E[e^{i\vartheta S}] = \exp\{-\frac{\lambda}{\beta_2}\int_0^\vartheta \frac{1-\psi_Y(\xi)}{\xi}\,d\xi\} \tag{3.2}$$

where $\psi_Y(\xi) = E[e^{i\xi Y}]$ is the characteristic function of Y. Using the well-known inversion formula for the characteristic function — some technical condition has to be verified — we find

$$f_3(x - c/\beta_2) = \frac{1}{\pi}\int_0^\infty \big(\frac{1-\cos\vartheta x}{\vartheta}\sin D(\vartheta) + \frac{\sin\vartheta x}{\vartheta}\cos D(\vartheta)\big)e^{-C(\vartheta)}\,d\vartheta \tag{3.3}$$

where

$$C(\vartheta) = \frac{\lambda}{\beta_2}\int_0^\vartheta\int_0^\infty \frac{1-\cos\xi z}{\xi}\,dG(z)\,d\xi$$

and

$$D(\vartheta) = \frac{\lambda}{\beta_2}\int_0^\vartheta\int_0^\infty \frac{\sin\xi z}{\xi}\,dG(z)\,d\xi\,.$$

This formula may be useful for numerical calculation of f_3.

3.3. Diffusion approximations

Because it is difficult to obtain explicit expressions for the ruin probabilities, approximations are called for. A nice approximation is the *diffusion approximation*, introduced to actuarial mathematics by Iglehart [35], see below. Unfortunately, diffusion approximations do not work well, unless the safety loading $(c - \lambda\mu)/(\lambda\mu)$ is very small. In queueing theory such approximations are called *heavy traffic approximations*. In [45] it is, however, shown that for the risk model with interest and borrowing, diffusion approximations work reasonably well. These approximations were improved in [III]. The reason that diffusion approximations work better in the case with interest than without interest is that the ruin probabilities decrease very fast. Hence approaching the region, where the diffusion approximation is far from the correct value, the ruin probability is very small anyway.

Let now $((C_t^{(n)}) : n \in \mathbb{N})$ be a sequence of classical risk models. Assume that $(C_t^{(n)})$ *converges weakly* to a diffusion process (C_t), that is $\lim_{n\to\infty} E[f(C^{(n)})] = E[f(C)]$ for all bounded continuous (with respect to the topology on the space of cadlag functions) functionals f. For *classical diffusion approximations* the parameters are chosen as $c^{(n)} = c + \lambda\mu(\sqrt{n} - 1)$, $\lambda^{(n)} = n\lambda$ and $G^{(n)}(x) = G(x\sqrt{n})$. If $E[Y^2] < \infty$ then the limit process (C_t) exists and is a Brownian motion with drift parameter $c - \lambda\mu$ and diffusion parameter $\lambda E[Y^2]$.

Consider now the corresponding processes with the possibility of investment and borrowing

$$dX_t^{(n)} = \delta(X_t^{(n)})\,dt + dC_t^{(n)}\,,$$
$$dX_t = \delta(X_t)\,dt + dC_t\,,$$

where $\delta(x) = 1\!\!1_{x<0}\beta_2 x + 1\!\!1_{x\ge\Delta}\beta_1(x-\Delta)$. It is shown in [45] that then $(X_t^{(n)})$ converges weakly to (X_t). Denoting the absolute ruin times by $\tau^{(n)} = \inf\{t \ge 0 : X_t^{(n)} < c^{(n)}/\beta_2\}$ we observe that almost surely $\{\tau^{(n)} < \infty\} = \{X_t^{(n)} \to -\infty\}$. Because for a non-degenerated diffusion process absolute ruin is difficult to define, we consider $\{X_t \to -\infty\}$ as the event that ruin occurs. Then, see [45], it follows that $\lim_{n\to\infty} P[X_t^{(n)} \to -\infty] = P[X_t \to -\infty]$ provided that $\overline{\lim}_{n\to\infty} \lambda^{(n)} E[(Y_1^{(n)})^2] < \infty$. The latter condition is very weak. Indeed, if $\lambda^{(n)} E[(Y_1^{(n)})^2]$ is unbounded, the variance of $C_1^{(n)}$ would be unbounded and a diffusion approximation would not make sense.

Let $\theta(r) = \lambda(M_Y(r)-1)-cr$ and assume that there is a R_0 such that $\theta'(R_0) = 0$. We assume that $M_Y''(R_0) < \infty$. R_0 is uniquely defined, because $\theta(r)$ is a strictly convex function. Siegmund [58] suggested the following *corrected diffusion approximation*. Let $c^{(n)} = \sqrt{n}c$, $\lambda^{(n)} = \lambda n M_Y((1-n^{-\frac{1}{2}})R_0)$ and

$$G^{(n)}(x) = \frac{\int_0^{\sqrt{n}x} e^{(1-n^{-1/2})R_0 y}\,dG(y)}{M_Y((1-n^{-1/2})R_0)}\,.$$

The idea is the following. Assume for the moment that the adjustment coefficient R exists. For simplicity let $u = 0$. For $-R_0 \le r \le R - R_0$ we have that the process $(L_t = e^{-(r+R_0)C_t - \theta(r+R_0)t})$ is a martingale with mean value one. We can define the new measure P_r via $dP_r/dP = L_t$ on $\mathcal{F}_t$. This measure is independent of the choice of t and can be extended to a measure on $\mathcal{F}$. On $\mathcal{F}$, however, P_r and P are singular, except if $r = -R_0$. Note that $P = P_{-R_0}$. For an introduction to change of measure techniques see [41]. It turns out that under the measure P_r the process (C_t) remains a classical risk model. Under the measure P_r the intensity is $\lambda_r = \lambda M_Y(R_0 + r)$ and the claim size distribution is $G_r(x) = \int_0^x e^{(R_0+r)y}\,dG(y)/M_Y(R_0+r)$. The ruin probability in finite time can be expressed as

$$P[\tau \le t] = E_r[e^{(R_0+r)C_\tau + \theta(R_0+r)\tau}; \tau \le t]\, e^{-(R_0+r)u}\,.$$

In particular, the expression is useful, if $r = R - R_0$ because $\theta(R) = 0$. For the measure P_r the corresponding function $\theta_r(s)$ is given by $\theta_r(s) = \theta(R_0 + r + s) - \theta(R_0 + r)$. For each $r < 0$ there exists an $r' > 0$ such that $\theta(R_0 + r) = \theta(R_0 + r')$. Assume now r changes also with n, that is we have a sequence $r_n < 0$. In a classical diffusion approximation $r_n = -R_0$ for all n. Inspired by the classical case we choose $\lambda^{(n)} = n\lambda_{r_n}$ and $G^{(n)}(x) = G_{r_n}(\sqrt{n}x)$. The ruin probability is then expressed as

$$P_{r_n}[\tau^{(n)} \le t] = E_{r_n'}[e^{(r_n'-r_n)C^{(n)}(\tau^{(n)})}; \tau^{(n)} \le t]\,. \tag{3.4}$$

Now

$$E_{r'_n}[C_t^{(n)}] = c^{(n)} - \lambda^{(n)}\mu^{(n)} = c^{(n)} - \sqrt{n}\lambda M'_Y(r_n)\,.$$

This indicates that $c^{(n)}$ should increase at rate $\sqrt{n}$. Otherwise, we cannot get weak convergence. This is fulfilled for a classical diffusion approximation. The idea of the corrected diffusion approximation is, that the exponent in (3.4) does not explode. Therefore $r'_n - r_n$ should decrease at rate $n^{-1/2}$. This is obtained by the choice above.

We get that $(C^{(n)})$ converges weakly to a Brownian motion C with drift coefficient $\lambda R_0 M''_Y(R_0)$ and diffusion coefficient $\lambda M''_Y(R_0)$. This limiting process has a larger drift and a larger diffusion coefficient than the classical diffusion approximation. Indeed,

$$\lambda\int_0^\infty R_0 x^2 e^{R_0 x}\,dG(x) > \lambda\int_0^\infty x(e^{R_0 x}-1)\,dG(x) = c - \lambda\mu$$

and $M''(r)$ is strictly increasing in r.

In [III] the ruin probabilities of the Cramér-Lundberg model with exponential claim sizes are compared with ruin probabilities of the corresponding diffusion approximations. It turns out that the error is quite small as long as the exact ruin probability is not too small.

Unfortunately, the method can only be applied if the claim size distribution allows exponential moments. In many cases of interest, this is not the case. Therefore the method can only be used in the small claim case.

4. Perturbed risk models

Let (C_t) be a risk model. We introduce another source of randomness, a Lévy process (B_t), and consider the *perturbed risk model* X defined as $X_t = C_t + \eta B_t$ where $\eta > 0$. It is assumed that C and B are independent. The model with $\eta = 0$ is called the *unperturbed risk model*.

The case where C is a *Cramér-Lundberg model* and B is a Brownian motion was considered in [25], [17], and [60]. Exponential inequalities for the probability of ruin for more general models C were obtained in [24]. More general perturbation processes B were considered in [23] and [43].

4.1. Cramér-Lundberg approximations

For the perturbed Cramér-Lundberg model the Cramér-Lundberg approximation was already obtained in [25]. Furrer and Schmidli [24] conjectured, that a Cramér-Lundberg approximation also holds for other models, where a Cramér-Lundberg approximation in the unperturbed case holds. Indeed, this approximation can be proved via change of measure. Let us explain the method in the case of a *perturbed Cramér-Lundberg model*.

In order to avoid $\psi(u) = 1$ for all u we have to assume the *net profit condition* $c > \lambda\mu$. The *adjustment coefficient* is the strictly positive solution R to

$$\lambda(M_Y(R) - 1) - cR + \tfrac{1}{2}\eta^2 R^2 = 0$$

provided such a solution exists. The process $(e^{-R(X_t-u)})$ is then a martingale with mean value one. Let $(\mathcal{F}_t)$ be the smallest (uncompleted) right-continuous filtration such that $((C_t, B_t))$ is adapted. On $\mathcal{F}_t$ the new measure Q defined as $Q[A] = E_P[e^{-R(X_t-u)}; A]$ is well-defined. Moreover, the martingale property implies that the definition is independent of t. The choice of the filtration implies that these measures can be extended to a measure Q on $\mathcal{F}$. However, P and Q are singular on $\mathcal{F}$. Moreover, if T is a stopping-time and $A \in \mathcal{F}_T$, $A \subset \{T < \infty\}$, then one obtains from the optional sampling theorem the useful formula $Q[A] = E_P[e^{-R(X_T-u)}; A]$. Thus the ruin probability can be expressed as

$$\psi(u) = E_P[e^{-R(X_\tau-u)}e^{R(X_\tau-u)}; \tau < \infty] = E_Q[e^{RX_\tau}; \tau < \infty]e^{-Ru}\,.$$

For an introduction to change of measure techniques see for instance [41]. Investigation on the law of X under Q shows that the process is a perturbed classical risk model with parameters $\tilde{\lambda} = \lambda M_Y(R)$, $\tilde{G}(x) = \int_0^x e^{Rx}\,dG(x)/M_Y(R)$ and $\tilde{c} = c - \eta^2 R$. Note that, in contrast to the unperturbed case, the premium rate c changes. This is a consequence of Girsanov's theorem. The process has negative safety loading, implying $Q[\tau < \infty] = 1$. This simplifies the ruin probability to

$$\psi(u) = E_Q[e^{RX_\tau}]e^{-Ru}\,.$$

As a consequence, *Lundberg's inequality* $\psi(u) \le e^{-Ru}$ follows immediately.

Dufresne and Gerber [17] considered the probabilities $\psi_d(u) = P[\tau < \infty, X_\tau < 0]$ and $\psi_c(u) = P[\tau < \infty, X_\tau = 0]$. Then by change of measure we find the expressions

$$\begin{aligned}\psi_d(u) &= E_Q[e^{RX_\tau}; X_\tau < 0]e^{-Ru}\,,\\ \psi_c(u) &= E_Q[e^{RX_\tau}; X_\tau = 0]e^{-Ru} = Q[X_\tau = 0]e^{-Ru}\,.\end{aligned}$$

In order to obtain *Cramér-Lundberg approximations* we only have to show that $f_d(u) = E_Q[e^{RX_\tau}; X_\tau < 0]$ and $f_c(u) = Q[X_\tau = 0]$ converge to non-zero limits as $u \to \infty$.

Let $T_1, T_2, \dots$ be the claim arrival times. We define $\tau_+ = \inf\{T_i : i > 0, X_{T_i} < \inf\{X_s : 0 \le s < T_i\}\}$, the *(modified) ladder epoch*, $L_c = \sup\{u - X_t : 0 \le t < \tau_+\}$ and if $\tau_+ < \infty$ let $L_d = u - X_{\tau_+} - L_c$. Then $L_c + L_d$ is the *(modified) ladder-height* if $\tau_+ < \infty$, L_c is the part of the ladder-height due to the perturbation, L_d the part due to the jump. Note that L_c is defined also on $\{\tau_+ = \infty\}$, and that $Q[\tau_+ < \infty] = 1$. Denoting by $H(x,y) = Q[L_c \le x, L_d \le y]$, $H_c(x) = Q[L_c \le x]$, $H_d(x) = Q[L_d \le x]$ and $\tilde{H}(x) = Q[L_d + L_c \le x]$ the *ladder-height distributions* we obtain the following *renewal equations*

$$\begin{aligned}f_d(u) &= \int_0^u f_d(u-x)\,d\tilde{H}(x) + \int_0^u \int_{u-x}^\infty e^{R(u-x-y)}\,H(dx,dy)\,,\\ f_c(u) &= \int_0^u f_c(u-x)\,d\tilde{H}(x) + 1 - H_c(u)\,.\end{aligned}$$

That the limits exist is then obtained from the key renewal theorem, see [22] or [41], and can be expressed as

$$\begin{aligned}\lim_{u\to\infty} E_Q[e^{RX_\tau}; X_\tau < 0] &= \frac{1 - E_Q[e^{-RL_d}]}{RE_Q[L_d + L_c]}\,,\\ \lim_{u\to\infty} Q[X_\tau = 0] &= \frac{E_Q[L_c]}{E_Q[L_d + L_c]}\,.\end{aligned}$$

In order that the limits will be different from zero we have to assume that $E_Q[Y] < \infty$. The constants can be obtained from

$$E_Q[f(L_c, L_d)] = E_P[f(L_c, L_d)e^{R(L_c+L_d)}; \tau_+ < \infty]\,.$$

Note that the explicit distribution of (L_c, L_d) under P is obtained in Section 4.2.

If one goes away from the Cramér-Lundberg model for the unperturbed risk process the situation becomes harder. The process X is not Markovian anymore. The trick is to *Markovize* the process, i.e. to introduce new variables, such that the extended process becomes Markovian. But then the martingale becomes more complicated. In order that a Cramér-Lundberg approximation exists extra conditions have to be introduced.

Consider a *renewal risk model*. This is a model C of the form (1.1) where (Y_k) is an iid sequence of positive random variables independent of N and N is a renewal process, see [22] or [41]. Let $T_1, T_2, \ldots$ denote the claim arrival times. For simplicity we only consider the ordinary case, that is we assume there is a claim at $T_0 = 0$. Let $T = T_1$ and denote its distribution function by F. The natural way to Markovize the process would be to consider the process $((X_t, W_t))$ where $W_t = t - T_{N_t}$. But it turns out that this is not convenient. In order to apply the method one would have to assume that F is (piecewise) absolutely continuous and the martingale to consider would be very complicated. An alternative is to consider $((X_t, V_t))$ with $V_t = T_{N_t+1} - t$. Note that the filtration $(\mathcal{F}_t)$ is then different from the natural filtration $(\mathcal{F}_t^X)$ of X. In fact, at any time it is known when the next claim will arrive. Even though this filtration is not observable the method yields the desired results.

Let R be the strictly positive solution to $M_Y(R)M_T(\eta^2 R^2/2 - cR) = 1$, provided such a solution exists. Here $M_T(r) = E[e^{rT}]$ is the moment generating function of T. R is then called the *adjustment coefficient*. The process

$$L_t = M_Y(R)\, e^{-(cR-\eta^2 R^2/2)V_t} e^{-R(X_t-u)}$$

is a martingale with mean value one. As before we define the new measure Q via $Q[A] = E_P[L_t; A]$ for $A \in \mathcal{F}_t$. Note that Q can be extended to $\mathcal{F}$. Under the measure Q the process X is again a perturbed renewal risk model, with parameters $\tilde{c} = c - \eta^2 R$, $\tilde{G}(x) = M_T(\eta^2 R^2/2 - cR) \int_0^x e^{Ry}\, dG(y)$ and $\tilde{F}(t) = M_Y(R) \int_0^t e^{-(cR-\eta^2 R^2/2)s}\, dF(s)$. This implies that, under the measure Q the process X has negative drift, and therefore $Q[\tau < \infty] = 1$.

As in the perturbed Cramér-Lundberg model the ruin probabilities can be expressed as

$$\begin{aligned}\psi_d(u) &= M_Y(R) E_Q[e^{(cR-\eta^2 R^2/2)V_\tau} e^{RX_\tau}; X_\tau < 0] e^{-Ru} = E_Q[e^{RX_\tau}; X_\tau < 0] e^{-Ru}\,,\\ \psi_c(u) &= M_Y(R) E_Q[e^{(cR-\eta^2 R^2/2)V_\tau}; X_\tau = 0] e^{-Ru}\,.\end{aligned}$$

As in the Cramér-Lundberg case it follows that

$$\lim_{u\to\infty} \psi_d(u) e^{Ru} = \frac{1 - E_Q[e^{-RL_d}]}{R E_Q[L_d + L_c]} = C_d$$

and the limit is different from zero if $E_Q[Y] < \infty$. The limit for $\psi_c(u)e^{Ru}$ is harder to obtain. The problem is, that we have to estimate $E_Q[e^{(cR-\eta^2 R^2/2)V_\tau}; X_\tau = 0]$, but we do not know the distribution of V_τ on the set $\{X_\tau = 0\}$. In order to get around the problem, we condition on τ, N_τ and $\{T_1, T_2, \ldots, T_{N_\tau}\}$. Then V_τ has conditionally the same distribution as $T - x$ given $T > x$ with $x = \tau - T_{N_\tau}$. This yields the condition

$$\sup_{x\ge 0} E_Q[e^{(cR-\eta^2 R^2/2)(T-x)} \mid T > x] < \infty$$

or equivalently

$$\inf_{x\ge 0} E_P[e^{-(cR-\eta^2 R^2/2)(T-x)} \mid T > x] > 0\,. \tag{4.1}$$

In [IV] the stronger (but easier to verify) condition

$$\sup_{x\geq 0} E_P[T-x \mid T>x] < \infty$$

is used. Under condition (4.1) we find that the limit

$$\lim_{u\to\infty} \psi_c(u)e^{Ru} = C_c$$

exists. Note that in general neither C_c nor C_d can be found in closed form. The above results are the *Cramér-Lundberg approximations.*

If we only assume that R exists it is without further assumptions possible to show that

$$\lim_{u\to\infty} \psi(u)e^{(R-\epsilon)u} = 0\,,$$
$$\lim_{u\to\infty} \psi(u)e^{(R+\epsilon)u} = \infty\,,$$

for any $\epsilon > 0$. The first result is obtained by considering inter-arrival times with distribution $F^{(n)}(x) = P[T \wedge n \leq x]$, see [24]. The second result is proved by considering the ruin time $\tilde{\tau} = \inf\{T_k : X_{T_k} < 0\}$. It readily follows that $\tilde{\tau} \geq \tau$. If $E_Q[Y] < \infty$ then the approach above shows that

$$\lim_{u\to\infty} P[\tilde{\tau} < \infty]e^{Ru}$$

exists and is strictly positive. If $E_Q[Y] = \infty$ the result follows from considering the model $X^{(n)}$ with claim size distribution $G^{(n)}(x) = P[Y \wedge n \leq x]$. The corresponding adjustment coefficients fulfil $R_n \downarrow R$ as $n \to \infty$. Clearly the ruin probabilities fulfil $\psi_n(u) \leq \psi(u)$. If we now choose n such that $R_n - R < \epsilon$ then it follows that

$$\psi(u)e^{(R+\epsilon)u} \geq \psi_n(u)e^{(R_n+\epsilon-(R_n-R))u} \to \infty$$

which extends Theorem 2 of [IV].

Next we consider the perturbed *Björk-Grandell model.* For simplicity we only consider the ordinary case. In general, the variable (L_1, σ_1) considered below could have a distribution different from (L_i, σ_i), where $i \geq 2$. Let $((L_i, \sigma_i) : i \in \mathbb{N})$ be a sequence of iid vectors with distribution function $F(\ell, s)$. By $(L, \sigma) = (L_1, \sigma_1)$ we denote a generic vector. By $S_i = \sum_{j=1}^{i} \sigma_j$ we denote the times where the intensity changes and the intensity process λ is defined as $\lambda_t = L_i$ for $S_{i-1} \leq t < S_i$. Let $\tilde{N}$ be a Poisson process with rate one independent of $((L_i, \sigma_i))$. Then the claim arrival process N is $N_t = \tilde{N}(\int_0^t \lambda_s\, ds)$. The claim sizes $(Y_i : i \in \mathbb{N})$ are assumed to be iid and independent of λ and N. The *net profit condition* $cE_P[\sigma] > E_P[L\sigma]$ is assumed in the sequel. The process C is again defined by (1.1).

The *adjustment coefficient* R is the strictly positive solution to

$$E_P[\exp\{(L(M_Y(R)-1) - cR + \eta^2R^2/2)\sigma\}] = 1\,,$$

provided such a solution exists. We now assume that R exists. It can be shown that R is uniquely determined. Let $V_t = S_i - t$ for $S_{i-1} \le t < S_i$ be the time remaining till the next change of the intensity. The process L defined as

$$L_t = e^{(\lambda_t(M_Y(R)-1)-cR+\eta^2R^2/2)V_t} e^{-R(X_t-u)}$$

is then a martingale, with respect to the filtration $(\mathcal{F}_t)$ generated by (X_t, λ_t, V_t). Again we use L to change the measure and get that X is a perturbed Björk-Grandell model under the measure Q with negative drift, i.e. $Q[\tau < \infty] = 1$. The parameters are $\tilde{G}(x) = \int_0^x e^{Ry}\, dG(y)/M_Y(R)$, $\tilde{c} = c - \eta^2 R$ and $\tilde{F}(\ell, s) = \int_0^{\ell/M_Y(R)} \int_0^s e^{(\ell(M_Y(R)-1)-cR+\eta^2R^2/2)w}\, F(dl, dw)$.

The ruin function can now be expressed as

$$\psi(u) = E_Q[e^{-(\lambda_\tau(M_Y(R)-1)-cR+\eta^2R^2/2)V_\tau} e^{RX_\tau}]e^{-Ru}$$

and a *Cramér-Lundberg approximation* is possible if the limit of the expected value exists and is non-zero. This problem is solved, via a different technique in [VI]. In [IV] it is shown that for any $\epsilon > 0$

$$\lim_{u\to\infty} \psi(u)e^{(R-\epsilon)u} = 0\,,$$
$$\lim_{u\to\infty} \psi(u)e^{(R+\epsilon)u} = \infty\,,$$

provided that

$$\inf_{v\ge 0} E_P[e^{(L(M_Y(R)-1)-cR+\eta^2R^2/2)(\sigma-v)} \mid \sigma > v] > 0$$

and $E_Q[|X_{\sigma_1}|] < \infty$. In [IV] stronger conditions are used because also finite time Lundberg inequalities are considered.

As a last model we consider the perturbed *Markov modulated risk model.* This model is similar to the Björk-Grandell model, but the process λ is a Markov chain in continuous time and the size of a claim at time t can depend on λ_t. More specifically, let Z be an irreducible Markov chain in continuous time on the state space $\{1, 2, \ldots, \mathcal{J}\}$ with intensity matrix Λ and stationary distribution π. Let $S_0 = 0, S_1, S_2, \ldots$ denote the times where Z changes. Let $(\lambda_i : i \le \mathcal{J})$ be non-negative numbers and $(G_i(x) : i \le \mathcal{J})$ be distribution functions. The claim number process is then defined by

$$N_t = \tilde{N}\Big(\int_0^t \lambda_{Z_s}\, ds\Big)$$

where $\tilde{N}$ is a Poisson process with rate 1. Let (T_i) denote the claim times. Let $(\tilde{Y}_i)$ be a sequence of iid uniformly on $(0, 1)$ distributed random variables independent of Z and N. Then the claim sizes are $Y_i = G^{\leftarrow}_{Z_{T_i}}(\tilde{Y}_i)$ where $G^{\leftarrow}_j(y) = \inf\{x : G_j(x) \ge y\}$ is the *generalized inversion* of the distribution function G_j. We assume the net profit condition $c > \sum_{j=1}^{\mathcal{J}} \pi_j \lambda_j \int_0^\infty y\, dG_j(y)$. The process C is then defined by (1.1).

We denote by $\boldsymbol{S}(r)$ the diagonal matrix with $S_{ii} = \lambda_i(M_i(r) - 1) - cr + \eta^2 r/2$ where $M_i(r) = \int_0^\infty e^{ry}\, dG_i(r)$ is the moment generating function of the claim sizes in state i and let $\boldsymbol{L}(r) = \Lambda + \boldsymbol{S}(r)$. Letting $\theta(r)$ be the logarithm of the spectral radius of $\exp\{\boldsymbol{L}(r)\}$ it can be shown that $\theta(r)$ is a convex function and that $\theta'(0) < 0$ under the net profit condition. Because $\boldsymbol{L}(0) = \Lambda$ we have $\theta(0) = 0$. A possible second solution R to $\theta(R) = 0$ must therefore be positive. Let us assume that such a solution exists. We call R the *adjustment coefficient*. By the Frobenius theorem one is an eigenvalue of $\exp\{\boldsymbol{L}(R)\}$, it is the only eigenvalue with absolute value one and the corresponding eigenvector (C_i) is the only eigenvector with strictly positive entries. We normalize (C_i) such that $\sum_{j=1}^{\mathcal{J}} \pi_j C_j = 1$. Then the process L defined via

$$L_t = C_{Z_t} e^{-R(X_t-u)}/E[C_{Z_0}]$$

is a martingale with mean one. As before, the martingale is used to change the measure. Under the new measure Q the process X is a perturbed Markov modulated risk model with negative drift, i.e. $Q[\tau < \infty] = 1$. The parameters are $\tilde{\Lambda}_{ij} = C_i^{-1} C_j \Lambda_{ij}$, $i \neq j$, $\tilde{\lambda}_i = \lambda_i M_i(R)$, $\tilde{G}_i(x) = \int_0^x e^{Ry}\, dG_i(y)/M_i(R)$ and $\tilde{c} = c - \eta^2 R$. Then we find the expressions

$$P[\tau < \infty, X_\tau = 0] = E_P[C_{Z_0}] E_Q[1/C_{Z_\tau}] e^{-Ru}\,,$$
$$P[\tau < \infty, X_\tau < 0] = E_P[C_{Z_0}] E_Q[e^{RX_\tau}/C_{Z_\tau}] e^{-Ru}\,.$$

These are the *Cramér-Lundberg approximations*. Because there are only a finite number of states it is possible to show that there are constants $C^{(c)}$ and $C^{(d)}$ such that

$$\lim_{u\to\infty} P[\tau < \infty, X_\tau = 0] e^{Ru} = E_P[C_{Z_0}] C^{(c)}$$
$$\lim_{u\to\infty} P[\tau < \infty, X_\tau < 0] e^{Ru} = E_P[C_{Z_0}] C^{(d)}\,.$$

Here the modified ladder-heights are defined as in the perturbed Cramér-Lundberg model but with the additional condition that Z has to be in a fixed state i at a ladder time. The state i is chosen such that $\lambda_i > 0$. In order that the constants $C^{(c)}$ and $C^{(d)}$ are different from zero one needs that $E_Q[Y_j] < \infty$ for all j such that $\lambda_j \neq 0$.

4.2. The distribution of the ladder-heights

We here assume for simplicity that $u = 0$. Let τ_+, L_c and L_d be as in Section 4.1 and $Z_+ = L_c + X_{(\tau_+)-}$ if $\tau_+ < \infty$. The claim leading to a new ladder-height has then size $U_+ = L_d + Z_+$. We here want to study the distributions of the above quantities in a quite general model. Note that in [V] the process $(-X_t)$ is considered, so that the formulae look a little bit different here.

For simplicity let us assume that $c = 1$. We consider a quite general model. Assume $\mathcal{M} = ((T_i, Y_i, M_i) : i \in \mathbb{Z})$ is an ergodic *stationary marked point process (smpp)* with event times $\cdots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \cdots$ and marks $(Y_i, M_i) \in$

$(0,\infty)\times E$. Here (Y_i) are the claim sizes and E is a Polish space with Borel σ-algebra $\mathcal{E}$. The M_i are interpreted as environmental marks. The claim number process is

$$N_t = \begin{cases} \sum_{i=1}^{\infty} \mathbb{I}_{0<T_i\le t}, & \text{if } t>0, \\ -\sum_{i=-\infty}^{0} \mathbb{I}_{t<T_i\le 0}, & \text{if } t\le 0. \end{cases}$$

The environmental mark at the ladder time is $M_+ = M_{N_{\tau_+}}$ if $\tau_+ < \infty$. We let $\lambda = E[N_1]$ be the intensity of the smpp and $\mu = \lambda^{-1}E[\sum_{i=1}^{N_1} Y_i]$ be the mean of a typical claim. We assume the *net profit condition* $\rho = \lambda\mu \le c = 1$, which implies that $\overline{\lim}_{t\to\infty} X_t = \infty$.

We also allow for a more general perturbation process B than Brownian motion. Here B is a Lévy process with no upward jumps, $E[|B_1|] < \infty$ and $E[B_1] = 0$. The main result of [V] are the following formulae. Let H be the distribution function of $-\inf_{t>0}(t+\eta B_t)$. Let P^0 denote the Palm probability measure, see for instance [41]. Intuitively, this is the conditional measure given $T_0 = 0$. Let $Y = Y_0$ and $M = M_0$. Then

$$\begin{aligned} &P[\tau_+ < \infty, M_+ \in F, L_c > \ell_c, L_d > \ell_d, Z_+ > z] \\ &\quad = \lambda(1-H(\ell_c)) \int_{\ell_d+z}^{\infty} P^0[Y > x, M \in F]\, dx \end{aligned} \tag{4.2}$$

for every $\ell_d, z \ge 0$ and $F \in \mathcal{E}$. Note that the formula also holds in the unperturbed case $\eta = 0$ where $P[L_c = 0] = 1$. For the event $\tau_+ = \infty$ we obtain

$$P[\tau_+ = \infty, L_c > \ell_c] = (1-\rho)(1-H(\ell_c))\,. \tag{4.3}$$

Note that (4.2) implies $P[\tau_+ < \infty] = \rho$.

The proof of (4.2) is based on Campbell's formula. For any measurable non-negative measurable functional ϕ we have

$$E\big[\sum_{k=-\infty}^{\infty} \phi(\mathcal{M},\sigma_k)\big] = \lambda E^0\big[\int_{-\infty}^{\infty} \phi(\mathcal{M}\circ\Theta_{-t}, t)\, dt\big]\,,$$

where Θ_{-t} is the *shift operator*, i.e. $-t$ is the new origin. If we choose the functional

$$\phi(\mathcal{M}, s) = P[M_+ \in F, L_c > \ell_c, L_d > \ell_d, Z_+ > z, \tau_+ = s \mid \mathcal{M}]$$

then

$$P[M_+ \in F, L_c > \ell_c, L_d > \ell_d, Z_+ > z, \tau_+ < \infty] = E\big[\sum_{k=-\infty}^{\infty} \phi(\mathcal{M},\sigma_k)\big]\,.$$

The latter can then by Campbell's formula be written as

$$\lambda \int_0^{\infty} P^0[A_t]\, dt = \lambda E^0\big[\int_0^{\infty} \mathbb{I}_{A_t}\, dt\big]$$

for some events A_t. After reversion of time $\tilde{S}_t = -S_{t-}$ the event A_t can be expressed with the process $(\tilde{S}_t)$. What we are interested in, is the expected Lebesgue measure of the times where the condition A_t is fulfilled. A_t is of the form $A_t = (\hat{A}_t, \tilde{A}_t)$ for some events $\hat{A}_t$ and $\tilde{A}_t$. One of the conditions, $\tilde{A}_t$ say, is not fulfilled after a jump until the time, the process reaches the level again at which it was immediately before the jump. Because all jumps are downwards, the process will reach this level level exactly. Thus it is possible to cut out all the pieces where condition $\tilde{A}_t$ is not fulfilled. Because B has independent and stationary increments, the process left follows the same law as $(t + \eta B_t)$. This procedure has then removed the smpp. The rest of the proof is just to calculate the remaining expression for the perturbation process B.

The proof of (4.3) uses similar ideas. The main idea is to consider the time reversed process $(\tilde{S}_t)$. Then the pieces just after a jump are cut out. We start with the removing procedure in $-\infty$. Then it is observed, that $\tau_+ = \infty$ if and only if the origin is not cut out. The remaining process has then again the same law as $(t + \eta B_t)$.

A special case is if (C_t) is a *Cramér-Lundberg model.* Because a Cramér-Lundberg model is in its stationary state at any time point, one can define ladder-heights $(L_c^{(k)})$ and $(L_d^{(k)})$, which all have the same distribution and are independent. The number K of ladder-heights has then a geometric distribution with parameter ρ. The ruin probability can therefore be expressed as

$$\psi(u) = 1 - (1 - \rho) \sum_{n=0}^{\infty} \rho^n (G_I^{*n} * H^{*(n+1)})(u)$$

where $G_I(x) = \mu^{-1} \int_0^x \overline{G}(y)\, dy$ denotes the integrated tail distribution. This expression was obtained in [17] for the perturbation by Brownian motion and in [23] for perturbation by α-stable Lévy motion. However, in [23] the interpretation in terms of ladder-heights could not be obtained by the methods used in [23].

As an application of the ladder-height distribution we show that, under some technical conditions, ruin is more likely in a perturbed Markov modulated risk model than in the perturbed classical model with the same intensity and the same marginal claim size distribution. The proof is analogous to the proof of the result for the unperturbed model in [3].

5. Cox risk processes

A *Cox risk process* or *doubly stochastic risk process* is constructed in the following way. There is an *intensity* process λ with state space $[0, \infty)$ and an independent Poisson process $\tilde{N}$ with rate 1. The claim number process N is defined as $N_t = \tilde{N}(\int_0^t \lambda_s \, ds)$. The risk process is of the form (1.1), where the claim sizes may depend on λ. Usually, it is assumed that λ is ergodic. If λ is not ergodic it will be difficult to consider Cramér-Lundberg approximations and exponential inequalities will always be determined by the worst case.

Most Cox risk models considered in the literature have a piecewise constant intensity. The first model of this type was considered by *Ammeter* [1]. His model was generalized by *Björk and Grandell* [8], see Section 4.1. Another model of this type is the Markov modulated risk model, see Section 4.1.

In [VI] a result is proved, that for instance is useful for obtaining Cramér-Lundberg approximations in Cox risk models. In [VII] a Cox risk model with a piecewise constant intensity is considered, that contains both the Björk-Grandell model and the Markov modulated risk model as special cases.

5.1. An extension to the renewal theorem

In applied probability one often has to deal with equations of the form

$$Z(u) = \int_0^u Z(u-y)(1-p(u,y)) \, dB(y) + z(u) \tag{5.1}$$

where $B(y)$ is a proper distribution function on $(0, \infty)$, $z(u)$ is a measurable function and the perturbation factor $p(u, y)$ converges to zero as $u \to \infty$. We call (5.1) an *ordinary renewal equation* if $p(u, y) = 0$ for all u, y, a *perturbed renewal equation* otherwise. A situation, where a perturbed renewal equation occurs is when a stochastic process with imbedded *regeneration points* is considered. Recall that at a regeneration point the process is dependent on its past via the present state only, and follows the same law afterwards. If we are interested in a certain event, this event may or may not occur in a regeneration epoch. But we are not able to decide whether the event has occurred or not by considering the process at the regeneration points only. In this case $p(u, y)$ is the probability that the event of interest has occurred in the first regeneration epoch, but cannot be observed from the state at the regeneration point. The function $z(u)$ is then the part of the equation that corresponds to occurrence of the event of interest before the regeneration point. Because (5.1) is quite close to an ordinary renewal equation one would expect, that under appropriate conditions $\lim_{u \to \infty} Z(u)$ exists.

Let us assume that $0 \le p(u, y) \le 1$. Then one can show that there is a unique solution to (5.1) that is bounded on bounded intervals. Uniqueness is proved as for the ordinary renewal theorem. A solution is constructed as the limit of a recursion sequence, where the recursion equation has a single fixed point. Moreover, $z(u) \ge 0$ implies $Z(u) \ge 0$ and the solution is bounded by the solution to the ordinary renewal equation. If $z(u)$ is continuous then $Z(u)$ is cadlag.

Assume now in addition that $p(u, y)$ is continuous in u and that $\int_0^u p(u, y)\, dB(y)$ is *directly Riemann integrable*, see [22] or [41]. If $z(u)$ is directly Riemann integrable then there exist the limits $\lim_{u\to\infty} Z(u)$ if $B(u)$ is not arithmetic and $\lim_{n\to\infty} Z(x + n\gamma)$ if $B(u)$ is arithmetic with span γ. *Arithmetic* with span γ means that all points of increase of the distribution function $B(x)$ are in the set $\{\dots, -2\gamma, -\gamma, 0, \gamma, 2\gamma, \dots\}$ and γ is the largest number with this property. It is an open question how the limiting value of $Z(u)$ can be determined in general.

It is enough to prove the result for $z(u) \ge 0$. First it is proved for a continuous function $z(u)$. This follows readily from rearranging the terms in (5.1)

$$Z(u) = \int_0^u Z(u-y)\, dB(y) + \bigl(z(u) - \int_0^u Z(u-y) p(u, y)\, dB(y)\bigr)$$

by noting that $z(u) - \int_0^u Z(u-y)p(u, y)\, dB(y)$ is directly Riemann integrable under the present assumptions. For arbitrary directly Riemann integrable functions $z(u)$ one only has to approximate $z(u)$ appropriately.

As an application we consider the *Björk-Grandell model*. For the definition and the notation see Section 4.1. Let

$$\phi(\vartheta, r) = E[\exp\{(L(M_Y(r) - 1) - cr - \vartheta)\sigma\}]\,.$$

For $r \in \mathbb{R}$ let $\theta(r)$ be the solution to $\phi(\theta(r), r) = 1$ if such a solution exists. We assume that there is a strictly positive solution R to $\phi(0, R) = 1$ and that there is an $r > R$ and $B > 0$ such that $\phi(0, r) < \infty$ and almost surely

$$E[\exp\{(L(M_Y(r) - 1) - cr - \theta(r))(\sigma - v)\} \mid \sigma > v, L] \ge B\,.$$

Let us denote by (S_i) the times where the intensity changes. In order to get a renewal type equation we have to define regeneration points. The natural choice for the regeneration points are the times (S_i). Let therefore $\tau_1 = \inf\{S_i : X_{S_i} < u\}$ and $B(x) = P[\tau_1 < \infty, u - X_{\tau_1} \le x]$. Then for $p(u, y) = P[\tau \le \tau_1 \mid \tau_1 < \infty, X_0 = u, X_{\tau_1} = u - y]$ we find

$$\psi(u) = \int_0^u \psi(u-y)(1 - p(u, y))\, dB(y) + P[\tau \le \tau_1, \tau < \infty \mid X_0 = u]\,.$$

This is not an equation of type (5.1) because $B(x)$ is not a proper distribution. But $\int_0^\infty e^{Ry}\, dB(y) = 1$, so

$$\psi(u)e^{Ru} = \int_0^u \psi(u-y)e^{R(u-y)}(1-p(u, y))e^{Ry}\, dB(y) + P[\tau \le \tau_1, \tau < \infty \mid X_0 = u]e^{Ru}$$

is a renewal equation of type (5.1). It is now just a technical matter to prove that the conditions on $p(u, y)$ and $z(u, y)$ are fulfilled. Therefore the limit of $\psi(u)e^{Ru}$ exists.

5.2. A cox model with a piecewise constant intensity

The obvious way to generalize the Björk-Grandell model, such that the Markov modulated risk model is contained as a special case is to introduce an environmental process with an infinite state space. For simplicity we let in [VII] the intensity levels (L_i) take over this rôle. The (discrete time) process $((L_i,\sigma_i))$ is defined via

$$P[(L_i,\sigma_i)\in A\times B\mid L_{i-1}=\ell]=\int_A\int_B dF_l(v)f(l;\ell)\,dF^0(l)$$

and

$$P[(L_i,\sigma_i)\in A\times B\mid (L_k,\sigma_k);k<i]=P[(L_i,\sigma_i)\in A\times B\mid L_{i-1}]\,.$$

This means in particular that the conditional distribution of L_i given L_{i-1} is absolutely continuous with respect to F^0. We assume that (L_i) is ergodic, and assume that F^0 is its stationary distribution. Let E_0 be the support of F^0. For $\ell\in E_0$ let $X^{(\ell)}$ be a Cramér-Lundberg model with initial capital 0, premium rate c, intensity ℓ and claim size distribution $G_\ell(x)$. The corresponding claim arrival process is $N^{(\ell)}$. We let $S_i=\sigma_1+\cdots+\sigma_i$ be the times where the intensity changes and $\lambda_t=L_i$, $X_t=X_{(S_{i-1})-}+X_t^{(L_i)}-X_{S_{i-1}}^{(L_i)}$ and $N_t=N_{(S_{i-1})-}+N_t^{(L_i)}-N_{S_{i-1}}^{(L_i)}$ if $S_{i-1}\le t<S_i$. We define $X_{0-}=u$ and $N_{0-}=0$. Then X is a Cox risk model with intensity process λ and claim arrival process N.

Let $M_\ell(r)=\int_0^\infty e^{ry}\,dG_\ell(y)$ be the moment generating function of the claim sizes if the intensity level is ℓ. We define the following quantity

$$M(\vartheta,r)=\operatorname*{ess\,sup}_{\ell\in E_0}\int_{E_0}\int_0^\infty \exp\{(l(M_\ell(r)-1)-\vartheta-cr)v\}\,dF_l(v)f(l;\ell)dF^0(l)\,.$$

Let $\mathcal{L}(E_0)$ be the space of all bounded continuous real functions on E_0. For (ϑ,r) such that $M_\ell(r)<\infty$ (F^0-a.s.) and $M(\vartheta,r)<\infty$ we define the operator $\mathcal{K}(\vartheta,r)$ on $\mathcal{L}(E_0)$ by

$$\mathcal{K}(\vartheta,r)h(\ell)=\int_{E_0}h(l)\int_0^\infty \exp\{(l(M_l(r)-1)-\vartheta-cr)v\}\,dF_l(v)f(l;\ell)dF^0(l)\,.$$

We need the following technical assumptions:

- the family $(f(\ell;\cdot):\ell\in E_0)$ is equicontinuous,
- there is no sequence (ℓ_n) in E_0 such that $f(l;\ell_n)$ converges to 0 for all $l\in E_0$.

These assumptions assure that $\mathcal{K}(\vartheta,r)$ is a continuous compact linear operator. Jentzsch's theorem, see [42, p.337], implies that the spectral radius $\operatorname{spr}(\mathcal{K}(\vartheta,r))$ of $\mathcal{K}(\vartheta,r)$ is an eigenvalue and that there is a unique eigenfunction $\tilde h(\ell;\vartheta,r)$, which can be chosen such that $\tilde h(\ell;\vartheta,r)>0$, F^0-a.s. Moreover, $\tilde h(\cdot;\vartheta,r)$ is bounded away from zero on compact subsets of E_0 by continuity.

We denote by $V(t)$ the time remaining at time t till the next change of the intensity, i.e. $V(t)=S_i-t$ if $S_{i-1}\le t<S_i$. We try to find a martingale $f(X_t,\lambda_t,V(t),t)$

of the form $f(x,\ell,v,t) = g(\ell,v)\exp\{-\theta t - rx\}$. Markov process theory gives then that

$$-\theta g(\ell,v) - crg(\ell,v) - \frac{\partial}{\partial v} g(\ell,v) + \ell(M_\ell(r)-1)g(\ell,v) = 0\,.$$

The above equation has the solution

$$g(\ell,v) = h(\ell)\exp\{-(\theta + cr - \ell(M_\ell(r)-1))v\}$$

where $h(\ell)$ is an arbitrary function. Moreover, because of the previsible jumps at S_i, the boundary condition $h(\ell) = \mathcal{K}(\theta,r)h(\ell)$ has to be fulfilled. This means, $h(\ell)$ is an eigenfunction and 1 is an eigenvalue. For our purposes we need a positive function. This means that θ has to be chosen in such a way that $\operatorname{spr}(\mathcal{K}(\theta,r)) = 1$. One can show that there is at most one $\theta(r)$ such that $\operatorname{spr}(\mathcal{K}(\theta(r),r)) = 1$. If we write $h(\ell;r)$ for the corresponding eigenfunction then

$$h(\lambda_t;r)\exp\{(\lambda_t(M_{\lambda_t}(r)-1) - \theta(r) - cr)V_t - \theta(r)t - rX_t\} \tag{5.2}$$

is a martingale, provided its initial value is integrable. We normalize the function $h(\ell;r)$ such that $\int_{E_0} h(\ell;r)\,dF^0(\ell) = 1$.

The function $\theta(r)$ is convex and $\theta(0) = 0$. If the *net profit condition*

$$\int_{E_0} (c - \ell\mu_\ell) \int_0^\infty v\,dF_\ell(v)\,dF^0(\ell) > 0$$

is fulfilled, where $\mu_\ell = \int_0^\infty y\,dG_\ell(y)$, we also have $\theta'(0) < 0$. Thus there might be a strictly positive solution R to $\theta(R) = 0$. Let us assume that R, called the *adjustment coefficient*, exists. Because for proving *Lundberg's inequality* in the Björk-Grandell model there is an additional condition needed we also need an additional condition here. For instance, this condition can be formulated in the following way. Assume there exists $B > 0$ such that

$$\inf_{v\geq 0} E[\exp\{(\ell(M_\ell(R)-1) - cr)(\sigma_1 - v)\} \mid \sigma_1 > v, L_1 = \ell] \geq B$$

for F^0 almost all ℓ. Then there exits a constant $\overline{C}$ such that

$$\psi(u) \leq \overline{C}e^{-Ru}\,.$$

Under a similar condition also a lower Lundberg bound

$$\psi(u) \geq \underline{C}e^{-Ru}$$

can be obtained.

As a last topic we consider *Cramér-Lundberg approximations*. In order to apply the results of [VI] one needs regeneration points, (τ_i) say. Such times can for instance be obtained

- if there exists ℓ_0 such that $F^0(\ell_0) - F^0(\ell_0-) > 0$. Then the times S_i with $L_i = \ell_0$ are regeneration points. If, moreover, $\ell_0 > 0$ and the corresponding σ is exponentially distributed, then $\tau_i = \inf\{t > \tau_{i-1} : \lambda_t = \ell_0, X_t < \inf_{s<t} X_s\}$ can be chosen. In the latter case an ordinary renewal approach may lead to the Cramér-Lundberg approximation.

- if there exists a petite set for the Markov process (λ_t, V_t). For the definition of *petite sets* see for instance [39].

In these cases one can verify the conditions given in [VI]. The function $0 \le p(u, x) \le 1$ will automatically be continuous in u. A condition like $\theta(r)$ exists for some $r > R$ usually yields the direct Riemann integrability conditions.

Unfortunately, there is an error in [VII]. The approach used to prove Theorem 4 does not work. However, the result holds. A similar proof as in [VI] in the case of a Björk-Grandell model applies.

6. Estimation of the adjustment coefficient

Let us consider a process (X_t) of the form (1.1) with some arbitrary claim number process (N_t) and claims sizes (Y_i). We assume that there exist constants $C, R > 0$ such that $\psi(u)e^{Ru} \to C$ as $u \to \infty$. We consider here the problem of estimating R. The adjustment coefficient R can be seen as a measure of risk. Waters [61] maximizes R considered as a function of the retention level in order to optimize reinsurance treaties. In fact, many decision problems have to be decided at the beginning for the present surplus. Short time later the surplus has changed and the decision may be different. Maximizing the adjustment coefficient can therefore be seen as finding the asymptotically best decision.

Often, the calculation of R depends strongly on the choice of the model and the distributions chosen. Thus choosing a model, estimating the distributions and then calculating the adjustment coefficient may lead to an error. Therefore, procedures for estimating the coefficient directly from data are called for.

The case of Cox risk model with an ergodic intensity process (λ_t) is of particular interest. For simplicity assume that (λ_t) is a Markov process. Such a model can be approximated by Cox models with a piecewise constant intensity, as considered in [VII]. If (λ_t) is not a Markov process, but can be Markovized, then we should use an environment process as underlying Markov process, see also the remark in Section 6 of [VII]. If R exists, one may hope that the adjustment coefficients of better and better approximations converge to R. We therefore consider models of the type considered in [VII].

We make the slightly stronger assumption that the martingale (5.2) with $r = R$ exists and that for each initial value of (λ_0, V_0) the Cramér-Lundberg approximation with a non-zero constant holds. Let us consider the following cycles. Define $W_0 = 0$ and the stopping-times

$$w_{k+1} = \inf\{t > W_k : N_t > N_{W_k}\}, \qquad W_{k+1} = \inf\{t > w_{k+1} : X_t = X_{(w_{k+1})-}\}.$$

The times (w_k) are just defined in order to start a cycle at a claim arrival time. The end of the cycle W_k is the first time the process reaches the level again the process was at just before the jump. The quantity of interest is then

$$Z_k = \sup\{X_{w_k} - X_t : w_k \le t \le W_k\}\,.$$

This procedure is similar to the one considered in [11]. It is then shown that

$$\lim_{x\to\infty} P[Z_i > x \mid \Lambda_i = \ell, U_i = v]e^{Rx} = B(\ell, v) \tag{6.1}$$

for some constants $B(\ell, v) \in (0, \infty)$ where $\Lambda_k = \lambda_{w_k}$ and $U_k = V_{w_k}$. Thus the problem looks similar to the problem of estimating the coefficient of regular variation. Indeed, it would be the same problem if we considered the variables $\exp\{Z_k\}$ instead. The problem is extensively studied in the case where (Z_k) is an iid sequence. This is not the case in our situation. But intuitively, if Z_k is large, then $W_k - w_k$ will be large. The next excursion Z_i that will be large will not very strongly depend on Z_k. To prove such a statement seems, however, to be hard. We anyway suggest the following *Hill type estimator* for R

$$\hat{R} = \Big(\frac{1}{k(n)} \sum_{j=1}^{k(n)} Z_{j:n} - Z_{k(n)+1:n}\Big)^{-1}$$

where $Z_{1:n} \geq Z_{2:n} \geq \cdots \geq Z_{n:n}$ is the order statistics of $\{Z_1, \ldots, Z_n\}$ and $k(n)$ is a sequence such that $\log\log n = o(k(n))$ and $k(n) = o(n)$. It is conjectured, that $\hat{R}$ is a consistent estimator for R.

Consider now the special case of a *Markov modulated risk model* described in Section 4.1. This is a special case of the model considered in Section 5.2. Here the intensity levels only take values in a finite set $\{\ell_1, \ell_2, \ldots, \ell_J\}$ and the conditional distributions of the length of the interval in which the intensity is constant given the intensity level is ℓ_j is exponentially distributed with parameter η_j. This has first the consequence, that (6.1) can be sharpened to

$$\lim_{x\to\infty} P[Z_i > x \mid \Lambda_i = \ell_j] e^{Rx} = C_j$$

where

$$C_j = \int_0^\infty B(\ell_j, v)\eta_j e^{-\eta_j v}\, dv\,.$$

Because there is only a finite number of limits we obtain uniform convergence. For a large threshold, x_0 say, $P[Z_i - x_0 > x \mid Z_k > x_0, \Lambda_i = \ell_j]e^{Rx} \approx 1$. The strong consistency follows now from comparison with exponentially distributed random variables.

7. Compound sums and subexponentiality

Recall that a positive distribution function G is called subexponential if (1.7) is fulfilled. Working with processes (X_t) where $X_{\tau_1} - X_{\tau_2}$ has a subexponential tail and $\tau_1 < \tau_2$ are stopping-times, the problem may appear whether

$$S_N = \sum_{i=1}^{N} Y_i$$

is subexponential or not, see for instance [6]. Here N is a positive integer valued random variable and the (Y_i) are iid independent of N. Let us denote the class of subexponential distributions by $\mathcal{S}$. Sometimes, one needs not only $G \in \mathcal{S}$ but also $G_I \in \mathcal{S}$ where $G_I(x) = (\int_0^\infty \overline{G}(y)\,dy)^{-1} \int_0^x \overline{G}(y)\,dy$ provided G has a finite mean. Klüppelberg [37] introduced the class $\mathcal{S}^*$ of distribution functions G with finite mean μ_G such that

$$\lim_{x\to\infty} \int_0^x \frac{\overline{G}(x-y)}{\overline{G}(x)} \overline{G}(y)\,dy = 2 \int_0^\infty \overline{G}(y)\,dy\,.$$

For $G \in \mathcal{S}^*$ one can show that both $G \in \mathcal{S}$ and $G_I \in \mathcal{S}$. Let now G be the distribution function of Y and F be the distribution function of S_N. We assume that $Y > 0$, i.e. $G(0) = 0$. The distribution of N is denoted by $P[N = n] = p_n$. A special case of is the *mixed Poisson* case where

$$p_n = \int_0^\infty \frac{\ell^n}{n!} e^{-\ell}\,dH(\ell)$$

for some *mixing distribution* function H with $H(x) = 0$ for all $x < 0$.

Let $\mathcal{R} \subset \mathcal{S}^*$ denote the subclass of distribution functions with a *regularly varying* tail, i.e. $\overline{G}(x) = x^{-\alpha} L(x)$ where $L(x)$ is *slowly varying*, that is $L(tx)/L(x) \to 1$ as $x \to \infty$ for all $t > 0$. The following was proved in [59]. Let $L(x)$ be a slowly varying function. Assume

$$\lim_{x\to\infty} L(x) x^\alpha \overline{G}(x) = \beta\,, \qquad \lim_{n\to\infty} L(n) n^\alpha P[N > n] = \gamma\,,$$

for some $\beta, \gamma \in [0, \infty)$. If $E[Y], E[N] < \infty$ (this implies $\alpha \ge 1$), or if $0 \le \alpha < 1$ and $E[N] < \infty$ (this implies $\gamma = 0$), or if $0 \le \alpha < 1$ and $E[Y] < \infty$ (this implies $\beta = 0$), then

$$\lim_{x\to\infty} L(x) x^\alpha \overline{F}(x) = \gamma E[Y]^\alpha + \beta E[N]\,.$$

If the tail of the distribution of N is thicker than the tail of the distribution of Y we have in the case $N \in \mathcal{R}$

$$P[S_N > x] \sim \gamma (x/E[Y])^{-\alpha}/L(x) \sim \gamma (x/E[Y])^{-\alpha}/L(x/E[Y]) \sim P[N > x/E[Y]]\,.$$

This result tells us that S_N only can become large if N becomes large, and that, conditioned on $S_N > x$, the conditional mean of Y_i is asymptotically $E[Y]$. Indeed, for a large N the strong law of large number implies $S_N/N \approx E[Y_i \mid S_N > x]$ given

$S_N > x$. In [31] (Proposition 8.4 and Corollary 8.5) it is shown that, for $\alpha \neq 1$, $L(n)n^\alpha P[N > n] \to \gamma$ as $n \to \infty$ holds if N is mixed Poisson distributed with a mixing distribution H satisfying $\overline{H}(\ell)L(\ell)\ell^\alpha \to \gamma$ as $\ell \to \infty$, i.e. $P[N > n] \sim \overline{H}(n)$. Some related results can also be found in [40].

It seems natural to expect $P[S_N > x] \sim P[N > x/E[Y]]$ also in the case $N \in \mathcal{S}$ or $P[N > n] \sim \overline{H}(n)$ also in the case $H \in \mathcal{S}$. But intuition fails, as it often happens for subexponential distributions. A counterexample is given in [4], see also [IX]. However, it is possible to give conditions under which $F \in \mathcal{S}$ or $F \in \mathcal{S}^*$. But the explicit behaviour of the tail of F is not obtained.

We denote by Γ the class of distributions G with the property that either $G(x_0) = 1$ for some $x_0 \in (0, \infty)$ or

$$\varliminf_{x\to\infty} \frac{\overline{G^{*(m+1)}}(x)}{\overline{G^{*m}}(x)} \geq a$$

for some $a > 1$ and all $m \in \mathbb{N}$. All light-tailed distribution functions of practical interest belong to Γ. Note that $\mathcal{S} \cap \Gamma = \emptyset$.

In [IX] conditions are found to assure that $F \in \mathcal{S}$ or $F \in \mathcal{S}^*$. The following conditions imply that $F \in \mathcal{S}$.

- If $G \in \mathcal{S}$ and $E[(1+\epsilon)^N] < \infty$, for some $\epsilon > 0$. In this case $\overline{F}(x) \sim E[N]\overline{G}(x)$.
- If $G \in \Gamma$ and $N \in \mathcal{S}$.
- If N is mixed Poisson distributed with mixing distribution $H \in \mathcal{S}$ then $N \in \mathcal{S}$. If in addition $G \in \Gamma$ then $F \in \mathcal{S}$.

If we consider the class S^* then the following conditions imply $F \in \mathcal{S}^*$.

- If $G \in \mathcal{S}^*$ and $E[(1+\epsilon)^N] < \infty$, for some $\epsilon > 0$.
- If $G \in \Gamma$, $E[Y - x \mid Y > x] \leq B < \infty$ for all x such that $P[Y > x] > 0$, and $N \in \mathcal{S}^*$.
- If N is mixed Poisson distributed with mixing distribution $H \in \mathcal{S}^*$ then $N \in \mathcal{S}^*$. If in addition $G \in \Gamma$ and $E[Y - x \mid Y > x] \leq B < \infty$ for all x such that $P[Y > x] > 0$, then $F \in \mathcal{S}^*$.

The proof of the case $G \in \mathcal{S}$ is well-known and the case $G \in \mathcal{S}^*$ follows readily.

Assume that $N \in \mathcal{S}$. The proof in this case is based on the representation

$$P[S_N > x] = \sum_{n=0}^{\infty} P[N > n] \left(G^{*n}(x) - G^{*(n+1)}(x)\right) .$$

The quantity to consider is therefore

$$\frac{P[\sum_{i=1}^{N_1+N_2} Y_i > x]}{P[\sum_{i=1}^{N} Y_i > x]} = \frac{\sum_{n=0}^{\infty} P[N_1 + N_2 > n] \left(G^{*n}(x) - G^{*(n+1)}(x)\right)}{\sum_{n=0}^{\infty} P[N > n] \left(G^{*n}(x) - G^{*(n+1)}(x)\right)}$$

where N_1, N_2 are two independent copies of N. In a first step one shows that for each fixed $M \in \mathbb{N}$ the limit of

$$\frac{\sum_{n=0}^{M} P[N_1 + N_2 > n] \left(G^{*n}(x) - G^{*(n+1)}(x)\right)}{\sum_{n=0}^{\infty} P[N > n] \left(G^{*n}(x) - G^{*(n+1)}(x)\right)}$$

as $x \to \infty$ is zero. Given $\epsilon > 0$ the estimate $P[N_1 + N_2 > n] < (2+\epsilon)P[N > n]$ holds for n large enough. This leads to $\overline{\lim}_{x\to\infty} \overline{F^{*2}}(x)/\overline{F}(x) \leq 2 + \epsilon$. Because ϵ is arbitrary one has $F \in \mathcal{S}$. The proof in the case $N \in \mathcal{S}^*$ is similar.

The proof in the mixed Poisson case with $H \in \mathcal{S}$ is based on the representation

$$P[N > n] = \int_0^\infty \frac{x^n}{n!} e^{-x} \overline{H}(x)\, dx\,.$$

In order to show that $N \in \mathcal{S}$ one observes that

$$\frac{P[N_1 + N_2 > n]}{P[N_1 > n]} = \frac{\int_0^\infty (x^n/n!) e^{-x} \overline{H^{*2}}(x)\, dx}{\int_0^\infty (x^n/n!) e^{-x} \overline{H}(x)\, dx}\,.$$

One first shows that for any fixed ℓ_0 the limit of

$$\frac{\int_0^{\ell_0} (x^n/n!) e^{-x} \overline{H^{*2}}(x)\, dx}{\int_0^\infty (x^n/n!) e^{-x} \overline{H}(x)\, dx}$$

as $x \to \infty$ is zero. The estimate $\overline{H^{*2}}(x) < (2+\epsilon)\overline{H}(x)$ for any $\epsilon > 0$ and x large enough yields then the result. For $H \in \mathcal{S}^*$ the proof is similar.

The result is applied to a *Björk-Grandell model* with subexponential intensity level distribution and light-tailed claim size distribution. Then the asymptotic behaviour of the ruin probabilities can be expressed in terms of the aggregate claims in an interval with constant intensity.

8. Optimal reinsurance

For an insurance company it is important to reinsure the claims, see for instance [56]. A very popular reinsurance form is *proportional reinsurance*. For this form, the insurer pays the proportion b of each claim, the reinsurer pays the proportion $1-b$. This is the most natural form of reinsurance. The idea of insurance is that a number of people share their risks. The strong law of large numbers tells us, if an insurance company has a lot of customers, then the aggregate claim amount per customer is (almost) deterministic. In this sense, an insurance contract is something like a reinsurance contract. The insurance company takes over the claims of a single customer, but the customer pays a small part of the claims in the portfolio. With proportional reinsurance, a larger number of customers participate in this game.

An insurance company has the possibility to choose between several retention levels b offered by a reinsurance company. One therefore would like to choose the optimal level. Often, a company will be interested to maximize the profit. Højgaard and Taksar [33], [34] maximized the "expected future surplus" in the sense that

$$E\Big[\int_0^\tau X_s e^{-\delta s}\,ds\Big]$$

became maximal. δ was a strictly positive discounting factor. Because the problem is difficult to solve for a classical risk process they considered a diffusion approximation to a risk model, see Section 3.3.

From a theoretical point of view it seems more natural to minimize the ruin probability. Waters [61] minimized the ruin probability for large values of the initial capital u in cases where the adjustment coefficient exists. He considered a general model where $(X_k - X_{k-1} : k \in \mathbb{N})$ was iid distributed. Here k must not necessarily denote time. He assumed that for each retention level b in a certain set there are strictly positive constants $R(b)$, $\underline{C}(b)$ and $\overline{C}(b)$ such that

$$\underline{C}(b)e^{-R(b)u} \le \psi_b(u) \le \overline{C}(b)e^{-R(b)u}$$

where $\psi_b(u)$ is the ruin probability under reinsurance with retention level b. The he showed that there is always a unique b_0 maximizing $R(b)$. This means that for each $b \ne b_0$ we have

$$\lim_{u\to\infty} \frac{\psi_{b_0}(u)}{\psi_b(u)} = 0\,.$$

Two questions arise in this context: What to do if the adjustment coefficient does not exist — as in the case of large claims — and what happens if the insurance company can change the retention level periodically?

Instead of minimizing the ruin probability we can maximize the survival probability $\delta_b(u) = 1-\psi_b(u)$. We allow in this work any reinsurance strategy (b_t), i.e. any previsible process. The corresponding surplus process is denoted by (X_t^b) and the survival probability by $\delta_b(u)$. Our aim is to find

$$\delta(u) = \sup_{(b_t)} \delta_b(u)$$

and, if it exists, an optimal reinsurance strategy.

We assume as in [61] that insurer and reinsurer use expected value principles with safety loadings η and θ, respectively. That is the premium income of the insurer is $1+\eta$ times the expected outflow and the reinsurance premium is $1+\theta$ times the expected outflow of the reinsurer. In order not to have an arbitrage possibility we have to assume $\theta \geq \eta$. Otherwise, the insurer would choose $b_t = 0$ and would have a profit without any risk. In order not to get the trivial solution $\psi_0(u) = 0$ we have to assume $\theta > \eta$.

8.1. The diffusion case

We first consider the case of a *diffusion approximation*. Then we consider η to be the drift of the surplus process without reinsurance ($b_t = 1$) and θ to be the drift the surplus of the reinsurer with maximal reinsurance ($b_t = 0$). If a reinsurance strategy (b_t) is chosen, the corresponding surplus of the insurer becomes

$$X_t^b = u + \int_0^t (b_s\theta - (\theta - \eta))\, ds + \sigma \int_0^t b_s\, dW_s$$

where $\sigma > 0$ denotes the diffusion coefficient in the approximation for the process without reinsurance. The Hamilton-Jacobi-Bellman equation corresponding to this problem is

$$\sup_{b\in[0,1]} (b\theta - (\theta - \eta))\delta'(u) + \frac{\sigma^2 b^2}{2}\delta''(u) = 0\,. \tag{8.1}$$

The solution to the above equation is $\delta(u) = 1 - e^{-\kappa u}$ where

$$\kappa = \begin{cases} \frac{\theta^2}{2\sigma^2(\theta-\eta)}, & \text{if } \eta < \theta < 2\eta, \\ \frac{2\eta}{\sigma^2}, & \text{if } \theta \geq 2\eta. \end{cases}$$

This suggests that the optimal strategy is constant over time $b_t^* = 2(1 - \eta/\theta) \wedge 1$. Using Itô's formula and the fact that the suggested $\delta(u)$ solves (8.1) one can show that indeed $\delta(u)$ solves our problem and (b_t^*) is an optimal strategy.

8.2. The Cramér-Lundberg case

In order to avoid technical difficulties we assume in this section that the claim size distribution $G(x)$ is continuous.

Let (T_i) be the occurrence times of the claims. Then, using the reinsurance strategy (b_t) in a classical risk model, the surplus process becomes

$$X_t^b = u + \int_0^t (b_s(1+\theta) - (\theta - \eta))\lambda\mu\, ds - \sum_{i=1}^{N_t} b_{T_i} Y_i\,.$$

In order that ruin does not occur almost surely we need that the income process is strictly increasing. Otherwise, the process (X_t^b) will have a bounded state space and therefore there will almost surely be a sequence of claims leading to ruin. Thus we can restrict to strategies such that $b_t \in (\underline{b}, 1]$ for $\underline{b} = (\theta - \eta)/(1 + \theta)$. The corresponding Hamilton-Jacobi-Bellman equation is

$$\sup_{b \in (\underline{b}, 1]} (b(1+\theta) - (\theta - \eta))\mu\delta'(u) + \int_0^{u/b} \delta(u - by)\, dG(y) - \delta(u) = 0\,. \qquad (8.2)$$

The solution of the above equation is hard. Let us reformulate the problem. If the function $\delta(u)$ we are looking for is indeed a solution to (8.2) then the optimal strategy will be of the form $(b(X_t^b))$, where $b(u)$ is the argument maximizing the left-hand side of (8.2). Because $\delta(u)$ is strictly increasing it follows that $b(u) \neq \underline{b}$. We can reformulate (8.2) to

$$\delta'(u) = \frac{\delta(u) - \int_0^{u/b(u)} \delta(u - b(u)y)\, dG(y)}{(b(u)(1+\theta) - (\theta - \eta))\mu}\,.$$

Because $\delta_b(u) = 1 - \int_u^\infty \delta_b'(x)\, dy$, to maximize $\delta_b(u)$ is the same as to minimize $\delta_b'(u)$. We therefore conjecture that $\delta(u)$ satisfies

$$\delta'(u) = \inf_{b \in (\underline{b}, 1]} \frac{\delta(u) - \int_0^{u/b} \delta(u - by)\, dG(y)}{(b(1+\theta) - (\theta - \eta))\mu}\,. \qquad (8.3)$$

The above equation only determines a solution up to a multiplicative constant. The solution looked for is determined by the boundary condition $\delta(\infty) = \lim_{u\to\infty} \delta(u) = 1$. In order to solve (8.3) we can therefore fix an initial condition $f(0)$, for example $f(0) = \delta_1(0) = \eta/(1 + \eta)$.

We discuss two examples. For exponentially distributed claims the optimal strategy seems to have the form $b(x) = 1\!\!1_{x<m} + b_R 1\!\!1_{x\geq m}$. Here b_R is the value of b that maximizes the adjustment coefficient corresponding to the risk process with constant reinsurance strategy $b(x) = b$. To prove that the optimal strategy really has this form is however not trivial. The ruin probability can be reduced considerably. A comparison with the strategy $b(x) = b_R$ shows that the optimal strategy reduces the ruin probability for small capital x, whereas for large capital x, $b(x) = b^*(x)$ and $b(x) = b_R$ yield almost the same ruin probability.

For large claims the situation is completely different. We consider Pareto distributed claim sizes. Here the optimal strategy $b^*(x)$ seems to be continuous and seems to approach a limiting value only very slowly. Also in this situation the ruin probabilities of the optimal strategy and of the case with no reinsurance differ considerably. If we choose the strategy $b(x) = b_a$ where b_a is the value of b that minimizes the ruin probability for very large x, $b(x) = b_a$ leads for not too large x to a ruin probability that is even larger than for the case of no reinsurance. So for surpluses of interest, the asymptotically optimal value of b is far from being an optimal choice.

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